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Complexity for holographic superconductors with the nonlinear electrodynamics

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Abstract

We systematically study the complexity of a strip-shaped subregion in a fully backreacted holographic model of a superconductor with the nonlinear electrodynamics by the "complexity=volume" (CV) conjecture, and compare it with the holographic entanglement entropy. We consider three types of typical nonlinear electrodynamics and find that the holographic complexity can be utilized as a good probe of the superconductor phase transition in the nonlinear electrodynamics like the holographic entanglement entropy does. For the operator $\mathcal{O}_-$, the complexity decreases (or increases) monotonically as the absolute value of the nonlinear parameter $|b|$ grows in the superconducting (or normal) phase, which is the opposite of the behavior of the holographic entanglement entropy, and this property holds for various types of the nonlinear electrodynamics. For the operator $\mathcal{O}_+$, in the superconducting phase, it is interesting to note that the complexity is a monotonic decreasing function of $|b|$ for the Logarithmic nonlinear electrodynamics (LNE), but in systems with the Born-Infeld nonlinear electrodynamics (BINE) and Exponential nonlinear electrodynamics (ENE), as the parameter $|b|$ increases, the complexity first decreases and arrives at its minimum at some threshold, then increases monotonously. Whereas the non-monotonic variation of the holographic entanglement entropy can be seen in all the three types of the nonlinear electrodynamics, concretely, it first rises, then descends with larger $|b|$, and has a peak at the inflection point. Furthermore, comparing with the BINE and LNE, we find that the ENE has stronger effect on the condensation formation, the subregion complexity and the entanglement entropy of the holographic superconductors with backreaction.

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1. Introduction

In the last decades, the anti-de Sitter/conformal field theories (AdS/CFT) correspondence [1–3], which can relate the strongly correlated conformal field theory living on the AdS boundary with a weakly coupled gravity theory in the bulk, has made great development and been successfully applied in various areas of modern physics. High-temperature superconductivity involving the strong correlation is one of the unsolved mysteries in modern condensed matter physics. Since the AdS/CFT correspondence was originally applied to the construction of superconductinglike phase transitions [4,5], this duality has been known as a powerful approach to explore the high-temperature superconductor systems and provided some meaningful theoretical insights to understand the physics behind these systems. Reviews of the holographic superconductors can be found in Refs. [6,7] and references therein. Most of the studies on the holographic superconductors are carried out in the framework of Maxwell electrodynamics. Along with the conventional Maxwell electrodynamic theory, the nonlinear electrodynamics theories, which essentially imply the higher derivative corrections of the gauge field, have also got a lot of attention. It is also of great interest and importance to investigate the effects of nonlinear electrodynamics on the holographic superconductors, which can help to understand the influences of the $1/N$ or $1/\lambda$ (λ is the 't Hooft coupling) corrections on the holographic dual models. The Born-Infeld electrodynamics (BINE) [8] proposed in 1934 is the pioneering study on the nonlinear electrodynamics to address the problem of the infinite self energies for charged point particles arising in the Maxwell theory, and it is an important electric-magnetic duality invariance theory [9]. Jing and Chen introduced the first holographic superconductor model in the Born-Infeld electrodynamics [10]. Along this line, other types of nonlinear electrodynamics in the context of gravitational field have also been introduced, which can also remove the divergence of the electric field at the origin, similar to Born-Infeld nonlinear electrodynamics [11]. Among them, two kinds of well-known nonlinear electrodynamics are Logarithmic nonlinear electrodynamics (LNE) and Exponential nonlinear electrodynamics (ENE). The nonlinear electromagnetic field in logarithmic form appears in the description of vacuum polarization effects, which was obtained as exact 1-loop corrections for electrons in a uniform electromagnetic field background [12]. It was introduced by Soleng as a Born-Infeld-like Lagrangian with a logarithmic term which can be added as a correction to the original Born-Infeld one [13]. The ENE was introduced in Ref. [14] to obtain the black hole solution whose asymptotic properties are the same as the charged BTZ solution, and its singularity is much weaker than the Einstein-Maxwell theory [15,16]. The two typical forms of Born-Infeld-like electrodynamics possess the special properties such as the absence of the shock waves, the birefringence phenomena [17] as well as enjoying an electric-magnetic duality [18]. And they would reduce to Maxwell electrodynamics in the region of the weak field or the small nonlinear parameter. Various models of holographic superconductors in the regime of nonlinear electrodynamics have been explored widely in the literatures (see e.g. [19–28]). In [20], the authors investigated systematically the holographic superconductors with the three kinds of typical Born-Infeld-like nonlinear electrodynamics in the probe limit. It was disclosed from the behavior of the boundary operator in the CFT that the nonlinear corrections to the usual Maxwell field will make it harder for the holographic superconductor to form. In addition, different types of nonlinear electrodynamics display different effects on formation of the scalar hair, the phase transition point, and the gap frequency. Compared with the BINE and LNE, the ENE has stronger effect on the condensation formation and conductivity for the holographic superconductors. In this paper, we would like to do more discussions on fully backreacted holographic superconductors in the frame of nonlinear electrodynamics theory from the aspects of the entanglement entropy and the complexity which connect the geometric quantities in the bulk spacetime with the entanglement properties of the boundary field theory.

The entanglement entropy is an important non-local quantity in the quantum information theory. As it turns out that this quantity is reflected in the bulk geometry according to the AdS/CFT duality. The computation of the entanglement entropy of the boundary field theory in a holographic framework is initiated by Ryu and Takayanagi [29,30]. Specifically, the Ryu-Takayanagi conjecture states that the holographic entanglement entropy of a subregion $\mathcal{A}$ with its complement on the boundary is proportional to the area of a minimal surface $\gamma_{\mathcal{A}}$ in the bulk which ends on the boundary of $\mathcal{A}$, i.e., $\partial\mathcal{A} = \partial\gamma_{\mathcal{A}}$, the formula is given by, in the units $c = \hbar = k_b = 1$,

$$S_{\mathcal{A}} = \frac{Area(\gamma_{\mathcal{A}})}{4G_N}, \tag{1}$$

where G_N is the Newton's constant. Since the entanglement entropy is sensitive to the number of available degrees of freedom for the strongly coupled system, keeping track of them, it was introduced as a tool to describe different phases and their corresponding phase transitions. The holographic entanglement entropy was firstly evaluated in the s-wave holographic superconducting transition in [31]. Starting with Ref. [31], there have been further efforts to employ the entanglement entropy as a probe of phase transitions in various holographic models [32–40].

