



High-precision four-loop mass and wave function renormalization in QED



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ARTICLE INFO

Article history:

Received 15 January 2020
 Received in revised form 27 January 2020
 Accepted 28 January 2020
 Available online 31 January 2020
 Editor: G.F. Giudice

Keywords:

Quantum electrodynamics
 Quantum chromodynamics
 Renormalization
 Feynman diagram
 Master integral

ABSTRACT

The 4-loop QED mass and wave function renormalization constants Z_2 and Z_m have been evaluated in the on-shell subtraction scheme with 1100 digits of precision. We also worked out the coefficients of the five color structures of the QCD renormalization constants Z_2^{OS} and Z_m^{OS} which can be obtained from QED-like diagrams. The results agree with lower precision results available in the literature. Analytical fits were also obtained for all these quantities.

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1. Introduction

The 4-loop QED contribution to the electron $g-2$ was calculated with 1100 digits of precision in Ref. [1]. In that paper high-precision numerical and analytical fits were also obtained for all the master integrals of the 4-loop self-mass QED diagrams. Therefore, other 4-loop quantities that can be expressed in terms of the same master integrals can be known with such precision. In Ref. [2] we used these results to obtain a high-precision value and an analytical fit of the 4-loop first derivative of the Dirac form factor $F_1'(0)$.

In this third paper, following the same approach as in Ref. [1,2], we calculate high-precision numerical values and analytical expression of the 4-loop QED wave function and mass renormalization constants Z_2 and Z_m , in the on-shell subtraction scheme. We infer also the values of the coefficients of five color structures of the QCD renormalization constants Z_2^{OS} and Z_m^{OS} which can be obtained from the QED result.

2. Definitions

The wave function renormalization constant Z_2 is defined as $\psi_0 = \sqrt{Z_2}\psi$, where ψ_0 and ψ are the bare and the renormalized electron field, respectively. The mass renormalization constant Z_m is defined as $m_0 = Z_m m$, where m_0 and m are the bare and the physical electron mass, respectively. Two-loop QED and QCD analytical results for on-shell Z_2 and Z_m were obtained in Ref. [3]; three-loop analytical results were obtained in Refs. [4,5].

In QCD $m_0 = Z_m^{\text{OS}} m^{\text{OS}} = Z_m^{\overline{\text{MS}}} m^{\overline{\text{MS}}}$. Being $Z_m^{\overline{\text{MS}}}$ known to sufficiently high degree of perturbative expansion [6–8], the ratio $m^{\text{OS}}/m^{\overline{\text{MS}}}$ can be used to determine the on-shell Z_m^{OS} ; the ratio $m^{\text{OS}}/m^{\overline{\text{MS}}}$ is known at two loops analytically [9], and at three loops it was obtained in numerical form in Ref. [10,11] and in analytical form in Ref. [5,12].

Four-loop QED and QCD numerical results of Z_2 were obtained in Ref. [13–15]; the four-loop numerical result of Z_m can be worked out from the QCD results for z_m and Z_m^{OS} of Ref. [13,16,17]; see also Ref. [18–20]. Both results have a precision of a few digits; see section 3.2 for more details.

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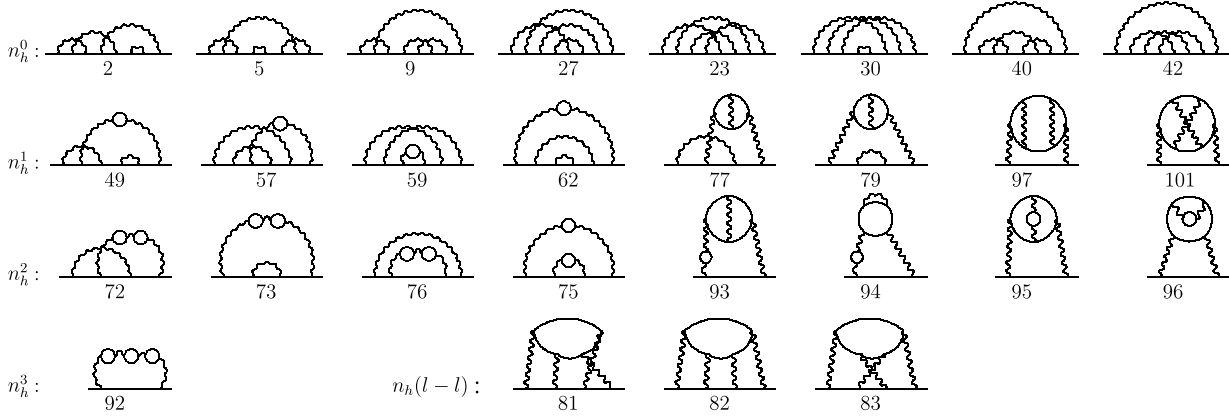


Fig. 1. A selection of some of 4-loop self-mass diagrams, with the indication of the part of Eq. (2) to which the diagram contributes. The numbering of the diagrams follows Ref. [1,2].

Z_2 and Z_m are gauge parameter independent (for Z_2 see Ref. [21–23]); in order to simplify the calculations, we choose the Feynman gauge.

We expand Z_2 and Z_m in power series of the bare coupling constant α_0 :

$$Z_2 = 1 + \sum_i \left(\frac{\alpha_0}{\pi}\right)^i \left(\Gamma(1 + \epsilon)(4\pi)^\epsilon m^{-2\epsilon}\right)^i Z_2^{(i)}, \quad Z_m = 1 + \sum_i \left(\frac{\alpha_0}{\pi}\right)^i \left(\Gamma(1 + \epsilon)(4\pi)^\epsilon m^{-2\epsilon}\right)^i Z_m^{(i)}. \quad (1)$$

We decompose the 4-loop coefficients $Z_2^{(4)}$ and $Z_m^{(4)}$ by considering QED with n_h different leptons of mass m ; in this way we separate the contributions coming from diagrams with different number of insertions of vacuum polarizations and light-light scattering diagrams. We use the notation n_h in order to adapt to the usual QCD convention of calling the number of massive fermions n_h and the number of massless fermions n_l . In this paper we do not consider diagrams with massless leptons. Therefore

$$\begin{aligned} Z_2^{(4)}(n_h) &= Z_2^{(4,0)} + n_h Z_2^{(4,1)} + n_h^2 Z_2^{(4,2)} + n_h^3 Z_2^{(4,3)} + n_h Z_2^{(4,l-l)}, \\ Z_m^{(4)}(n_h) &= Z_m^{(4,0)} + n_h Z_m^{(4,1)} + n_h^2 Z_m^{(4,2)} + n_h^3 Z_m^{(4,3)} + n_h Z_m^{(4,l-l)}. \end{aligned} \quad (2)$$

Clearly

$$Z_2^{(4)}(n_h = 1) = Z_2^{(4)}, \quad Z_m^{(4)}(n_h = 1) = Z_m^{(4)}. \quad (3)$$

In Fig. 1 we show a selection of some of the 4-loop self-mass diagrams, with the indication of the part of Eq. (2) to which the diagram contributes. The complete set of 104 diagrams is shown in Ref. [1,2].

