



# General relativity as a biconformal gauge theory

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Received 25 August 2018; received in revised form 16 April 2019; accepted 21 April 2019

Available online 26 April 2019

Editor: Clay Córdova

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## Abstract

We consider the conformal group of a space of dim  $n = p + q$ , with  $SO(p, q)$  metric. The quotient of this group by its homogeneous Weyl subgroup gives a principal fiber bundle with  $2n$ -dim base manifold and Weyl fibers. The Cartan generalization to a curved  $2n$ -dim geometry admits an action functional linear in the curvatures. Because symmetry is maintained between the translations and the special conformal transformations in the construction, these spaces are called biconformal; this same symmetry gives biconformal spaces overlapping structures with double field theories, including manifest T-duality. We establish that biconformal geometry is a form of double field theory, showing how general relativity with integrable local scale invariance arises from its field equations. While we discuss the relationship between biconformal geometries and the double field theories of T-dual string theories, our principal interest is the study of the gravity theory. We show that vanishing torsion and vanishing co-torsion solutions to the field equations overconstrain the system, implying a trivial biconformal space. With co-torsion unconstrained, we show that (1) the torsion-free solutions are foliated by copies of an  $n$ -dim Lie group, (2) torsion-free solutions generically describe locally scale-covariant general relativity with symmetric, divergence-free sources on either the co-tangent bundle of  $n$ -dim  $(p, q)$ -spacetime or the torus of double field theory, and (3) torsion-free solutions admit a subclass of spacetimes with  $n$ -dim non-abelian Lie symmetry. These latter cases include the possibility of a gravity-electroweak unification. It is notable that the field equations reduce all curvature components to dependence only on the solder form of an  $n$ -dim Lagrangian submanifold, despite the increased number of curvature components and doubled number of initial independent variables.

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<https://doi.org/10.1016/j.nuclphysb.2019.114624>

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## 1. Introduction

### 1.1. Biconformal spaces and the biconformal action

It was shown in the 1950s and 1960s that general relativity may be cast as a Lorentz [1] or Poincaré [2] gauge theory. Subsequent approaches [3–9] refined the methods and extended the initial symmetry to Weyl, deSitter and conformal. A systematic approach to the resulting gauge theories of gravity shows that it is possible to formulate general relativity in several ways [10].

#### 1.1.1. Conformally based theories of gravity

Generally, the use of conformal symmetries for gravity theories (and in the MacDowell-Mansouri case, de Sitter) leads to action functionals which are quadratic in the curvature and apply only to 4-dimensional spacetimes. This is because conformal scaling by  $e^\phi$  changes the volume form by  $e^{n\phi}$  in  $n$ -dimensions. In four dimensions this factor may be offset by two factors of the curvature, but in even dimension  $n = 2k$ , we require  $k$  factors of the curvature to make the action dimensionless. Various techniques allow these quadratic theories to nonetheless reduce to general relativity [4,8,9,11]. The most studied case is that of Weyl (conformal) gravity, in which the difficulty of higher order field equations has been alternatively exploited and overcome. When only the metric is varied, the field equations are fourth order, and include solutions not found in general relativity [12–14]. Mannheim [15] attempts to use the additional scaling properties to explain galactic rotation curves (but also see [16]). Alternatively, it has been shown in [17] that by reformulating Weyl gravity as a gauge theory and varying all of the gauge fields, the additional field equations give the integrability conditions needed to reduce the order and exactly reproduce locally scale covariant general relativity.

There are two exceptions to these higher-order curvature requirements, which have actions linear in the curvature and can be formulated in any dimension. In the first of the two approaches, Dirac [3] builds on previous work with scalar-tensor theories [18–21], achieving a curvature-linear action by including a scalar field to help offset the scaling of the volume form. Dirac considers a Weyl geometry in which the curvature is coupled to the Weyl vector and a scalar field. Generalizing his action functional to  $n$ -dimensions, it takes the form

