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High-precision four-loop mass and wave function renormalization in QED

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ABSTRACT

The 4-loop QED mass and wave function renormalization constants Z_2 and Z_m have been evaluated in the on-shell subtraction scheme with 1100 digits of precision. We also worked out the coefficients of the five color structures of the QCD renormalization constants Z_2^{OS} and Z_m^{OS} which can be obtained from QED-like diagrams. The results agree with lower precision results available in the literature. Analytical fits were also obtained for all these quantities.

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1. Introduction

The 4-loop QED contribution to the electron g-2 was calculated with 1100 digits of precision in Ref. [1]. In that paper high-precision numerical and analytical fits were also obtained for all the master integrals of the 4-loop self-mass QED diagrams. Therefore, other 4-loop quantities that can be expressed in terms of the same master integrals can be known with such precision. In Ref. [2] we used these results to obtain a high-precision value and an analytical fit of the 4-loop first derivative of the Dirac form factor $F_1'(0)$.

In this third paper, following the same approach as in Ref. [1,2], we calculate high-precision numerical values and analytical expression of the 4-loop QED wave function and mass renormalization constants Z_2 and Z_m, in the on-shell subtraction scheme. We infer also the values of the coefficients of five color structures of the QCD renormalization constants Z_2^{OS} and Z_m^{OS} which can be obtained from the QED result.

2. Definitions

The wave function renormalization constant Z_2 is defined as $\psi_0 = \sqrt{Z_2}\psi$, where ψ_0 and ψ are the bare and the renormalized electron field, respectively. The mass renormalization constant Z_m is defined as $m_0 = Z_m m$, where m_0 and m are the bare and the physical electron mass, respectively. Two-loop QED and QCD analytical results for on-shell Z_2 and Z_m were obtained in Ref. [3]; three-loop analytical results were obtained in Refs. [4,5].

In QCD $m_0 = Z_m^{\mathrm{OS}} m^{\mathrm{OS}} = Z_m^{\overline{\mathrm{MS}}} m^{\overline{\mathrm{MS}}}$. Being $Z_m^{\overline{\mathrm{MS}}}$ known to sufficiently high degree of perturbative expansion [6–8], the ratio $m^{\mathrm{OS}}/m^{\overline{\mathrm{MS}}}$ can be used to determine the on-shell Z_m^{OS}; the ratio $m^{\mathrm{OS}}/m^{\overline{\mathrm{MS}}}$ is known at two loops analytically [9], and at three loops it was obtained in numerical form in Ref. [10,11] and in analytical form in Ref. [5,12].

Four-loop QED and QCD numerical results of Z_2 were obtained in Ref. [13–15]; the four-loop numerical result of Z_m can be worked out from the QCD results for z_m and Z_m^{OS} of Ref. [13,16,17]; see also Ref. [18–20]. Both results have a precision of a few digits; see section 3.2 for more details.

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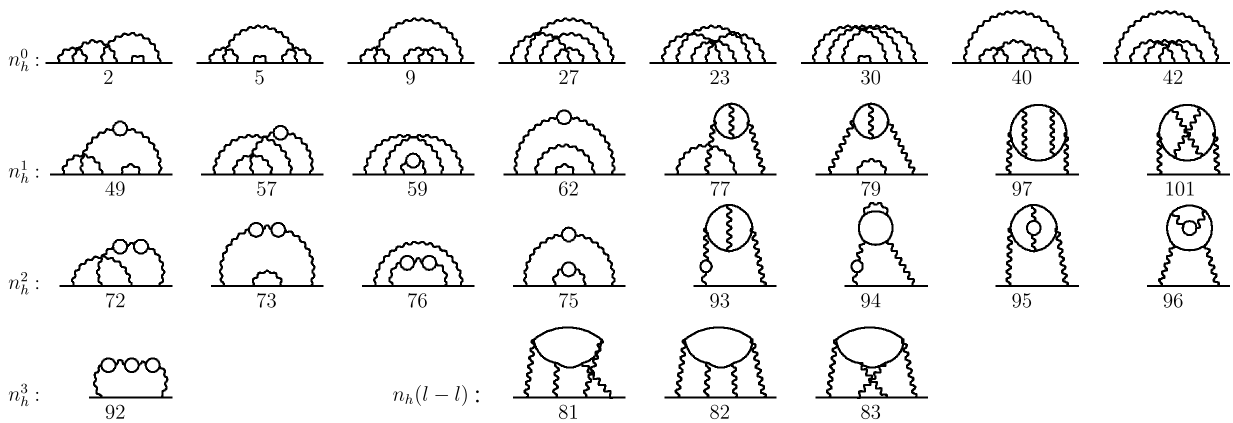

Fig. 1. A selection of some of 4-loop self-mass diagrams, with the indication of the part of Eq. (2) to which the diagram contributes. The numbering of the diagrams follows Ref. [1,2].

Z_2 and Z_m are gauge parameter independent (for Z_2 see Ref. [21–23]); in order to simplify the calculations, we choose the Feynman gauge.

We expand Z_2 and Z_m in power series of the bare coupling constant α_0:

$$Z_2 = 1 + \sum_i \left(\frac{\alpha_0}{\pi}\right)^i \left(\Gamma(1+\epsilon)(4\pi)^\epsilon m^{-2\epsilon}\right)^i Z_2^{(i)} , \qquad Z_m = 1 + \sum_i \left(\frac{\alpha_0}{\pi}\right)^i \left(\Gamma(1+\epsilon)(4\pi)^\epsilon m^{-2\epsilon}\right)^i Z_m^{(i)} . \tag{1}$$

We decompose the 4-loop coefficients $Z_2^{(4)}$ and $Z_m^{(4)}$ by considering QED with n_h different leptons of mass m; in this way we separate the contributions coming from diagrams with different number of insertions of vacuum polarizations and light-light scattering diagrams. We use the notation n_h in order to adapt to the usual QCD convention of calling the number of massive fermions n_h and the number of massless fermions n_l. In this paper we do not consider diagrams with massless leptons. Therefore

$$\begin{aligned} Z_2^{(4)}(n_h) &= Z_2^{(4,0)} + n_h Z_2^{(4,1)} + n_h^2 Z_2^{(4,2)} + n_h^3 Z_2^{(4,3)} + n_h Z_2^{(4,l-l)} , \\ Z_m^{(4)}(n_h) &= Z_m^{(4,0)} + n_h Z_m^{(4,1)} + n_h^2 Z_m^{(4,2)} + n_h^3 Z_m^{(4,3)} + n_h Z_m^{(4,l-l)} . \end{aligned} \tag{2}$$

Clearly

$$Z_2^{(4)}(n_h = 1) = Z_2^{(4)} , \qquad Z_m^{(4)}(n_h = 1) = Z_m^{(4)} . \tag{3}$$

In Fig. 1 we show a selection of some of the 4-loop self-mass diagrams, with the indication of the part of Eq. (2) to which the diagram contributes. The complete set of 104 diagrams is shown in Ref. [1,2].

2.1. Method

We briefly describe here the method used to obtain our results. At 4-loop level there are 104 QED self-mass diagrams. The contribution to $Z_2^{(4)}$ and $Z_m^{(4)}$ from each self-mass diagram is extracted by using projectors and taking traces with a FORM [24,25] program; it is a linear combination of Feynman integrals. Then, for each self-mass diagram a system of integration-by-parts (I.B.P.) identities [26,27] is build and solved [28] in order to reduce the Feynman integrals to linear combinations of master integrals. We used the systems of I.B.P. identities generated in Ref. [1] for the 4-loop g-2. Counterterms were added where needed, excluding vacuum polarizations, since we expand in the bare coupling constant α_0. The renormalization constants are reduced to linear combinations of the 4-loop g-2 master integrals:

$$Z_2^{(4)} = \sum_{i=1}^{N} C_{2,i}(\epsilon) M_i(\epsilon) , \qquad Z_m^{(4)} = \sum_{i=1}^{N} C_{m,i}(\epsilon) M_i(\epsilon) , \tag{4}$$

where $C_{2,i}$ and $C_{m,i}$ are rational functions in ϵ, $N = 334$. We use the numerical values and analytical fits of the master integrals M_i worked out in Ref. [1]. As a simple consistency check, we extracted $Z_1^{(4)}$ from the 891 4-loop vertex diagrams, and we checked numerically and analytically the Ward identity $Z_1^{(4)} = Z_2^{(4)}$.