2.1. Method

We briefly describe here the method used to obtain our results. At 4-loop level there are 104 QED self-mass diagrams. The contribution to $Z_2^{(4)}$ and $Z_m^{(4)}$ from each self-mass diagram is extracted by using projectors and taking traces with a FORM [24,25] program; it is a linear combination of Feynman integrals. Then, for each self-mass diagram a system of integration-by-parts (I.B.P.) identities [26,27] is build and solved [28] in order to reduce the Feynman integrals to linear combinations of master integrals. We used the systems of I.B.P. identities generated in Ref. [1] for the 4-loop $g=2$. Counterterms were added where needed, excluding vacuum polarizations, since we expand in the bare coupling constant α_0 . The renormalization constants are reduced to linear combinations of the 4-loop $g=2$ master integrals:

$$Z_2^{(4)} = \sum_{i=1}^N C_{2,i}(\epsilon) M_i(\epsilon), \quad Z_m^{(4)} = \sum_{i=1}^N C_{m,i}(\epsilon) M_i(\epsilon), \quad (4)$$

where $C_{2,i}$ and $C_{m,i}$ are rational functions in ϵ , $N = 334$. We use the numerical values and analytical fits of the master integrals M_i worked out in Ref. [1]. As a simple consistency check, we extracted $Z_1^{(4)}$ from the 891 4-loop vertex diagrams, and we checked numerically and analytically the Ward identity $Z_1^{(4)} = Z_2^{(4)}$.

3. Results

3.1. Numerical results

By substituting the numerical values of $M_i(\epsilon)$ in Eq. (4), we have obtained 1100-digits numerical values for $Z_2^{(4,x)}$ and $Z_m^{(4,x)}$. We show here results truncated to 40 digits for the sake of space. Full-precision results are available from the author. The numerical value of the coefficients of the powers of n_h in Eq. (2) are:

3.2.3. QCD mass renormalization constant

Now we consider the ratio between the QCD mass renormalization constants in the OS and $\overline{\text{MS}}$ scheme:

$$z_m(\mu) = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} = 1 + \sum_{n \geq 1} \left(\frac{\alpha_s(\mu)}{\pi} \right)^n z_m^{(n)}(\mu); \quad (22)$$

Using the notation of Ref. [17], z_m is decomposed in 23 color structures, and as above we can infer the coefficients of the five structures which involve only QED-like diagrams.

$$z_m^{(4)} = C_F^4 Z_m^{FFFF} + C_F^3 T n_h Z_m^{FFFH} + C_F^2 T^2 n_h^2 Z_m^{FFHH} + C_F T^3 n_h^3 Z_m^{FHHH} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c} z_m^{dFFH} + \dots \quad (23)$$

Our high-precision values are

$$\begin{aligned} z_m^{FFFF} &= -6.943004942063674366729783085730127607323, \\ z_m^{FFFH} &= -1.364670155556599206268818327382565081078, \\ z_m^{FFHH} &= 1.657513434712808758859954030175433075697, \\ z_m^{FHHH} &= -0.1490239616711777449135125845999523299403, \\ z_m^{dFFH} &= -3.947552272425901748117750000830748333155. \end{aligned} \quad (24)$$

The results of Ref. [17]

$$\begin{aligned} z_m^{FFFF} &= -6.983(805), \\ z_m^{FFFH} &= -1.3625(132), \\ z_m^{FFHH} &= 1.65752(31), \\ z_m^{FHHH} &= -0.14902, \\ z_m^{dFFH} &= -3.924(642), \end{aligned} \quad (25)$$

are in agreement with ours at the level of 0.16σ at worst.

3.3. Analytical fits

By substituting the analytical fits of $M_i(\epsilon)$ in Eq. (4), we have obtained the following analytical expressions:

