PTEP

Prog. Theor. Exp. Phys. **2019**, 043B05 (19 pages)
DOI: 10.1093/ptep/ptz018

Yukawa interactions, flavor symmetry, and non-canonical Kähler potential

Yoshiharu Kawamura*

Department of Physics, Shinshu University, Matsumoto 390-8621, Japan
*E-mail: haru@azusa.shinshu-u.ac.jp

Received August 24, 2018; Revised January 4, 2019; Accepted February 14, 2019; Published April 30, 2019

We study the origin of fermion mass hierarchy and flavor mixing in the standard model, paying attention to flavor symmetries and fermion kinetic terms. There is a possibility that the hierarchical flavor structure of quarks and charged leptons originates from non-canonical types of fermion kinetic terms in the presence of flavor-symmetric Yukawa interactions. A flavor symmetry can be hidden in the form of non-unitary bases in the standard model. The structure of the Kähler potential could become a touchstone of new physics.

Subject Index B40, B55, B57

1. Introduction

The origin of the fermion mass hierarchy and flavor mixing has been a big mystery, which comes from the fact that there is no powerful principle to determine Yukawa couplings in the standard model (SM). Yukawa couplings are expressed as general square matrices taking complex values, and they are diagonalized by bi-unitary transformations. Their eigenvalues become quark and charged lepton masses after multiplying the vacuum expectation value (VEV) of the neutral component in the Higgs doublet. The mixing of flavors occurs from the difference between mass eigenstates and weak interaction ones [1–3].

There have been many intriguing attempts to explain the values of physical parameters concerning fermion masses and flavor mixing matrices. Most of them are based on the top-down approach [4–9], i.e., Yukawa couplings are constructed or given in the form of ansatzes based on high-energy physics such as grand unified theories (GUTs) and superstring theories (SSTs) or extensions of SM with some flavor symmetry, and the analyses have been carried out model-dependently and/or independently of the phenomenological point of view.

At present, any evidence from new physics except for neutrinos has not yet been discovered, and new physics might be beyond all imagination. Hence, it would be interesting to see flavor physics through a different lens, with the expectation that it offers some hints of a fundamental theory. We adopt several reasonable assumptions in a theory beyond the SM. (a) *The field variables are not necessarily the same as those in the SM.* (b) *There is a symmetry relating to the flavor or family of the SM (a flavor or family symmetry). The symmetry is broken down by the VEVs of some scalar fields called flavons.* (c) *Flavons couple to matter fields through matter kinetic terms dominantly.* The second assumption is based on the idea that the family number is naturally understood as a dimension of representation and predictability is improved by the reduction of free parameters, in the presence of a flavor or family symmetry. The last one is based on the fact that various fields

Funded by SCOAP³

are easy to couple to among them in a Kähler potential, compared with a superpotential controlled by holomorphy, and the Kähler potential can change by receiving radiative corrections in contrast with the superpotential, in supersymmetric (SUSY) theories. We expect that the SUSY exists in an underlying theory, even if it is broken down at a high-energy scale.

Supposing that a flavor symmetry exists, we have several questions, such as "what type of symmetry exists?", "what is the breaking mechanism?", and "how is it hidden in the SM?" Here, we focus on the last one. There is a possibility that a flavor symmetry is hidden in the form of non-unitary bases, i.e., matter fields in the SM are transformed by non-unitary matrices. In Appendix A, we give an illustration of a realization of $\mathrm{U}(N)$ symmetry using non-unitary matrices.

Our approach is summarized as follows. We suppose field variables respecting a flavor symmetry (that the corresponding transformation is realized by unitary matrices) and rewrite the Lagrangian density in the SM using such variables. We investigate the structure of terms violating the flavor symmetry, and attempt to conjecture physics beyond the SM. Although physics remains by a choice of field variables and representations, there can be a difference in the understandability of physical phenomena. For instance, in relativistic quantum mechanics, the Dirac representation of γ matrices is useful to analyze non-relativistic phenomena and the chiral representation is suitable to investigate high-energy physics. It is desirable to find helpful field variables in order to envisage a mechanism of flavor symmetry breaking in an underlying theory. We expect that unitary bases of flavor symmetries are suitable to describe physics right after the breakdown of flavor symmetries, although they are unfit for perturbative calculations due to the presence of non-canonical kinetic terms. One of the best plans would be to attack a flavor structure with both bottom-up and top-down approaches. Knowledge and information obtained by the bottom-up approach can provide a new procedure based on the top-down approach.

In this paper, we study the origin of fermion mass hierarchy and flavor mixing in the SM, using the above-mentioned approach. We examine whether the hierarchical flavor structure of quarks and charged leptons can originate from specific forms of their kinetic terms in the presence of flavor-symmetric Yukawa interactions or not. We also propose a variant procedure based on the top-down approach.

The outline of this paper is as follows. In the next section, we review quark Yukawa interactions and a no-go theorem on flavor symmetries in the SM. We explore the origin of the hierarchical structure of quarks and charged leptons, paying attention to flavor symmetries and fermion kinetic terms in Sect. 3. In the last section, we give conclusions and discussions.

2. Yukawa interactions and flavor symmetry

We review quark Yukawa interactions and the absence of exact flavor symmetries in the SM.

2.1. Quark Yukawa interactions

Let us start with the Lagrangian densities of the quark sector,

$$\mathscr{L}^{\mathrm{quark}}_{\mathrm{kinetic}} = \overline{q}_{\mathrm{L}i} i\not{D} q_{\mathrm{L}i} + \overline{u}_{\mathrm{R}i} i\not{D} u_{\mathrm{R}i} + \overline{d}_{\mathrm{R}i} i\not{D} d_{\mathrm{R}i}, \tag{1}$$

$$\mathscr{L}^{\mathrm{quark}}_{\mathrm{Yukawa}} = -y^{(u)}_{ij} \overline{q}_{\mathrm{L}i} \tilde{\phi} u_{\mathrm{R}j} - y^{(d)}_{ij} \overline{q}_{\mathrm{L}i} \phi d_{\mathrm{R}j} + \mathrm{h.c.}, \tag{2}$$

where $q_{\mathrm{L}i}$ are left-handed quark doublets, $u_{\mathrm{R}i}$ and $d_{\mathrm{R}i}$ are right-handed up- and down-type quark singlets, $i, j (= 1, 2, 3)$ are family labels, summation over repeated indices is understood throughout this paper, $y^{(u)}_{ij}$ and $y^{(d)}_{ij}$ are Yukawa couplings, ϕ is the Higgs doublet, $\tilde{\phi} = i\tau_2\phi^*$, and h.c.

stands for the Hermitian conjugation of former terms. The Yukawa couplings are diagonalized as $V_{\mathrm{L}}^{(u)} y^{(u)} V_{\mathrm{R}}^{(u)\dagger} = y_{\mathrm{diag}}^{(u)}$ and $V_{\mathrm{L}}^{(d)} y^{(d)} V_{\mathrm{R}}^{(d)\dagger} = y_{\mathrm{diag}}^{(d)}$ by bi-unitary transformations and the quark masses are obtained as

$$V_{\mathrm{L}}^{(u)} y^{(u)} V_{\mathrm{R}}^{(u)\dagger} \frac{v}{\sqrt{2}} = y_{\mathrm{diag}}^{(u)} \frac{v}{\sqrt{2}} = M_{\mathrm{diag}}^{(u)} = \mathrm{diag}\,(m_u, m_c, m_t), \tag{3}$$

$$V_{\mathrm{L}}^{(d)} y^{(d)} V_{\mathrm{R}}^{(d)\dagger} \frac{v}{\sqrt{2}} = y_{\mathrm{diag}}^{(d)} \frac{v}{\sqrt{2}} = M_{\mathrm{diag}}^{(d)} = \mathrm{diag}\,(m_d, m_s, m_b), \tag{4}$$

where $V_{\mathrm{L}}^{(u)}$, $V_{\mathrm{L}}^{(d)}$, $V_{\mathrm{R}}^{(u)}$, and $V_{\mathrm{R}}^{(d)}$ are unitary matrices, $v/\sqrt{2}$ is the VEV of the neutral component in the Higgs doublet, family labels are omitted, and m_u, m_c, m_t, m_d, m_s, and m_b are masses of up, charm, top, down, strange, and bottom quarks, respectively.

The Yukawa couplings are expressed by

$$y^{(u)} = V_{\mathrm{L}}^{(u)\dagger} y_{\mathrm{diag}}^{(u)} V_{\mathrm{R}}^{(u)}, \ \ y^{(d)} = V_{\mathrm{L}}^{(d)\dagger} y_{\mathrm{diag}}^{(d)} V_{\mathrm{R}}^{(d)} = V_{\mathrm{L}}^{(u)\dagger} V_{\mathrm{KM}} y_{\mathrm{diag}}^{(d)} V_{\mathrm{R}}^{(d)}, \tag{5}$$

using $V_{\mathrm{L}}^{(u)}$, $V_{\mathrm{R}}^{(u)}$, $V_{\mathrm{R}}^{(d)}$, $y_{\mathrm{diag}}^{(u)}$, $y_{\mathrm{diag}}^{(d)}$, and the Kobayashi–Maskawa matrix defined by [3]

$$V_{\mathrm{KM}} \equiv V_{\mathrm{L}}^{(u)} V_{\mathrm{L}}^{(d)\dagger}. \tag{6}$$

Information on physics beyond the SM is hidden in $V_{\mathrm{L}}^{(u)}$, $V_{\mathrm{R}}^{(u)}$, and $V_{\mathrm{R}}^{(d)}$ besides observable parameters $y_{\mathrm{diag}}^{(u)}$, $y_{\mathrm{diag}}^{(d)}$, and V_{KM}. The matrices $V_{\mathrm{L}}^{(u)}$, $V_{\mathrm{R}}^{(u)}$, and $V_{\mathrm{R}}^{(d)}$ are completely unknown in the SM, because they can be eliminated by the global $\mathrm{U}(3) \times \mathrm{U}(3) \times \mathrm{U}(3)/\mathrm{U}(1)$ symmetry that the quark kinetic term $\mathscr{L}_{\mathrm{kinetic}}^{\mathrm{quark}}$ possesses.

From Eqs. (3), (4) and experimental values of quark masses, the eigenvalues of $y^{(u)}$ and $y^{(d)}$ are roughly estimated at the weak scale as

$$y_{\mathrm{diag}}^{(u)} \doteqdot \mathrm{diag}\left(1.3 \times 10^{-5},\ 7.3 \times 10^{-3},\ 1.0\right), \tag{7}$$

$$y_{\mathrm{diag}}^{(d)} \doteqdot \mathrm{diag}\left(2.7 \times 10^{-5},\ 5.5 \times 10^{-4},\ 2.4 \times 10^{-2}\right). \tag{8}$$

We find that there is a large hierarchy among the size of Yukawa couplings, and it has thrown up the big mystery of its origin. From Eq. (5), we derive the relation:

$$y^{(d)} V_{\mathrm{R}}^{(d)\dagger} = y^{(u)} V_{\mathrm{R}}^{(u)\dagger} y_{\mathrm{diag}}^{(u)-1} V_{\mathrm{KM}} y_{\mathrm{diag}}^{(d)}, \tag{9}$$

where $y_{\mathrm{diag}}^{(u)-1}$ is the inverse matrix of $y_{\mathrm{diag}}^{(u)}$. The matrix $y_{\mathrm{diag}}^{(u)-1} V_{\mathrm{KM}} y_{\mathrm{diag}}^{(d)}$ can be a barometer of the difference between $y^{(u)} V_{\mathrm{R}}^{(u)\dagger}$ and $y^{(d)} V_{\mathrm{R}}^{(d)\dagger}$, and it is roughly estimated at the weak scale as

$$\begin{aligned} y_{\mathrm{diag}}^{(u)-1} V_{\mathrm{KM}} y_{\mathrm{diag}}^{(d)} &\doteqdot \begin{pmatrix} \left(1 - \frac{\lambda^2}{2}\right) \frac{m_d}{m_u} & \lambda \frac{m_s}{m_u} & A\lambda^3(\rho - i\eta) \frac{m_b}{m_u} \\ -\lambda \frac{m_d}{m_c} & \left(1 - \frac{\lambda^2}{2}\right) \frac{m_s}{m_c} & A\lambda^2 \frac{m_b}{m_c} \\ A\lambda^3(1 - \rho - i\eta) \frac{m_d}{m_t} & -A\lambda^2 \frac{m_s}{m_t} & \frac{m_b}{m_t} \end{pmatrix} \\ &= \begin{pmatrix} O(1) & O(10) & O(10) \\ O\left(10^{-3}\right) & O\left(10^{-1}\right) & O\left(10^{-1}\right) \\ O\left(10^{-7}\right) & O\left(10^{-5}\right) & O\left(10^{-2}\right) \end{pmatrix}, \end{aligned} \tag{10}$$

where we use the Wolfenstein parametrization [10], i.e., $\lambda = \sin\theta_{\mathrm{C}} \doteqdot 0.225$ (θ_{C} is the Cabibbo angle [2]), $A \doteqdot 0.811$, and ρ and η are real parameters [11].