$$S = \int \left( \kappa^2 g^{\alpha\beta} R_{\alpha\beta} - \frac{\beta}{2} \kappa^{2\left(\frac{n-4}{n-2}\right)} g^{\alpha\mu} g^{\beta\nu} \Omega_{\alpha\beta} \Omega_{\mu\nu} - \frac{4(n-1)}{n-2} g^{\alpha\beta} D_{\alpha\kappa} D_{\beta\kappa} - \lambda \kappa^{\frac{2n}{n-2}} \right) \sqrt{-g} d^n x$$

where  $\kappa$  is a scalar field of conformal weight  $w_\kappa = -\frac{n-2}{2}$  and  $\Omega_{\alpha\beta} = W_{\alpha,\beta} - W_{\beta,\alpha}$  is the dilatational curvature. The only occurrence of the Weyl vector is in the dilatational curvature,  $\Omega_{\alpha\beta}$ , which must vanish or be unmeasurably small to avoid unphysical size changes. Dirac interpreted  $\Omega_{\alpha\beta}$  as the electromagnetic field and the time dependence of the scalar field  $\kappa$  as a time-dependent gravitational constant. The electromagnetic interpretation is untenable, but solutions with  $\Omega_{\alpha\beta} = 0$  or with other interpretations remain to be explored.

The second curvature-linear action arises when the volume element is like that of a phase space, since the “momentum” directions have opposite conformal weight from the “space” dimensions. Such a space arises naturally from the quotient of the conformal group by its Weyl subgroup ( $SO(p, q)$  and dilatations), with the most general curvature-linear action built from the  $SO(p, q)$  and dilatational curvatures [11] being

$$S = \int e_{ac\dots d}{}^{be\dots f} (\alpha \Omega^a{}_b + \beta \delta^a{}_b \Omega + \gamma \mathbf{e}^a \wedge \mathbf{f}_b) \wedge \mathbf{e}^c \wedge \dots \wedge \mathbf{e}^d \wedge \mathbf{f}_e \wedge \dots \wedge \mathbf{f}_f \quad (1)$$

Here,  $\Omega^a{}_b$  is the curvature of the  $SO(p, q)$  gauge field,  $\Omega$  is the dilatational curvature, and  $\alpha, \beta$  and  $\gamma$  are dimensionless constants. The differential forms  $\mathbf{e}^a$  and  $\mathbf{f}_a$  are the gauge fields of translations and special conformal transformations, respectively. Together the latter give an orthonormal frame field on a  $2n$ -dimensional manifold. Because of the symmetry maintained between the translations of the conformal group and the special conformal transformations, these spaces are called *biconformal*. Here we explore the large class of torsion-free biconformal spaces, showing that they reduce to general relativity on Lagrangian submanifolds. It is the consequences of the action, Eq. (1) that will occupy our present inquiry.

### 1.1.2. The biconformal gauging

The construction of a biconformal space begins with a flat,  $n$ -dimensional space with  $SO(p, q)$  invariant metric,  $p + q = n$ . Compactifying this space by adding an appropriate point or null cones at infinity (See Appendix A) allows us to define its conformal group,  $SO(p + 1, q + 1)$ . The biconformal quotient [8,9] is then  $SO(p + 1, q + 1)/SO(p, q) \times SO(1, 1)$ , where  $SO(1, 1)$  transformations represent dilatations and the full subgroup  $\mathcal{W} \equiv SO(p, q) \times SO(1, 1)$  is the homogeneous Weyl group. The quotient gives rise to a principal fiber bundle with  $2n$  dimensional base manifold and homogeneous Weyl fibers. The connection of this *flat biconformal space* is then generalized, giving rise to conformal Lie algebra-valued 2-form curvatures. These are required to be horizontal and the resulting Cartan equations integrable. The  $2n$ -dimensional curved base manifolds are *biconformal spaces* while local  $SO(p, q)$  and dilatational invariance remain, together comprising the *biconformal bundle*.

*Biconformal gravity* is the gravity theory following from variation of the action, Eq. (1), with respect to each of the conformal gauge fields, together with the Cartan structure equations to define the curvatures in terms of the connection, and the generalized Bianchi identities arising as integrability conditions. The construction of these models is described in full detail in [22]. See also [8,9,11,23].

### 1.2. Relationship to double field theory

Biconformal spaces share many features in common with double field theories.