$$\begin{aligned} Z_2^{(4)} &= \frac{3779}{18432\epsilon^4} + \frac{33053}{55296\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{515315}{73728} - \frac{7205}{768} \zeta(2) - \frac{131}{64} \zeta(3) + \frac{131}{16} \zeta(2) \ln 2 \right) + \frac{1}{\epsilon} \left(\frac{19571293}{663552} \right. \\ &+ \frac{154747}{2160} \zeta(2) - \frac{11521}{2304} \zeta(3) - \frac{29539}{192} \zeta(2) \ln 2 + \frac{3115}{64} \zeta(4) - \frac{215}{8} \zeta(2) \ln^2 2 - \frac{19}{4} t_4 - \frac{5}{64} \zeta(5) - \frac{21}{16} \zeta(3) \zeta(2) \Big) \\ &+ \frac{9565004502941}{87787929600} + \frac{1535743349}{691200} \zeta(2) + \frac{12128503957}{25401600} \zeta(3) + \frac{10500647}{17280} \zeta(2) \ln 2 \\ &- \frac{535034261}{1451520} \zeta(4) + \frac{5631023}{9072} \zeta(2) \ln^2 2 + \frac{7932313}{9072} t_4 - \frac{2050259}{17280} \zeta(5) \\ &+ \frac{9296423}{8640} \zeta(3) \zeta(2) - \frac{609737}{480} \zeta(4) \ln 2 + \frac{1739}{24} \zeta(2) \ln^3 2 - \frac{2671}{30} t_5 - \frac{715229459}{62208} \zeta(6) \\ &- \frac{4006421}{11520} \zeta^2(3) + \frac{1780957}{240} \zeta(3) \zeta(2) \ln 2 - \frac{4689809}{720} \zeta(4) \ln^2 2 - \frac{9674}{45} t_{61} + \frac{10276}{45} t_{62} + \frac{642767}{90} t_{63} \\ &+ \frac{1495323863}{580608} \zeta(7) - \frac{5775661427}{161280} \zeta(5) \zeta(2) - \frac{7455877}{10752} \zeta(4) \zeta(3) + \frac{2153119}{128} \zeta(6) \ln 2 \\ &- \frac{561331}{126} \zeta(3) \zeta(2) \ln^2 2 - \frac{19454}{63} t_{71} + \frac{9144}{7} t_{72} + \frac{4306}{7} t_4 \zeta(3) + \frac{553037}{15} t_5 \zeta(2) + \frac{135039}{5} t_{73} \\ &+ \sqrt{3} \left(\frac{15489}{320} \text{Cl}_4 \left(\frac{\pi}{3} \right) - \frac{1311089}{960} \zeta(2) \text{Cl}_2 \left(\frac{\pi}{3} \right) + \frac{3071}{60} v_{61} + \frac{2109}{50} v_{62} + \frac{7472227}{34320} v_{63} - \frac{8978057}{6480} v_{64} \right) \\ &+ \frac{5797}{16} v_{65} + \frac{115735}{96} \zeta(2) \text{Cl}_2^2 \left(\frac{\pi}{3} \right) + 18 v_{71} + 6 v_{72} - 36 \zeta(2) \text{Cl}_2 \left(\frac{\pi}{2} \right) - \frac{12606}{5} \zeta(2) \text{Cl}_2 \left(\frac{\pi}{2} \right)^2 \\ &+ \sqrt{3} \pi \left(-\frac{10163659}{230400} B_3 + \frac{224075873}{6220800} C_3 - \frac{11863}{7776} f_2(0, 0, 1) + \frac{56207}{23328} e_{51} - \frac{14615}{1728} e_{52} - \frac{45499}{20736} e_{61} \right. \\ &\left. + \frac{30961}{41472} e_{62} \right) + \zeta(2) \left(-\frac{2354}{243} f_1(0, 0, 1) - \frac{30961}{3456} e_{53} - \frac{673}{162} e_{54} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{507}{80}C_{81a} - 26C_{81b} + \frac{11}{2}C_{81c} - \frac{1057}{320}C_{83a} + \frac{91}{24}C_{83b} + \frac{7}{2}C_{83c} + O(\epsilon), \\
Z_m^{(4)} = & \frac{1547}{18432\epsilon^4} + \frac{25175}{55296\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{202951}{73728} - \frac{2737}{768}\zeta(2) - \frac{71}{128}\zeta(3) + \frac{119}{32}\zeta(2)\ln 2 \right) + \frac{1}{\epsilon} \left(\frac{27750271}{1990656} \right. \\
& + \frac{132763}{5760}\zeta(2) - \frac{1625}{768}\zeta(3) - \frac{10807}{192}\zeta(2)\ln 2 + \frac{6173}{384}\zeta(4) - \frac{553}{48}\zeta(2)\ln^2 2 + \frac{35}{24}t_4 - \frac{5}{8}\zeta(5) \\
& + \frac{21}{32}\zeta(3)\zeta(2) \left. \right) + \frac{1885204711}{23887872} + \frac{5804091169}{6220800}\zeta(2) + \frac{169571}{768}\zeta(3) + \frac{87397}{5760}\zeta(2)\ln 2 - \frac{1866527}{6912}\zeta(4) \\
& + \frac{74477}{288}\zeta(2)\ln^2 2 + \frac{62645}{144}t_4 - \frac{15459}{256}\zeta(5) + \frac{20351}{48}\zeta(3)\zeta(2) - \frac{33367}{64}\zeta(4)\ln 2 + \frac{357}{16}\zeta(2)\ln^3 2 \\
& + \frac{35}{4}t_5 - \frac{8229601}{1728}\zeta(6) - \frac{5605}{32}\zeta^2(3) + \frac{12297}{4}\zeta(3)\zeta(2)\ln 2 - \frac{10975}{4}\zeta(4)\ln^2 2 + 3040t_{63} - \frac{161}{128}\zeta(7) \\
& - \frac{27786101}{1920}\zeta(5)\zeta(2) - \frac{272627}{384}\zeta(4)\zeta(3) + \frac{252105}{32}\zeta(6)\ln 2 - \frac{3675}{2}\zeta(3)\zeta(2)\ln^2 2 + \frac{74368}{5}t_5\zeta(2) \\
& + \frac{53368}{5}t_{73} + \sqrt{3} \left(45\text{Cl}_4\left(\frac{\pi}{3}\right) - \frac{12757}{24}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{98}{5}v_{62} - \frac{4949}{9}v_{64} \right) + 180v_{65} \\
& + \frac{993}{2}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{3}\right) - 1008\zeta(2)\text{Cl}_2\left(\frac{\pi}{2}\right)^2 + \sqrt{3}\pi \left(-\frac{754571}{46080}B_3 + \frac{17787301}{1244160}C_3 - \frac{503}{1728}f_2(0, 0, 1) + \frac{895}{324}e_{51} \right. \\
& - \frac{1295}{216}e_{52} - \frac{1493}{1728}e_{61} + \frac{671}{3456}e_{62} \left. \right) + \zeta(2) \left(-\frac{100}{27}f_1(0, 0, 1) - \frac{671}{288}e_{53} - \frac{346}{27}e_{54} \right) - \frac{21}{8}C_{81a} - 10C_{81b} \\
& - \frac{21}{16}C_{83a} + \frac{5}{3}C_{83b} + O(\epsilon).
\end{aligned} \tag{27}$$

The analytical expressions of the separate contributions of Eq. (2) are:

$$\begin{aligned}
Z_2^{(4,0)} = & \frac{27}{2048\epsilon^4} + \frac{171}{2048\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{5835}{8192} - \frac{351}{256}\zeta(2) - \frac{27}{64}\zeta(3) + \frac{27}{16}\zeta(2)\ln 2 \right) + \frac{1}{\epsilon} \left(\frac{23865}{8192} + \frac{4171}{128}\zeta(2) \right. \\
& + \frac{2527}{256}\zeta(3) - \frac{4713}{64}\zeta(2)\ln 2 + \frac{909}{64}\zeta(4) - \frac{81}{8}\zeta(2)\ln^2 2 + \frac{57}{4}t_4 - \frac{25}{64}\zeta(5) - \frac{9}{16}\zeta(3)\zeta(2) \left. \right) + \frac{2033213}{98304} \\
& + \frac{6966313}{15360}\zeta(2) + \frac{352489}{11520}\zeta(3) + \frac{642189}{640}\zeta(2)\ln 2 + \frac{9612919}{23040}\zeta(4) + \frac{574357}{1440}\zeta(2)\ln^2 2 + \frac{2341}{720}t_4 \\
& - \frac{1149787}{5760}\zeta(5) + \frac{108635}{192}\zeta(3)\zeta(2) - \frac{83371}{160}\zeta(4)\ln 2 + \frac{161}{8}\zeta(2)\ln^3 2 + \frac{1727}{10}t_5 - \frac{38991047}{5184}\zeta(6) \\
& - \frac{756779}{11520}\zeta^2(3) + \frac{414137}{120}\zeta(3)\zeta(2)\ln 2 - \frac{1227247}{360}\zeta(4)\ln^2 2 - \frac{19994}{45}t_{61} + \frac{25156}{45}t_{62} + \frac{24104}{9}t_{63} \\
& - \frac{9389399}{82944}\zeta(7) - \frac{8309201}{512}\zeta(5)\zeta(2) + \frac{4441247}{4608}\zeta(4)\zeta(3) + \frac{1096627}{128}\zeta(6)\ln 2 - \frac{72457}{36}\zeta(3)\zeta(2)\ln^2 2 \\
& - \frac{136}{9}t_{71} - \frac{32}{3}t_{72} - \frac{1}{3}t_4\zeta(3) + \frac{50705}{3}t_5\zeta(2) + 12331t_{73} + \sqrt{3} \left(-\frac{6949}{192}\text{Cl}_4\left(\frac{\pi}{3}\right) - \frac{563899}{576}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) \right. \\
& + \frac{3071}{60}v_{61} + \frac{2109}{50}v_{62} + \frac{7472227}{34320}v_{63} - \frac{8978057}{6480}v_{64} \left. \right) + \frac{26125}{32}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{3}\right) - \frac{1995}{16}v_{65} - 18v_{71} - 6v_{72} \\
& + \sqrt{3}\pi \left(-\frac{32021}{25600}B_3 + \frac{3656149}{230400}C_3 - \frac{4775}{1296}f_2(0, 0, 1) - \frac{965143}{23328}e_{51} + \frac{32885}{576}e_{52} - \frac{10705}{6912}e_{61} + \frac{585}{512}e_{62} \right) \\
& + \zeta(2) \left(-\frac{2414}{243}f_1(0, 0, 1) - \frac{1755}{128}e_{53} + \frac{3565}{54}e_{54} \right) + O(\epsilon),
\end{aligned} \tag{28}$$