Nevertheless, the entanglement entropy is not enough to understand the rich geometric structures that exist behind the horizon because it only grows for a very short time [41]. Thus, another important concept from the quantum computation theory has been introduced, the complexity, which is by definition the minimum number of simple unitary operations (gates) that are required to obtain a desired target state from a reference state. Two main conjectures for the holographic dual of the complexity have recently been proposed. One is known as “complexity=volume” (CV) conjecture [42,43], which suggests that the complexity of a state is in proportion to the maximal volume V of the codimension-one surface connecting the codimension-two time slices on two AdS boundaries ($C_V \sim V/G_N\mathcal{L}$). The other is the “complexity=action” (CA) conjecture [44,45], which says that the complexity of the state corresponds to the on-shell bulk gravitational action I_{WDW} within the Wheeler-DeWitt (WDW) patch ($C_A \sim I_{WDW}/\pi\hbar$). Here we will focus on the CV conjecture. Though the choice of the length scale $\mathcal{L}$ in the CV conjecture for different backgrounds is ambiguous, with the right choice of $\mathcal{L}$, the CV conjecture yields significant results and avoids dealing with the region of the singularity involved in the on-shell action computation which is the main difficulty arising in the CA conjecture. Concretely, we evaluate the holographic complexity of subregions following Alishahiha's proposal [46]. Thus, the holographic complexity for a subregion $\mathcal{A}$ is proportional to the volume enclosed by the minimal surface (Ryu-Takayanagi surface):

$$C_{\mathcal{A}} = \frac{Volume(\gamma_{\mathcal{A}})}{8\pi L G_N}, \tag{2}$$

where L is the AdS curvature radius. The complexity essentially measures how difficult it is to turn a quantum state into another and so it is expected that the holographic complexity can reflect a phase transition on the boundary field theory. Momeni et al. [47] and Zangeneh et al. [48] firstly discussed the holographic complexity in one dimensional s-wave superconductor by the analytical and numerical methods, respectively. Later, the holographic complexity has successively been studied in the $2+1$ dimensional superconductor [49], $1+1$ dimensional p-wave superconductor [50], Stückelberg superconductor [51], unbalanced superconductors [52] and QCD phase transition [53]. As it turns out, the holographic complexity indeed captures the phase transition like the holographic entanglement entropy would, and these two quantities do not behave in the same manner which means that the complexity may provide the information different from that reflected by the entanglement entropy.

In this paper we perform a calculation of the subregion complexity in holographic superconductors with full backreaction in the Exponential, Born-Infeld and Logarithmic nonlinear electrodynamics, respectively. The authors of [36] have discussed the holographic entanglement entropy in the metal/superconductor phase transition for the Born-Infeld electrodynamics. Recently, the holographic subregion complexity in the same model has also been studied in Ref. [54]. It was found that the entropy and the complexity conjectured with the volume can be a good probe to study the properties of the phase transition in this holographic superconducting system. Particularly, for the operator $\mathcal{O}_+$, with the fixed temperature below the critical value, an interesting non-monotonic behavior in terms of the Born-Infeld factor can be seen in both the entanglement entropy and complexity. So the motivation for completing this work is two fold. On one level, it is of great interesting to see the effects of the exponential and logarithmic parameter on the behavior of the complexity in the fully backreacted holographic superconductor models, and to examine whether the holographic subregion complexity is useful in probing the properties of the phase transition system with the ENE and LNE. On another more speculative level, it would be important to systematically investigate the holographic complexity in the holographic dual models with several typical nonlinear electrodynamics and see some universal feature for the effects of the higher derivative corrections to the gauge field on the complexity, which may help us to enhance and supplement the description of the phase transition in holographic superconductors in the nonlinear electromagnetic generalization.

The plan of the work is the following. In the next section, we introduce the construction of a fully backreacted holographic superconductor with the Exponential, Born-Infeld and Logarithmic nonlinear electrodynamics in the AdS black hole spacetime. In section 3, we compute the entanglement entropy and the complexity for a strip subregion in the holographic dual models with the ENE, BINE and LNE, respectively, and comparing the numerical results in the three types of nonlinear electrodynamics and analyzing the detail behaviors. Finally, section 4 is devoted to conclusions.

2. Holographic superconductor model in the nonlinear electrodynamics with backreaction

The background geometry of a black hole and nonlinear gauge field coupling with a charged scalar field is governed by the action

$$\begin{aligned} S = & \int d^d x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) \right] \\ & + \int d^d x \sqrt{-g} \left[\mathcal{L}(F^2) - |\nabla_\mu \psi - i q A_\mu \psi|^2 - m^2 |\psi|^2 \right], \end{aligned} \tag{3}$$

where $\Lambda = -(d-1)(d-2)/2L^2$ is the cosmological constant with the asymptotic AdS curvature radius L, and R and g are the Ricci scalar and the determinant of the metric, respectively. The scalar field ψ with the charge q and mass m as well as the gauge field A_μ are real. The term $\mathcal{L}(F^2)$ represents the Lagrangian of three classes of Born-Infeld-like nonlinear electrodynamics

$$\mathcal{L}(F^2)=\begin{cases}\frac{1}{4b^2}(e^{-b^2F^2}-1), & \text{ENE},\\ \frac{1}{b^2}\left(1-\sqrt{1+\frac{1}{2}b^2F^2}\right), & \text{BINE},\\ -\frac{2}{b^2}\ln\left(1+\frac{1}{8}b^2F^2\right), & \text{LNE},\end{cases} \tag{4}$$

where $F^2=F_{\mu\nu}F^{\mu\nu}$ is the quadratic term with the strength of the electromagnetic field $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Here b is the nonlinear parameter which can give us the higher derivative corrections of the gauge fields. With the same value of b, we can discuss the difference in the three types of the holographic dual models with the nonlinear electrodynamics quantitatively. In the limit $b\to 0$, the Lagrangian $\mathcal{L}(F^2)$ will reduce to the Maxwell form $\mathcal{L}(F^2)=-F^2/4$.

Since we will take the full backreaction into account, we consider the metric in the form

$$ds^2=-f(r)e^{-\chi(r)}dt^2+\frac{dr^2}{f(r)}+r^2h_{ij}dx^idx^j. \tag{5}$$

The function $f(r)$ gives the position of the event horizon of the black hole, i.e., $f(r_+)=0$. The Hawking temperature is given by

$$T_H=\frac{f'(r_+)e^{-\chi(r_+)/2}}{4\pi}. \tag{6}$$

Taking the following ansatz for the matter fields

$$A=\phi(r)dt\,,\quad \psi=\psi(r)\,, \tag{7}$$

we can get the equations of motion for the scalar field ψ and metric function χ

$$\begin{aligned}&\psi''+\left(\frac{f'}{f}+\frac{d-2}{r}-\frac{\chi'}{2}\right)\psi'+\frac{1}{f}\left(\frac{q^2e^\chi\phi^2}{f}-m^2\right)\psi=0,\\ &\chi'+\frac{2r}{d-2}\left(\psi'^2+\frac{q^2e^\chi\phi^2\psi^2}{f^2}\right)=0,\end{aligned} \tag{8}$$

which are the same forms for the three types of nonlinear electrodynamics (4). For the gauge field ϕ and function f, however, the corresponding equations of motion turn out to be

$$\begin{cases}\left(1+4b^2e^\chi\phi'^2\right)\phi''+\left(\frac{d-2}{r}+\frac{\chi'}{2}\right)\phi'+2b^2e^\chi\chi'\phi'^3-\frac{2q^2\psi^2e^{-2b^2e^\chi\phi'^2}}{f}\phi=0,\\ f'-\left[\frac{(d-1)r}{L^2}-\frac{(d-3)f}{r}\right]\\ \quad+\frac{r}{d-2}\left[m^2\psi^2+f\left(\psi'^2+\frac{q^2e^\chi\phi^2\psi^2}{f^2}\right)+\frac{1-e^{2b^2e^\chi\phi'^2}(1-4b^2e^\chi\phi'^2)}{4b^2}\right]=0,\quad \text{ENE},\end{cases} \tag{9}$$

$$\begin{cases}\phi''+\left(\frac{d-2}{r}+\frac{\chi'}{2}\right)\phi'-\frac{(d-2)b^2e^\chi}{r}\phi'^3-\frac{2q^2\psi^2(1-b^2e^\chi\phi'^2)^{\frac{3}{2}}}{f}\phi=0,\\ f'-\left[\frac{(d-1)r}{L^2}-\frac{(d-3)f}{r}\right]\\ \quad+\frac{r}{d-2}\left[m^2\psi^2+f\left(\psi'^2+\frac{q^2e^\chi\phi^2\psi^2}{f^2}\right)+\frac{1-\sqrt{1-b^2e^\chi\phi'^2}}{b^2\sqrt{1-b^2e^\chi\phi'^2}}\right]=0,\quad \text{BINE},\end{cases} \tag{10}$$