2.2. *No unbroken flavor symmetry*

We explain that there is no unbroken flavor-dependent symmetry respecting the $SU(2)_L$ gauge symmetry [12,13]. If the quark sector is invariant under a global transformation (a low-energy remnant of some flavor symmetries):

$$q_L \to F_L q_L, \ \ u_R \to F_R^{(u)} u_R, \ \ d_R \to F_R^{(d)} d_R, \ \ \phi \to e^{i\theta}\phi, \tag{11}$$

the quark Yukawa couplings should satisfy the relations:

$$e^{-i\theta} F_L^{\dagger} y^{(u)} F_R^{(u)} = y^{(u)}, \ \ e^{i\theta} F_L^{\dagger} y^{(d)} F_R^{(d)} = y^{(d)}, \tag{12}$$

where F_L, $F_R^{(u)}$, and $F_R^{(d)}$ are 3×3 unitary matrices, and θ is a real number. From Eq. (12), we have the relations:

$$\left[y^{(u)} y^{(u)\dagger}, \ F_L\right] = 0, \ \ \left[y^{(u)\dagger} y^{(u)}, \ F_R^{(u)}\right] = 0, \tag{13}$$

$$\left[y^{(d)} y^{(d)\dagger}, \ F_L\right] = 0, \ \ \left[y^{(d)\dagger} y^{(d)}, \ F_R^{(d)}\right] = 0, \tag{14}$$

and then F_L can also be diagonalized by the unitary matrices $V_L^{(u)}$ and $V_L^{(d)}$, which diagonalize $y^{(u)} y^{(u)\dagger}$ and $y^{(d)} y^{(d)\dagger}$ such that

$$V_L^{(u)} F_L V_L^{(u)\dagger} = F_{L\,\mathrm{diag}}^{(u)}, \ \ V_L^{(d)} F_L V_L^{(d)\dagger} = F_{L\,\mathrm{diag}}^{(d)}. \tag{15}$$

In the same way, $F_R^{(u)}$ and $F_R^{(d)}$ can also be diagonalized by the unitary matrices $V_R^{(u)}$ and $V_R^{(d)}$, which diagonalize $y^{(u)\dagger} y^{(u)}$ and $y^{(d)\dagger} y^{(d)}$ such that

$$V_R^{(u)} F_R^{(u)} V_R^{(u)\dagger} = F_{R\,\mathrm{diag}}^{(u)}, \ \ V_R^{(d)} F_R^{(d)} V_R^{(d)\dagger} = F_{R\,\mathrm{diag}}^{(d)}. \tag{16}$$

By multiplying both sides of each relation in Eq. (12) by $V_L^{(u)}$ and $V_L^{(d)}$ from the left and $V_R^{(u)\dagger}$ and $V_R^{(d)\dagger}$ from the right and using Eqs. (5), (15), and (16), the following relations are obtained:

$$e^{-i\theta} F_{L\,\mathrm{diag}}^{(u)\dagger} y_{\mathrm{diag}}^{(u)} F_{R\,\mathrm{diag}}^{(u)} = y_{\mathrm{diag}}^{(u)}, \ \ e^{i\theta} F_{L\,\mathrm{diag}}^{(d)\dagger} y_{\mathrm{diag}}^{(d)} F_{R\,\mathrm{diag}}^{(d)} = y_{\mathrm{diag}}^{(d)}, \tag{17}$$

and they lead to $F_{L\,\mathrm{diag}}^{(u)} = e^{-i\theta} F_{R\,\mathrm{diag}}^{(u)}$ and $F_{L\,\mathrm{diag}}^{(d)} = e^{i\theta} F_{R\,\mathrm{diag}}^{(d)}$.

From Eq. (15), we obtain the relation:

$$F_{L\,\mathrm{diag}}^{(u)} V_{KM} = V_{KM} F_{L\,\mathrm{diag}}^{(d)}. \tag{18}$$

Then, we find that $F_{L\,\mathrm{diag}}^{(u)} = F_{L\,\mathrm{diag}}^{(d)} = e^{i\varphi} I$ (where φ is a real number and I is the 3×3 identity matrix) from the fact that all mixing angles of V_{KM} are nonzero; this means that any exact flavor-dependent symmetries do not exist in the quark sector of the SM. In the same way, it is shown that any exact flavor-dependent symmetries also do not survive in the lepton sector of the SM.

3. Kähler structure in SM and beyond

Based on feasible assumptions in a theory beyond the SM that the field variables are not necessarily the same as those in the SM, there is a flavor symmetry broken down by the VEVs of flavons, and flavons couple to matter fields in matter kinetic terms dominantly, we rewrite the Lagrangian density in the SM using unitary bases of a flavor symmetry, investigate the structure of terms violating the flavor symmetry, and attempt to conjecture physics beyond the SM. Here, unitary bases mean sets of fields that are transformed by unitary matrices. For more details, see Appendix A.

3.1. Change of variables and matching conditions

We assume that a theory beyond the SM has a flavor symmetry[1] and the symmetry is broken down by the VEVs of flavons at some high-energy scale near $M_{\rm BSM}$. Here, $M_{\rm BSM}$ is an energy scale of new physics or the upper limit of a scale where the SM holds. We assume that $M_{\rm BSM}$ is much bigger than the weak scale, for simplicity. In this case, there is a possibility that we obtain useful information on flavor physics from the matching conditions at $M_{\rm BSM}$.

We denote unitary bases of a flavor group $\mathrm{G_F}$ for quarks by $q'_{\rm L}$, $u'_{\rm R}$, and $d'_{\rm R}$. They transform as

$$q'_{\rm L} \to F_{\rm L} q'_{\rm L},\ \ u'_{\rm R} \to F_{\rm R}^{(u)} u'_{\rm R},\ \ d'_{\rm R} \to F_{\rm R}^{(d)} d'_{\rm R},\ \ \phi \to e^{i\theta}\phi, \tag{19}$$

under the $\mathrm{G_F}$ transformation, where $F_{\rm L}$, $F_{\rm R}^{(u)}$, and $F_{\rm R}^{(d)}$ are 3×3 unitary matrices. Then, the Yukawa interaction terms are rewritten as

$$\mathscr{L}^{\rm quark}_{\rm Yukawa} = -\,(y_1)_{ij}\,\overline{q}'_{{\rm L}i}\tilde{\phi} u'_{{\rm R}j} - (y_2)_{ij}\,\overline{q}'_{{\rm L}i}\phi d'_{{\rm R}j} + \text{h.c.}, \tag{20}$$

where $(y_1)_{ij}$ and $(y_2)_{ij}$ are Yukawa couplings in the unitary bases of flavor symmetry. These couplings, in general, consist of two parts, i.e., $(y_1)_{ij} = \left(y_1^{\rm F}\right)_{ij} + (\Delta y_1)_{ij}$ and $(y_2)_{ij} = \left(y_2^{\rm F}\right)_{ij} + (\Delta y_2)_{ij}$. Here, $\left(y_1^{\rm F}\right)_{ij}$ and $\left(y_2^{\rm F}\right)_{ij}$ are $\mathrm{G_F}$-invariant couplings satisfying $e^{-i\theta}F_{\rm L}^{\dagger} y_1^{\rm F} F_{\rm R}^{(u)} = y_1^{\rm F}$ and $e^{i\theta}F_{\rm L}^{\dagger} y_2^{\rm F} F_{\rm R}^{(d)} = y_2^{\rm F}$, respectively, and $(\Delta y_1)_{ij}$ and $(\Delta y_2)_{ij}$ are non-invariant ones showing the breakdown of $\mathrm{G_F}$ due to the VEVs of flavons.

The unitary bases of $\mathrm{G_F}$ are related to the SM ones $q_{\rm L}$, $u_{\rm R}$, and $d_{\rm R}$ by the change of variables as

$$q_{\rm L} = V_q J_q U_q q'_{\rm L},\ \ \left(\overline{q}_{\rm L} = \overline{q}'_{\rm L} U_q^{\dagger} J_q V_q^{\dagger}\right), \tag{21}$$

$$u_{\rm R} = \left(y^{(u)}\right)^{-1} V_q J_q^{-1} U_q y_1 u'_{\rm R} = V_{\rm R}^{(u)\dagger}\left(y^{(u)}_{\rm diag}\right)^{-1} V_{\rm L}^{(u)} V_q J_q^{-1} U_q y_1 u'_{\rm R}, \tag{22}$$

$$d_{\rm R} = \left(y^{(d)}\right)^{-1} V_q J_q^{-1} U_q y_2 d'_{\rm R} = V_{\rm R}^{(d)\dagger}\left(y^{(d)}_{\rm diag}\right)^{-1} V_{\rm KM}^{\dagger} V_{\rm L}^{(u)} V_q J_q^{-1} U_q y_2 d'_{\rm R}, \tag{23}$$

where V_q and U_q are 3×3 unitary matrices and J_q is a real 3×3 diagonal matrix. Note that V_q can be eliminated by the redefinition of $V_{\rm L}^{(u)}$ and $V_{\rm L}^{(d)}$.

Using new variables, the quark kinetic terms in the SM are rewritten as

$$\mathscr{L}^{\rm quark}_{\rm kinetic} = k_{ij}^{(q)}\overline{q}'_{{\rm L}i} i\not{D} q'_{{\rm L}j} + k_{ij}^{(u)}\overline{u}'_{{\rm R}i} i\not{D} u'_{{\rm R}j} + k_{ij}^{(d)}\overline{d}'_{{\rm R}i} i\not{D} d'_{{\rm R}j}, \tag{24}$$

where the kinetic coefficients $k_{ij}^{(q)}$, $k_{ij}^{(u)}$, and $k_{ij}^{(d)}$ are given by

$$k_{ij}^{(q)} = \left(U_q^{\dagger}(J_q)^2 U_q\right)_{ij}, \tag{25}$$

$$k_{ij}^{(u)} = \left(y_1^{\dagger} W^{(u)\dagger}\left(y^{(u)-1}_{\rm diag}\right)^2 W^{(u)} y_1\right)_{ij}, \tag{26}$$

$$k_{ij}^{(d)} = \left(y_2^{\dagger} W^{(d)\dagger}\left(y^{(d)-1}_{\rm diag}\right)^2 W^{(d)} y_2\right)_{ij} = \left(y_2^{\dagger} W^{(u)\dagger} V_{\rm KM}\left(y^{(d)-1}_{\rm diag}\right)^2 V_{\rm KM}^{\dagger} W^{(u)} y_2\right)_{ij}. \tag{27}$$

[1] The flavor structure of quarks and leptons has been studied intensively, based on various flavor symmetries [9,14–20].

Here, $W^{(u)} = V_{\rm L}^{(u)} V_q J_q^{-1} U_q$, $W^{(d)} = V_{\rm L}^{(d)} V_q J_q^{-1} U_q$, and we use the feature that the kinetic coefficients are positive definite. Note that $W^{(u)}$ and $W^{(d)}$ are not necessarily unitary matrices. If J_q is the identity matrix, $k_{ij}^{(q)}$ is the canonical one (the identity matrix) and $W^{(u)}$ and $W^{(d)}$ become unitary matrices.