3. Results

3.1. Numerical results

By substituting the numerical values of $M_i(\epsilon)$ in Eq. (4), we have obtained 1100-digits numerical values for $Z_2^{(4,x)}$ and $Z_m^{(4,x)}$. We show here results truncated to 40 digits for the sake of space. Full-precision results are available from the author. The numerical value of the coefficients of the powers of n_h in Eq. (2) are:

$$
\begin{aligned}
Z_2^{(4,0)} = {} & 0.01318359375\epsilon^{-4} + 0.08349609375\epsilon^{-3} - 0.12614017034082527654676150723162704408 15\epsilon^{-2} \\
& - 2.2188295538072904721561623648263224874 55\epsilon^{-1} \\
& - 3.5729103878128353006549335354404200399 48 + O(\epsilon) \,, \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{(4,1)} = {} & 0.0703125\epsilon^{-4} + 0.255859375\epsilon^{-3} - 0.8863236793443281079737642171893946198198\epsilon^{-2} \\
& - 4.529505177334142374400047352337054775442\epsilon^{-1} \\
& - 3.084249643446330546559819112760212296306 + O(\epsilon) \,, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{(4,2)} = {} & 0.09375\epsilon^{-4} + 0.2109375\epsilon^{-3} - 0.6375168808614659083936734252065216879251\epsilon^{-2} \\
& - 1.118089921229380504506879422676331183925\epsilon^{-1} \\
& - 6.170238285159180066980183430726643994320 + O(\epsilon) \,, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{(4,3)} = {} & 0.02777777777777777777777777777777777777777\epsilon^{-4} \\
& + 0.04745370370370370370370370370370370370370\epsilon^{-3} \\
& + 0.08219673321229348679616766668042765163181\epsilon^{-2} \\
& + 0.1653060637464923649096813755300430397436\epsilon^{-1} \\
& + 0.2373760011149585346199605381719332090858 + O(\epsilon) \,, \qquad (8)
\end{aligned}
$$

$$
Z_2^{(4,l-l)} = -0.125\epsilon^{-1} + 0.1053076031438024383339691487092823499443 + O(\epsilon) \,, \qquad (9)
$$

$$
\begin{aligned}
Z_m^{(4,0)} = {} & 0.01318359375\epsilon^{-4} + 0.05712890625\epsilon^{-3} + 0.2149248550926856846841693020449834317223\epsilon^{-2} \\
& - 0.6311216133387257157783705667342740710866\epsilon^{-1} \\
& - 6.640775996670789293945443649244052753483 + O(\epsilon) \,, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,1)} = {} & 0.03515625\epsilon^{-4} + 0.0171875\epsilon^{-3} + 0.2255475542195506536638492553197210259614\epsilon^{-2} \\
& - 3.655124674567472080082766471095472176020\epsilon^{-1} \\
& - 1.052445900250388864170227694635348345572 + O(\epsilon) \,, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,2)} = {} & 0.02864583333333333333333333333333333333333\epsilon^{-4} + 0.171875\epsilon^{-3} \\
& + 0.02125506230176173571324393225701890093679\epsilon^{-2} \\
& - 1.803410367796313396577956494832032506831\epsilon^{-1} \\
& + 1.933766152908452159688148893799688168422 + O(\epsilon) \,, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,3)} = {} & 0.006944444444444444444444444444444444444444\epsilon^{-4} \\
& + 0.05439814814814814814814814814814814814814\epsilon^{-3} \\
& + 0.002035866606146743398083833340213825815905\epsilon^{-2} \\
& + 0.4351198769462674456781824350266903384281\epsilon^{-1} \\
& - 0.8481242413000520746932873847887178671417 + O(\epsilon) \,, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,l-l)} = {} & 0.5009641733560598212811272632084921831710\epsilon^{-1} \\
& - 3.947552272425901748117750000830748333155 + O(\epsilon) \,. \qquad (14)
\end{aligned}
$$

From Eq. (3) we obtain the values of the renormalization constants:

$$
\begin{aligned}
Z_2^{(4)} = {} & 0.2050238715277777777777777777777777777778\epsilon^{-4} \\
& + 0.5977466724537037037037037037037037037037\epsilon^{-3} \\
& - 1.567783997334325806118031482947115700194\epsilon^{-2} \\
& - 7.826118588624320986153407764309665407079\epsilon^{-1} \\
& - 12.48471471215958494124100639204606077154 + O(\epsilon) \,, \qquad (15)
\end{aligned}
$$

Table 1
Comparison between our values of δZ_2 and the QED and QCD results of Ref. [14].

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\delta Z_2^{(4)}$	0.205023871527777	0.597746672453703	−0.893282495748014	−6.188211339005751	−17.26913874640770
$\delta Z_2^{(4)}$ [14]	0.20500(37)	0.5980(27)	−0.895(21)	−6.18(17)	−17.4(16)
$\delta Z_2^{(4,0)}$	0.01318359375	0.08349609375	−0.082767885375100	−1.965268322191886	−4.036104196851619
δZ_2^{FFFF} [14]	0.01317(25)	0.0836(19)	−0.084(71)	−1.96(16)	−4.1(15)
$\delta Z_2^{(4,l-l)}$	0	0	0	−0.125	0.105307603143802
δZ_2^{dFFH} [14]	−0.00001(23)	0.0001(15)	−0.001(11)	−0.120(76)	0.10(50)
$\delta Z_2^{(4,1)}$	0.0703125	0.255859375	−0.655004826193796	−3.800454407485363	−5.953609261982407
δZ_2^{FFFH} [14]	0.070313(23)	0.255860(93)	−0.65497(55)	−3.8002(36)	−5.953(19)
$\delta Z_2^{(4,2)}$	0.09375	0.2109375	−0.329091743327423	−0.574390474672734	−7.996856441249995
δZ_2^{FFHH} [14]	0.0937498(14)	0.2109378(59)	−0.329095(35)	−0.57438(13)	−7.99681(79)
$\delta Z_2^{(4,3)}$	0.027777777777777	0.047453703703703	0.173581959148306	0.276901865344232	0.612123550532518
δZ_2^{FHHH} [14]	0.0277778	0.047454	0.173582	0.276902	0.61212

$$\begin{aligned}
Z_m^{(4)} = {} & 0.083930121527777777777777777777777777\epsilon^{-4} \\
& + 0.455277054398148148148148148148148148148\epsilon^{-3} \\
& + 0.463763338220144817459346322961937184436 4\epsilon^{-2} \\
& - 5.153572605400183925479783834426596232339\epsilon^{-1} \\
& - 10.555132257738679821238559835699179130 93 + O(\epsilon)\,.
\end{aligned} \tag{16}$$

3.2. Comparisons

3.2.1. QED wave function renormalization constant

Now we compare our results Eqs. (5)-(10) and Eqs. (15)-(16) with the numerical results of Ref. [14]; In Ref. [14] Z_2 is written as:

$$Z_2 = 1 + \sum_i \left(\frac{\alpha_0(\mu)}{\pi}\right)^i \left((4\pi)^\epsilon e^{-\gamma\epsilon}\left(\frac{\mu^2}{m^2}\right)^\epsilon\right)^i \delta Z_2^{(i)}\,. \tag{17}$$