$$\begin{cases} \left(1+\frac{b^2}{4}e^{\chi}\phi'^2\right)\phi''+\left(\frac{d-2}{r}+\frac{\chi'}{2}\right)\phi' \\ \quad +\left(\frac{\chi'}{2}-\frac{d-2}{r}\right)\frac{b^2}{4}e^{\chi}\phi'^3-\left(1-\frac{b^2}{4}e^{\chi}\phi'\right)^2\frac{2q^2\psi^2}{f}\phi=0, \\ f'-\left[\frac{(d-1)r}{L^2}-\frac{(d-3)f}{r}\right]+\frac{r}{d-2}\left[m^2\psi^2+f\left(\psi'^2+\frac{q^2e^{\chi}\phi^2\psi^2}{f^2}\right)\right. \\ \quad \left.+\frac{2}{b^2}\ln\left(1-\frac{b^2}{4}e^{\chi}\phi'^2\right)+\frac{e^{\chi}\phi'^2}{1-\frac{b^2}{4}e^{\chi}\phi'^2}\right]=0, \quad \text{LNE}. \end{cases} \tag{11}$$

In the Eqs. (8)-(11), the prime denotes the derivative with respect to r, and we scale $16\pi G=1$.

We are interested in both the normal phase and superconducting phase to study the holographic subregion complexity of the fully backreacted superconductor with three types of nonlinear electrodynamics, so we have to solve the above equations. For this, we need the boundary conditions at the black hole horizon $r=r_+$ and the asymptotic AdS boundary $r\to\infty$. The behaviors of various fields at the horizon can be given by the following analytic expansion near $r=r_+$

$$\begin{aligned} &\psi(r)=\psi_0+\psi_1(r-r_+)+\psi_2(r-r_+)^2+\cdots, \\ &\phi(r)=\phi_1(r-r_+)+\phi_2(r-r_+)^2+\cdots, \\ &f(r)=f_1(r-r_+)+f_2(r-r_+)^2+\cdots, \\ &\chi(r)=\chi_0+\chi_1(r-r_+)+\chi_2(r-r_+)^2+\cdots, \end{aligned} \tag{12}$$

where the regular boundary condition $\phi(r_+)=0$ was used, for $\phi(r)dt$ to have finite norm at the horizon. As the spacetime is asymptotically AdS, at infinity ($r\to\infty$), the asymptotic behaviors of the solutions are

$$\chi\to 0, \quad f\approx\frac{r^2}{L^2}, \quad \phi\approx\mu-\frac{\rho}{r^{d-3}}, \quad \psi\approx\frac{\psi_-}{r^{\Delta_-}}+\frac{\psi_+}{r^{\Delta_+}}, \tag{13}$$

with $\Delta_\pm=\left[(d-1)\pm\sqrt{(d-1)^2+4m^2L^2}\right]/2$ is the conformal dimension of the operators, μ and ρ represent the chemical potential and charge density in the dual field theory respectively. From the AdS/CFT correspondence, ψ_- and ψ_+ can be dual to the vacuum expectation value of an operator $\mathcal{O}$ in the dual field theory while the other is the source. Note that, provided Δ_- is larger than the unitarity bound, both ψ_- and ψ_+ are normalizable and can be used to define the dual scalar operator, $\psi_-=\langle\mathcal{O}_-\rangle$ and $\psi_+=\langle\mathcal{O}_+\rangle$, respectively. Thus, in the following discussion, we will choose either ψ_- or ψ_+ as the source and turn it off, then the other corresponds to the condensate. Specifically, we impose either the condition

$$\psi_-=0, \quad and \quad \psi_+=\langle\mathcal{O}_+\rangle, \tag{14}$$

or

$$\psi_+=0, \quad and \quad \psi_-=\langle\mathcal{O}_-\rangle. \tag{15}$$

At high temperature, the state is normal and there is no condensate. The solution is simply the black hole with the nonlinear electrodynamics. For the temperature below some critical value, the condensation operator is turned on, i.e., $\psi\neq 0$. Then, there are no analytical solutions for the field equations, but we can solve them numerically. This new type of solution with the scalar hair represents the superconducting phase of the system.

For convenience, we apply a new dimensionless coordinate $z=r_+/r$ with $0\le z\le 1$ corresponding to the region $r_+<r<\infty$. On the other hand, in this system, there are some useful scaling symmetries obtained from Eqs. (8)-(11)

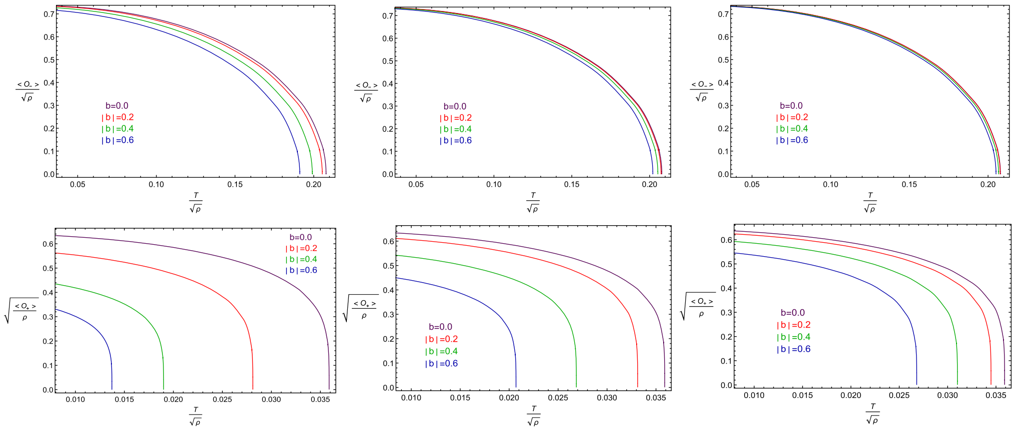

Fig. 1. (Color online) The condensation operators $\mathcal{O}_-$ and $\mathcal{O}_+$ with the ENE (left two panels), BINE (middle two panels) and LNE (right two panels) in terms of temperature in the backreacted holographic superconductor model for $d = 4$ dimension and the mass of the scalar field $m^2 = -2$. In each panel, the various colors represent different value of the nonlinear parameter, i.e. $|b| = 0$ (purple), 0.2 (red), 0.4 (green) and 0.6 (blue) respectively.

$$r \to \lambda r\,, \quad (t, x, y) \to \frac{1}{\lambda}(t, x, y)\,, \quad \phi \to \lambda \phi\,, \quad f \to \lambda^2 f, \tag{16}$$

$$L \to \lambda L\,, \quad r \to \lambda r\,, \quad t \to \lambda t\,, \quad q \to \frac{1}{\lambda} q, \tag{17}$$

where λ is a real positive number. So in the following discussion, we can use them to scale L unity for simplicity, and also set $q = 1$ and $r_+ = 1$ in the concrete numerical computations without any loss of generality. Hereafter, we focus on the case of the four-dimensional AdS black hole spacetime, i.e., $d = 4$. And, considering that the choices of the scalar field mass will not qualitatively modify our results, we choose $m^2 = -2$ above the Breitenlohner-Freedman (BF) bound $m^2_{BF} = -\frac{(d-1)^2}{4} = -\frac{9}{4}$ when performing calculations.