We briefly give an alternative proof on the absence of exact flavor symmetries in the SM. Under the assumption that $\mathscr{L}_{\rm Yukawa}^{\rm quark}$ given in Eq. (20) is invariant under the transformation (19), i.e., $e^{-i\theta} F_{\rm L}^{\dagger} y_1 F_{\rm R}^{(u)} = y_1$ and $e^{i\theta} F_{\rm L}^{\dagger} y_2 F_{\rm R}^{(d)} = y_2$, it is shown that no exact flavor symmetries exist from the invariance of $\mathscr{L}_{\rm kinetic}^{\rm quark}$ given in Eq. (24) under the transformation (19), in the following. Eigenvalues of $F_{\rm R}^{(u)}$ and $F_{\rm R}^{(d)}$ are given by those of $F_{\rm L}$ multiplied by $e^{i\theta}$ and $e^{-i\theta}$, respectively, as estimated from Eq. (17). Using Eq. (25) and $F_{\rm L}^{\dagger} k^{(q)} F_L = k^{(q)}$ derived from the invariance of $k_{ij}^{(q)} \overline{q}'_{{\rm L}i} i\not{D} q'_{{\rm L}j}$, we find that $F_{\rm L}$ is diagonalized by U_q as $U_q F_{\rm L} U_q^{\dagger} = F_{\rm L\ diag}$. Here, we omit the labels of flavor. From Eqs. (26), (27), and $F_{\rm R}^{(u)\dagger} k^{(u)} F_{\rm R}^{(u)} = k^{(u)}$ and $F_{\rm R}^{(d)\dagger} k^{(d)} F_{\rm R}^{(d)} = k^{(d)}$ derived from the invariance of other kinetic terms, we obtain the relations $\tilde{F}_{\rm L\ diag} V_{\rm L}^{(u)} = V_{\rm L}^{(u)} F_{\rm L\ diag}$ and $\tilde{F}_{\rm L\ diag} V_{\rm L}^{(d)} = V_{\rm L}^{(d)} F_{\rm L\ diag}$, using $e^{-i\theta} F_{\rm L}^{\dagger} y_1 F_{\rm R}^{(u)} = y_1$, $e^{i\theta} F_{\rm L}^{\dagger} y_2 F_{\rm R}^{(d)} = y_2$, and $U_q F_{\rm L} U_q^{\dagger} = F_{\rm L\ diag}$. Here, $\tilde{F}_{\rm L\ diag}$ is a diagonal unitary matrix. These relations lead to $\tilde{F}_{\rm L\ diag} V_{\rm KM} = V_{\rm KM} \tilde{F}_{\rm L\ diag}$, which means that $\tilde{F}_{\rm L\ diag}$ and $F_{\rm L\ diag}$ should be proportional to the identity matrix or the non-existence of exact flavor-dependent symmetries.

From Eqs. (7) and (8), $\left(y_{\rm diag}^{(u)-1}\right)^2$, $\left(y_{\rm diag}^{(d)-1}\right)^2$, and $V_{\rm KM} \left(y_{\rm diag}^{(d)-1}\right)^2 V_{\rm KM}^{\dagger}$ are roughly estimated at the weak scale as

$$\left(y_{\rm diag}^{(u)-1}\right)^2 \doteqdot {\rm diag}\left(5.9 \times 10^9,\ 1.9 \times 10^4,\ 1.0\right) = 5.9 \times 10^9 \times {\rm diag}\left(1,\ 3.2 \times 10^{-6},\ 1.7 \times 10^{-10}\right), \quad (28)$$

$$\left(y_{\rm diag}^{(d)-1}\right)^2 \doteqdot {\rm diag}\left(1.4 \times 10^9,\ 3.8 \times 10^6,\ 2.1 \times 10^3\right) = 1.4 \times 10^9 \times {\rm diag}\left(1,\ 2.7 \times 10^{-3},\ 1.5 \times 10^{-6}\right), \quad (29)$$

$$V_{\rm KM} \left(y_{\rm diag}^{(d)-1}\right)^2 V_{\rm KM}^{\dagger} \doteqdot 1.4 \times 10^9 \times \begin{pmatrix} 1-\lambda^2 & -\lambda & O\left(\lambda^3\right) \\ -\lambda & \lambda^2 & O\left(\lambda^4\right) \\ O\left(\lambda^3\right) & O\left(\lambda^4\right) & O\left(\lambda^6\right) \end{pmatrix}. \quad (30)$$

Physical parameters, in general, receive radiative corrections, and the above values should be evaluated by considering renormalization effects and should match their counterparts at $M_{\rm BSM}$. From Eqs. (20) and (24), information on the flavor structure in the SM is transferred to $k_{ij}^{(q)}$, $k_{ij}^{(u)}$, and $k_{ij}^{(d)}$ in the kinetic terms.

To speculate on a theory of quarks beyond the SM, let us describe it by

$$\mathscr{L}_{\rm BSM}^{\rm quark} = K_{ij}^{(q)} \overline{q}'_{{\rm L}i} i\not{D} q'_{{\rm L}j} + K_{ij}^{(u)} \overline{u}'_{{\rm R}i} i\not{D} u'_{{\rm R}j} + K_{ij}^{(d)} \overline{d}'_{{\rm R}i} i\not{D} d'_{{\rm R}j} - (Y_1)_{ij}\, \overline{q}'_{{\rm L}i} \tilde{\phi} u'_{{\rm R}j} - (Y_2)_{ij}\, \overline{q}'_{{\rm L}i} \phi d'_{{\rm R}j} + {\rm h.c.}, \quad (31)$$

where $K_{ij}^{(q)}$, $K_{ij}^{(u)}$, $K_{ij}^{(d)}$, $(Y_1)_{ij}$, and $(Y_2)_{ij}$ contain fields such that $\mathscr{L}_{\rm BSM}^{\rm quark}$ is invariant under the $\rm G_F$ transformation. The $\mathscr{L}_{\rm BSM}^{\rm quark}$ describes only the part relating to quarks in new physics, and chiral anomalies are supposed to be canceled by other contributions if the $\rm G_F$ symmetry is local.

When $\mathscr{L}^{\text{quark}}_{\text{kinetic}}$ and $\mathscr{L}^{\text{quark}}_{\text{Yukawa}}$ are obtained by getting the VEVs after the breakdown of G_F symmetry, the following matching conditions should be imposed on

$$k^{(q)}_{ij} = \left\langle K^{(q)}_{ij} \right\rangle,\ k^{(u)}_{ij} = \left\langle K^{(u)}_{ij} \right\rangle,\ k^{(d)}_{ij} = \left\langle K^{(d)}_{ij} \right\rangle,\ (y_1)_{ij} = \left\langle (Y_1)_{ij} \right\rangle,\ (y_2)_{ij} = \left\langle (Y_2)_{ij} \right\rangle, \tag{32}$$

at M_{BSM}, from Eqs. (20), (24), and (31).

3.2. *Examples*

As we have few hints on flavor symmetry, we study two examples, i.e., a case with a U(3) symmetry and that with an S_3 one. Here S_3 is the permutation group of order 3.

3.2.1. *U(3) case*

In the case that a U(3) family symmetry is hidden in the SM, the Yukawa interactions are written by $\mathscr{L}^{\text{quark}}_{\text{Yukawa}} = -y_1 \overline{q}'_{\text{L}i} \tilde{\phi} u'_{\text{R}i} - y_2 \overline{q}'_{\text{L}i} \phi d'_{\text{R}i} + \text{h.c.}$, where y_1 and y_2 are complex numbers. We assume that U(3) symmetric terms dominate in Yukawa interactions. This is justified; in the case that M_{BSM} is much bigger than the weak scale, other terms including fermions contain non-renormalizable higher-dimensional operators and they can be suppressed by a power of M_{BSM}.

Now, we conjecture the structure of the Kähler metric, based on Eqs. (25)–(30). There are many possibilities to realize the quark masses and flavor mixing consistent with experimental data. For simplicity, we assume that $J_q = I$, i.e., $k^{(q)}_{ij} = \delta_{ij}$. Then, $k^{(u)}_{ij}$ and $k^{(d)}_{ij}$ are written by

$$k^{(u)}_{ij} = |y_1|^2 \left(U^{(u)\dagger}_{\text{L}} \left(y^{(u)-1}_{\text{diag}} \right)^2 U^{(u)}_{\text{L}} \right)_{ij},\quad k^{(d)}_{ij} = |y_2|^2 \left(U^{(u)\dagger}_{\text{L}} V_{\text{KM}} \left(y^{(d)-1}_{\text{diag}} \right)^2 V^{\dagger}_{\text{KM}} U^{(u)}_{\text{L}} \right)_{ij}, \tag{33}$$

where $U^{(u)}_{\text{L}}$ is a unitary matrix. $U^{(u)}_{\text{L}}$ is written by $U^{(u)}_{\text{L}} \equiv V^{(u)}_{\text{L}} V_q$ where V_q is a unitary matrix reflecting the U(3) invariance of q_{L}'s kinetic term. There is a possibility that $k^{(u)}_{ij}$ and $k^{(d)}_{ij}$ take forms whose every component has almost the same magnitude of $O(1)$, if $|y_1|^2 = O\left(10^{-10}\right)$ and $|y_2|^2 = O\left(10^{-9}\right)$. This is suggested from the formulas

$$U \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \varepsilon_1 \begin{pmatrix} 1 & \varpi & \omega \\ \omega & 1 & \varpi \\ \varpi & \omega & 1 \end{pmatrix} + \varepsilon_2 \begin{pmatrix} 1 & \omega & \varpi \\ \varpi & 1 & \omega \\ \omega & \varpi & 1 \end{pmatrix} \right\} U^\dagger = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix} \tag{34}$$

and

$$U \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 & \omega & \varpi \\ \varpi & 1 & -2\omega \\ \omega & -2\varpi & 1 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & \varpi - 1 & \omega - 1 \\ \omega - 1 & 0 & \varpi - 1 \\ \varpi - 1 & \omega - 1 & 0 \end{pmatrix} \right\} U^\dagger = 3 \begin{pmatrix} 1 - \lambda^2 & -\lambda & 0 \\ -\lambda & \lambda^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

with the unitary matrix

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varpi & \omega \\ 1 & \omega & \varpi \end{pmatrix}, \tag{36}$$

where ε_1 and ε_2 are arbitrary numbers, $\omega = e^{2\pi i/3}$, and $\overline{\omega} = \omega^2 = e^{4\pi i/3}(= -1 - \omega)$. The above formulas are merely examples. Quark kinetic coefficients and unitary matrices might take complicated forms and contain tiny parameters intricately. At any rate, a large mass hierarchy and mixing can originate from a tiny variance of the democratic form whose every component has a common value. In other words, *the hierarchical structure can be realized in the case that Kähler metrics* $K_{ij}^{(u)}$ *and* $K_{ij}^{(d)}$ *acquire the VEVs of semi-democratic forms as*

$$\left\langle K_{ij}^{(u)} \right\rangle = \xi^{(u)} S_{ij} + O(\varepsilon_i), \quad \left\langle K_{ij}^{(d)} \right\rangle = \xi^{(d)} S_{ij} + O(\varepsilon_i, \lambda) \tag{37}$$

with some constants $\xi^{(u)}$ *and* $\xi^{(d)}$*, after the breakdown of the family symmetry, and the reception of tiny corrections.* Here, S_{ij} is the democratic matrix defined by

$$S_{ij} \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{38}$$

It is hard to derive semi-democratic forms (37) dynamically at a level of perturbation, from a U(3)-invariant Kähler potential $K = |\Phi_i|^2 + \cdots$, as suggested by a model in Appendix B. Here the ellipsis stands for higher-dimensional terms that are sub-leading-order ones. We need a mechanism to realize semi-democratic forms and small Yukawa couplings such as $|y_1|^2 = O\left(10^{-10}\right)$ and $|y_2|^2 = O\left(10^{-9}\right)$.

3.2.2. S_3 case

Based on an S_3-invariant Kähler potential containing the democratic form and Yukawa couplings with the democratic form and small S_3 breaking ones, it was pointed that the heavy top quark mass can be attributed to a singular normalization of its kinetic term [21]. Sfermion masses were also studied using the S_3-invariant Kähler potential [22].