The relation between $\delta Z_2^{(4)}$ and our $Z_2^{(4)}$ for $\mu = 1$ is

$$\delta Z_2^{(4)} = \left(\frac{e^{-\gamma\epsilon}}{\Gamma(1+\epsilon)}\right)^4 Z_2^{(4)}\,. \tag{18}$$

In the first row of Table 1 we show our high-precision numerical values of the coefficients of the powers of ϵ of $\delta Z_2^{(4)}$, truncated to 15 digits for reason of space, and the corresponding values from Ref. [14]. They are in good agreement, the worst error being 0.1σ. In the next subsection we will need to decompose δZ_2 in terms with different n_h,

$$\delta Z_2^{(4)} = \delta Z_2^{(4,0)} + n_h \delta Z_2^{(4,1)} + n_h^2 \delta Z_2^{(4,2)} + n_h^3 \delta Z_2^{(4,3)} + n_h \delta Z_2^{(4,l-l)}\,; \tag{19}$$

our numerical values of $\delta Z_2^{(4)}$ are listed in Table 1; preliminar values were presented in Ref. [29].

3.2.2. QCD wave function renormalization constant

We consider now the QCD renormalization constant Z_2^{QCD} in the OS renormalization scheme. In the notation of Ref. [14] Z_2^{QCD} is decomposed in 23 color structures:

$$\delta Z_2^{QCD} = C_F^4 \delta Z_2^{FFFF} + C_F^3 T n_h \delta Z_2^{FFFH} + C_F^2 T^2 n_h^2 \delta Z_2^{FFHH} + C_F T^3 n_h^3 \delta Z_2^{FHHH} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c} \delta Z_2^{d_{FF}H} + \dots\,. \tag{20}$$

The coefficients $Z_2^{(4,x)}$ of Eq. (19) must coincide with the corresponding coefficients of the color structures which can be obtained from QED-like diagrams:

$$\delta Z_2^{4,0} = \delta Z_2^{FFFF}\,, \quad \delta Z_2^{4,1} = \delta Z_2^{FFFH}\,, \quad \delta Z_2^{4,2} = \delta Z_2^{FFHH}\,, \qquad \delta Z_2^{4,3} = \delta Z_2^{FHHH}\,, \quad \delta Z_2^{4,l-l} = \delta Z_2^{d_{FF}H}\,. \tag{21}$$

In Table 1 we compare our high-precision results and the corresponding ones from Ref. [14]. They are in good agreement, the worst error being 0.08σ.

3.2.3. QCD mass renormalization constant

Now we consider the ratio between the QCD mass renormalization constants in the OS and $\overline{\mathrm{MS}}$ scheme:

$$z_m(\mu) = \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}} = 1 + \sum_{n\geq 1} \left(\frac{\alpha_s(\mu)}{\pi}\right)^n z_m^{(n)}(\mu) \; ; \tag{22}$$

Using the notation of Ref. [17], z_m is decomposed in 23 color structures, and as above we can infer the coefficients of the five structures which involve only QED-like diagrams.

$$z_m^{(4)} = C_F^4 z_m^{FFFF} + C_F^3 T n_h z_m^{FFFH} + C_F^2 T^2 n_h^2 z_m^{FFHH} + C_F T^3 n_h^3 z_m^{FHHH} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c} z_m^{d_{FF}H} + \ldots \tag{23}$$

Our high-precision values are

$$\begin{aligned}
z_m^{FFFF} &= -6.943004942063674366729783085730127607323 \, , \\
z_m^{FFFH} &= -1.364670155556599206268818327382565081078 \, , \\
z_m^{FFHH} &= 1.657513434712808758859954030175433075697 \, , \\
z_m^{FHHH} &= -0.1490239616711777449135125845999523299403 \, , \\
z_m^{d_{FF}H} &= -3.947552272425901748117750000830748333155 \, .
\end{aligned} \tag{24}$$

The results of Ref. [17]

$$\begin{aligned}
z_m^{FFFF} &= -6.983(805) \, , \\
z_m^{FFFH} &= -1.3625(132) \, , \\
z_m^{FFHH} &= 1.65752(31) \, , \\
z_m^{FHHH} &= -0.14902 \, , \\
z_m^{d_{FF}H} &= -3.924(642) \, ,
\end{aligned} \tag{25}$$

are in agreement with ours at the level of 0.16σ at worst.

3.3. Analytical fits

By substituting the analytical fits of $M_i(\epsilon)$ in Eq. (4), we have obtained the following analytical expressions:

$$\begin{aligned}
Z_2^{(4)} = {} & \frac{3779}{18432\epsilon^4} + \frac{33053}{55296\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{515315}{73728} - \frac{7205}{768}\zeta(2) - \frac{131}{64}\zeta(3) + \frac{131}{16}\zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{19571293}{663552}\right. \\
& \left. + \frac{154747}{2160}\zeta(2) - \frac{11521}{2304}\zeta(3) - \frac{29539}{192}\zeta(2)\ln 2 + \frac{3115}{64}\zeta(4) - \frac{215}{8}\zeta(2)\ln^2 2 - \frac{19}{4}t_4 - \frac{5}{64}\zeta(5) - \frac{21}{16}\zeta(3)\zeta(2)\right) \\
& + \frac{9565004502941}{87787929600} + \frac{1535743349}{691200}\zeta(2) + \frac{12128503957}{25401600}\zeta(3) + \frac{10500647}{17280}\zeta(2)\ln 2 \\
& - \frac{535034261}{1451520}\zeta(4) + \frac{5631023}{9072}\zeta(2)\ln^2 2 + \frac{7932313}{9072}t_4 - \frac{2050259}{17280}\zeta(5) \\
& + \frac{9296423}{8640}\zeta(3)\zeta(2) - \frac{609737}{480}\zeta(4)\ln 2 + \frac{1739}{24}\zeta(2)\ln^3 2 - \frac{2671}{30}t_5 - \frac{715229459}{62208}\zeta(6) \\
& - \frac{4006421}{11520}\zeta^2(3) + \frac{1780957}{240}\zeta(3)\zeta(2)\ln 2 - \frac{4689809}{720}\zeta(4)\ln^2 2 - \frac{9674}{45}t_{61} + \frac{10276}{45}t_{62} + \frac{642767}{90}t_{63} \\
& + \frac{1495323863}{580608}\zeta(7) - \frac{5775661427}{161280}\zeta(5)\zeta(2) - \frac{7455877}{10752}\zeta(4)\zeta(3) + \frac{2153119}{128}\zeta(6)\ln 2 \\
& - \frac{561331}{126}\zeta(3)\zeta(2)\ln^2 2 - \frac{19454}{63}t_{71} + \frac{9144}{7}t_{72} + \frac{4306}{7}t_4\zeta(3) + \frac{553037}{15}t_5\zeta(2) + \frac{135039}{5}t_{73} \\
& + \sqrt{3}\left(\frac{15489}{320}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{1311089}{960}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3071}{60}v_{61} + \frac{2109}{50}v_{62} + \frac{7472227}{34320}v_{63} - \frac{8978057}{6480}v_{64}\right) \\
& + \frac{5797}{16}v_{65} + \frac{115735}{96}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + 18v_{71} + 6v_{72} - 36\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) - \frac{12606}{5}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 \\
& + \sqrt{3}\pi\left(-\frac{10163659}{230400}B_3 + \frac{224075873}{6220800}C_3 - \frac{11863}{7776}f_2(0,0,1) + \frac{56207}{23328}e_{51} - \frac{14615}{1728}e_{52} - \frac{45499}{20736}e_{61}\right. \\
& \left. + \frac{30961}{41472}e_{62}\right) + \zeta(2)\left(-\frac{2354}{243}f_1(0,0,1) - \frac{30961}{3456}e_{53} - \frac{673}{162}e_{54}\right)
\end{aligned}$$