$$\begin{aligned}
Z_2^{(4,l)} = & -\frac{1}{8\epsilon} - \frac{205}{128} + \frac{2563}{2}\zeta(2) + \frac{35933}{192}\zeta(3) + \frac{1419}{2}\zeta(2)\ln 2 - \frac{15155}{192}\zeta(4) - \frac{679}{3}\zeta(2)\ln^2 2 + \frac{628}{3}t_4 - \frac{51259}{192}\zeta(5) \\
& + \frac{95239}{120}\zeta(3)\zeta(2) - \frac{20975}{16}\zeta(4)\ln 2 - 16t_5 - \frac{56584517}{10368}\zeta(6) - \frac{541547}{1920}\zeta^2(3) + \frac{46507}{10}\zeta(3)\zeta(2)\ln 2 \\
& - \frac{20947}{6}\zeta(4)\ln^2 2 + \frac{688}{3}t_{61} - \frac{992}{3}t_{62} + \frac{81632}{15}t_{63} + \frac{195131207}{72576}\zeta(7) - \frac{394782889}{20160}\zeta(5)\zeta(2) \\
& - \frac{6682045}{4032}\zeta(4)\zeta(3) + \frac{264123}{32}\zeta(6)\ln 2 - \frac{615463}{252}\zeta(3)\zeta(2)\ln^2 2 - \frac{18502}{63}t_{71} + \frac{27656}{21}t_{72} + \frac{12925}{21}t_4\zeta(3) \\
& + \frac{299512}{15}t_5\zeta(2) + \frac{73384}{5}t_{73} + \sqrt{3} \left(\frac{693}{8}\text{Cl}_4\left(\frac{\pi}{3}\right) - \frac{2523}{8}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) \right) + \frac{4969}{16}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{3}\right) + \frac{3873}{8}v_{65}
\end{aligned}$$

$$\begin{aligned}
 &+ 36v_{71} + 12v_{72} - 36\zeta(2)\text{Cl}_2\left(\frac{\pi}{2}\right) - \frac{12606}{5}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{2}\right) + \sqrt{3}\pi\left(-\frac{15399}{320}B_3 + \frac{1477}{192}C_3 + \frac{857}{288}f_2(0, 0, 1)\right) \\
 &+ \frac{39785}{972}e_{51} - \frac{13235}{216}e_{52} - \frac{1673}{2592}e_{61} - \frac{2053}{5184}e_{62} + \zeta(2)\left(\frac{20}{81}f_1(0, 0, 1) + \frac{2053}{432}e_{53} - \frac{5684}{81}e_{54}\right) \\
 &- \frac{507}{80}C_{81a} - 26C_{81b} + \frac{11}{2}C_{81c} - \frac{1057}{320}C_{83a} + \frac{91}{24}C_{83b} + \frac{7}{2}C_{83c} + O(\epsilon), \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 Z_2^{(4,1)} &= \frac{9}{128\epsilon^4} + \frac{131}{512\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{1747}{768} - \frac{135}{32}\zeta(2) - \frac{9}{8}\zeta(3) + \frac{9}{2}\zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{142385}{18432} + \frac{50989}{1152}\zeta(2)\right) \\
 &+ \frac{2777}{768}\zeta(3) - \frac{747}{8}\zeta(2)\ln 2 + \frac{891}{32}\zeta(4) - \frac{63}{4}\zeta(2)\ln^2 2 + t_4 + \frac{5}{16}\zeta(5) - \frac{3}{4}\zeta(3)\zeta(2) + \frac{934395461}{37324800} \\
 &+ \frac{8610967}{19440}\zeta(2) + \frac{809922361}{3110400}\zeta(3) - \frac{394219}{432}\zeta(2)\ln 2 - \frac{24640541}{34560}\zeta(4) + \frac{57655}{144}\zeta(2)\ln^2 2 + \frac{26603}{36}t_4 \\
 &+ \frac{450661}{1440}\zeta(5) - \frac{8861}{27}\zeta(3)\zeta(2) + \frac{136613}{240}\zeta(4)\ln 2 + \frac{121}{3}\zeta(2)\ln^3 2 - \frac{1886}{15}t_5 + \frac{92170207}{62208}\zeta(6) \\
 &- \frac{1}{32}\zeta^2(3) - \frac{10899}{16}\zeta(3)\zeta(2)\ln 2 + \frac{6185}{16}\zeta(4)\ln^2 2 - \frac{1957}{2}t_{63} + \sqrt{3}\left(-\frac{487}{240}\text{Cl}_4\left(\frac{\pi}{3}\right)\right) \\
 &- \frac{51373}{720}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3773}{48}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{3}\right) + \frac{23}{8}v_{65} + \sqrt{3}\pi\left(\frac{237613}{34560}B_3 + \frac{6777463}{933120}C_3\right) \\
 &- \frac{1733}{1944}f_2(0, 0, 1) + \frac{3695}{1296}e_{51} - \frac{3695}{864}e_{52} + O(\epsilon), \tag{30}
 \end{aligned}$$

$$\begin{aligned}
 Z_2^{(4,2)} &= \frac{3}{32\epsilon^4} + \frac{27}{128\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{4337}{1536} - \frac{25}{8}\zeta(2) - \frac{1}{2}\zeta(3) + 2\zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{167545}{13824} - \frac{3827}{2160}\zeta(2)\right) \\
 &- \frac{31843}{2304}\zeta(3) + \frac{55}{6}\zeta(2)\ln 2 + \frac{53}{8}\zeta(4) - \zeta(2)\ln^2 2 - 20t_4 + \frac{40705891003}{1828915200} + \frac{3264023}{48600}\zeta(2) \\
 &+ \frac{3697035983}{152409600}\zeta(3) - \frac{2133}{10}\zeta(2)\ln 2 - \frac{5625569}{362880}\zeta(4) + \frac{4638449}{90720}\zeta(2)\ln^2 2 - \frac{964289}{22680}t_4 + \frac{1889}{54}\zeta(5) \\
 &+ \frac{1609}{36}\zeta(3)\zeta(2) - \frac{15}{2}\zeta(4)\ln 2 + 12\zeta(2)\ln^3 2 - 120t_5 + \sqrt{3}\pi\left(-\frac{111683}{69120}B_3 + \frac{9696589}{1866240}C_3\right) \\
 &+ \frac{145}{1944}f_2(0, 0, 1) + O(\epsilon), \tag{31}
 \end{aligned}$$