Let us re-examine a case with the S_3 symmetry using our formulation. Strictly speaking, the flavor group is $S_3 \times S_3 \times S_3$, and $q_{\mathrm{L}i}$, $u_{\mathrm{R}i}$, and $d_{\mathrm{R}i}$ are transformed as 3D representations of the first, second, and third S_3, respectively. These 3D representations are reducible and are decomposed into two irreducible ones such as 1D ones and 2D ones. In the presence of S_3 symmetry, the Yukawa couplings are written by

$$(y_1)_{ij} = y_1^{\mathrm{F}} S_{ij} + \Delta y_1 T_{ij}^{(u)}, \quad (y_2)_{ij} = y_2^{\mathrm{F}} S_{ij} + \Delta y_2 T_{ij}^{(d)}, \tag{39}$$

where y_a^{F} and Δy_a $(a = 1, 2)$ are complex numbers, and $T_{ij}^{(u)}$ and $T_{ij}^{(d)}$ are complex matrices (whose components take values of at most $O(1)$) that originate from S_3 breaking effects. We cannot derive realistic quark masses without $T_{ij}^{(u)}$ and $T_{ij}^{(d)}$. We assume that $\left|y_a^{\mathrm{F}}\right| = O(1)$ tentatively, according to Dirac's naturalness. Here, Dirac's naturalness means that *the magnitude of dimensionless parameters on terms allowed by symmetries should be* $O(1)$ *in a fundamental theory*. In contrast, we suppose that $|\Delta y_a| \ll \left|y_a^{\mathrm{F}}\right|$ from a conjecture that the S_3 breaking terms stem from non-renormalizable interactions suppressed by a power of M_{BSM}.

In the following, we examine whether the magnitudes of components in $k_{ij}^{(u)}$ and $k_{ij}^{(d)}$ can be at most $O(1)$ or not under the above assumptions, i.e., $\left|y_a^{\mathrm{F}}\right| = O(1)$ and $|\Delta y_a| \ll \left|y_a^{\mathrm{F}}\right|$. In other words, $k_{ij}^{(u)}$ and $k_{ij}^{(d)}$ are, in general, written by

$$k_{ij}^{(u)} = k_1^{(u)} \delta_{ij} + k_2^{(u)} S_{ij} + k_3^{(u)} Z_{ij}^{(u)}, \quad k_{ij}^{(d)} = k_1^{(d)} \delta_{ij} + k_2^{(d)} S_{ij} + k_3^{(d)} Z_{ij}^{(d)}, \tag{40}$$

where $k_b^{(u)}$ and $k_b^{(d)}$ $(b = 1, 2, 3)$ are real numbers, and $Z_{ij}^{(u)}$ and $Z_{ij}^{(d)}$ are Hermitian matrices (whose components take values of at most $O(1)$) that represent S_3 breaking effects. Then, can the magnitudes of $k_b^{(u)}$ and $k_b^{(d)}$ be at most $O(1)$ or not?

By inserting the first relation of Eq. (39) into Eq. (26), the following relation is derived:

$$k_{ij}^{(u)} = \left|y_1^{\mathrm{F}}\right|^2 \left(S W^{(u)\dagger} \left(y_{\mathrm{diag}}^{(u)-1}\right)^2 W^{(u)} S\right)_{ij} + |\Delta y_1|^2 \left(T^{(u)\dagger} W^{(u)\dagger} \left(y_{\mathrm{diag}}^{(u)-1}\right)^2 W^{(u)} T^{(u)}\right)_{ij} + \left(y_1^{\mathrm{F}}\right)^* \Delta y_1 \left(S W^{(u)\dagger} \left(y_{\mathrm{diag}}^{(u)-1}\right)^2 W^{(u)} T^{(u)}\right)_{ij} + \mathrm{h.c.} \tag{41}$$

Using the formula $SXS = (\sum_{i,j=1}^{3} X_{ij})S$, we find that the following condition should be fulfilled:

$$\sum_{i,j=1}^{3} \left(W^{(u)\dagger} \left(y_{\mathrm{diag}}^{(u)-1}\right)^2 W^{(u)}\right)_{ij} = O(1), \tag{42}$$

in order to make the magnitudes of the first term in Eq. (41) at most $O(1)$. For simplicity, let us take an ansatz of $W^{(u)}$ such as[2]

$$W^{(u)} \equiv \begin{pmatrix} w_{11}^{(u)} & w_{12}^{(u)} & -w_{11}^{(u)} - w_{12}^{(u)} \\ w_{21}^{(u)} & w_{22}^{(u)} & -w_{21}^{(u)} - w_{22}^{(u)} \\ w_{31}^{(u)} & w_{32}^{(u)} & w_{33}^{(u)} \end{pmatrix}, \tag{43}$$

where $w_{ij}^{(u)}$ are complex numbers of at most $O(1)$. Then, we obtain the relation:

$$k_{ij}^{(u)} \doteqdot \left|y_1^{\mathrm{F}}\right|^2 \left|w_{31}^{(u)} + w_{32}^{(u)} + w_{33}^{(u)}\right|^2 S_{ij} + |\Delta y_1|^2 \left(T^{(u)\dagger} W^{(u)\dagger} \left(y_{\mathrm{diag}}^{(u)-1}\right)^2 W^{(u)} T^{(u)}\right)_{ij} + O(|\Delta y_1|). \tag{44}$$

If $|\Delta y_1|^2 = O\left(10^{-10}\right)$, the magnitude of every component in the second term of Eq. (44) can also be at most $O(1)$, and $k_{ij}^{(u)}$ can take the form given by the first relation of Eq. (40) with $\left|y_1^{\mathrm{F}}\right| = O(1)$.

In the same way, by inserting the second relation of Eq. (39) into Eq. (27) and using the relation:

$$W^{(u)} = V_{\mathrm{KM}} W^{(d)}, \tag{45}$$

we obtain the relation:

$$k_{ij}^{(d)} \doteqdot \left|y_2^{\mathrm{F}}\right|^2 \times O(\lambda^6/y_d^2) \times \left|w_{31}^{(u)} + w_{32}^{(u)} + w_{33}^{(u)}\right|^2 S_{ij} + |\Delta y_2|^2 \left(T^{(d)\dagger} W^{(d)\dagger} \left(y_{\mathrm{diag}}^{(d)-1}\right)^2 W^{(d)} T^{(d)}\right)_{ij} + O(|\Delta y_2|). \tag{46}$$

Hence, we need $\left|y_2^{\mathrm{F}}\right|^2 = O(y_d^2/\lambda^6) = O(10^{-6})$ and $|\Delta y_2|^2 = O(10^{-9})$ in order to make the magnitude of every component in the first and second terms of Eq. (46) at most $O(1)$, respectively.

[2] In the case that $W_{13}^{(u)}$ and $W_{23}^{(u)}$ are replaced into $-w_{11}^{(u)} - w_{12}^{(u)} + O(\varepsilon_u)$ and $-w_{21}^{(u)} - w_{22}^{(u)} + O(\varepsilon_c)$, we also get $\left|y_1^{\mathrm{F}}\right| = O(1)$ and $|\Delta y_1|^2 = O\left(10^{-10}\right)$, although extra contributions exist in $k_{ij}^{(u)}$.

3.3. Lepton sector

We study the lepton sector in the SM. In the absence of Majorana masses of right-handed neutrino singlets, the same argument as the quarks holds in the replacement of fields and couplings. Here, we consider the case with large Majorana masses and a flavor symmetry in a theory beyond the SM.

The lepton sector is described by the Lagrangian densities:

$$\mathscr{L}^{\rm lepton}_{\rm kinetic} = \bar{l}_{{\rm L}i} i\partial\!\!\!/ l_{{\rm L}i} + \bar{e}_{{\rm R}i} i\partial\!\!\!/ e_{{\rm R}i} + \bar{\nu}_{{\rm R}i} i\partial\!\!\!/ \nu_{{\rm R}i} - \frac{1}{2} M_{ij} \nu^{\rm t}_{{\rm R}i} C \nu_{{\rm R}j}, \tag{47}$$

$$\mathscr{L}^{\rm lepton}_{\rm Yukawa} = -y^{(e)}_{ij} \bar{l}_{{\rm L}i} \phi e_{{\rm R}j} - y^{(\nu)}_{ij} \bar{l}_{{\rm L}i} \tilde{\phi} \nu_{{\rm R}j} + {\rm h.c.}, \tag{48}$$

where $l_{{\rm L}i}$ are left-handed lepton doublets, $e_{{\rm R}i}$ and $\nu_{{\rm R}i}$ are right-handed electron- and neutrino-type lepton singlets, M_{ij} are Majorana masses, $\nu^{\rm t}_{{\rm R}i}$ is a transpose of $\nu_{{\rm R}i}$, $C = i\gamma^2\gamma^0$, and $y^{(e)}_{ij}$ and $y^{(\nu)}_{ij}$ are Yukawa couplings. The $y^{(e)}_{ij}$ and $y^{(\nu)}_{ij}$ are diagonalized as $V^{(e)}_{\rm L} y^{(e)} V^{(e)\dagger}_{\rm R} = y^{(e)}_{\rm diag}$ and $V^{(\nu)}_{\rm L} y^{(\nu)} V^{(\nu)\dagger}_{\rm R} = y^{(\nu)}_{\rm diag}$ by bi-unitary transformations, and M_{ij} is also diagonalized by $V^{(\nu)}_{\rm R}$ as $V^{(\nu)*}_{\rm R} M V^{(\nu)\dagger}_{\rm R} = M_{\rm diag} = {\rm diag}\,(M_1, M_2, M_3)$ under the assumption that the flavor symmetry exists beyond the SM. Lepton masses are obtained as

$$V^{(e)}_{\rm L} y^{(e)} V^{(e)\dagger}_{\rm R} \frac{v}{\sqrt{2}} = y^{(e)}_{\rm diag} \frac{v}{\sqrt{2}} = M^{(e)}_{\rm diag} = {\rm diag}\left(m_e, m_\mu, m_\tau\right), \tag{49}$$

$$V^{(\nu)}_{\rm L} y^{(\nu)} M^{-1} y^{(\nu){\rm t}} V^{(\nu){\rm t}}_{\rm L} \frac{v^2}{2} = M^{(\nu)}_{\rm diag} = {\rm diag}\left(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\right), \tag{50}$$

where $V^{(e)}_{\rm L}$, $V^{(\nu)}_{\rm L}$, and $V^{(e)}_{\rm R}$ are unitary matrices, m_e, m_μ, and m_τ are masses of electron, muon, and tauon, respectively, and the seesaw mechanism is used to obtain tiny neutrino masses m_{ν_1}, m_{ν_2}, and m_{ν_3} [23–25]. The lepton Yukawa couplings are expressed by

$$y^{(e)} = V^{(e)\dagger}_{\rm L} y^{(e)}_{\rm diag} V^{(e)}_{\rm R},\ \ y^{(\nu)} = V^{(\nu)\dagger}_{\rm L} y^{(\nu)}_{\rm diag} V^{(\nu)}_{\rm R} = V^{(e)\dagger}_{\rm L} V_{\rm MNS} y^{(\nu)}_{\rm diag} V^{(\nu)}_{\rm R}, \tag{51}$$

using $V^{(e)}_{\rm L}$, $V^{(e)}_{\rm R}$, $V^{(\nu)}_{\rm R}$, $y^{(e)}_{\rm diag}$, $y^{(\nu)}_{\rm diag}$, and the Maki–Nakagawa–Sakata matrix $V_{\rm MNS} \equiv V^{(e)}_{\rm L} V^{(\nu)\dagger}_{\rm L}$.

From Eq. (49) and experimental values of charged lepton masses, the magnitude of $y^{(e)}_{\rm diag}$ is roughly estimated at the weak scale as

$$y^{(e)}_{\rm diag} \doteqdot {\rm diag}\left(2.9\times 10^{-6},\ 6.1\times 10^{-4},\ 1.0\times 10^{-2}\right). \tag{52}$$

We find that there is a hierarchy among charged lepton Yukawa couplings.