For different nonlinear parameter b, the scalar operators $\mathcal{O}_-$ and $\mathcal{O}_+$ with the ENE (left two panels), BINE (middle two panels) and LNE (right two panels) as a function of temperature are explicitly presented in Fig. 1. We can see that the condensations of both operators appear at a critical temperature T_c, which corresponds to a superconducting phase. Tables 1 and 2 show that the critical temperature T_c for the operators $\mathcal{O}_-$ and $\mathcal{O}_+$ becomes lower with the increasing absolute value of the nonlinear parameter $|b|$, and this feature can be shown in the ENE, BINE and LNE. This implies that the higher nonlinear electrodynamics correction to the usual Maxwell field makes the scalar hair harder to be developed in the full-backreaction model. In addition, comparing the numerical results of the condensation in the three kinds of the nonlinear electrodynamics considered here, we find that for a fixing b, the ENE leads to the smaller critical temperature T_c than the BINE and LNE, that is to say, the phase transition is more difficult to take place in a model with the ENE. And the LNE has the minimal effect on the condensation formation in the backreacted model among the three types of Born-Infeld-like nonlinear electrodynamics.

3. Holographic subregion complexity

In this section, we will use the shooting method to numerically investigate the complexity of the subregion in the holographic superconductor models with the exponential nonlinear elec-

Table 1
The critical temperature T_c for the operator $\mathcal{O}_-$ with different values of b for $d=4$ and $m^2=-2$.

$\lvert b \rvert$	0	0.1	0.2	0.3	0.4	0.5	0.6
ENE	0.208017	0.207334	0.205417	0.202563	0.199082	0.195218	0.191142
BINE	0.208017	0.207844	0.207326	0.206473	0.205301	0.203828	0.202079
LNE	0.208017	0.207931	0.207672	0.207245	0.206659	0.205922	0.205044

Table 2
The critical temperature T_c for the operator $\mathcal{O}_+$ with different values of b for $d=4$ and $m^2=-2$.

$\lvert b \rvert$	0	0.1	0.2	0.3	0.4	0.5	0.6
ENE	0.0358912	0.0333035	0.0281052	0.0230542	0.0190007	0.0159609	0.0137244
BINE	0.0358912	0.0351617	0.0331269	0.0301907	0.0268639	0.0236021	0.0207002
LNE	0.0358912	0.035525	0.0344856	0.0329215	0.0310134	0.0289297	0.0268066

trodynamics, Born-Infeld electrodynamics and the logarithmic nonlinear electrodynamics. We expect that the holographic complexity would suggest more deep physics about the superconducting phase transitions in the dual systems. The computation of the holographic entanglement entropy is also presented for completeness and compared with the results of the subregion complexity.

Now we consider strip subregion $\mathcal{A}$ defined by $-\frac{l}{2} \le x \le \frac{l}{2}$ and $-\frac{R}{2} < y < \frac{R}{2}$, where l is the size of region $\mathcal{A}$ and R is a regulator which can be set to infinity. According to the Ryu-Takayanagi proposal [29], the holographic entanglement entropy $\mathcal{S}_{\mathcal{A}}$ is proportional to the area of the minimal extended surface $\gamma_{\mathcal{A}}$ whose boundary matches with that of the stripe $\mathcal{A}$, i.e., $\partial\gamma_{\mathcal{A}} = \partial\mathcal{A}$. Thus, with the induced metric of the hypersurface $\gamma_{\mathcal{A}}$

$$ds^2_{induced} = \frac{1}{z^2}\left[\frac{1}{z^2 f(z)} + \left(\frac{dx}{dz}\right)^2\right]dz^2 + \frac{1}{z^2}dy^2, \tag{18}$$

the holographic entanglement entropy associated with the surface area reads

$$4G_4 S_{\mathcal{A}} = Area(\gamma_{\mathcal{A}}) = \int dy \int_{-l/2}^{l/2} \frac{dx}{z^2}\sqrt{1 + \frac{1}{z^2 f(z)}\left(\frac{dz}{dx}\right)^2}. \tag{19}$$

Since none of the coordinates (z, x) is independent of the other, z is considered as a function of x in the above. Using the Hamiltonian approach, we obtain the minimality condition

$$\frac{dz}{dx} = \pm\sqrt{z^2 f(z)\left(\frac{z_*^4}{z^4} - 1\right)}, \tag{20}$$

in which z_* denotes the turning point of the extremal surface and fulfills the condition $\frac{dz}{dx}|_{z_*} = 0$. Taking $x(z_*) = 0$, the integral of the condition (20) gives

$$x(z) = \int_z^{z_*} \frac{z^2 dz}{\sqrt{(z_*^4 - z^4)z^2 f(z)}}. \tag{21}$$

The width l of the interval can be now evaluated as

$$\frac{l}{2} = x(\epsilon \to 0) = \int_{\epsilon}^{z_*} \frac{z^2 dz}{\sqrt{(z_*^4 - z^4) z^2 f(z)}}, \tag{22}$$

where ϵ is a UV cutoff. After minimizing the area, we arrive at the resulting entanglement entropy of the form

$$S_{\mathcal{A}} = \frac{R}{2G_4} \int_{\epsilon}^{z_*} \frac{z_*^2}{z^2} \frac{dz}{\sqrt{(z_*^4 - z^4) z^2 f(z)}}, \tag{23}$$

with $R = \int dy$.

In holographic framework of the CV (complexity=volume) conjecture, following the proposal by Alishahiha [46], the holographic complexity for this subregion $\mathcal{A}$ is related to the volume surrounded by the aforementioned minimal surface (Ryu-Takayanagi surface) $\gamma_{\mathcal{A}}$. So we have

$$C_{\mathcal{A}} = \frac{R}{4\pi G_4} \int_{\epsilon}^{z_*} dz \frac{x(z)}{z^4 \sqrt{f(z)}}. \tag{24}$$

Here, the holographic subregion complexity is time-independent, i.e., the minimal surface $\gamma_{\mathcal{A}}$ is totally outside of the black hole horizon.

It is worth noting that both the holographic entanglement entropy $S_{\mathcal{A}}$ and subregion complexity $C_{\mathcal{A}}$ are formally divergent. The UV divergent term of $S_{\mathcal{A}}$ behaves like $R/(2G_4\epsilon)$ and will not change after the operator condensation because it is only sensitive to UV quantities. Therefore, subtracting this divergence from $S_{\mathcal{A}}$ in Eq. (23), we have the finite part s of the entanglement entropy which is independent of the cutoff and is physical important. For the holographic complexity, the divergent part of $C_{\mathcal{A}}$ is associated with a function of z_* as $F(z_*)/\epsilon^2$, and the function $F(z_*)$ has different form under different situations, so it is hard to find a general form of the divergence analytically. However, in view of the fact that the value of the universal term is invariant for different cutoffs, we can find the value of $F(z_*)$ numerically by subtracting the holographic complexity in Eq. (24) with two different values of cutoff ϵ_1 and ϵ_2, that is,

$$F(z_*) = \frac{4\pi G_4 [C_{\mathcal{A}}(\epsilon_1) - C_{\mathcal{A}}(\epsilon_2)]}{R(\epsilon_1^{-2} - \epsilon_2^{-2})}. \tag{25}$$

Taking advantage of the values of $F(z_*)$ in various situations, we can confirm the singular part in $C_{\mathcal{A}}$ and pick up the finite term c.