Double field theory is a means of making the  $O(d, d)$  symmetry of  $T$ -duality manifest. By introducing scalars to produce an additional  $d$  dimension, Duff [27] doubled the  $X(\sigma, \tau)$  string variables to make this  $O(d, d)$  symmetry manifest. Siegel brought the idea to full fruition by deriving results from superstring theory [24–26]. Allowing fields to depend on all  $2d$  coordinates, Siegel introduced generalized Lie brackets, gauge transformations, covariant derivatives, and a section condition on the full doubled space, thereby introducing torsions and curvatures in addition to the manifest  $T$ -duality.

There has been substantial subsequent development. Much of this development is reviewed in [28]; the introduction to [29] gives a concise summary. Briefly, double field theory arises by making  $T$ -duality manifest in string theory. When we compactify  $n$  dimensions on a torus, the windings of string about the torus can be interpreted as momenta.  $T$ -duality is a mapping between the original spatial directions and these momenta. Double field theory arises when these two  $n$ -spaces are kept present simultaneously, making  $T$ -duality manifest and leading to an overall  $O(n, n)$  symmetry. The  $T$ -duality is identified with the Weyl group of  $O(n, n)$ , consisting of permutations of the distinct circles of the maximum torus and interchange of phases.

### 1.2.1. Invariant tensors in double field theory and biconformal spaces

In double field theory, doubled coordinates are introduced, extending the spacetime coordinate  $x^\alpha$  by an equal number of momenta,

$$x^M = \begin{pmatrix} x^\alpha \\ y_\beta \end{pmatrix}$$

where  $M, N, \dots = 1, \dots, 2n$  and  $\alpha, \beta, \dots = 1, \dots, n$  are coordinate indices. There are at least two important invariant tensors identified in [29]. Defining the  $O(n, n)$  symmetry, there is the  $2n \times 2n$  quadratic form

$$K_{AB} = \begin{pmatrix} 0 & \delta_b^a \\ \delta_a^b & 0 \end{pmatrix}$$

with  $A, B, \dots, L = 1, \dots, 2n$  and lower case Latin indices,  $a, b, \dots = 1, \dots, n$  orthonormal. The second invariant object is the spacetime/dual generalized metric,  $M_{ab}$ , built from the spacetime metric and the Kalb-Ramond potential, which takes the orthonormal form

$$M_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta^{ab} \end{pmatrix}$$

where  $\eta_{ab}$  is either Euclidean or Lorentzian, depending on the model considered.

These are only half the invariant structures in biconformal spaces, all of which arise from natural invariances of the conformal group. Again letting  $\eta_{ab}$  be Euclidean or Lorentzian (or in our main development, any  $(p, q)$  metric), we make use of the Killing form of the conformal group of a compactified  $(p, q)$  space,

$$K_{\Sigma\Delta} = \begin{pmatrix} \delta_b^a \delta_d^c - \eta^{ac} \eta_{bd} & 0 & 0 & 0 \\ 0 & 0 & \delta_b^a & 0 \\ 0 & \delta_a^b & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the upper left block is the norm on Lorentz or Euclidean transformations, the next  $n$  rows and columns arise from translations and the next  $n$  from special conformal transformations. The final 1 in the lower right gives the Killing norm on dilatations. Upper case Greek indices run over the dimension of the conformal group. When  $K_{\Sigma\Delta}$  is restricted to the biconformal manifold, we have only the translations and special conformal portion, and this is precisely the  $O(n, n)$  metric,

$$K_{AB} = \begin{pmatrix} 0 & \delta_b^a \\ \delta_a^b & 0 \end{pmatrix}$$

Use of the Killing form as metric was first mentioned in [9] with explicit use in biconformal spaces in [30,31,22] where the orthonormal basis  $(\mathbf{e}^a, \mathbf{f}_b)$  is taken to satisfy

$$\langle \mathbf{e}^a, \mathbf{e}^b \rangle = 0 \tag{2}$$

$$\langle \mathbf{e}^a, \mathbf{f}_b \rangle = \delta_b^a \tag{3}$$

$$\langle \mathbf{f}_a, \mathbf{f}_b \rangle = 0 \tag{4}$$

General linear changes of the original  $(\mathbf{e}^a, \mathbf{f}_b)$  basis are allowed,

$$\chi^a = A^a{}_b \mathbf{e}^b + B^{ab} \mathbf{f}_b \tag{5}$$

$$\psi_a = C_{ab} \mathbf{e}^b + D_a{}^b \mathbf{f}_b \tag{6}$$

These become  $O(n, n)$  transformations when they are required to preserve the inner product given in Eqs. (2)–(4). These basis forms  $(\chi^a, \psi_b)$  are local, but may be defined globally when the structure equations and field equations provide an appropriate involution. Such alternative choices of basis have been explored in [31,22,32].