Using field variables $l'_{\rm L}$, $e'_{\rm R}$, and $\nu'_{\rm R}$ defined by

$$l'_{\rm L} \equiv U^\dagger_l J^{-1}_l V^\dagger_l l_{\rm L},\ \ \left(\bar{l}'_{\rm L} \equiv \bar{l}_{\rm L} V_l J^{-1}_l U_l\right), \tag{53}$$

$$e'_{\rm R} \equiv y^{-1}_3 U^\dagger_l J_l V^\dagger_l y^{(e)} e_{\rm R} = y^{-1}_3 U^\dagger_l J_l V^\dagger_l V^{(e)\dagger}_{\rm L} y^{(e)}_{\rm diag} V^{(e)}_{\rm R} e_{\rm R}, \tag{54}$$

$$\begin{aligned}\nu'_{\rm R} &\equiv y^{-1}_4 U^\dagger_l J_l V^\dagger_l y^{(\nu)} \nu_{\rm R} = y^{-1}_4 U^\dagger_l J_l V^\dagger_l V^{(\nu)\dagger}_{\rm L} y^{(\nu)}_{\rm diag} V^{(\nu)}_{\rm R} \nu_{\rm R} \\ &= y^{-1}_4 U^\dagger_l J_l V^\dagger_l V^{(e)\dagger}_{\rm L} V_{\rm MNS} y^{(\nu)}_{\rm diag} V^{(\nu)}_{\rm R} \nu_{\rm R},\end{aligned} \tag{55}$$

the Lagrangian densities are rewritten as

$$\mathscr{L}^{\rm lepton}_{\rm kinetic} = k^{(l)}_{ij} \bar{l}'_{{\rm L}i} i\partial\!\!\!/ l'_{{\rm L}j} + k^{(e)}_{ij} \bar{e}'_{{\rm R}i} i\partial\!\!\!/ e'_{{\rm R}j} + k^{(\nu)}_{ij} \bar{\nu}'_{{\rm R}i} i\partial\!\!\!/ \nu'_{{\rm R}j} - \frac{1}{2} M^{(\nu)}_{ij} \nu'^{\rm t}_{{\rm R}i} C \nu'_{{\rm R}j}, \tag{56}$$

$$\mathscr{L}^{\text{lepton}}_{\text{Yukawa}} = -(y_3)_{ij}\,\overline{l}'_{\text{L}i}\phi e'_{\text{R}j} - (y_4)_{ij}\,\overline{l}'_{\text{L}i}\tilde{\phi}\nu'_{\text{R}j} + \text{h.c.}, \tag{57}$$

where V_l and U_l are unitary matrices, J_l is a real diagonal matrix, J_l^{-1} is the inverse matrix of J_l, $(y_3)_{ij}$ and $(y_4)_{ij}$ are lepton Yukawa couplings in the unitary bases of flavor symmetry, and $k_{ij}^{(l)}$, $k_{ij}^{(e)}$, $k_{ij}^{(\nu)}$, and $M_{ij}^{(\nu)}$ are given by

$$k_{ij}^{(l)} = \left(U_l^\dagger (J_l)^2 U_l\right)_{ij}, \tag{58}$$

$$k_{ij}^{(e)} = \left(y_3^\dagger W^{(e)\dagger}\left(y_{\text{diag}}^{(e)-1}\right)^2 W^{(e)} y_3\right)_{ij}, \tag{59}$$

$$k_{ij}^{(\nu)} = \left(y_4^\dagger W^{(\nu)\dagger}\left(y_{\text{diag}}^{(\nu)-1}\right)^2 W^{(\nu)} y_4\right)_{ij} = \left(y_4^\dagger W^{(e)\dagger} V_{\text{MNS}}\left(y_{\text{diag}}^{(\nu)-1}\right)^2 V_{\text{MNS}}^\dagger W^{(e)} y_4\right)_{ij}, \tag{60}$$

$$\begin{aligned} M_{ij}^{(\nu)} &= \left(y_4^{\text{t}} W^{(\nu)\text{t}} y_{\text{diag}}^{(\nu)-1} M_{\text{diag}} y_{\text{diag}}^{(\nu)-1} W^{(\nu)} y_4\right)_{ij} \\ &= \left(y_4^{\text{t}} W^{(e)\text{t}} V_{\text{MNS}}^* y_{\text{diag}}^{(\nu)-1} M_{\text{diag}} y_{\text{diag}}^{(\nu)-1} V_{\text{MNS}}^\dagger W^{(e)} y_4\right)_{ij}. \end{aligned} \tag{61}$$

Here, $W^{(e)} = V_{\text{L}}^{(e)} V_l J_l^{-1} U_l$ and $W^{(\nu)} = V_{\text{L}}^{(\nu)} V_l J_l^{-1} U_l$. Note that V_l can be eliminated by the redefinition of $V_{\text{L}}^{(e)}$ and $V_{\text{L}}^{(\nu)}$. From Eq. (52), $\left(y_{\text{diag}}^{(e)-1}\right)^2$ is roughly estimated at the weak scale as

$$\begin{aligned} \left(y_{\text{diag}}^{(e)-1}\right)^2 &\doteqdot \text{diag}\left(1.2\times 10^{11},\ 2.7\times 10^{6},\ 1.0\times 10^{4}\right) \\ &= 1.2\times 10^{11} \times \text{diag}\left(1,\ 2.2\times 10^{-5},\ 8.3\times 10^{-8}\right). \end{aligned} \tag{62}$$

When a theory of leptons beyond the SM can be described by

$$\begin{aligned} \mathscr{L}^{\text{lepton}}_{\text{BSM}} &= K_{ij}^{(l)} \overline{l}'_{\text{L}i} i\not{D} l'_{\text{L}j} + K_{ij}^{(e)} \overline{e}'_{\text{R}i} i\not{D} e'_{\text{R}j} + K_{ij}^{(\nu)} \overline{\nu}'_{\text{R}i} i\not{D} \nu'_{\text{R}j} - \frac{1}{2}\hat{M}_{ij}^{(\nu)} \nu'^{\text{t}}_{\text{R}i} C \nu'_{\text{R}j} \\ &\quad - (Y_3)_{ij}\,\overline{l}'_{\text{L}i}\phi e'_{\text{R}j} - (Y_4)_{ij}\,\overline{l}'_{\text{L}i}\tilde{\phi}\nu'_{\text{R}j} + \text{h.c.}, \end{aligned} \tag{63}$$

we have the relations:

$$k_{ij}^{(l)} = \left\langle K_{ij}^{(l)}\right\rangle,\ k_{ij}^{(e)} = \left\langle K_{ij}^{(e)}\right\rangle,\ k_{ij}^{(\nu)} = \left\langle K_{ij}^{(\nu)}\right\rangle,\ M_{ij}^{(\nu)} = \left\langle \hat{M}_{ij}^{(\nu)}\right\rangle, \tag{64}$$

$$(y_3)_{ij} = \left\langle (Y_3)_{ij}\right\rangle,\quad (y_4)_{ij} = \left\langle (Y_4)_{ij}\right\rangle, \tag{65}$$

as the matching conditions at M_{BSM}, from Eqs. (56), (57), and (63).

In the case that the U(3) family symmetry exists and $|y_3|^2 = O\left(10^{-11}\right)$, the VEV of $K_{ij}^{(e)}$ can be the form whose every component has almost the same magnitude of $O(1)$ and a mass hierarchy can originate from a tiny variance of the democratic form. We need a mechanism to realize semi-democratic forms and a small Yukawa coupling. In the case that the S_3 flavor symmetry exists, we find that a Yukawa coupling is written by

$$(y_3)_{ij} = y_3^{\text{F}} S_{ij} + \Delta y_3 T_{ij}^{(e)}, \tag{66}$$

and it is compatible with the Kähler metric:

$$k_{ij}^{(e)} = k_1^{(e)}\delta_{ij} + k_2^{(e)} S_{ij} + k_3^{(e)} Z_{ij}^{(e)} \tag{67}$$

with a suitable $W^{(e)}$. Here, y_3^{F} and Δy_3 are complex numbers whose magnitudes are $\left|y_3^{\mathrm{F}}\right|^2 = O(10^{-4})$ and $|\Delta y_3|^2 = O\left(10^{-11}\right)$, $T_{ij}^{(e)}$ is a complex matrix whose components take values of at most $O(1)$, $k_b^{(e)}$ $(b = 1, 2, 3)$ are real numbers of at most $O(1)$, and $Z_{ij}^{(e)}$ is a Hermitian matrix whose components take values of at most $O(1)$.

3.4. Top-down approach

We have developed a strategy taking the SM as a starting point. There are limitations on such a bottom-up approach. It is desirable to combine use of the bottom-up and top-down ones. Here, we propose a new procedure based on the top-down one, using knowledge and information obtained in the previous subsections.

First, we construct a theory with a flavor symmetry, extract fermion parts from it, write down a Lagrangian density as

$$\begin{aligned}\mathscr{L}_{\mathrm{BSM}}^{\mathrm{fermion}} = & K_{ij}^{(q)}\overline{q}'_{\mathrm{L}i} i\partial\!\!\!/ q'_{\mathrm{L}j} + K_{ij}^{(u)}\overline{u}'_{\mathrm{R}i} i\partial\!\!\!/ u'_{\mathrm{R}j} + K_{ij}^{(d)}\overline{d}'_{\mathrm{R}i} i\partial\!\!\!/ d'_{\mathrm{R}j} \\ & + K_{ij}^{(l)}\overline{l}'_{\mathrm{L}i} i\partial\!\!\!/ l'_{\mathrm{L}j} + K_{ij}^{(e)}\overline{e}'_{\mathrm{R}i} i\partial\!\!\!/ e'_{\mathrm{R}j} + K_{ij}^{(\nu)}\overline{\nu}'_{\mathrm{R}i} i\partial\!\!\!/ \nu'_{\mathrm{R}j} - \frac{1}{2}\hat{M}_{ij}^{(\nu)}\nu'^{\mathrm{t}}_{\mathrm{R}i} C \nu'_{\mathrm{R}j} \\ & - (Y_1)_{ij}\,\overline{q}'_{\mathrm{L}i}\tilde{\phi} u'_{\mathrm{R}j} - (Y_2)_{ij}\,\overline{q}'_{\mathrm{L}i}\phi d'_{\mathrm{R}j} - (Y_3)_{ij}\,\overline{l}'_{\mathrm{L}i}\phi e'_{\mathrm{R}j} - (Y_4)_{ij}\,\overline{l}'_{\mathrm{L}i}\tilde{\phi}\nu'_{\mathrm{R}j} + \mathrm{h.c.},\end{aligned} \tag{68}$$

and obtain the VEVs of flavons from the minimum of a scalar potential. Then, we calculate $\left\langle K_{ij}^{(q)}\right\rangle$, $\left\langle K_{ij}^{(u)}\right\rangle$, $\left\langle K_{ij}^{(d)}\right\rangle$, $\left\langle K_{ij}^{(l)}\right\rangle$, $\left\langle K_{ij}^{(e)}\right\rangle$, $\left\langle K_{ij}^{(\nu)}\right\rangle$, $\left\langle \hat{M}_{ij}^{(\nu)}\right\rangle$, $\langle (Y_1)_{ij}\rangle$, $\langle (Y_2)_{ij}\rangle$, $\langle (Y_3)_{ij}\rangle$, and $\langle (Y_4)_{ij}\rangle$. If the SUSY or its remnant exists, $\tilde{\phi}$ and ϕ should be treated as independent fields.

Second, we diagonalize $\left\langle K_{ij}^{(q)}\right\rangle$ and $\left\langle K_{ij}^{(l)}\right\rangle$ by unitary transformations as

$$\left(\tilde{U}_q\right)_{ii'}\left\langle K_{i'j'}^{(q)}\right\rangle\left(\tilde{U}_q^\dagger\right)_{j'j} = \left(\tilde{J}_q\right)^2_{ij}, \quad \left(\tilde{U}_l\right)_{ii'}\left\langle K_{i'j'}^{(l)}\right\rangle\left(\tilde{U}_l^\dagger\right)_{j'j} = \left(\tilde{J}_l\right)^2_{ij}, \tag{69}$$

where $\tilde{U}_q$ and $\tilde{U}_l$ are unitary matrices and $\tilde{J}_q$ and $\tilde{J}_l$ are real diagonal matrices. These matrices are counterparts of U_q, U_l, J_q, and J_l, and they should equate with each other if experimental data on flavor physics are completely explained by them.