$$\begin{aligned}
 Z_2^{(4,3)} &= \frac{1}{36\epsilon^4} + \frac{41}{864\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{679}{576} - \frac{2}{3}\zeta(2)\right) + \frac{1}{\epsilon}\left(\frac{71143}{10368} - \frac{103}{30}\zeta(2) - \frac{14}{3}\zeta(3) + 4\zeta(2)\ln 2\right) \\
 &+ \frac{166888903}{3919104} - \frac{31451}{1350}\zeta(2) - \frac{37691}{1512}\zeta(3) + \frac{103}{5}\zeta(2)\ln 2 + \frac{259}{12}\zeta(4) - \frac{10}{3}\zeta(2)\ln^2 2 - \frac{104}{3}t_4 + O(\epsilon), \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 Z_m^{(4,0)} &= \frac{27}{2048\epsilon^4} + \frac{117}{2048\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{3063}{8192} - \frac{135}{256}\zeta(2) - \frac{27}{128}\zeta(3) + \frac{27}{32}\zeta(2)\ln 2\right) + \frac{1}{\epsilon}\left(\frac{28653}{8192} + \frac{1479}{128}\zeta(2)\right) \\
 &+ \frac{471}{256}\zeta(3) - \frac{1773}{64}\zeta(2)\ln 2 + \frac{747}{128}\zeta(4) - \frac{63}{16}\zeta(2)\ln^2 2 + \frac{45}{8}t_4 - \frac{15}{32}\zeta(5) + \frac{9}{32}\zeta(3)\zeta(2) + \frac{2907301}{98304} \\
 &+ \frac{166719}{1024}\zeta(2) + \frac{1109}{128}\zeta(3) + \frac{56537}{128}\zeta(2)\ln 2 + \frac{7613}{128}\zeta(4) + \frac{4149}{32}\zeta(2)\ln^2 2 + \frac{765}{16}t_4 - \frac{11123}{256}\zeta(5) \\
 &+ \frac{3429}{16}\zeta(3)\zeta(2) - \frac{13867}{64}\zeta(4)\ln 2 + \frac{81}{16}\zeta(2)\ln^3 2 + \frac{135}{4}t_5 - \frac{22615}{8}\zeta(6) + \frac{219}{32}\zeta^2(3) + 1235\zeta(3)\zeta(2)\ln 2 \\
 &- 1390\zeta(4)\ln^2 2 + 1018t_{63} - \frac{189}{64}\zeta(7) + \frac{110845}{256}\zeta(4)\zeta(3) - \frac{1658509}{256}\zeta(5)\zeta(2) + \frac{108045}{32}\zeta(6)\ln 2 \\
 &- \frac{1575}{2}\zeta(3)\zeta(2)\ln^2 2 + 6698t_5\zeta(2) + 4898t_{73} + \sqrt{3}\left(-\frac{2983}{8}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{98}{5}v_{62} - \frac{4949}{9}v_{64}\right) \\
 &+ \frac{675}{2}\zeta(2)\text{Cl}_2^2\left(\frac{\pi}{3}\right) + \sqrt{3}\pi\left(-\frac{3123}{5120}B_3 + \frac{33383}{5120}C_3 - \frac{781}{576}f_2(0, 0, 1) - \frac{5245}{324}e_{51} + \frac{1615}{72}e_{52}\right) \\
 &- \frac{721}{1152}e_{61} + \frac{1075}{2304}e_{62} + \zeta(2)\left(-\frac{100}{27}f_1(0, 0, 1) - \frac{1075}{192}e_{53} + \frac{245}{9}e_{54}\right) + O(\epsilon), \tag{33}
 \end{aligned}$$

$$\begin{aligned}
Z_m^{(4,l)} = & \frac{1}{\epsilon} \left(-\frac{1}{16} + \frac{15}{32} \zeta(3) \right) - \frac{135}{128} + 555 \zeta(2) + \frac{18625}{192} \zeta(3) + 48 \zeta(2) \ln 2 - \frac{803}{64} \zeta(4) - 20 \zeta(2) \ln^2 2 + 80 t_4 \\
& - \frac{7205}{96} \zeta(5) + \frac{1273}{4} \zeta(3) \zeta(2) - 603 \zeta(4) \ln 2 - \frac{2134769}{864} \zeta(6) - \frac{1453}{8} \zeta^2(3) + 2086 \zeta(3) \zeta(2) \ln 2 + 2384 t_{63} \\
& - 1490 \zeta(4) \ln^2 2 + \frac{217}{128} \zeta(7) - \frac{877789}{768} \zeta(4) \zeta(3) - \frac{30694567}{3840} \zeta(5) \zeta(2) + \frac{36015}{8} \zeta(6) \ln 2 - 1050 \zeta(3) \zeta(2) \ln^2 2 \\
& + \frac{40878}{5} t_5 \zeta(2) + \frac{28878}{5} t_{73} + \sqrt{3} \left(45 \text{Cl}_4 \left(\frac{\pi}{3} \right) - 129 \zeta(2) \text{Cl}_2 \left(\frac{\pi}{3} \right) \right) + 126 \zeta(2) \text{Cl}_2^2 \left(\frac{\pi}{3} \right) + 180 v_{65} \\
& - 1008 \zeta(2) \text{Cl}_2^2 \left(\frac{\pi}{2} \right) + \sqrt{3} \pi \left(-\frac{2859}{160} B_3 + \frac{1253}{480} C_3 + \frac{299}{216} f_2(0, 0, 1) + \frac{1450}{81} e_{51} - \frac{725}{27} e_{52} - \frac{823}{3456} e_{61} \right. \\
& \left. - \frac{1883}{6912} e_{62} \right) + \zeta(2) \left(\frac{1883}{576} e_{53} - \frac{1081}{27} e_{54} \right) - \frac{21}{8} C_{81a} - 10 C_{81b} - \frac{21}{16} C_{83a} + \frac{5}{3} C_{83b} + O(\epsilon), \tag{34}
\end{aligned}$$

$$\begin{aligned}
Z_m^{(4,1)} = & \frac{9}{256 \epsilon^4} + \frac{11}{64 \epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{2507}{3072} - \frac{93}{64} \zeta(2) - \frac{9}{32} \zeta(3) + \frac{15}{8} \zeta(2) \ln 2 \right) + \frac{1}{\epsilon} \left(\frac{4363}{1024} + \frac{91}{6} \zeta(2) \right) \\
& + \frac{51}{128} \zeta(3) - 35 \zeta(2) \ln 2 + \frac{71}{8} \zeta(4) - \frac{25}{4} \zeta(2) \ln^2 2 + \frac{5}{2} t_4 - \frac{5}{32} \zeta(5) + \frac{3}{8} \zeta(3) \zeta(2) + \frac{1273135}{36864} \\
& + \frac{1312775}{6912} \zeta(2) + \frac{87181}{768} \zeta(3) - \frac{3187}{8} \zeta(2) \ln 2 - \frac{2063917}{6912} \zeta(4) + 133 \zeta(2) \ln^2 2 + 320 t_4 \\
& + \frac{1567}{32} \zeta(5) - \frac{257}{2} \zeta(3) \zeta(2) + \frac{4653}{16} \zeta(4) \ln 2 + \frac{45}{4} \zeta(2) \ln^3 2 + 15 t_5 + \frac{34251}{64} \zeta(6) \\
& - \frac{3}{8} \zeta^2(3) - \frac{987}{4} \zeta(3) \zeta(2) \ln 2 + \frac{545}{4} \zeta(4) \ln^2 2 - 362 t_{63} - \frac{89}{3} \sqrt{3} \zeta(2) \text{Cl}_2 \left(\frac{\pi}{3} \right) + 33 \zeta(2) \text{Cl}_2^2 \left(\frac{\pi}{3} \right) \\
& + \sqrt{3} \pi \left(\frac{3599}{1280} B_3 + \frac{97783}{34560} C_3 - \frac{227}{648} f_2(0, 0, 1) + \frac{85}{81} e_{51} - \frac{85}{54} e_{52} \right) + O(\epsilon), \tag{35}
\end{aligned}$$