$$
-\frac{507}{80}C_{81a}-26C_{81b}+\frac{11}{2}C_{81c}-\frac{1057}{320}C_{83a}+\frac{91}{24}C_{83b}+\frac{7}{2}C_{83c}+O(\epsilon)\,, \tag{26}
$$

$$
\begin{aligned}
Z_m^{(4)} = {} & \frac{1547}{18432\epsilon^4}+\frac{25175}{55296\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{202951}{73728}-\frac{2737}{768}\zeta(2)-\frac{71}{128}\zeta(3)+\frac{119}{32}\zeta(2)\ln 2\right)+\frac{1}{\epsilon}\left(\frac{27750271}{1990656}\right.\\
& +\frac{132763}{5760}\zeta(2)-\frac{1625}{768}\zeta(3)-\frac{10807}{192}\zeta(2)\ln 2+\frac{6173}{384}\zeta(4)-\frac{553}{48}\zeta(2)\ln^2 2+\frac{35}{24}t_4-\frac{5}{8}\zeta(5)\\
& \left.+\frac{21}{32}\zeta(3)\zeta(2)\right)+\frac{1885204711}{23887872}+\frac{5804091169}{6220800}\zeta(2)+\frac{169571}{768}\zeta(3)+\frac{87397}{5760}\zeta(2)\ln 2-\frac{1866527}{6912}\zeta(4)\\
& +\frac{74477}{288}\zeta(2)\ln^2 2+\frac{62645}{144}t_4-\frac{15459}{256}\zeta(5)+\frac{20351}{48}\zeta(3)\zeta(2)-\frac{33367}{64}\zeta(4)\ln 2+\frac{357}{16}\zeta(2)\ln^3 2\\
& +\frac{35}{4}t_5-\frac{8229601}{1728}\zeta(6)-\frac{5605}{32}\zeta^2(3)+\frac{12297}{4}\zeta(3)\zeta(2)\ln 2-\frac{10975}{4}\zeta(4)\ln^2 2+3040t_{63}-\frac{161}{128}\zeta(7)\\
& -\frac{27786101}{1920}\zeta(5)\zeta(2)-\frac{272627}{384}\zeta(4)\zeta(3)+\frac{252105}{32}\zeta(6)\ln 2-\frac{3675}{2}\zeta(3)\zeta(2)\ln^2 2+\frac{74368}{5}t_5\zeta(2)\\
& +\frac{53368}{5}t_{73}+\sqrt{3}\left(45\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{12757}{24}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\frac{98}{5}v_{62}-\frac{4949}{9}v_{64}\right)+180v_{65}\\
& +\frac{993}{2}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)-1008\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2+\sqrt{3}\pi\left(-\frac{754571}{46080}B_3+\frac{17787301}{1244160}C_3-\frac{503}{1728}f_2(0,0,1)+\frac{895}{324}e_{51}\right.\\
& \left.-\frac{1295}{216}e_{52}-\frac{1493}{1728}e_{61}+\frac{671}{3456}e_{62}\right)+\zeta(2)\left(-\frac{100}{27}f_1(0,0,1)-\frac{671}{288}e_{53}-\frac{346}{27}e_{54}\right)-\frac{21}{8}C_{81a}-10C_{81b}\\
& -\frac{21}{16}C_{83a}+\frac{5}{3}C_{83b}+O(\epsilon)\,.
\end{aligned} \tag{27}
$$

The analytical expressions of the separate contributions of Eq. (2) are:

$$
\begin{aligned}
Z_2^{(4,0)} = {} & \frac{27}{2048\epsilon^4}+\frac{171}{2048\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{5835}{8192}-\frac{351}{256}\zeta(2)-\frac{27}{64}\zeta(3)+\frac{27}{16}\zeta(2)\ln 2\right)+\frac{1}{\epsilon}\left(\frac{23865}{8192}+\frac{4171}{128}\zeta(2)\right.\\
& \left.+\frac{2527}{256}\zeta(3)-\frac{4713}{64}\zeta(2)\ln 2+\frac{909}{64}\zeta(4)-\frac{81}{8}\zeta(2)\ln^2 2+\frac{57}{4}t_4-\frac{25}{64}\zeta(5)-\frac{9}{16}\zeta(3)\zeta(2)\right)+\frac{2033213}{98304}\\
& +\frac{6966313}{15360}\zeta(2)+\frac{352489}{11520}\zeta(3)+\frac{642189}{640}\zeta(2)\ln 2+\frac{9612919}{23040}\zeta(4)+\frac{574357}{1440}\zeta(2)\ln^2 2+\frac{2341}{720}t_4\\
& -\frac{1149787}{5760}\zeta(5)+\frac{108635}{192}\zeta(3)\zeta(2)-\frac{83371}{160}\zeta(4)\ln 2+\frac{161}{8}\zeta(2)\ln^3 2+\frac{1727}{10}t_5-\frac{38991047}{5184}\zeta(6)\\
& -\frac{756779}{11520}\zeta^2(3)+\frac{414137}{120}\zeta(3)\zeta(2)\ln 2-\frac{1227247}{360}\zeta(4)\ln^2 2-\frac{19994}{45}t_{61}+\frac{25156}{45}t_{62}+\frac{24104}{9}t_{63}\\
& -\frac{9389399}{82944}\zeta(7)-\frac{8309201}{512}\zeta(5)\zeta(2)+\frac{4441247}{4608}\zeta(4)\zeta(3)+\frac{1096627}{128}\zeta(6)\ln 2-\frac{72457}{36}\zeta(3)\zeta(2)\ln^2 2\\
& -\frac{136}{9}t_{71}-\frac{32}{3}t_{72}-\frac{1}{3}t_4\zeta(3)+\frac{50705}{3}t_5\zeta(2)+12331t_{73}+\sqrt{3}\left(-\frac{6949}{192}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{563899}{576}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right.\\
& \left.+\frac{3071}{60}v_{61}+\frac{2109}{50}v_{62}+\frac{7472227}{34320}v_{63}-\frac{8978057}{6480}v_{64}\right)+\frac{26125}{32}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)-\frac{1995}{16}v_{65}-18v_{71}-6v_{72}\\
& +\sqrt{3}\pi\left(-\frac{32021}{25600}B_3+\frac{3656149}{230400}C_3-\frac{4775}{1296}f_2(0,0,1)-\frac{965143}{23328}e_{51}+\frac{32885}{576}e_{52}-\frac{10705}{6912}e_{61}+\frac{585}{512}e_{62}\right)\\
& +\zeta(2)\left(-\frac{2414}{243}f_1(0,0,1)-\frac{1755}{128}e_{53}+\frac{3565}{54}e_{54}\right)+O(\epsilon)\,,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
Z_2^{(4,l)} = {} & -\frac{1}{8\epsilon}-\frac{205}{128}+\frac{2563}{2}\zeta(2)+\frac{35933}{192}\zeta(3)+\frac{1419}{2}\zeta(2)\ln 2-\frac{15155}{192}\zeta(4)-\frac{679}{3}\zeta(2)\ln^2 2+\frac{628}{3}t_4-\frac{51259}{192}\zeta(5)\\
& +\frac{95239}{120}\zeta(3)\zeta(2)-\frac{20975}{16}\zeta(4)\ln 2-16t_5-\frac{56584517}{10368}\zeta(6)-\frac{541547}{1920}\zeta^2(3)+\frac{46507}{10}\zeta(3)\zeta(2)\ln 2\\
& -\frac{20947}{6}\zeta(4)\ln^2 2+\frac{688}{3}t_{61}-\frac{992}{3}t_{62}+\frac{81632}{15}t_{63}+\frac{195131207}{72576}\zeta(7)-\frac{394782889}{20160}\zeta(5)\zeta(2)\\
& -\frac{6682045}{4032}\zeta(4)\zeta(3)+\frac{264123}{32}\zeta(6)\ln 2-\frac{615463}{252}\zeta(3)\zeta(2)\ln^2 2-\frac{18502}{63}t_{71}+\frac{27656}{21}t_{72}+\frac{12925}{21}t_4\zeta(3)\\
& +\frac{299512}{15}t_5\zeta(2)+\frac{73384}{5}t_{73}+\sqrt{3}\left(\frac{693}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{2523}{8}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)+\frac{4969}{16}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)+\frac{3873}{8}v_{65}
\end{aligned}
$$