Third, we change $\left\langle K_{ij}^{(u)}\right\rangle$, $\left\langle K_{ij}^{(d)}\right\rangle$, $\left\langle K_{ij}^{(e)}\right\rangle$, and $\left\langle K_{ij}^{(\nu)}\right\rangle$ into the following:

$$\left\langle \tilde{K}_{ij}^{(u)}\right\rangle \equiv \left(\tilde{J}_q\right)_{ii'}\left(\tilde{U}_q\right)_{i'i''}\left\langle\left(Y_1^{\dagger-1}\right)_{i''i'''}\right\rangle\left\langle K_{i'''j'''}^{(u)}\right\rangle\left\langle\left(Y_1^{-1}\right)_{j'''j''}\right\rangle\left(\tilde{U}_q^\dagger\right)_{j''j'}\left(\tilde{J}_q\right)_{j'j}, \tag{70}$$

$$\left\langle \tilde{K}_{ij}^{(d)}\right\rangle \equiv \left(\tilde{J}_q\right)_{ii'}\left(\tilde{U}_q\right)_{i'i''}\left\langle\left(Y_2^{\dagger-1}\right)_{i''i'''}\right\rangle\left\langle K_{i'''j'''}^{(d)}\right\rangle\left\langle\left(Y_2^{-1}\right)_{j'''j''}\right\rangle\left(\tilde{U}_q^\dagger\right)_{j''j'}\left(\tilde{J}_q\right)_{j'j}, \tag{71}$$

$$\left\langle \tilde{K}_{ij}^{(e)}\right\rangle \equiv \left(\tilde{J}_l\right)_{ii'}\left(\tilde{U}_l\right)_{i'i''}\left\langle\left(Y_3^{\dagger-1}\right)_{i''i'''}\right\rangle\left\langle K_{i'''j'''}^{(e)}\right\rangle\left\langle\left(Y_3^{-1}\right)_{j'''j''}\right\rangle\left(\tilde{U}_l^\dagger\right)_{j''j'}\left(\tilde{J}_l\right)_{j'j}, \tag{72}$$

$$\left\langle \tilde{K}_{ij}^{(\nu)}\right\rangle \equiv \left(\tilde{J}_l\right)_{ii'}\left(\tilde{U}_l\right)_{i'i''}\left\langle\left(Y_4^{\dagger-1}\right)_{i''i'''}\right\rangle\left\langle K_{i'''j'''}^{(\nu)}\right\rangle\left\langle\left(Y_4^{-1}\right)_{j'''j''}\right\rangle\left(\tilde{U}_l^\dagger\right)_{j''j'}\left(\tilde{J}_l\right)_{j'j}, \tag{73}$$

using $\tilde{U}_q$, $\tilde{U}_l$, $\tilde{J}_q$, $\tilde{J}_l$, the inverse matrices $\left\langle Y_a^{-1}\right\rangle$ of $\langle Y_a\rangle$ $(a = 1, 2, 3, 4)$, and their Hermitian conjugations.

Fourth, we diagonalize $\left\langle \tilde{K}_{ij}^{(u)}\right\rangle$, $\left\langle \tilde{K}_{ij}^{(d)}\right\rangle$, $\left\langle \tilde{K}_{ij}^{(e)}\right\rangle$, and $\left\langle \tilde{K}_{ij}^{(\nu)}\right\rangle$ by unitary transformations as

$$\left(\tilde{V}_{\mathrm{L}}^{(u)}\right)_{ii'}\left\langle \tilde{K}_{i'j'}^{(u)}\right\rangle\left(\tilde{V}_{\mathrm{L}}^{(u)\dagger}\right)_{j'j} = \left(\tilde{k}_{\mathrm{diag}}^{(u)}\right)_{ij}, \quad \left(\tilde{V}_{\mathrm{L}}^{(d)}\right)_{ii'}\left\langle \tilde{K}_{i'j'}^{(d)}\right\rangle\left(\tilde{V}_{\mathrm{L}}^{(d)\dagger}\right)_{j'j} = \left(\tilde{k}_{\mathrm{diag}}^{(d)}\right)_{ij}, \tag{74}$$

$$\left(\tilde{V}^{(e)}_{\rm L}\right)_{ii'}\left\langle\tilde{K}^{(e)}_{i'j'}\right\rangle\left(\tilde{V}^{(e)\dagger}_{\rm L}\right)_{j'j}=\left(\tilde{k}^{(e)}_{\rm diag}\right)_{ij},\quad\left(\tilde{V}^{(\nu)}_{\rm L}\right)_{ii'}\left\langle\tilde{K}^{(\nu)}_{i'j'}\right\rangle\left(\tilde{V}^{(\nu)\dagger}_{\rm L}\right)_{j'j}=\left(\tilde{k}^{(\nu)}_{\rm diag}\right)_{ij}. \tag{75}$$

Last, we examine whether the following relations hold or not:

$$\left(\tilde{k}^{(u)}_{\rm diag}\right)_{ij}=\left(y^{(u)-1}_{\rm diag}\right)^2_{ij},\quad\left(\tilde{k}^{(d)}_{\rm diag}\right)_{ij}=\left(y^{(d)-1}_{\rm diag}\right)^2_{ij},\quad\left(\tilde{k}^{(e)}_{\rm diag}\right)_{ij}=\left(y^{(e)-1}_{\rm diag}\right)^2_{ij}, \tag{76}$$

$$\tilde{V}^{(u)}_{\rm L}\tilde{V}^{(d)\dagger}_{\rm L}=V_{\rm KM},\quad\tilde{V}^{(e)}_{\rm L}\tilde{V}^{(\nu)\dagger}_{\rm L}=V_{\rm MNS}. \tag{77}$$

Note that we need to diagonalize six Hermitian matrices in total by unitary transformations in our procedure. As explained in Appendix C, we need ten Hermitian matrices in total by unitary transformations, using an ordinary procedure.

As was described previously, we should consider renormalization effects when we match theoretical predictions to experimental data. We also need some modifications in the presence of mixing with extra particles, in the case with a large flavor symmetry and/or many matter fields.

3.5. *Unification*

We discuss whether realistic mass hierarchies and flavor mixing are realized or nor, based on a grand unification and a family unification.

First, we consider a model based on $SU(5)\times S_3\times S_3$ where SU(5) is the GUT group and $S_3\times S_3$ is the flavor group. We assume that these symmetries are broken down to the SM one $G_{\rm SM}$ at the GUT scale $M_{\rm U}$. Matter fields $l'_{{\rm L}i}$ and $(d'_{{\rm R}i})^c$ belong to $\psi'^{(\bar{\mathbf{5}})}_i$ in the representation $(\bar{\mathbf{5}},\mathbf{3},\mathbf{1})$ and $q'_{{\rm L}i}$, $(u'_{{\rm R}i})^c$, and $(e'_{{\rm R}i})^c$ belong to $\psi'^{(\mathbf{10})}_i$ in $(\mathbf{10},\mathbf{1},\mathbf{3})$, where $\mathbf{3}$ is a 3D reducible representation of S_3. The Lagrangian density of matter fields (except for neutrino singlets) is given by

$$\begin{aligned}\mathscr{L}^{\rm fermion}_{\rm SU(5)}&=K^{(\psi^{(\bar{\mathbf{5}})})}_{ij}\overline{\psi}'^{(\bar{\mathbf{5}})}_i i\not{D}\psi'^{(\bar{\mathbf{5}})}_j+K^{(\psi^{(\mathbf{10})})}_{ij}\overline{\psi}'^{(\mathbf{10})}_i i\not{D}\psi'^{(\mathbf{10})}_j\\&\quad-\left(Y^{\rm U}_1\right)_{ij}\psi'^{(\mathbf{10})^t}_i C\psi'^{(\bar{\mathbf{5}})}_j\phi^{(\bar{\mathbf{5}})}-\left(Y^{\rm U}_2\right)_{ij}\psi'^{(\mathbf{10})^t}_i C\psi'^{(\mathbf{10})}_j\phi^{(\mathbf{5})}+\text{h.c.},\end{aligned} \tag{78}$$

where $\left(Y^{\rm U}_1\right)_{ij}$ and $\left(Y^{\rm U}_2\right)_{ij}$ are Yukawa couplings, and $\phi^{(\bar{\mathbf{5}})}$ and $\phi^{(\mathbf{5})}$ are scalar fields in $(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$ and $(\mathbf{5},\mathbf{1},\mathbf{1})$, respectively. If $K^{(\psi^{(\bar{\mathbf{5}})})}_{ij}$, $K^{(\psi^{(\mathbf{10})})}_{ij}$, $\left(Y^{\rm U}_1\right)_{ij}$, and $\left(Y^{\rm U}_2\right)_{ij}$ are SU(5) singlets, we have the relations:

$$\left\langle K^{(\psi^{(\bar{\mathbf{5}})})}_{ij}\right\rangle=\left\langle K^{(l)}_{ij}\right\rangle=\left\langle K^{(d)}_{ij}\right\rangle,\quad\left\langle K^{(\psi^{(\mathbf{10})})}_{ij}\right\rangle=\left\langle K^{(q)}_{ij}\right\rangle=\left\langle K^{(u)}_{ij}\right\rangle=\left\langle K^{(e)}_{ij}\right\rangle, \tag{79}$$

$$\left\langle\left(Y^{\rm U}_1\right)_{ij}\right\rangle=\left\langle(y_2)_{ij}\right\rangle=\left\langle(y_3)_{ji}\right\rangle,\quad\left\langle\left(Y^{\rm U}_2\right)_{ij}\right\rangle=\left\langle(y_1)_{ij}\right\rangle, \tag{80}$$

at $M_{\rm U}$. From Eqs. (79) and (80), we derive the usual GUT relation among down-type quark and charged lepton Yukawa couplings:

$$\left(y^{(d)}\right)_{ij}=\left(y^{(e)}\right)_{ji}. \tag{81}$$

In the case that $\left(Y^{\rm U}_1\right)_{ij}$ and $\left(Y^{\rm U}_2\right)_{ij}$ contain SU(5) non-singlet parts, realistic mass hierarchies and mixing can be realized with suitable VEVs of non-singlet parts.

Next, we consider a model based on $SO(10)\times S_3$. Matter fields $l'_{{\rm L}i}$, $(d'_{{\rm R}i})^c$, $q'_{{\rm L}i}$, $(u'_{{\rm R}i})^c$, $(e'_{{\rm R}i})^c$, and $(\nu'_{{\rm R}i})^c$ belong to $\psi'^{(\mathbf{16})}_i$ in $(\mathbf{16},\mathbf{3})$. The matter sector is described by

$$\mathscr{L}^{\rm fermion}_{\rm SO(10)}=K^{(\psi^{(\mathbf{16})})}_{ij}\overline{\psi}'^{(\mathbf{16})}_i i\not{D}\psi'^{(\mathbf{16})}_j-\left(\left(Y^{\rm U}\right)_{ij}\psi'^{(\mathbf{16})^t}_i C\psi'^{(\mathbf{16})}_j\phi^{(\mathbf{10})}+\text{h.c.}\right), \tag{82}$$

where $\left(Y^{\mathrm{U}}\right)_{ij}$ is a Yukawa coupling and $\phi^{(\mathbf{10})}$ is a scalar field in $(\mathbf{10}, \mathbf{1})$. If $K_{ij}^{(\psi^{(\mathbf{16})})}$ and $\left(Y^{\mathrm{U}}\right)_{ij}$ are SO(10) singlets, we have the relations:

$$\left\langle K_{ij}^{(\psi^{(\mathbf{16})})}\right\rangle = \left\langle K_{ij}^{(l)}\right\rangle = \left\langle K_{ij}^{(d)}\right\rangle = \left\langle K_{ij}^{(q)}\right\rangle = \left\langle K_{ij}^{(u)}\right\rangle = \left\langle K_{ij}^{(e)}\right\rangle = \left\langle K_{ij}^{(\nu)}\right\rangle, \tag{83}$$

$$\left\langle \left(Y^{\mathrm{U}}\right)_{ij}\right\rangle = \left\langle (y_1)_{ij}\right\rangle = \left\langle (y_2)_{ij}\right\rangle = \left\langle (y_3)_{ij}\right\rangle = \left\langle (y_4)_{ij}\right\rangle, \tag{84}$$

at M_{U}. In this case, without extra matters and/or extra contributions, quark and lepton masses and flavor mixing cannot be explained. In the case that $\left(Y^{\mathrm{U}}\right)_{ij}$ contain SO(10) non-singlet parts, we also need extra contributions if Dirac's naturalness is adopted.