$$\begin{aligned}
Z_m^{(4,2)} = & \frac{11}{384 \epsilon^4} + \frac{11}{64 \epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{1555}{1536} - \frac{5}{4} \zeta(2) - \frac{1}{16} \zeta(3) + \zeta(2) \ln 2 \right) + \frac{1}{\epsilon} \left(\frac{3343}{1152} - \frac{329}{180} \zeta(2) - \frac{97}{32} \zeta(3) \right) \\
& + \frac{53}{12} \zeta(2) \ln 2 + \frac{131}{96} \zeta(4) - \frac{4}{3} \zeta(2) \ln^2 2 - \frac{20}{3} t_4 - \frac{183577}{55296} + \frac{1777157}{48600} \zeta(2) + \frac{7043}{576} \zeta(3) \\
& - \frac{31397}{360} \zeta(2) \ln 2 - \frac{44239}{1728} \zeta(4) + \frac{335}{18} \zeta(2) \ln^2 2 + \frac{5}{9} t_4 + \frac{439}{48} \zeta(5) + \frac{239}{12} \zeta(3) \zeta(2) \\
& + \frac{15}{2} \zeta(4) \ln 2 + 6 \zeta(2) \ln^3 2 - 40 t_5 + \sqrt{3} \pi \left(-\frac{8159}{11520} B_3 + \frac{726817}{311040} C_3 + \frac{5}{162} f_2(0, 0, 1) \right) + O(\epsilon), \tag{36}
\end{aligned}$$

$$\begin{aligned}
Z_m^{(4,3)} = & \frac{1}{144 \epsilon^4} + \frac{47}{864 \epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{317}{576} - \frac{1}{3} \zeta(2) \right) + \frac{1}{\epsilon} \left(\frac{103963}{31104} - \frac{83}{45} \zeta(2) - \frac{43}{24} \zeta(3) + 2 \zeta(2) \ln 2 \right) \\
& + \frac{3579989}{186624} - \frac{7622}{675} \zeta(2) - \frac{1529}{144} \zeta(3) + \frac{166}{15} \zeta(2) \ln 2 + \frac{347}{48} \zeta(4) - \frac{8}{3} \zeta(2) \ln^2 2 - \frac{40}{3} t_4 + O(\epsilon). \tag{37}
\end{aligned}$$

In the above expressions we use these combinations of constants:

$$t_4 = a_4 + \frac{1}{24} \ln^4 2, \quad t_5 = a_5 + \frac{1}{12} \zeta(2) \ln^3 2 - \frac{1}{120} \ln^5 2, \tag{38}$$

$$t_{61} = b_6 - a_5 \ln 2 + \zeta(5) \ln 2 + \frac{1}{6} \zeta(3) \ln^3 2 - \frac{1}{12} \zeta(2) \ln^4 2 + \frac{1}{144} \ln^6 2, \tag{39}$$

$$t_{62} = a_6 - \frac{1}{48} \zeta(2) \ln^4 2 + \frac{1}{720} \ln^6 2, \tag{40}$$

$$\begin{aligned}
t_{71} = & d_7 - 2b_6 \ln 2 + 4a_6 \ln 2 + 2a_5 \ln^2 2 - \frac{49}{32} \zeta^2(3) \ln 2 - \frac{95}{32} \zeta(5) \ln^2 2 + \frac{1}{8} \zeta(4) \ln^3 2 \\
& - \frac{1}{3} \zeta(3) \ln^4 2 + \frac{1}{12} \zeta(2) \ln^5 2 - \frac{1}{120} \ln^7 2, \tag{41}
\end{aligned}$$

$$t_{72} = b_7 - 3a_7 - a_6 \ln 2 - \frac{1}{2} \zeta(5) \ln^2 2 + \frac{1}{48} \zeta(4) \ln^3 2 - \frac{1}{24} \zeta(3) \ln^4 2 + \frac{1}{120} \zeta(2) \ln^5 2 - \frac{1}{1680} \ln^7 2, \tag{42}$$

$$t_{73} = \left(a_4 - \frac{1}{4} \zeta(2) \ln^2 2 + \frac{7}{16} \zeta(3) \ln 2 + \frac{1}{24} \ln^4 2 \right) \zeta(2) \ln 2, \tag{43}$$

$$v_{61} = \text{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \text{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \text{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{27}{26}\text{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{207}{104}\text{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10}{3}a_4\text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7}{4}\zeta(3)\text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{8}\zeta(3)\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{5}{72}\zeta(3)\zeta(2)\pi - \frac{5}{6}\text{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{5}{36}\text{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 - \frac{27413}{67392}\zeta(5)\pi + \frac{4975}{11583}\zeta(4)\text{Cl}_2\left(\frac{\pi}{3}\right), \tag{44}$$

$$v_{62} = \zeta(2)\left(\text{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{3}{2}\text{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{1}{6}\zeta(3)\pi + \frac{1}{108}\zeta(2)\pi \ln 2 - \frac{5}{2}\text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\ln 2 - \frac{15}{4}\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\ln 2 + \frac{25}{12}\text{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{661}{1188}\text{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\right), \tag{45}$$

$$v_{63} = \text{Cl}_6\left(\frac{\pi}{3}\right) - \frac{3}{4}\zeta(4)\text{Cl}_2\left(\frac{\pi}{3}\right), \quad v_{64} = \text{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{91}{66}\zeta(4)\text{Cl}_2\left(\frac{\pi}{3}\right), \tag{46}$$

$$v_{65} = \text{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \text{Cl}_2\left(\frac{\pi}{3}\right)\text{Cl}_4\left(\frac{\pi}{3}\right), \tag{47}$$

$$v_{71} = \text{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 4\text{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{27}{8}\text{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{135}{16}\text{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{27}{2}\text{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\text{Cl}_4\left(\frac{\pi}{3}\right) + \frac{3}{2}\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\text{Cl}_4\left(\frac{\pi}{3}\right) + \frac{145}{132}\text{Cl}_6\left(\frac{\pi}{3}\right)\pi, \tag{48}$$

$$v_{72} = \zeta(2)\left(\text{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 2\text{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{9}{4}\text{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{9}{2}\text{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3}{2}\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\text{Cl}_2\left(\frac{\pi}{3}\right)\right), \tag{49}$$