Under the scaling symmetries (16), the width l, the entanglement entropy s, and the subregion complexity c can be transformed into

$$l \to \frac{1}{\lambda} l, \quad s \to \lambda s, \quad c \to \lambda c, \tag{26}$$

which allow us to employ the following dimensionless quantities in the numerical computation for convenience

$$l\sqrt{\rho}, \quad \frac{s}{\sqrt{\rho}}, \quad \frac{c}{\sqrt{\rho}}. \tag{27}$$

3.1. Operator $\mathcal{O}_-$

Figs. 2, 3 and 4 show the holographic entanglement entropy s as well as the holographic subregion complexity c of the operator $\mathcal{O}_-$ as a function of temperature with the fixed width

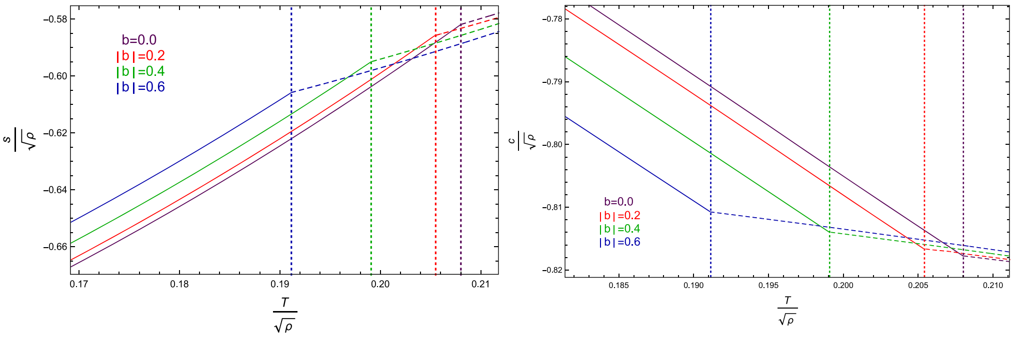

Fig. 2. (Color online) The holographic entanglement entropy (left) and subregion complexity (right) of the operator $\mathcal{O}_-$ versus temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the holographic superconductor with ENE. We scale $q = 1$ and $r_+ = 1$ in the numerical calculation.

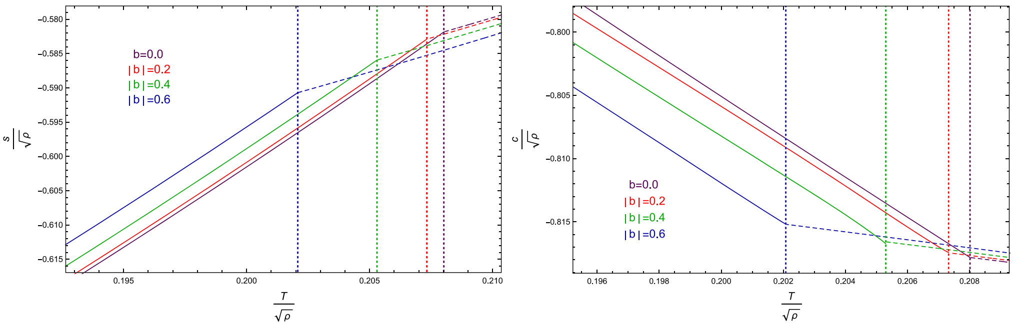

Fig. 3. (Color online) The holographic entanglement entropy (left) and subregion complexity (right) of the operator $\mathcal{O}_-$ versus temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the holographic superconductor with BINE.

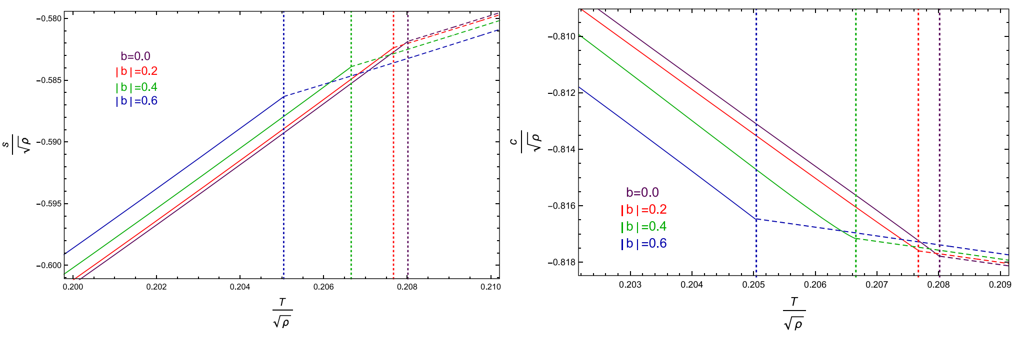

Fig. 4. (Color online) The holographic entanglement entropy (left) and subregion complexity (right) of the operator $\mathcal{O}_-$ versus temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the holographic superconductor with LNE.

of the strip subregion $l\sqrt{\rho} = 1$ for different strengths of the nonlinear parameter b in the ENE, BINE and LNE respectively. In each panel, the dashed lines are for the normal phases and the solid lines are for the superconducting phases.

As can be seen in Figs. 2, 3 and 4, both the holographic complexity and entanglement entropy are continuous but the slopes of them are discontinuous at the critical temperature T_c indicated by the vertical dot-dashed lines, which signalizes the occurrence of the second order phase transition. Also, for the three types of the nonlinear electrodynamics, we observe that the transition temperature read from the plots of the complexity c agrees with the critical one reflected by the plots of the entanglement entropy s for each given nonlinear parameter b, and the value of T_c decreases with the increase of the absolute value of the factor $|b|$, that is to say, the higher nonlinear electrodynamics corrections will make the scalar hair more difficult to be developed. Furthermore, we can find that, for a fixed strength of the nonlinear parameter b (except the case of $b = 0$, i.e., the usual Maxwell electrodynamics), the value of the critical temperature of the ENE system is the lowest and the one of the LNE system is the highest, which is consistent with the previous findings in the behaviors of the scalar operator $\mathcal{O}_-$ in Fig. 1 and Table 1. Apart from the above common characteristics, the holographic subregion complexity behaves quite differently from the holographic entanglement entropy. The complexity c monotonically decreases as the temperature grows up, and c always has a larger value in the superconducting phase than in the normal phase for various values of b. On the contrary, the entanglement entropy s is a monotonic increasing function of the temperature, then the superconducting phase always has a smaller s than the normal phase. Since the entanglement entropy is a measure of the degrees of freedom in the field theory, the physical interpretation for this behavior of the holographic entanglement entropy is that during the phase transition as the temperature becomes lower, the free charge carriers in the normal state are continuously condensed to Cooper pairs, so the formation of the Cooper pairs suppress the number of the effective degrees of freedom, which leads to the drop of the entropy. For the holographic complexity, which describes the difficulty of turning a certain reference state into a target state, the larger c at lower temperatures implies that the system in the superconducting phase becomes more complicated.

On the other hand, it is shown that the nonlinear parameter b has different effects on the holographic complexity and entanglement entropy. The complexity c increases as the absolute value of the nonlinear parameter $|b|$ increases for the normal phase, but decreases as $|b|$ increases for the superconducting phase. While an inverse behavior is observed in the entanglement entropy, namely, the entanglement entropy s reduces as $|b|$ grows in the normal state but increases with the rise of $|b|$ in the superconducting state. The dependence of the two quantities on b is explicitly demonstrated in Figs. 5 and 6, in which we depict the holographic complexity and entanglement entropy as a function of b with chosen values of the strip width for the ENE (red), BINE (green) and LNE (blue), at the temperature $T/\sqrt{\rho} = 0.10$ below the critical value, within the superconducting phase. We can see that, for the fixed width l, the larger $|b|$ results in a smaller complexity c, but makes for a larger entanglement entropy s. And, for a fixed nonlinear factor b, both the complexity and entanglement entropy go up by raising the width l. These features are shared by the models with the ENE, BINE and LNE. However, in each panel of both the entropy and complexity with the fixed l, we find that the ENE system has a smaller complexity and a larger entanglement entropy than the BINE system and LNE system for a given value of the nonlinear parameter, and the comparison of the slopes of the curves for the three kinds of the nonlinear electrodynamics indicates that the ENE correction to the usual Maxwell field has stronger effect on both the entanglement entropy and complexity of the holographic superconductor than the BINE and LNE.