There are further objects, discussed in detail in [31] and more comprehensively in [22], where it is shown that there exists a Kähler structure on biconformal space. The complex structure arises from the symmetry of the conformal Lie algebra given by interchanging translation and special conformal transformation generators and changing the sign of the dilatation generator. This is essentially an inversion, and when carried through to its effect as a linear operation on the basis forms, may be written as

$$J^A_B = \begin{pmatrix} 0 & -\eta^{ab} \\ \eta_{ab} & 0 \end{pmatrix}$$

Further, it has long been recognized that the Maurer-Cartan equation of dilatations and, generically, its Cartan generalization describe a symplectic form,

$$d\omega = e^a \wedge f_a$$

The symplectic character is manifest since the left side shows the 2-form to be closed while the right shows it to be non-degenerate. As a matrix in this basis, the symplectic form is

$$S_{AB} = \begin{pmatrix} 0 & \delta_b^a \\ -\delta_a^b & 0 \end{pmatrix}$$

The two of these may be used to define a Kähler metric via

$$\begin{aligned} M_{AB} &\equiv S_{AC} J^C_B \\ &= \begin{pmatrix} 0 & \delta_a^c \\ -\delta_c^a & 0 \end{pmatrix} \begin{pmatrix} 0 & -\eta^{cb} \\ \eta_{cb} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta^{ab} \end{pmatrix} \end{aligned}$$

which is exactly the  $M_{AB}$  of double field theory. All of these objects arise from properties of the conformal group. We note that the Killing metric  $K_{AB}$  is *not* the metric defined by the almost Kähler structure.

The change of basis of Eq. (5) and Eq. (6) will be restricted further depending on which of these objects the change is required to preserve. For example, the time theorem of [31] requires invariance of the inner product, Eqs. (2)–(4), and preservation of the symplectic form,  $S_{AB}$ , reducing the allowed change of basis to the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A')^{-1} \end{pmatrix}$$

This is simply an instance of spontaneous symmetry breaking. Solutions typically do not preserve the full symmetry of a system of equations.

### 1.2.2. Connection and action

The principal differences between the usual treatment of double field theories and biconformal gravity lie in the connection, the action, and the means by which the doubled dimension is reduced back to an  $n$ -dimensional spacetime.

There have been multiple proposals for a connection in dual field theory [29], including the Weitzenböch connection,  $\Gamma^A_{BC} = e_M^A \partial_B e_C^M$ . While this is compatible with the double field theory structures, it leads to vanishing curvature and nonvanishing torsion. Even its generalization has vanishing curvature. Constructing an action becomes problematic. One proposal, given in [29], is an action on the full doubled space given by

$$S = \int dx dy e^{-2d} L$$

where  $d = \Phi - \frac{1}{2} \ln(|\deg g_{ij}|)$  generalizes the dilaton  $\Phi$  and  $L$  is given by

$$L = \frac{1}{8} M^{AB} \partial_A M^{CD} \partial_B M_{CD} - \frac{1}{2} M^{AB} \partial_A M^{CD} \partial_C M_{BD} + 4M^{AB} \partial_A \partial_B d - 2\partial_A \partial_B M^{AB} \\ - 4M^{AB} \partial_A d \partial_B d + 4\partial_A M^{AB} \partial_B d + \frac{1}{2} \eta^{AB} \eta^{CD} \partial_A e_C^M \partial_B e_{DM}$$