$$
+36v_{71}+12v_{72}-36\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)-\frac{12606}{5}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)+\sqrt{3}\pi\left(-\frac{15399}{320}B_3+\frac{1477}{192}C_3+\frac{857}{288}f_2(0,0,1)\right.
+\frac{39785}{972}e_{51}-\frac{13235}{216}e_{52}-\frac{1673}{2592}e_{61}-\left.\frac{2053}{5184}e_{62}\right)+\zeta(2)\left(\frac{20}{81}f_1(0,0,1)+\frac{2053}{432}e_{53}-\frac{5684}{81}e_{54}\right)
-\frac{507}{80}C_{81a}-26C_{81b}+\frac{11}{2}C_{81c}-\frac{1057}{320}C_{83a}+\frac{91}{24}C_{83b}+\frac{7}{2}C_{83c}+O(\epsilon)\,, \tag{29}
$$

$$
Z_2^{(4,1)}=\frac{9}{128\epsilon^4}+\frac{131}{512\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{1747}{768}-\frac{135}{32}\zeta(2)-\frac{9}{8}\zeta(3)+\frac{9}{2}\zeta(2)\ln 2\right)+\frac{1}{\epsilon}\left(\frac{142385}{18432}+\frac{50989}{1152}\zeta(2)\right.
+\frac{2777}{768}\zeta(3)-\frac{747}{8}\zeta(2)\ln 2+\frac{891}{32}\zeta(4)-\frac{63}{4}\zeta(2)\ln^2 2+t_4+\frac{5}{16}\zeta(5)-\left.\frac{3}{4}\zeta(3)\zeta(2)\right)+\frac{934395461}{37324800}
+\frac{8610967}{19440}\zeta(2)+\frac{809922361}{3110400}\zeta(3)-\frac{394219}{432}\zeta(2)\ln 2-\frac{24640541}{34560}\zeta(4)+\frac{57655}{144}\zeta(2)\ln^2 2+\frac{26603}{36}t_4
+\frac{450661}{1440}\zeta(5)-\frac{8861}{27}\zeta(3)\zeta(2)+\frac{136613}{240}\zeta(4)\ln 2+\frac{121}{3}\zeta(2)\ln^3 2-\frac{1886}{15}t_5+\frac{92170207}{62208}\zeta(6)
-\frac{1}{32}\zeta^2(3)-\frac{10899}{16}\zeta(3)\zeta(2)\ln 2+\frac{6185}{16}\zeta(4)\ln^2 2-\frac{1957}{2}t_{63}+\sqrt{3}\left(-\frac{487}{240}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\right.
-\left.\frac{51373}{720}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)+\frac{3773}{48}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)+\frac{23}{8}v_{65}+\sqrt{3}\pi\left(\frac{237613}{34560}B_3+\frac{6777463}{933120}C_3\right.
-\left.\frac{1733}{1944}f_2(0,0,1)+\frac{3695}{1296}e_{51}-\frac{3695}{864}e_{52}\right)+O(\epsilon)\,, \tag{30}
$$

$$
Z_2^{(4,2)}=\frac{3}{32\epsilon^4}+\frac{27}{128\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{4337}{1536}-\frac{25}{8}\zeta(2)-\frac{1}{2}\zeta(3)+2\zeta(2)\ln 2\right)+\frac{1}{\epsilon}\left(\frac{167545}{13824}-\frac{3827}{2160}\zeta(2)\right.
-\frac{31843}{2304}\zeta(3)+\frac{55}{6}\zeta(2)\ln 2+\frac{53}{8}\zeta(4)-\zeta(2)\ln^2 2-\left.20t_4\right)+\frac{40705891003}{1828915200}+\frac{3264023}{48600}\zeta(2)
+\frac{3697035983}{152409600}\zeta(3)-\frac{2133}{10}\zeta(2)\ln 2-\frac{5625569}{362880}\zeta(4)+\frac{4638449}{90720}\zeta(2)\ln^2 2-\frac{964289}{22680}t_4+\frac{1889}{54}\zeta(5)
+\frac{1609}{36}\zeta(3)\zeta(2)-\frac{15}{2}\zeta(4)\ln 2+12\zeta(2)\ln^3 2-120t_5+\sqrt{3}\pi\left(-\frac{111683}{69120}B_3+\frac{9696589}{1866240}C_3\right.
+\left.\frac{145}{1944}f_2(0,0,1)\right)+O(\epsilon)\,, \tag{31}
$$

$$
Z_2^{(4,3)}=\frac{1}{36\epsilon^4}+\frac{41}{864\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{679}{576}-\frac{2}{3}\zeta(2)\right)+\frac{1}{\epsilon}\left(\frac{71143}{10368}-\frac{103}{30}\zeta(2)-\frac{14}{3}\zeta(3)+4\zeta(2)\ln 2\right)
+\frac{166888903}{3919104}-\frac{31451}{1350}\zeta(2)-\frac{37691}{1512}\zeta(3)+\frac{103}{5}\zeta(2)\ln 2+\frac{259}{12}\zeta(4)-\frac{10}{3}\zeta(2)\ln^2 2-\frac{104}{3}t_4+O(\epsilon)\,, \tag{32}
$$

$$
Z_m^{(4,0)}=\frac{27}{2048\epsilon^4}+\frac{117}{2048\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{3063}{8192}-\frac{135}{256}\zeta(2)-\frac{27}{128}\zeta(3)+\frac{27}{32}\zeta(2)\ln 2\right)+\frac{1}{\epsilon}\left(\frac{28653}{8192}+\frac{1479}{128}\zeta(2)\right.
+\frac{471}{256}\zeta(3)-\frac{1773}{64}\zeta(2)\ln 2+\frac{747}{128}\zeta(4)-\frac{63}{16}\zeta(2)\ln^2 2+\frac{45}{8}t_4-\frac{15}{32}\zeta(5)+\left.\frac{9}{32}\zeta(3)\zeta(2)\right)+\frac{2907301}{98304}
+\frac{166719}{1024}\zeta(2)+\frac{1109}{128}\zeta(3)+\frac{56537}{128}\zeta(2)\ln 2+\frac{7613}{128}\zeta(4)+\frac{4149}{32}\zeta(2)\ln^2 2+\frac{765}{16}t_4-\frac{11123}{256}\zeta(5)
+\frac{3429}{16}\zeta(3)\zeta(2)-\frac{13867}{64}\zeta(4)\ln 2+\frac{81}{16}\zeta(2)\ln^3 2+\frac{135}{4}t_5-\frac{22615}{8}\zeta(6)+\frac{219}{32}\zeta^2(3)+1235\zeta(3)\zeta(2)\ln 2
-1390\zeta(4)\ln^2 2+1018t_{63}-\frac{189}{64}\zeta(7)+\frac{110845}{256}\zeta(4)\zeta(3)-\frac{1658509}{256}\zeta(5)\zeta(2)+\frac{108045}{32}\zeta(6)\ln 2
-\frac{1575}{2}\zeta(3)\zeta(2)\ln^2 2+6698t_5\zeta(2)+4898t_{73}+\sqrt{3}\left(-\frac{2983}{8}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\frac{98}{5}v_{62}-\frac{4949}{9}v_{64}\right)
+\frac{675}{2}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)+\sqrt{3}\pi\left(-\frac{3123}{5120}B_3+\frac{33383}{5120}C_3-\frac{781}{576}f_2(0,0,1)-\frac{5245}{324}e_{51}+\frac{1615}{72}e_{52}\right.
-\left.\frac{721}{1152}e_{61}+\frac{1075}{2304}e_{62}\right)+\zeta(2)\left(-\frac{100}{27}f_1(0,0,1)-\frac{1075}{192}e_{53}+\frac{245}{9}e_{54}\right)+O(\epsilon)\,, \tag{33}
$$