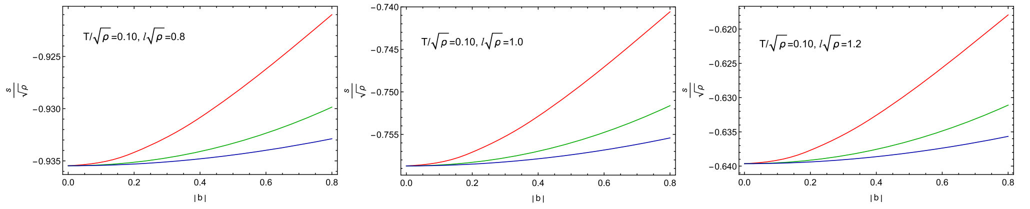

Fig. 5. (Color online) The holographic entanglement entropy of the operator $\mathcal{O}_-$ as a function of the nonlinear factor b with the width $l\sqrt{\rho} = 0.8$ (left), 1.0 (middle) and 1.2 (right), at $T/\sqrt{\rho} = 0.10$. In each panel the three lines from top to bottom correspond to the models with ENE (red), BINE (green) and LNE (blue), respectively.

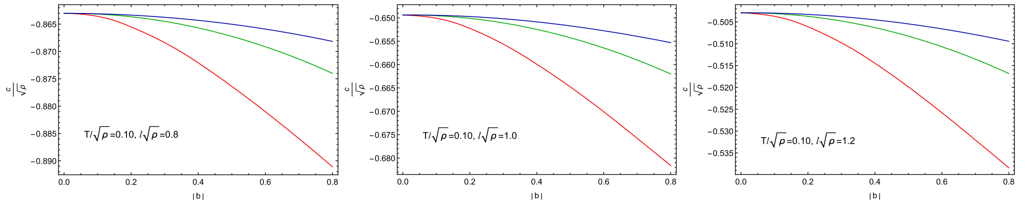

Fig. 6. (Color online) The holographic subregion complexity of the operator $\mathcal{O}_-$ as a function of the nonlinear factor b with the width $l\sqrt{\rho} = 0.8$ (left), 1.0 (middle) and 1.2 (right), at $T/\sqrt{\rho} = 0.10$. In each panel the three lines from bottom to top correspond to the models with ENE (red), BINE (green) and LNE (blue), respectively.

3.2. Operator $\mathcal{O}_+$

We fix the strip width $l\sqrt{\rho} = 1$ and plot the finite terms of the holographic entanglement entropy, s, and the holographic subregion complexity, c, against temperature $T/\sqrt{\rho}$ for the operator $\mathcal{O}_+$ with the ENE, BINE and LNE, respectively in Figs. 7, 8 and 9. There are several features similar to the $\mathcal{O}_-$ case, and which hold for each type of the nonlinear electrodynamics considered here. Firstly, the slopes of the entropy s and the complexity c have a jump at the same temperature reflecting the second order phase transition point from the normal state to the superconducting state. Secondly, the critical temperature T_c drops with the increasing nonlinear factor $|b|$, which means that the higher nonlinear electrodynamics corrections hinder the condensation for the operator $\mathcal{O}_+$ to be formed. Thirdly, the complexity c in the superconducting phase is larger than the one in the normal phase, whereas the superconducting phase has a smaller entanglement entropy s than the normal phase, and varying b does not change these properties of the two quantities. What is noteworthy and interesting is that, unlike the monotonic dependence of the holographic entanglement entropy and complexity on the parameter $|b|$ for the operator $\mathcal{O}_-$ for all cases in previous section, the entropy s and complexity c for the operator $\mathcal{O}_+$ show up the non-monotonic behavior in terms of $|b|$ for the superconducting phase in the models with ENE (Fig. 7) and BINE (Fig. 8), namely, the entanglement entropy first increases (top left plot) and then decreases (top right plot) as the parameter $|b|$ increases, while the complexity first drops (bottom left plot) and then rises (bottom right plot) with increasing $|b|$. Similar non-monotonic dependence of the entanglement entropy on $|b|$ in the superconducting phase can also be seen from Fig. 9 (top two plots) for the LNE.

For the sake of further studying the effect of the nonlinear parameter on the holographic entanglement entropy s and subregion complexity c for the operator $\mathcal{O}_+$ with the ENE, BINE and LNE, the both quantities as a function of b are displayed in Figs. 10 and 11, with a number of the chosen width $l\sqrt{\rho}$ at $T/\sqrt{\rho} = 0.01$ in the superconducting phase. In each picture, the

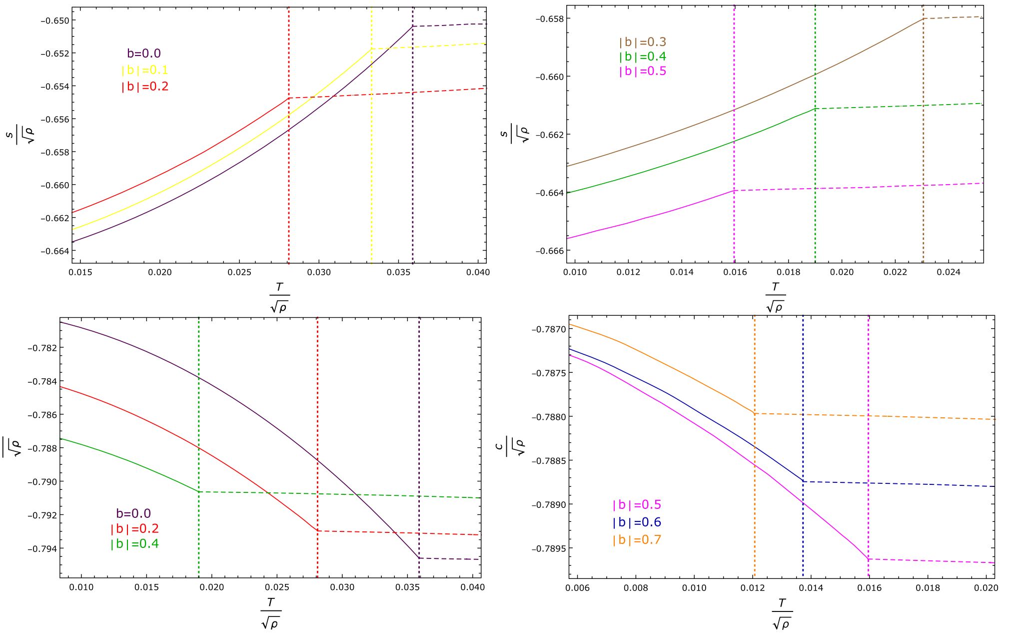

Fig. 7. (Color online) The holographic entanglement entropy (top two panels) and subregion complexity (bottom two panels) of the operator $\mathcal{O}_+$ behave as temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the ENE system. Their derivatives jump at the phase transition points represented by the vertical dot-dashed lines. The dashed lines are from the normal phase and the solid lines are from the superconducting phase.