There are also multiple proposals in double field theory for finding a condition that will reduce the full space back to an  $n$ -dimensional spacetime. One proposal [33], due to Scherk and Schwarz, proposes requiring the functional dependence of fields to be of the form

$$V^A_B(X, Y) = (W^{-1})^A_{\hat{A}}(\hat{Y}) W^{\hat{B}}_B(Y) [\hat{V}(x)]^{\hat{A}}_{\hat{B}}$$

with the scalar field additively separable,  $d(X, Y) = \hat{d}(X) + \lambda(Y)$ . Here the hatted indices,  $\hat{A}$ , refer to the gauged double field theory while unhatted  $A, B$  are associated to the double field theory before applying the Scherk-Schwarz reduction. Alternatively, Berman et al. [29] propose additive separability with the “section condition”  $\eta^{AB} \partial_A \partial_B = 0$  acting on all fields.

Faced with these divergent approaches, the authors of [29] summarize a set of desirable properties for a connection on double field theory. We quote (replacing their notation with ours and numbering the points for convenient reference below):

“... we might want it to

1. define a covariant derivative that maps generalized tensors into generalized tensors,
2. be compatible with the generalized metric  $M_{AB}$ ,
3. be compatible with the  $O(n, n)$  structure  $K_{AB}$ ,
4. be completely determined in terms of the physical fields, in particular the vielbein and its derivatives,
5. be torsion-free,
6. lead to a curvature that may be contracted with the metric to give the scalar which appears in the action.”

Their proposal satisfies conditions 1 – 4.

The situation is quite different in biconformal geometry because it has been developed first as a gravity theory, and all the relevant structures are present from the Cartan construction. In particular, the connection is automatically given by the  $SO(p, q)$  spin connection and the Weyl vector, and these are compatible with not only the generalized metric (the biconformal Kähler metric)  $M_{AB}$  and the  $O(n, n)$  structure  $K_{AB}$  present as the restricted conformal Killing form, but also the almost complex structure  $J^A_B$  and symplectic form  $S_{AB}$ . This satisfies points 1, 2 and 3.

While an  $O(n)$  rather than an  $O(n, n)$  connection may seem restrictive,  $O(n, n)$  transformations of the orthonormal basis still retain the larger symmetry. Moreover, the spin connection and Weyl vector start as general 1-forms on a  $2n$ -dimensional space,

$$\omega^a{}_b = \omega^a{}_{bc}(x^\alpha, y_\beta)\mathbf{e}^c(x^\alpha, y_\beta) + \omega^a{}_b{}^c(x^\alpha, y_\beta)\mathbf{f}_c(x^\alpha, y_\beta) \tag{7}$$

$$\omega = W_c(x^\alpha, y_\beta)\mathbf{e}^c(x^\alpha, y_\beta) + W^c(x^\alpha, y_\beta)\mathbf{f}_c(x^\alpha, y_\beta) \tag{8}$$

It is because the spin connection performs the same  $O(n)$  rotation simultaneously on each subspace that it is able to preserve the multiple structures. In fact, Eq. (7) displays far more generality than we ultimately want: we would like for all fields to be determined purely the spacetime solder form,  $\mathbf{e}^c(x^\alpha)$ , and this will require reduction of both components (e.g.,  $(\omega^a{}_{bc}, \omega^a{}_b{}^c) \rightarrow \omega^a{}_{bc}$ ) and of independent variables  $((x^\alpha, y_\beta) \rightarrow x^\alpha)$ . Accomplishing this will satisfy point 4 above, and this is the central accomplishment of the current presentation.

The only assumption we make, beyond the Cartan biconformal construction and the field equations following from the action (1), is vanishing torsion. This is a natural constraint for a spacetime gravity theory, is consistent with existing measurements in general relativity, and satisfies point 5.

Point 6 is satisfied by the action, (1), which despite retaining scale invariance, is linear in the biconformal curvatures. Notice that the  $\alpha$  term in Eq. (1) is completely analogous to the Einstein-Hilbert action written in similar language, i.e.,  $S_{EH} = \int \mathbf{R}^{ab} \wedge \mathbf{e}^c \wedge \dots \wedge \mathbf{e}^d \varepsilon_{abc\dots d}$ .

We therefore claim that the reduction presented here satisfies all six desired conditions.

### 1.3. Additional potential advantages

There are further potential advantages of biconformal models.