$$
\begin{aligned}
Z_m^{(4,l)} = & \frac{1}{\epsilon}\left(-\frac{1}{16}+\frac{15}{32}\zeta(3)\right) - \frac{135}{128} + 555\zeta(2) + \frac{18625}{192}\zeta(3) + 48\zeta(2)\ln 2 - \frac{803}{64}\zeta(4) - 20\zeta(2)\ln^2 2 + 80t_4 \\
& - \frac{7205}{96}\zeta(5) + \frac{1273}{4}\zeta(3)\zeta(2) - 603\zeta(4)\ln 2 - \frac{2134769}{864}\zeta(6) - \frac{1453}{8}\zeta^2(3) + 2086\zeta(3)\zeta(2)\ln 2 + 2384t_{63} \\
& - 1490\zeta(4)\ln^2 2 + \frac{217}{128}\zeta(7) - \frac{877789}{768}\zeta(4)\zeta(3) - \frac{30694567}{3840}\zeta(5)\zeta(2) + \frac{36015}{8}\zeta(6)\ln 2 - 1050\zeta(3)\zeta(2)\ln^2 2 \\
& + \frac{40878}{5}t_5\zeta(2) + \frac{28878}{5}t_{73} + \sqrt{3}\left(45\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - 129\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + 126\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + 180v_{65} \\
& - 1008\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) + \sqrt{3}\pi\left(-\frac{2859}{160}B_3 + \frac{1253}{480}C_3 + \frac{299}{216}f_2(0,0,1) + \frac{1450}{81}e_{51} - \frac{725}{27}e_{52} - \frac{823}{3456}e_{61}\right. \\
& \left. - \frac{1883}{6912}e_{62}\right) + \zeta(2)\left(\frac{1883}{576}e_{53} - \frac{1081}{27}e_{54}\right) - \frac{21}{8}C_{81a} - 10C_{81b} - \frac{21}{16}C_{83a} + \frac{5}{3}C_{83b} + O(\epsilon)\,, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,1)} = & \frac{9}{256\epsilon^4} + \frac{11}{64\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{2507}{3072} - \frac{93}{64}\zeta(2) - \frac{9}{32}\zeta(3) + \frac{15}{8}\zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{4363}{1024} + \frac{91}{6}\zeta(2)\right. \\
& \left. + \frac{51}{128}\zeta(3) - 35\zeta(2)\ln 2 + \frac{71}{8}\zeta(4) - \frac{25}{4}\zeta(2)\ln^2 2 + \frac{5}{2}t_4 - \frac{5}{32}\zeta(5) + \frac{3}{8}\zeta(3)\zeta(2)\right) + \frac{1273135}{36864} \\
& + \frac{1312775}{6912}\zeta(2) + \frac{87181}{768}\zeta(3) - \frac{3187}{8}\zeta(2)\ln 2 - \frac{2063917}{6912}\zeta(4) + 133\zeta(2)\ln^2 2 + 320t_4 \\
& + \frac{1567}{32}\zeta(5) - \frac{257}{2}\zeta(3)\zeta(2) + \frac{4653}{16}\zeta(4)\ln 2 + \frac{45}{4}\zeta(2)\ln^3 2 + 15t_5 + \frac{34251}{64}\zeta(6) \\
& - \frac{3}{8}\zeta^2(3) - \frac{987}{4}\zeta(3)\zeta(2)\ln 2 + \frac{545}{4}\zeta(4)\ln^2 2 - 362t_{63} - \frac{89}{3}\sqrt{3}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + 33\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) \\
& + \sqrt{3}\pi\left(\frac{3599}{1280}B_3 + \frac{97783}{34560}C_3 - \frac{227}{648}f_2(0,0,1) + \frac{85}{81}e_{51} - \frac{85}{54}e_{52}\right) + O(\epsilon)\,, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,2)} = & \frac{11}{384\epsilon^4} + \frac{11}{64\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{1555}{1536} - \frac{5}{4}\zeta(2) - \frac{1}{16}\zeta(3) + \zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{3343}{1152} - \frac{329}{180}\zeta(2) - \frac{97}{32}\zeta(3)\right. \\
& \left. + \frac{53}{12}\zeta(2)\ln 2 + \frac{131}{96}\zeta(4) - \frac{4}{3}\zeta(2)\ln^2 2 - \frac{20}{3}t_4\right) - \frac{183577}{55296} + \frac{1777157}{48600}\zeta(2) + \frac{7043}{576}\zeta(3) \\
& - \frac{31397}{360}\zeta(2)\ln 2 - \frac{44239}{1728}\zeta(4) + \frac{335}{18}\zeta(2)\ln^2 2 + \frac{5}{9}t_4 + \frac{439}{48}\zeta(5) + \frac{239}{12}\zeta(3)\zeta(2) \\
& + \frac{15}{2}\zeta(4)\ln 2 + 6\zeta(2)\ln^3 2 - 40t_5 + \sqrt{3}\pi\left(-\frac{8159}{11520}B_3 + \frac{726817}{311040}C_3 + \frac{5}{162}f_2(0,0,1)\right) + O(\epsilon)\,, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{(4,3)} = & \frac{1}{144\epsilon^4} + \frac{47}{864\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{317}{576} - \frac{1}{3}\zeta(2)\right) + \frac{1}{\epsilon}\left(\frac{103963}{31104} - \frac{83}{45}\zeta(2) - \frac{43}{24}\zeta(3) + 2\zeta(2)\ln 2\right) \\
& + \frac{3579989}{186624} - \frac{7622}{675}\zeta(2) - \frac{1529}{144}\zeta(3) + \frac{166}{15}\zeta(2)\ln 2 + \frac{347}{48}\zeta(4) - \frac{8}{3}\zeta(2)\ln^2 2 - \frac{40}{3}t_4 + O(\epsilon)\,. \qquad (37)
\end{aligned}
$$