red line is for the ENE, the green line is for the BINE and the blue line is for the LNE. In the entropy plots, the non-monotonic dependence on $|b|$ can be seen in the models with ENE, BINE and LNE for each given width. Specifically, there exists a threshold of the nonlinear parameter, which shows that the entanglement entropy in the condensed phase first rises, then descends as the absolute value of the parameter $|b|$ increases, and has a peak at the threshold. However, it should be noted that the absolute values of the threshold for the three models are not the same, i.e., the one in the ENE system is the smallest whereas that of the LNE system is the largest. In the complexity plots, the obvious non-monotonic behavior of c related to $|b|$ can only been found in the case of the ENE and BINE for $l\sqrt{\rho} = 0.8$, 1.0 and 1.2. The complexity of these two kinds of nonlinear electrodynamics systems first decreases and reaches its minimum at some threshold and then increases as $|b|$ increases, and the absolute value of the threshold in the ENE system is smaller than that of the BINE. While the complexity in the LNE system just diminishes monotonously by raising $|b|$ for all the given values of $l\sqrt{\rho}$. The non-monotonic variation of the complexity with the nonlinear parameter is originally found in Ref. [54] for the holographic superconductors with BINE. Here by considering three types of nonlinear electrodynamics, our numerical results clarify that this non-monotonic behavior of the complexity is not general for the nonlinear electrodynamics system. In addition, for the models in the ENE and BINE with the fixed $l\sqrt{\rho}$, we find that the complexity always reaches the inflection point at a greater absolute value of the nonlinear parameter $|b|$ than the entropy. On the other hand, comparing with the curves for the three types of the nonlinear electrodynamics in each panel of both the entanglement entropy and subregion complexity, it can be seen that the effect of the ENE on the entropy s and complexity c is more apparent than the BINE and LNE when we strengthen the nonlinearity $|b|$.

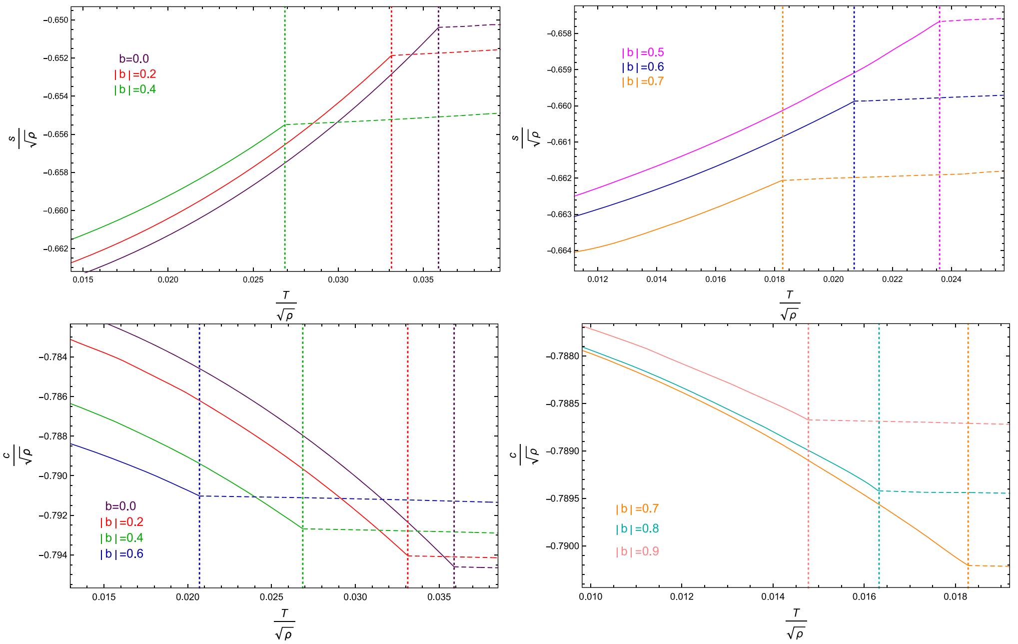

Fig. 8. (Color online) The holographic entanglement entropy (top two panels) and subregion complexity (bottom two panels) of the operator $\mathcal{O}_+$ behave as temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the BINE system.

Lastly, from Figs. 10 and 11, we find that the larger strip width $l\sqrt{\rho}$ shift the inflection points of both the non-monotonic changing entanglement entropy and complexity toward larger values of $|b|$. And with a fixed nonlinear parameter b, the increase of the strip width will make both the entanglement entropy and complexity increase.

4. Conclusions

In this paper, we have systematically investigated the holographic complexity of a strip-shaped subregion for the superconductor phase transition in three kinds of typical Born-Infeld-like nonlinear electrodynamics with full backreaction by employing the "complexity=volume" (CV) conjecture, and comparing with the results of the holographic entanglement entropy. We deal with the UV divergence of the complexity tactfully and confirm that the universal part of it is finite and significant to study during the phase transition in the superconductor model with the ENE, BINE and LNE. For the three types of nonlinear electrodynamics considered here, our analysis has shown that in both the cases of the operator $\mathcal{O}_-$ and operator $\mathcal{O}_+$, the complexity c and entanglement entropy s capture the superconductor phase transition which is of second order at the same critical temperature T_c, and the critical temperature T_c of the condensation decreases with increasing absolute value of the nonlinear parameter $|b|$, which implies that the higher nonlinear corrections to the usual Maxwell electrodynamics will make the phase transition more difficult to take place. Besides, for the fixed parameter b and given size l of the subregion, the complexity c reduces as the temperature T grows up and the value of it in the normal phase is always smaller than the one in the superconducting phase, whereas the entanglement entropy s is

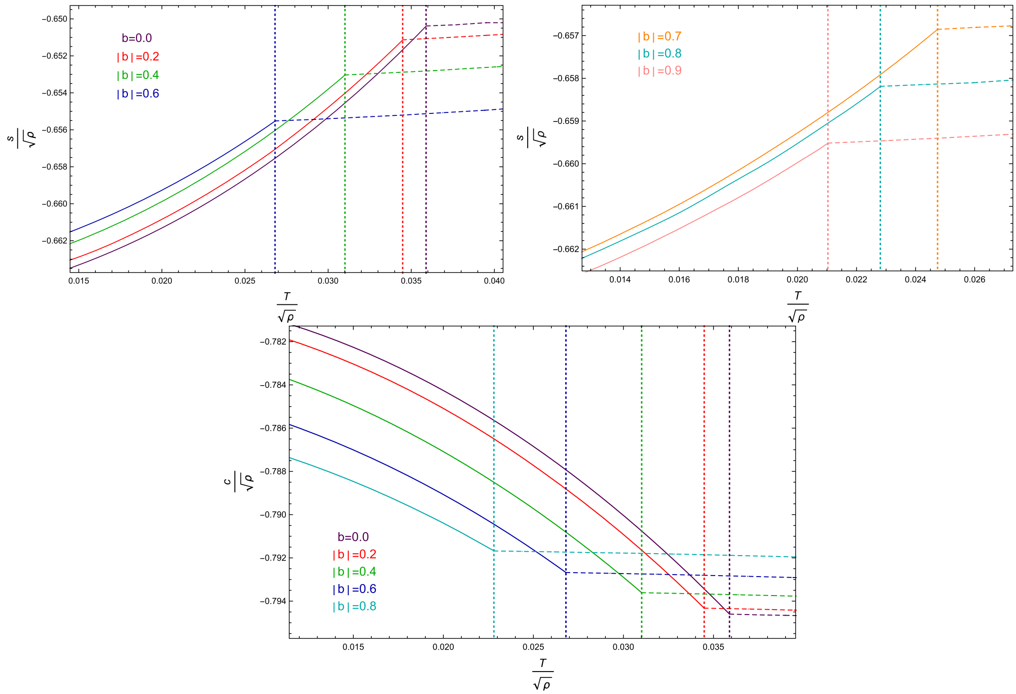

Fig. 9. (Color online) The holographic entanglement entropy (top two panels) and subregion complexity (bottom panel) of the operator $\mathcal{O}_+$ behave as temperature for different values of nonlinear parameter b with $l\sqrt{\rho} = 1$ and $m^2 = -2$ in the LNE system.