Last, we consider the family unification based on a simple gauge group $\mathrm{G_{FU}}$ whose maximal subgroup is $\mathrm{G_U} \times \mathrm{G_F}$. Here, $\mathrm{G_U}$ is a GUT group and $\mathrm{G_F}$ is a family group. We assume that a field Ψ with a vectorlike representation contains three families of SM fermions $\psi_i^I = (q'_{\mathrm{L}i}, (u'_{\mathrm{R}i})^c, (d'_{\mathrm{R}i})^c, l'_{\mathrm{L}i}, (e'_{\mathrm{R}i})^c, (\nu'_{\mathrm{R}i})^c)$ $(I = q, u, d, l, e, \nu)$ as its submultiplets. After the breakdown of $\mathrm{G_{FU}}$ into $\mathrm{G_{SM}}$, the kinetic term $K\overline{\Psi}i\not{D}\Psi$ changes into $\left\langle K_{ij}^{(I)}\right\rangle \overline{\psi}_i^I i\not{D}\psi_j^I$. In this case, $\left\langle K_{ij}^{(I)}\right\rangle$ are not, in general, common and there is a possibility to explain fermion masses and flavor mixing. However, it seems to be unnatural because we need a fine-tuning on a realization of semi-democratic type of Kähler metrics in order to generate fermion mass hierarchies, as explained in Appendix B. Another problem in the family unification is that extra particles including mirror particles appear; this is solved in the family unification on orbifold [26–28] and special GUTs [29,30].

4. Conclusions and discussions

We have studied the origin of fermion mass hierarchy and flavor mixing in the SM, using the bottom-up approach. The approach is based on the assumptions that the field variables in the SM are not necessarily the same as those in a theory beyond the SM, and there is a flavor symmetry and flavons couple to matter fields in the matter kinetic terms dominantly. We have supposed field variables respecting a flavor symmetry (unitary bases of a flavor symmetry) and rewritten the Lagrangian density in the SM using such variables. We have investigated the structure of terms violating the flavor symmetry, and conjectured physics beyond the SM. We have suggested that the hierarchical structure in the Yukawa interactions of quarks and charged leptons can originate from non-canonical matter kinetic terms in the presence of flavor-symmetric Yukawa interactions and a flavor symmetry can be hidden in the form of non-unitary bases in the SM. We have proposed a variant top-down procedure, using an insight and formulas obtained by our bottom-up approach.

In our approach, the problem of fermion masses and flavor mixing is deeply related to not only the determination of Yukawa coupling matrices but also the determination of matter kinetic terms and the VEVs of Kähler metric $K_{ij}^{(I)}$. If flavons couple to matter fields in the Kähler potential, the VEVs of $K_{ij}^{(I)}$ strongly depend on the dynamics of flavor symmetry breaking due to flavons. In a grand unification with a flavor symmetry, contributions of GUT group non-singlet parts in $K_{ij}^{(I)}$ can be essential to derive a realistic flavor structure.

We explain preceding works on the flavor physics based on matter kinetic terms, other than Refs. [21,22]. The problem of fermion mass hierarchies was investigated in supergravity and superstring models with non-canonical Kähler potentials including dilaton and moduli fields [31,32]. The Yukawa textures were obtained from non-canonical Kähler potentials in the extension of minimal SUSY SM with an anomalous horizontal symmetry [33]. In both works, the symmetry corresponding to a flavor symmetry is an Abelian one and the structure of Yukawa couplings resembles that derived from the Froggatt–Nielsen mechanism [9]. The effect of the Kähler potential on mixing matrices was studied in a model-independent way [34]. The flavor symmetry of kinetic terms was discussed in a SUSY SM [35]. The flavor problem was studied through contributions of higher-dimensional operators in the case with hierarchical fermion kinetic terms originating from hierarchical fermion wave functions, under the assumption that the energy scale of new physics is in the TeV range [36]. In our setup, the scale M_{BSM} can also be constrained by the suppression of flavor-changing transitions.

As fermion kinetic functions or the Kähler metric $K_{ij}^{(I)}$ contain flavons in our approach, they are regarded as counterparts of "Yukawaons" such that Yukawa couplings are not parameters but fields [37].

Our approach would be useful as a complementary one to explore physics beyond the SM and it would be worth studying flavor physics model-dependently and/or independently by paying close attention to matter kinetic terms, because the structure of the Kähler potential could play a vital role as a key test of new physics.

Acknowledgements

This work was supported in part by scientific grants from the Ministry of Education, Culture, Sports, Science and Technology under Grant No. 17K05413.

Funding

Open Access funding: SCOAP[3].

Appendix A. Unitary and non-unitary bases

We give an illustration of a realization of U(N) symmetry using unitary matrices and non-unitary ones based on a polynomial:

$$\mathscr{L} = \Phi^\dagger K \Phi + \Phi^\dagger \Phi, \tag{A.1}$$

where Φ is an N-plet of U(N), and K is an $N \times N$ Hermitian matrix. We consider a case in which K depends on a set of fields $\{\varphi\}$, i.e., $K = K(\varphi, \varphi^\dagger)$. If K changes into $K \to UKU^\dagger$ in accord with the U(N) transformation $\Phi \to U\Phi$ with an arbitrary unitary matrix U, $\mathscr{L}$ is invariant under the U(N) transformation. We call fields transformed by unitary matrices such as Φ "unitary bases".

The U(N) invariance can be spontaneously broken down to a smaller one, after some φ acquire the VEV $\langle\varphi\rangle$ and $\langle K\rangle (\equiv K(\langle\varphi\rangle, \langle\varphi^\dagger\rangle))$ takes a form that is not proportional to the identity matrix I. $\langle K\rangle$ is a Hermitian matrix and it is written as $\langle K\rangle = W^\dagger W$ with a general $N \times N$ complex matrix W. By using a redefinition of field as $\tilde{\Phi} \equiv W\Phi$ and $\tilde{\Phi}^\dagger \equiv \Phi^\dagger W^\dagger$, $\mathscr{L}$ is rewritten by

$$\tilde{\mathscr{L}} = \Phi^\dagger \langle K\rangle \Phi + \Phi^\dagger \Phi = \Phi^\dagger W^\dagger W \Phi + \Phi^\dagger \Phi = \tilde{\Phi}^\dagger \tilde{\Phi} + \tilde{\Phi}^\dagger (W^\dagger)^{-1} W^{-1} \tilde{\Phi}. \tag{A.2}$$

The previous U(N) transformation is realized by $\tilde{\Phi} \to \tilde{U}\tilde{\Phi}$ with $\tilde{U} = WUW^{-1}$. Note that $\tilde{U}$ is not necessarily a unitary matrix because W is not a unitary matrix, and the second term $\tilde{\Phi}^\dagger (W^\dagger)^{-1} W^{-1} \tilde{\Phi}$ is invariant under $\tilde{\Phi} \to \tilde{U}\tilde{\Phi}$, but the first one $\tilde{\Phi}^\dagger \tilde{\Phi}$ is not necessarily. The transformation of unbroken subgroup H is realized by a unitary matrix. We call fields transformed by non-unitary matrices such as $\tilde{\Phi}$ "non-unitary bases".

$\mathscr{L}$ and the final form of $\tilde{\mathscr{L}}$ can be regarded as counterparts of the Lagrangian density of the matter sector in a theory beyond the SM and the Lagrangian density of the matter sector in the SM, respectively. As the global U(3) × U(3) × U(3)/U(1) symmetry appears in the quark kinetic term $\mathscr{L}_{\rm kinetic}^{\rm quark}$, a U$(N)$ symmetry emerges in the first term of $\tilde{\mathscr{L}}$ or $\tilde{\Phi}^\dagger\tilde{\Phi}$ is invariant under the U(N) transformation $\tilde{\Phi} \to V\tilde{\Phi}$ with an arbitrary unitary matrix V.

Appendix B. Non-canonical Kähler potential

We consider a SUSY model with the flavor symmetry SU(3) × C_3 (where C_3 is the cyclic group of order 3) and a non-minimal Kähler potential:

$$K = \left(1 + \frac{a_1}{\Lambda^2}\varphi_k^\alpha\varphi_k^{\alpha\dagger} + \frac{a_2}{\Lambda^2}\sum_\alpha \varphi_k^\alpha \sum_\beta \varphi_k^{\beta\dagger}\right)|\phi_i|^2 + \left(\frac{a_3}{\Lambda^2}\varphi_i^\alpha\varphi_j^{\alpha\dagger} + \frac{a_4}{\Lambda^2}\sum_\alpha \varphi_i^\alpha \sum_\beta \varphi_j^{\beta\dagger}\right)\phi_i^\dagger\phi_j + \cdots, \tag{B.1}$$

where a_1, a_2, a_3, and a_4 are parameters, Λ is a high-energy scale, and φ_i^α and ϕ_i are the scalar components of the flavon chiral supermultiplets and matter chiral supermultiplet, respectively. The ellipsis stands for higher-dimensional terms with $O\left(1/\Lambda^4\right)$. The family labels are denoted by i, j, and k, and φ_i^α and ϕ_i belong to triplets of SU(3). The indices α and β are labels of C_3 and run from 1 to 3. From Eq. (B.1), the Kähler metric of matter fields is calculated as

$$K_{ij} = \frac{\partial^2 K}{\partial\phi_i^\dagger\partial\phi_j} = \left(1 + \frac{a_1}{\Lambda^2}\varphi_k^\alpha\varphi_k^{\alpha\dagger} + \frac{a_2}{\Lambda^2}\sum_\alpha \varphi_k^\alpha \sum_\beta \varphi_k^{\beta\dagger}\right)\delta_{ij} + \frac{a_3}{\Lambda^2}\varphi_i^\alpha\varphi_j^{\alpha\dagger} + \frac{a_4}{\Lambda^2}\sum_\alpha \varphi_i^\alpha \sum_\beta \varphi_j^{\beta\dagger} + \cdots. \tag{B.2}$$

If Λ is much bigger than the VEVs of φ_i^α, $|\phi_i|^2$ dominates in K and the matter kinetic terms take almost canonical forms with $\langle K_{ij}\rangle = \delta_{ij} + O\left(\left(\langle\varphi_i^\alpha\rangle/\Lambda\right)^2\right)$.

To obtain a semi-democratic form, we need $\langle\varphi_i^\alpha\rangle = O(\Lambda)$. In this case, other higher-order terms can contribute to the determination of $\langle K_{ij}\rangle$ and then the evaluation cannot be justified in a perturbation region. Although we have such a problem, we study a case with $\langle\varphi_i^\alpha\rangle = O(\Lambda)$ by taking the superpotential of flavons:

$$W^{(\varphi)} = c_1\varphi^3 + \frac{c_2}{\Lambda^3}\left(\varphi^3\right)^2, \tag{B.3}$$

where c_1 and c_2 are parameters and $\varphi^3 \equiv \varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma}\varphi_i^\alpha\varphi_j^\beta\varphi_k^\gamma$. One of the SUSY-preserving conditions is given by

$$\frac{\partial W^{(\varphi)}}{\partial\varphi_i^\alpha} = 3\varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma}\varphi_j^\beta\varphi_k^\gamma\left(c_1 + \frac{2c_2}{\Lambda^3}\varphi^3\right) = 0, \tag{B.4}$$

and there exist two kinds of vacuum solutions $\langle\varphi_i^\alpha\rangle = 0$ and $\langle\varphi_i^\alpha\rangle \neq 0$.

(a) Flavor-symmetric vacuum with $\langle \varphi_i^\alpha \rangle = 0$
By inserting $\langle \varphi_i^\alpha \rangle = 0$ into Eqs. (B.1) and (B.2), we obtain the canonical one for matter fields, i.e., $\langle K_{ij} \rangle = \delta_{ij}$.
(b) Broken vacuum of flavor symmetry with $\langle \varphi_i^\alpha \rangle \neq 0$
From Eq. (B.4), we find a broken vacuum of flavor symmetry represented by

$$\langle \varphi_i^\alpha \rangle = \left(\frac{-c_1}{2c_2} \right)^{1/3} \times \Lambda \delta_i^\alpha. \tag{B.5}$$

Then, by inserting these VEVs into Eq. (B.2), we obtain the VEV of K_{ij}:

$$\langle K_{ij} \rangle = \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \xi \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \cdots, \tag{B.6}$$

where η and ξ are given by

$$\eta = 1 + (3a_1 + 3a_2 + a_3) \left(\frac{-c_1}{2c_2} \right)^{2/3}, \ \ \xi = a_4 \left(\frac{-c_1}{2c_2} \right)^{2/3}, \tag{B.7}$$

respectively. From Eq. (B.6), $\langle K_{ij} \rangle$ can be semi-democratic with suitable values of parameters, but it seems to be unnatural with a fine-tuning among parameters (including ones from higher-order terms) based on a perturbative analysis. A Kähler potential from a non-perturbative effect can play a crucial role in the derivation of semi-democratic types of kinetic terms.