$$v_{73} = \zeta(2)\left(\text{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right) + \text{Cl}_2\left(\frac{\pi}{2}\right)\text{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right) - \frac{1}{2}\text{Cl}_4\left(\frac{\pi}{2}\right)\pi + \frac{1}{4}\text{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\right), \tag{50}$$

$$e_{51} = f_2(0, 2, 0) - \frac{9}{4}f_2(0, 0, 1)\ln 2, \quad e_{52} = f_2(0, 1, 1) - \frac{3}{8}f_2(0, 0, 2) - \frac{3}{2}f_2(0, 0, 1)\ln 2, \tag{51}$$

$$e_{53} = f_1(1, 0, 1) - f_1(0, 1, 1) + \frac{1}{4}f_1(0, 0, 2), \quad e_{54} = e_{51} - \frac{3}{2}e_{52}, \tag{52}$$

$$e_{61} = f_2(2, 1, 0) + \frac{7}{3}f_2(1, 2, 0) - 2f_2(1, 1, 1) + \frac{40}{27}f_2(0, 3, 0) - \frac{7}{3}f_2(0, 2, 1) + f_2(0, 1, 2) - 30e_{54}\ln 2, \tag{53}$$

$$e_{62} = f_2(2, 0, 1) + \frac{14}{3}f_2(1, 2, 0) - 2f_2(1, 1, 1) - 2f_2(1, 0, 2) - \frac{370}{27}f_2(0, 3, 0) + \frac{85}{3}f_2(0, 2, 1) - 22f_2(0, 1, 2) + 7f_2(0, 0, 3) + 11\zeta(2)f_2(0, 0, 1) - 20e_{54}\ln 2. \tag{54}$$

In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i} i^{-n}$, $b_6 = H_{0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $b_7 = H_{0,0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\text{Cl}_n(\theta) = \text{ImLi}_n(e^{i\theta})$. C_{8xy} are the ϵ^0 coefficients of the ϵ -expansion of six master integrals (see Ref. [1]). $H_{i_1, i_2, \dots}(x)$ are the harmonic polylogarithms [29–31]. The integrals f_j are defined as follows:

$$f_m(i, j, k) = \int_1^9 ds D_1(s)\text{Re}\left(\sqrt{3^{m-1}}D_m(s)\right)\left(s - \frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s), \tag{55}$$

$$D_m(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(m-1-(2m-3)\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right); \tag{56}$$

B_3 and C_3 have hypergeometric expressions [32,33]:

$$B_3 = \frac{\pi}{27}\sqrt{3}\left(4\tilde{F}_3\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}; 1\right) - 4\tilde{F}_3\left(\frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}; 1\right)\right), \tag{57}$$

$$C_3 = \frac{\pi}{27}\sqrt{3}\left(4\tilde{F}_3\left(\frac{1}{6}, \frac{1}{3}, \frac{4}{3}, -\frac{1}{2}; 1\right) - 4\tilde{F}_3\left(-\frac{7}{6}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{2}; 1\right)\right), \tag{58}$$

$$4\tilde{F}_3\left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix}; x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)} {}_4F_3\left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix}; x\right). \tag{59}$$

4. Z_2 and Z_m to three loops

For completeness we list here the analytical expressions of Z_2 and Z_m at one, two, and three loops, expanded in ϵ up to the level needed for five-loop renormalization (two powers in ϵ more than the results of Ref. [4] and one power more than Ref. [14,17]).

$$Z_2^{(1)} = Z_m^{(1)} = -\frac{3}{4\epsilon} - \frac{1}{1-2\epsilon}, \quad (60)$$

$$\begin{aligned} Z_2^{(2)} = & \frac{17}{32\epsilon^2} + \frac{229}{192\epsilon} + \frac{8453}{1152} - \frac{3}{2}\zeta(3) + 6\zeta(2)\ln 2 - \frac{55}{8}\zeta(2) + \epsilon \left(\frac{86797}{6912} - \frac{419}{16}\zeta(2) - \frac{203}{8}\zeta(3) \right. \\ & + \frac{93}{2}\zeta(2)\ln 2 + \frac{63}{2}\zeta(4) - 12\zeta(2)\ln^2 2 - 24t_4 \left. \right) + \epsilon^2 \left(\frac{2197589}{41472} - \frac{3393}{32}\zeta(2) - \frac{779}{8}\zeta(3) + 180\zeta(2)\ln 2 \right. \\ & + \frac{1073}{8}\zeta(4) - 93\zeta(2)\ln^2 2 - 186t_4 + \frac{609}{4}\zeta(5) + 18\zeta(3)\zeta(2) - 93\zeta(4)\ln 2 + 36\zeta(2)\ln^3 2 - 144t_5 \left. \right) \\ & + \epsilon^3 \left(\frac{22329277}{248832} - \frac{22387}{64}\zeta(2) - \frac{5769}{16}\zeta(3) + 594\zeta(2)\ln 2 + 526\zeta(4) - 360\zeta(2)\ln^2 2 - 720t_4 \right. \\ & + \frac{14919}{16}\zeta(5) + \frac{169}{2}\zeta(3)\zeta(2) - \frac{2883}{4}\zeta(4)\ln 2 + 279\zeta(2)\ln^3 2 - 1116t_5 + \frac{363}{4}\zeta(6) - \frac{579}{2}\zeta^2(3) \\ & \left. + 300\zeta(3)\zeta(2)\ln 2 + 324\zeta(4)\ln^2 2 - 54\zeta(2)\ln^4 2 - 720t_{61} + 576t_{62} \right) + O(\epsilon^4), \quad (61) \end{aligned}$$

$$\begin{aligned} Z_m^{(2)} = & \frac{13}{32\epsilon^2} + \frac{73}{64\epsilon} + \frac{475}{128} - \frac{3}{4}\zeta(3) - \frac{23}{8}\zeta(2) + 3\zeta(2)\ln 2 + \epsilon \left(\frac{2529}{256} - \frac{245}{16}\zeta(2) - \frac{47}{4}\zeta(3) \right. \\ & + 24\zeta(2)\ln 2 + \frac{63}{4}\zeta(4) - 6\zeta(2)\ln^2 2 - 12t_4 \left. \right) + \epsilon^2 \left(\frac{13379}{512} - \frac{1831}{32}\zeta(2) - \frac{437}{8}\zeta(3) + 96\zeta(2)\ln 2 \right. \\ & + 80\zeta(4) - 48\zeta(2)\ln^2 2 - 96t_4 + \frac{609}{8}\zeta(5) + 9\zeta(3)\zeta(2) - \frac{93}{2}\zeta(4)\ln 2 + 18\zeta(2)\ln^3 2 - 72t_5 \left. \right) \\ & + \epsilon^3 \left(\frac{69001}{1024} - \frac{12613}{64}\zeta(2) - \frac{3115}{16}\zeta(3) + 321\zeta(2)\ln 2 + 259\zeta(4) - 192\zeta(2)\ln^2 2 - 384t_4 \right. \\ & + \frac{1011}{2}\zeta(5) + 49\zeta(3)\zeta(2) - 372\zeta(4)\ln 2 + 144\zeta(2)\ln^3 2 - 576t_5 + \frac{363}{8}\zeta(6) - \frac{579}{4}\zeta^2(3) \\ & \left. + 150\zeta(3)\zeta(2)\ln 2 + 162\zeta(4)\ln^2 2 - 27\zeta(2)\ln^4 2 - 360t_{61} + 288t_{62} \right) + O(\epsilon^4), \quad (62) \end{aligned}$$