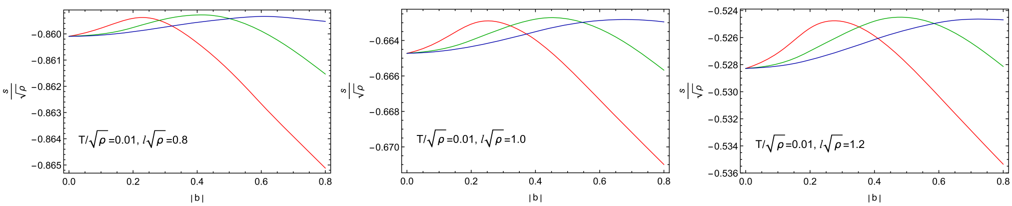

Fig. 10. (Color online) The holographic entanglement entropy with respect to the nonlinear parameter b for the width $l\sqrt{\rho} = 0.8$ (left), 1.0 (middle) and 1.2 (right), at $T/\sqrt{\rho} = 0.01$ in the superconducting phase. In each panel, the red curve is for the ENE, the green one is for the BINE and the blue one is for the LNE.

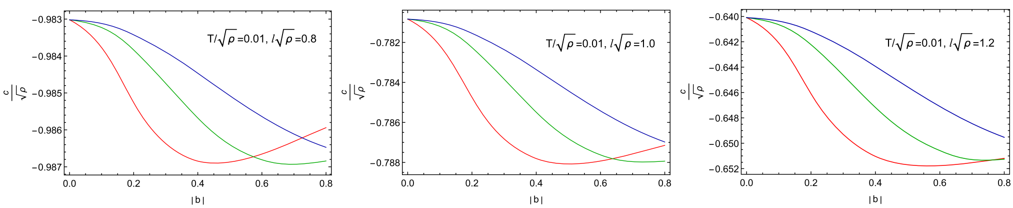

Fig. 11. (Color online) The holographic subregion complexity with respect to the nonlinear parameter b for the width $l\sqrt{\rho} = 0.8$ (left), 1.0 (middle) and 1.2 (right), at $T/\sqrt{\rho} = 0.01$ in the superconducting phase. In each panel, the various colors represent the three types of the nonlinear electrodynamics, i.e. the ENE (red), BINE (green) and LNE (blue).

an increasing function against T and has a larger value in the normal phase than in the superconducting phase. The fact that the emergence of the Cooper pairs in the condensed phase makes the reduction of the number of degrees of freedom cause the smaller entanglement entropy s at lower temperature. And the behavior of the complexity is related to the quantum state of the system becoming more complicated across the phase transition.

On the other hand, we concentrate on the effect of the nonlinear parameter b on the entanglement entropy and subregion complexity in the holographic dual models with the ENE, BINE and LNE. For a fixed T in the normal phase, we can see that when the absolute value of the nonlinear parameter $|b|$ increases, the entanglement entropy drops and the complexity rises for all cases. However, the situation in the superconducting phase is quite different and interesting. For the operator $\mathcal{O}_-$, at the temperature $T/\sqrt{\rho} = 0.1$, we observe that the changes of the complexity as well as the entanglement entropy with respect to factor $|b|$ are monotonic in the three kinds of nonlinear electrodynamics systems, and the larger entropy but the smaller complexity corresponds to the higher strength of the nonlinear parameter. For the operator $\mathcal{O}_+$, at $T/\sqrt{\rho} = 0.01$, it is shown that the holographic entanglement entropy behaves non-monotonically in terms of $|b|$ and has the maximum at some threshold for all the models with the three kinds of nonlinear electrodynamics. Particularly, the dependence of the holographic subregion complexity on the nonlinearity $|b|$ is non-monotonic for the ENE and BINE, but it turns out to be a monotonic decreasing function in the presence of LNE. Thus, we argue that this non-monotonic variation of the complexity for the operator $\mathcal{O}_+$ is not a universal feature in the holographic superconductors with nonlinear electrodynamics. In the ENE and BINE systems, strengthening the nonlinearity $|b|$ makes the complexity first tends to a minimum then goes up monotonously, which is opposite to the behavior of the entanglement entropy. And it should be noted that under the same conditions, the absolute value of parameter $|b|$ corresponding to the complexity reaching the inflection point is always greater than that of the entanglement entropy. The non-monotonic behavior of the entanglement entropy implies that there is some kind of the significant reorganization of the degrees of freedom. The change of the geometric structure is also shown up by the non-monotonic behavior of the complexity. And the non-monotonic dependences of the entanglement entropy and complexity on the nonlinearity $|b|$ are quite different, which means these two quantities reflect different spacetime properties, but the underlying mechanism remains to be further studied in the future.

Last but not least, comparing the results for the ENE, BINE and LNE, for a given b and l, we obtain that the critical temperature of the phase transition reflected by the entanglement entropy and complexity in the ENE system is always lower than that of the BINE system and LNE system, which implies that the scalar hair is more difficult to be developed in the exponential form of nonlinear electrodynamics. This is in good agreement with the results shown in Fig. 1 and Tables 1 and 2. Besides, for both the operators $\mathcal{O}_-$ and $\mathcal{O}_+$ at fixed T in the superconducting phase, as one tunes the strength of the nonlinearity b, the changes of the entanglement entropy and complexity are more obvious in the ENE than in the BINE and LNE. Specially, in the case of the operator $\mathcal{O}_+$, the ENE system always gives the smallest absolute value of factor $|b|$ corresponding to the inflection points of the non-monotonic changing entanglement entropy and complexity, which shows that the both quantities are more sensitive to the strengthening nonlinearity in the exponential form of nonlinear electrodynamics. Obviously, the ENE affects the spacetime geometry more strongly since both the entanglement entropy and complexity can describe the spacetime properties. As matter of fact, it is natural that, at leading order, the ENE correction to the usual Maxwell field is larger than those of the BINE and LNE. So we conclude that the ENE correction to the usual Maxwell field has stronger effects on the holographic superconductors with fully backreaction than the BINE and LNE, which is in agreement with the finding in Ref. [20].

The complexity is indeed a good probe to the phase transition in holographic superconductors with nonlinear electrodynamics and may give richer physical information which is different from that captured by the entanglement entropy, though the detailed underlying physics of it and deep insight from the field theory side remain to be further studied. In this paper, we mainly focused on the time-independent subregion holographic complexity. Recently, the authors of Ref. [55] have studied the time-dependent complexity of holographic superconductors by using the CV conjecture in an asymptotically AdS_{d+1} geometry. It would be interesting to extend the investigation to the holographic time-dependent complexity in our holographic dual models so as to further understand the full time evolution and the deep physics for the holographic superconductors in the presence of nonlinear electrodynamics. And, it would also be of interest to study the complexity of the holographic superconductor with the nonlinear electrodynamics by the "complexity=action" (CA) conjecture, and compare them with our results of the holographic subregion complexity and entanglement entropy. We will leave it for further study and expect that it provides even more nontrivial insights into the holographic superconductor as well as the deep and subtle connection between gravity and field theory.

CRediT authorship contribution statement

Chuyu Lai: Conceptualization, Formal analysis, Software, Validation, Writing – original draft. **Qiyuan Pan:** Methodology, Resources, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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