In the above expressions we use these combinations of constants:

$$t_4 = a_4 + \frac{1}{24}\ln^4 2\,, \qquad t_5 = a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\,, \qquad (38)$$

$$t_{61} = b_6 - a_5\ln 2 + \zeta(5)\ln 2 + \frac{1}{6}\zeta(3)\ln^3 2 - \frac{1}{12}\zeta(2)\ln^4 2 + \frac{1}{144}\ln^6 2\,, \qquad (39)$$

$$t_{62} = a_6 - \frac{1}{48}\zeta(2)\ln^4 2 + \frac{1}{720}\ln^6 2\,, \qquad (40)$$

$$
\begin{aligned}
t_{71} = & d_7 - 2b_6\ln 2 + 4a_6\ln 2 + 2a_5\ln^2 2 - \frac{49}{32}\zeta^2(3)\ln 2 - \frac{95}{32}\zeta(5)\ln^2 2 + \frac{1}{8}\zeta(4)\ln^3 2 \\
& - \frac{1}{3}\zeta(3)\ln^4 2 + \frac{1}{12}\zeta(2)\ln^5 2 - \frac{1}{120}\ln^7 2\,, \qquad (41)
\end{aligned}
$$

$$t_{72} = b_7 - 3a_7 - a_6\ln 2 - \frac{1}{2}\zeta(5)\ln^2 2 + \frac{1}{48}\zeta(4)\ln^3 2 - \frac{1}{24}\zeta(3)\ln^4 2 + \frac{1}{120}\zeta(2)\ln^5 2 - \frac{1}{1680}\ln^7 2\,, \qquad (42)$$

$$t_{73} = \left(a_4 - \frac{1}{4}\zeta(2)\ln^2 2 + \frac{7}{16}\zeta(3)\ln 2 + \frac{1}{24}\ln^4 2\right)\zeta(2)\ln 2\,, \qquad (43)$$

$$v_{61} = \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{27}{26}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+ \frac{207}{104}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10}{3}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7}{4}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{8}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$- \frac{5}{72}\zeta(3)\zeta(2)\pi - \frac{5}{6}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{5}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 - \frac{27413}{67392}\zeta(5)\pi + \frac{4975}{11583}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \tag{44}$$

$$v_{62} = \zeta(2)\Bigg(\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{1}{6}\zeta(3)\pi + \frac{1}{108}\zeta(2)\pi\ln 2 - \frac{5}{2}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\ln 2$$
$$- \frac{15}{4}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\ln 2 + \frac{25}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{661}{1188}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\Bigg), \tag{45}$$

$$v_{63} = \mathrm{Cl}_6\left(\frac{\pi}{3}\right) - \frac{3}{4}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \qquad v_{64} = \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{91}{66}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \tag{46}$$

$$v_{65} = \mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right), \tag{47}$$

$$v_{71} = \mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 4\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{27}{8}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$- \frac{135}{16}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{27}{2}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+ \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{145}{132}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi, \tag{48}$$

$$v_{72} = \zeta(2)\Bigg(\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 2\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{9}{4}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{9}{2}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+ \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg), \tag{49}$$

$$v_{73} = \zeta(2)\Bigg(\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{2}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right) - \frac{1}{2}\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\pi + \frac{1}{4}\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\Bigg), \tag{50}$$

$$e_{51} = f_2(0,2,0) - \frac{9}{4}f_2(0,0,1)\ln 2, \qquad e_{52} = f_2(0,1,1) - \frac{3}{8}f_2(0,0,2) - \frac{3}{2}f_2(0,0,1)\ln 2, \tag{51}$$

$$e_{53} = f_1(1,0,1) - f_1(0,1,1) + \frac{1}{4}f_1(0,0,2), \qquad e_{54} = e_{51} - \frac{3}{2}e_{52}, \tag{52}$$

$$e_{61} = f_2(2,1,0) + \frac{7}{3}f_2(1,2,0) - 2f_2(1,1,1) + \frac{40}{27}f_2(0,3,0) - \frac{7}{3}f_2(0,2,1) + f_2(0,1,2) - 30e_{54}\ln 2, \tag{53}$$

$$e_{62} = f_2(2,0,1) + \frac{14}{3}f_2(1,2,0) - 2f_2(1,1,1) - 2f_2(1,0,2) - \frac{370}{27}f_2(0,3,0)$$
$$+ \frac{85}{3}f_2(0,2,1) - 22f_2(0,1,2) + 7f_2(0,0,3) + 11\zeta(2)f_2(0,0,1) - 20e_{54}\ln 2. \tag{54}$$

In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i}\, i^{-n}$, $b_6 = H_{0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $b_7 = H_{0,0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\mathrm{Cl}_n(\theta) = \mathrm{ImLi}_n(e^{i\theta})$. C_{8xy} are the ϵ^0 coefficients of the $\epsilon-$expansion of six master integrals (see Ref. [1]). $H_{i_1,i_2,\ldots}(x)$ are the harmonic polylogarithms [29–31]. The integrals f_j are defined as follows:

$$f_m(i,j,k) = \int_1^9 ds\, D_1(s)\mathrm{Re}\left(\sqrt{3^{m-1}}D_m(s)\right)\left(s - \frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s), \tag{55}$$

$$D_m(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(m - 1 - (2m-3)\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right); \tag{56}$$

B_3 and C_3 have hypergeometric expressions [32,33]:

$$B_3 = \frac{\pi}{27}\sqrt{3}\left({}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right), \tag{57}$$

$$C_3 = \frac{\pi}{27}\sqrt{3}\left({}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2}\\ -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2}\\ -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}\end{matrix};1\right)\right), \tag{58}$$

$${}_4\tilde{F}_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right). \tag{59}$$

4. Z_2 and Z_m to three loops

For completeness we list here the analytical expressions of Z_2 and Z_m at one, two, and three loops, expanded in ϵ up to the level needed for five-loop renormalization (two powers in ϵ more than the results of Ref. [4] and one power more than Ref. [14,17]).

$$Z_2^{(1)} = Z_m^{(1)} = -\frac{3}{4\epsilon} - \frac{1}{1-2\epsilon} \,, \tag{60}$$

$$\begin{aligned}
Z_2^{(2)} &= \frac{17}{32\epsilon^2} + \frac{229}{192\epsilon} + \frac{8453}{1152} - \frac{3}{2}\zeta(3) + 6\zeta(2)\ln 2 - \frac{55}{8}\zeta(2) + \epsilon\bigg(\frac{86797}{6912} - \frac{419}{16}\zeta(2) - \frac{203}{8}\zeta(3) \\
&+ \frac{93}{2}\zeta(2)\ln 2 + \frac{63}{2}\zeta(4) - 12\zeta(2)\ln^2 2 - 24t_4\bigg) + \epsilon^2\bigg(\frac{2197589}{41472} - \frac{3393}{32}\zeta(2) - \frac{779}{8}\zeta(3) + 180\zeta(2)\ln 2 \\
&+ \frac{1073}{8}\zeta(4) - 93\zeta(2)\ln^2 2 - 186t_4 + \frac{609}{4}\zeta(5) + 18\zeta(3)\zeta(2) - 93\zeta(4)\ln 2 + 36\zeta(2)\ln^3 2 - 144t_5\bigg) \\
&+ \epsilon^3\bigg(\frac{22329277}{248832} - \frac{22387}{64}\zeta(2) - \frac{5769}{16}\zeta(3) + 594\zeta(2)\ln 2 + 526\zeta(4) - 360\zeta(2)\ln^2 2 - 720t_4 \\
&+ \frac{14919}{16}\zeta(5) + \frac{169}{2}\zeta(3)\zeta(2) - \frac{2883}{4}\zeta(4)\ln 2 + 279\zeta(2)\ln^3 2 - 1116t_5 + \frac{363}{4}\zeta(6) - \frac{579}{2}\zeta^2(3) \\
&+ 300\zeta(3)\zeta(2)\ln 2 + 324\zeta(4)\ln^2 2 - 54\zeta(2)\ln^4 2 - 720t_{61} + 576t_{62}\bigg) + O(\epsilon^4) \,,
\end{aligned} \tag{61}$$