$$\begin{aligned} Z_2^{(3)} = & -\frac{131}{384\epsilon^3} - \frac{2141}{2304\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{116489}{13824} + \frac{935}{96}\zeta(2) + \frac{17}{8}\zeta(3) - \frac{17}{2}\zeta(2)\ln 2 \right) - \frac{2121361}{82944} - \frac{197731}{8640}\zeta(2) \\ & + \frac{2803}{144}\zeta(3) + \frac{1367}{24}\zeta(2)\ln 2 - \frac{383}{8}\zeta(4) + 23\zeta(2)\ln^2 2 + 18t_4 - \frac{5}{16}\zeta(5) + \frac{3}{4}\zeta(3)\zeta(2) + \epsilon \left(-\frac{200754221}{2488320} \right. \\ & - \frac{14806511}{86400}\zeta(2) - \frac{216391}{960}\zeta(3) + \frac{5687}{5}\zeta(2)\ln 2 - \frac{141}{4}\zeta(4) - \frac{8405}{24}\zeta(2)\ln^2 2 - \frac{1712}{3}t_4 - \frac{3201}{16}\zeta(5) \\ & + \frac{835}{24}\zeta(3)\zeta(2) + \frac{689}{12}\zeta(4)\ln 2 - \frac{193}{3}\zeta(2)\ln^3 2 + \frac{388}{3}t_5 - \frac{899}{6}\zeta(6) + \frac{29}{32}\zeta^2(3) + 84\zeta(3)\zeta(2)\ln 2 \\ & \left. - 60\zeta(4)\ln^2 2 + 96t_{63} \right) + \epsilon^2 \left(-\frac{7232293891}{24883200} - \frac{8115586429}{7776000}\zeta(2) - \frac{384910567}{259200}\zeta(3) + \frac{18690613}{2700}\zeta(2)\ln 2 \right. \\ & + \frac{1946767}{576}\zeta(4) - \frac{3203351}{432}\zeta(2)\ln^2 2 - \frac{4052957}{540}t_4 + \frac{9351}{2}\zeta(5) + \frac{103579}{96}\zeta(3)\zeta(2) - \frac{9511}{4}\zeta(4)\ln 2 \\ & + \frac{24373}{12}\zeta(2)\ln^3 2 - \frac{20576}{3}t_5 - \frac{110993}{144}\zeta(6) + \frac{30301}{48}\zeta^2(3) - \frac{1339}{4}\zeta(3)\zeta(2)\ln 2 - \frac{4637}{12}\zeta(4)\ln^2 2 \\ & + \frac{763}{6}\zeta(2)\ln^4 2 + 988t_{61} - \frac{2704}{3}t_{62} + 62t_{63} + \frac{13801}{32}\zeta(7) - \frac{22743}{32}\zeta(4)\zeta(3) + \frac{9695}{32}\zeta(5)\zeta(2) \\ & \left. - \frac{15435}{32}\zeta(6)\ln 2 - 49\zeta(3)\zeta(2)\ln^2 2 - 98t_{71} + 392t_{72} + 196t_4\zeta(3) + 576t_5\zeta(2) + 576t_{73} \right) + O(\epsilon^3), \quad (63) \end{aligned}$$

$$\begin{aligned} Z_m^{(3)} = & -\frac{221}{1152\epsilon^3} - \frac{5561}{6912\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{154445}{41472} + \frac{391}{96}\zeta(2) + \frac{13}{16}\zeta(3) - \frac{17}{4}\zeta(2)\ln 2 \right) - \frac{3489365}{248832} - \frac{5783}{2880}\zeta(2) \\ & + \frac{719}{72}\zeta(3) + \frac{89}{6}\zeta(2)\ln 2 - \frac{979}{48}\zeta(4) + \frac{65}{6}\zeta(2)\ln^2 2 + \frac{23}{3}t_4 + \frac{5}{8}\zeta(5) - \frac{3}{8}\zeta(3)\zeta(2) + \epsilon \left(-\frac{410529217}{7464960} \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{6911609}{86400}\zeta(2) - \frac{240973}{4320}\zeta(3) + \frac{41969}{90}\zeta(2)\ln 2 - \frac{15215}{288}\zeta(4) - \frac{2011}{18}\zeta(2)\ln^2 2 - \frac{1756}{9}t_4 - \frac{6985}{96}\zeta(5) \\
& + \frac{91}{8}\zeta(3)\zeta(2) + \frac{51}{8}\zeta(4)\ln 2 - \frac{51}{2}\zeta(2)\ln^3 2 + 46t_5 - 25\zeta(6) - \frac{1}{2}\zeta^2(3) + \frac{63}{4}\zeta(3)\zeta(2)\ln 2 \\
& - \frac{45}{4}\zeta(4)\ln^2 2 + 18t_{63} \Big) + \epsilon^2 \left(-\frac{52076602061}{223948800} - \frac{1249645817}{2592000}\zeta(2) - \frac{69525299}{129600}\zeta(3) + \frac{2040043}{675}\zeta(2)\ln 2 \right. \\
& + \frac{553243}{432}\zeta(4) - \frac{166889}{54}\zeta(2)\ln^2 2 - \frac{397312}{135}t_4 + \frac{109385}{72}\zeta(5) + \frac{42119}{96}\zeta(3)\zeta(2) - \frac{3421}{6}\zeta(4)\ln 2 \\
& + 673\zeta(2)\ln^3 2 - 2408t_5 - \frac{57055}{96}\zeta(6) + \frac{2745}{16}\zeta^2(3) + \frac{235}{4}\zeta(3)\zeta(2)\ln 2 - \frac{509}{4}\zeta(4)\ln^2 2 + 230t_{61} \\
& - 184t_{62} + 150t_{63} + \frac{531}{4}\zeta(7) - 447\zeta(4)\zeta(3) - \frac{219}{2}\zeta(5)\zeta(2) - \frac{945}{8}\zeta(6)\ln 2 - 12\zeta(3)\zeta(2)\ln^2 2 \\
& \left. + \frac{153}{4}\zeta(2)\ln^4 2 - 24t_{71} + 96t_{72} + 48t_4\zeta(3) + 516t_5\zeta(2) + 516t_{73} \right) + O(\epsilon^3). \tag{64}
\end{aligned}$$

Acknowledgements

This work has been supported by the Supporting Talent in ReSearch at Padova University (UniPD STARS Grant 2017 “Diagrammalgebra”).

I wish to thank Pierpaolo Mastrolia for the encouragement and the support. I wish to thank Thomas Gehrmann for providing me the access to the computing facilities of the Institute for Theoretical Physics of Zurich.

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