Appendix C. Ordinary top-down procedure

For reference purposes, we explain an ordinary top-down procedure, starting from $\mathscr{L}_{\mathrm{BSM}}^{\mathrm{fermion}}$ of Eq. (68) with the VEVs $\langle K_{ij}^{(q)} \rangle$, $\langle K_{ij}^{(u)} \rangle$, $\langle K_{ij}^{(d)} \rangle$, $\langle K_{ij}^{(l)} \rangle$, $\langle K_{ij}^{(e)} \rangle$, $\langle K_{ij}^{(\nu)} \rangle$, $\langle \hat{M}_{ij}^{(\nu)} \rangle$, $\langle (Y_1)_{ij} \rangle$, $\langle (Y_2)_{ij} \rangle$, $\langle (Y_3)_{ij} \rangle$, and $\langle (Y_4)_{ij} \rangle$.

First, we diagonalize the Kähler metrics by unitary transformations as

$$\left(\tilde{U}_q\right)_{ii'} \langle K_{i'j'}^{(q)} \rangle \left(\tilde{U}_q^\dagger\right)_{j'j} = \left(\tilde{J}_q\right)_{ij}^2, \ \ \left(\tilde{U}_u\right)_{ii'} \langle K_{i'j'}^{(u)} \rangle \left(\tilde{U}_u^\dagger\right)_{j'j} = \left(\tilde{J}_u\right)_{ij}^2, \tag{C.1}$$

$$\left(\tilde{U}_d\right)_{ii'} \langle K_{i'j'}^{(d)} \rangle \left(\tilde{U}_d^\dagger\right)_{j'j} = \left(\tilde{J}_d\right)_{ij}^2, \ \ \left(\tilde{U}_l\right)_{ii'} \langle K_{i'j'}^{(l)} \rangle \left(\tilde{U}_l^\dagger\right)_{j'j} = \left(\tilde{J}_l\right)_{ij}^2, \tag{C.2}$$

$$\left(\tilde{U}_e\right)_{ii'} \langle K_{i'j'}^{(e)} \rangle \left(\tilde{U}_e^\dagger\right)_{j'j} = \left(\tilde{J}_e\right)_{ij}^2, \ \ \left(\tilde{U}_\nu\right)_{ii'} \langle K_{i'j'}^{(\nu)} \rangle \left(\tilde{U}_\nu^\dagger\right)_{j'j} = \left(\tilde{J}_\nu\right)_{ij}^2, \tag{C.3}$$

where $\tilde{U}_q$, $\tilde{U}_u$, $\tilde{U}_d$, $\tilde{U}_l$, $\tilde{U}_e$, and $\tilde{U}_\nu$ are unitary matrices and $\tilde{J}_q$, $\tilde{J}_u$, $\tilde{J}_d$, $\tilde{J}_l$, $\tilde{J}_e$, and $\tilde{J}_\nu$ are real diagonal matrices.

Second, we obtain the following Yukawa couplings from $(Y_1)_{ij}$, $(Y_2)_{ij}$, $(Y_3)_{ij}$, and $(Y_4)_{ij}$ such that

$$\tilde{y}_{ij}^{(u)} = \left(\tilde{J}_q^{-1}\right)_{ii'} \left(\tilde{U}_q\right)_{i'i''} \langle (Y_1)_{i''j''} \rangle \left(\tilde{U}_u^\dagger\right)_{j''j'} \left(\tilde{J}_u^{-1}\right)_{j'j}, \tag{C.4}$$

$$\tilde{y}_{ij}^{(d)} = \left(\tilde{J}_q^{-1}\right)_{ii'} \left(\tilde{U}_q\right)_{i'i''} \langle (Y_2)_{i''j''} \rangle \left(\tilde{U}_d^\dagger\right)_{j''j'} \left(\tilde{J}_d^{-1}\right)_{j'j}, \tag{C.5}$$

$$\tilde{y}_{ij}^{(e)} = \left(\tilde{J}_l^{-1}\right)_{ii'} \left(\tilde{U}_l\right)_{i'i''} \langle (Y_3)_{i''j''} \rangle \left(\tilde{U}_e^\dagger\right)_{j''j'} \left(\tilde{J}_e^{-1}\right)_{j'j}, \tag{C.6}$$

$$\tilde{y}^{(\nu)}_{ij} = \left(\tilde{J}_l^{-1}\right)_{ii'} \left(\tilde{U}_l\right)_{i'i''} \langle (Y_4)_{i''j''}\rangle \left(\tilde{U}^\dagger_\nu\right)_{j''j'} \left(\tilde{J}_\nu^{-1}\right)_{j'j}, \tag{C.7}$$

where $\tilde{J}_q^{-1}$, $\tilde{J}_u^{-1}$, $\tilde{J}_d^{-1}$, $\tilde{J}_l^{-1}$, $\tilde{J}_e^{-1}$, and $\tilde{J}_\nu^{-1}$ are the inverse matrices of $\tilde{J}_q$, $\tilde{J}_u$, $\tilde{J}_d$, $\tilde{J}_l$, $\tilde{J}_e$, and $\tilde{J}_\nu$, respectively.

Third, we diagonalize $\left(\tilde{y}^{(u)}\tilde{y}^{(u)\dagger}\right)_{ij}$, $\left(\tilde{y}^{(d)}\tilde{y}^{(d)\dagger}\right)_{ij}$, $\left(\tilde{y}^{(e)}\tilde{y}^{(e)\dagger}\right)_{ij}$, and $\left(\tilde{y}^{(\nu)}\tilde{y}^{(\nu)\dagger}\right)_{ij}$ by unitary transformations as

$$\left(\tilde{V}_{\mathrm{L}}^{(u)}\right)_{ii'} \left(\tilde{y}^{(u)}\tilde{y}^{(u)\dagger}\right)_{i'j'} \left(\tilde{V}_{\mathrm{L}}^{(u)\dagger}\right)_{j'j} = \left(\tilde{y}^{(u)2}_{\mathrm{diag}}\right)_{ij}, \tag{C.8}$$

$$\left(\tilde{V}_{\mathrm{L}}^{(d)}\right)_{ii'} \left(\tilde{y}^{(d)}\tilde{y}^{(d)\dagger}\right)_{i'j'} \left(\tilde{V}_{\mathrm{L}}^{(d)\dagger}\right)_{j'j} = \left(\tilde{y}^{(d)2}_{\mathrm{diag}}\right)_{ij}, \tag{C.9}$$

$$\left(\tilde{V}_{\mathrm{L}}^{(e)}\right)_{ii'} \left(\tilde{y}^{(e)}\tilde{y}^{(e)\dagger}\right)_{i'j'} \left(\tilde{V}_{\mathrm{L}}^{(e)\dagger}\right)_{j'j} = \left(\tilde{y}^{(e)2}_{\mathrm{diag}}\right)_{ij}, \tag{C.10}$$

$$\left(\tilde{V}_{\mathrm{L}}^{(\nu)}\right)_{ii'} \left(\tilde{y}^{(\nu)}\tilde{y}^{(\nu)\dagger}\right)_{i'j'} \left(\tilde{V}_{\mathrm{L}}^{(\nu)\dagger}\right)_{j'j} = \left(\tilde{y}^{(\nu)2}_{\mathrm{diag}}\right)_{ij}, \tag{C.11}$$

where $\tilde{y}^{(u)2}_{\mathrm{diag}}$, $\tilde{y}^{(d)2}_{\mathrm{diag}}$, $\tilde{y}^{(e)2}_{\mathrm{diag}}$, and $\tilde{y}^{(\nu)2}_{\mathrm{diag}}$ are $\tilde{y}^{(u)}_{\mathrm{diag}}$ squared, $\tilde{y}^{(d)}_{\mathrm{diag}}$ squared, $\tilde{y}^{(e)}_{\mathrm{diag}}$ squared, and $\tilde{y}^{(\nu)}_{\mathrm{diag}}$ squared, respectively.

Last, we examine whether the following relations hold or not:

$$\left(\tilde{y}^{(u)}_{\mathrm{diag}}\right)_{ij} = \left(y^{(u)}_{\mathrm{diag}}\right)_{ij}, \quad \left(\tilde{y}^{(d)}_{\mathrm{diag}}\right)_{ij} = \left(y^{(d)}_{\mathrm{diag}}\right)_{ij}, \quad \left(\tilde{y}^{(e)}_{\mathrm{diag}}\right)_{ij} = \left(y^{(e)}_{\mathrm{diag}}\right)_{ij}, \tag{C.12}$$

$$\tilde{V}_{\mathrm{L}}^{(u)}\tilde{V}_{\mathrm{L}}^{(d)\dagger} = V_{\mathrm{KM}}, \quad \tilde{V}_{\mathrm{L}}^{(e)}\tilde{V}_{\mathrm{L}}^{(\nu)\dagger} = V_{\mathrm{MNS}}. \tag{C.13}$$

References

[1] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
[2] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
[3] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[4] M. S. Chanowitz, J. Ellis, and M. K. Gaillard, Nucl. Phys. B **128**, 506 (1977).
[5] H. Fritzsch, Phys. Lett. B **73**, 317 (1978).
[6] H. Harari, H. Haut, and J. Weyers, Phys. Lett. B **78**, 459 (1978).
[7] H. Fritzsch, Nucl. Phys. B **155**, 189 (1979).
[8] H. Georgi and C. Jarlskog, Phys. Lett. B **86**, 297 (1979).
[9] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **147**, 277 (1979).
[10] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).
[11] C. Patrignani et al. [Particle Data Group], Chin. Phys. C **40**, 100001 (2016).
[12] M. Leurer, Y. Nir, and N. Seiberg, Nucl. Phys. B **398**, 319 (1993).
[13] Y. Koide, Phys. Rev. D **71**, 016010 (2005).
[14] T. Maehara and T. Yanagida, Prog. Theor. Phys. **60**, 822 (1978).
[15] K. Inoue, Prog. Theor. Phys. **93**, 403 (1995).
[16] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002).
[17] P. F. Harrison and W. G. Scott, Phys. Lett. B **535**, 163 (2002).
[18] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010).
[19] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Prog. Theor. Phys. Suppl. **183**, 1 (2010).
[20] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, Lect. Notes Phys. **858**, 1 (2012) and references therein.
[21] M. Kakizaki and M. Yamaguchi, Phys. Lett. B **573**, 123 (2003).
[22] K. Hamaguchi, M. Kakizaki, and M. Yamaguchi, Phys. Rev. D **68**, 056007 (2003).
[23] P. Minkowski, Phys. Lett. B **67**, 421 (1977).
[24] T. Yanagida, in *Proceedings of Workshop on the Unified Theory and the Baryon Number in the Universe*, eds. O. Sawada and A. Sugamoto (National laboratory for high energy physics, Tsukuba, Japan, 1979), KEK report 79-18.

[25] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, eds. P. van Nieuwenhuizen and D. Z. Freedman (North Holland, Amsterdam, 1979).
[26] Y. Kawamura, T. Kinami, and K.-y. Oda, Phys. Rev. D . **76**, 035001 (2007).
[27] Y. Goto, Y. Kawamura, and T. Miura, Phys. Rev. D **88**, 055016 (2013).
[28] Y. Goto and Y. Kawamura, Phys. Rev. D **98**, 035039 (2018) [arXiv:1712.06444 [hep-ph]] [Search INSPIRE].
[29] N. Yamatsu, Prog. Theor. Exp. Phys. **2017**, 061B01 (2017).
[30] N. Yamatsu, Prog. Theor. Exp. Phys. **2017**, 101B01 (2017).
[31] P. Binétruy and E. Dudas, Nucl. Phys. B **442**, 21 (1995).
[32] P. Binetruy and E. Dudas, Nucl. Phys. B **451**, 31 (1995).
[33] P. Binétruy, S. Lavignac, and P. Ramond, Nucl. Phys. B **477**, 353 (1996).
[34] J. R. Espinosa and A. Ibarra, J. High Energy Phys. **0408**, 010 (2004).
[35] C. Liu, Commun. Theor. Phys. **47**, 1088 (2007).
[36] S. Davidson, G. Isidori, and S. Uhlig, Phys. Lett. B **663**, 73 (2008).
[37] Y. Koide, Phys. Rev. D **78**, 093006 (2008).