$$\begin{aligned}
Z_m^{(2)} &= \frac{13}{32\epsilon^2} + \frac{73}{64\epsilon} + \frac{475}{128} - \frac{3}{4}\zeta(3) - \frac{23}{8}\zeta(2) + 3\zeta(2)\ln 2 + \epsilon\bigg(\frac{2529}{256} - \frac{245}{16}\zeta(2) - \frac{47}{4}\zeta(3) \\
&+ 24\zeta(2)\ln 2 + \frac{63}{4}\zeta(4) - 6\zeta(2)\ln^2 2 - 12t_4\bigg) + \epsilon^2\bigg(\frac{13379}{512} - \frac{1831}{32}\zeta(2) - \frac{437}{8}\zeta(3) + 96\zeta(2)\ln 2 \\
&+ 80\zeta(4) - 48\zeta(2)\ln^2 2 - 96t_4 + \frac{609}{8}\zeta(5) + 9\zeta(3)\zeta(2) - \frac{93}{2}\zeta(4)\ln 2 + 18\zeta(2)\ln^3 2 - 72t_5\bigg) \\
&+ \epsilon^3\bigg(\frac{69001}{1024} - \frac{12613}{64}\zeta(2) - \frac{3115}{16}\zeta(3) + 321\zeta(2)\ln 2 + 259\zeta(4) - 192\zeta(2)\ln^2 2 - 384t_4 \\
&+ \frac{1011}{2}\zeta(5) + 49\zeta(3)\zeta(2) - 372\zeta(4)\ln 2 + 144\zeta(2)\ln^3 2 - 576t_5 + \frac{363}{8}\zeta(6) - \frac{579}{4}\zeta^2(3) \\
&+ 150\zeta(3)\zeta(2)\ln 2 + 162\zeta(4)\ln^2 2 - 27\zeta(2)\ln^4 2 - 360t_{61} + 288t_{62}\bigg) + O(\epsilon^4) \,,
\end{aligned} \tag{62}$$

$$\begin{aligned}
Z_2^{(3)} &= -\frac{131}{384\epsilon^3} - \frac{2141}{2304\epsilon^2} + \frac{1}{\epsilon}\bigg(-\frac{116489}{13824} + \frac{935}{96}\zeta(2) + \frac{17}{8}\zeta(3) - \frac{17}{2}\zeta(2)\ln 2\bigg) - \frac{2121361}{82944} - \frac{197731}{8640}\zeta(2) \\
&+ \frac{2803}{144}\zeta(3) + \frac{1367}{24}\zeta(2)\ln 2 - \frac{383}{8}\zeta(4) + 23\zeta(2)\ln^2 2 + 18t_4 - \frac{5}{16}\zeta(5) + \frac{3}{4}\zeta(3)\zeta(2) + \epsilon\bigg(-\frac{200754221}{2488320} \\
&- \frac{14806511}{86400}\zeta(2) - \frac{216391}{960}\zeta(3) + \frac{5687}{5}\zeta(2)\ln 2 - \frac{141}{4}\zeta(4) - \frac{8405}{24}\zeta(2)\ln^2 2 - \frac{1712}{3}t_4 - \frac{3201}{16}\zeta(5) \\
&+ \frac{835}{24}\zeta(3)\zeta(2) + \frac{689}{12}\zeta(4)\ln 2 - \frac{193}{3}\zeta(2)\ln^3 2 + \frac{388}{3}t_5 - \frac{899}{6}\zeta(6) + \frac{29}{32}\zeta^2(3) + 84\zeta(3)\zeta(2)\ln 2 \\
&- 60\zeta(4)\ln^2 2 + 96t_{63}\bigg) + \epsilon^2\bigg(-\frac{7232293891}{24883200} - \frac{8115586429}{7776000}\zeta(2) - \frac{384910567}{259200}\zeta(3) + \frac{18690613}{2700}\zeta(2)\ln 2 \\
&+ \frac{1946767}{576}\zeta(4) - \frac{3203351}{432}\zeta(2)\ln^2 2 - \frac{4052957}{540}t_4 + \frac{9351}{2}\zeta(5) + \frac{103579}{96}\zeta(3)\zeta(2) - \frac{9511}{4}\zeta(4)\ln 2 \\
&+ \frac{24373}{12}\zeta(2)\ln^3 2 - \frac{20576}{3}t_5 - \frac{110993}{144}\zeta(6) + \frac{30301}{48}\zeta^2(3) - \frac{1339}{4}\zeta(3)\zeta(2)\ln 2 - \frac{4637}{12}\zeta(4)\ln^2 2 \\
&+ \frac{763}{6}\zeta(2)\ln^4 2 + 988t_{61} - \frac{2704}{3}t_{62} + 62t_{63} + \frac{13801}{32}\zeta(7) - \frac{22743}{32}\zeta(4)\zeta(3) + \frac{9695}{32}\zeta(5)\zeta(2) \\
&- \frac{15435}{32}\zeta(6)\ln 2 - 49\zeta(3)\zeta(2)\ln^2 2 - 98t_{71} + 392t_{72} + 196t_4\zeta(3) + 576t_5\zeta(2) + 576t_{73}\bigg) + O(\epsilon^3) \,,
\end{aligned} \tag{63}$$

$$\begin{aligned}
Z_m^{(3)} &= -\frac{221}{1152\epsilon^3} - \frac{5561}{6912\epsilon^2} + \frac{1}{\epsilon}\bigg(-\frac{154445}{41472} + \frac{391}{96}\zeta(2) + \frac{13}{16}\zeta(3) - \frac{17}{4}\zeta(2)\ln 2\bigg) - \frac{3489365}{248832} - \frac{5783}{2880}\zeta(2) \\
&+ \frac{719}{72}\zeta(3) + \frac{89}{6}\zeta(2)\ln 2 - \frac{979}{48}\zeta(4) + \frac{65}{6}\zeta(2)\ln^2 2 + \frac{23}{3}t_4 + \frac{5}{8}\zeta(5) - \frac{3}{8}\zeta(3)\zeta(2) + \epsilon\bigg(-\frac{410529217}{7464960}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{6911609}{86400}\zeta(2)-\frac{240973}{4320}\zeta(3)+\frac{41969}{90}\zeta(2)\ln 2-\frac{15215}{288}\zeta(4)-\frac{2011}{18}\zeta(2)\ln^2 2-\frac{1756}{9}t_4-\frac{6985}{96}\zeta(5)\\
&+\frac{91}{8}\zeta(3)\zeta(2)+\frac{51}{8}\zeta(4)\ln 2-\frac{51}{2}\zeta(2)\ln^3 2+46t_5-25\zeta(6)-\frac{1}{2}\zeta^2(3)+\frac{63}{4}\zeta(3)\zeta(2)\ln 2\\
&-\frac{45}{4}\zeta(4)\ln^2 2+18t_{63}\Big)+\epsilon^2\Big(-\frac{52076602061}{223948800}-\frac{1249645817}{2592000}\zeta(2)-\frac{69525299}{129600}\zeta(3)+\frac{2040043}{675}\zeta(2)\ln 2\\
&+\frac{553243}{432}\zeta(4)-\frac{166889}{54}\zeta(2)\ln^2 2-\frac{397312}{135}t_4+\frac{109385}{72}\zeta(5)+\frac{42119}{96}\zeta(3)\zeta(2)-\frac{3421}{6}\zeta(4)\ln 2\\
&+673\zeta(2)\ln^3 2-2408t_5-\frac{57055}{96}\zeta(6)+\frac{2745}{16}\zeta^2(3)+\frac{235}{4}\zeta(3)\zeta(2)\ln 2-\frac{509}{4}\zeta(4)\ln^2 2+230t_{61}\\
&-184t_{62}+150t_{63}+\frac{531}{4}\zeta(7)-447\zeta(4)\zeta(3)-\frac{219}{2}\zeta(5)\zeta(2)-\frac{945}{8}\zeta(6)\ln 2-12\zeta(3)\zeta(2)\ln^2 2\\
&+\frac{153}{4}\zeta(2)\ln^4 2-24t_{71}+96t_{72}+48t_4\zeta(3)+516t_5\zeta(2)+516t_{73}\Big)+O(\epsilon^3)\,. \qquad (64)
\end{aligned}
$$

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