The biconformal theory developed here also overlaps strongly with calculations in twistor space. Twistor space, in arbitrary dimension, is the space of spinors of the conformal group, up to projection by an overall factor. Witten showed in [34] (see also [35–37]) that when a topological string theory is formulated in twistor space it is equivalent to  $N = 4$  supersymmetric Yang-Mills theory. Most notably, twistor string theory provided a string/gauge theory equivalence that allowed remarkably efficient calculations of scattering amplitudes for gauge theory, reducing months of supercomputer calculations to fewer than two dozen integrals. The effort slowed considerably when it was thought that it would necessarily lead to fourth order Weyl gravity instead of general relativity. While a few alternative formulations were found, Mason was led to conclude [37]:

Clearly, more work is required to discover what other twistor–string theories can be constructed. In particular, one would like to have twistor–string theories that give rise to Poincaré supergravities, or to pure super-Yang–Mills, or that incorporate other representations of the gauge and Lorentz groups.

Biconformal gravity might be an ideal ground state for twistor string since it arises from conformal symmetry, maintains scale invariance, and reduces to general relativity. It is therefore of interest to formulate twistor string theory in a biconformal space. These will naturally use the spinor representation, as in the supergravity extension of biconformal space [38].

Biconformal spaces also seem well suited to string compactification. In the present work, we show that these  $2n$ -dimensional gravity theories reduce via their field equations to  $n$ -dimensional

general relativity. As a result, a string theory written in a 10-dimensional biconformal background will require only two dimensional compactification to describe 4-dimensional general relativity. There are only a countable number of 2-dimensional topologies, compared to the truly huge number of 6-dimensional compact spaces available when going from 10 directly to 4-dimensions.

The situation is even more restrictive than all 2-dimensional compact spaces because the compactification is required to go between two biconformal spaces. It therefore must include one basis direction of each conformal weight. This necessarily restricts the compactification to a 2-torus or possibly a 2-sphere.

#### 1.4. Organization

The organization of the paper begins with the basic equations of biconformal gravity, including some notational conventions and ending with the field equations. In Section 3, we show the effects of vanishing torsion on the remaining curvatures, using the Bianchi identities and field equations to reduce the number of components. From the effect of vanishing torsion on the curvatures, it is immediate to see by symmetry that if the co-torsion were also to be set to zero, the additional constraints would force the solution to be trivial. We begin solving for the connection in Section 4 by making use of the Frobenius theorem on the involution of the solder form. This clarifies the meaning of the doubled dimension, showing that the biconformal space is foliated by an  $n$ -dimensional Lie group. This foliation may be interpreted as the translation group of the co-tangent bundle, the torus of double field theory, or as a new, nonabelian internal symmetry.

In Section 5, we extend the partial solution for the connection from the involution back to the full biconformal space and substitute into each structure equation to find the resulting form of the curvatures, then use these results to reduce the field equations. From this point on, the solution divides into two cases depending on whether the Lie group of the foliation is abelian or non-abelian. Each of these cases merits a Section 6, 7. Finally, we summarize our results in Section 8.

## 2. The field equations of biconformal gravity

The first construction of the biconformal quotient was carried out by Ivanov and Niederle [8], who used it to describe a gravity theory using a curvature-quadratic action. Subsequently, the geometry was revived [9] and a curvature-linear action was introduced [11] to give biconformal gravity. The details of the construction are given in [22], along with a demonstration of the signature-changing properties derived in [31]. Here, we rely on the specifics given in [22], providing only a basic description and introducing some convenient nomenclature, then moving quickly to the Cartan structure equations, Bianchi identities, and the linear action.

From the action, we find the field equations and study their consequences with only the assumption of vanishing torsion. Throughout, we work in arbitrary dimension with arbitrary signature for the conformal metric class.

### 2.1. Building the structure equations

Consider a space of dimension  $n = p + q$ , with an  $SO(p, q)$ -symmetric orthonormal metric  $\eta$ . We compactify with appropriate null cones at infinity, to permit the inversions that give the space a well-defined conformal symmetry,  $\mathcal{C} = SO(p + 1, q + 1)$ . The homogeneous Weyl subgroup  $\mathcal{W} = SO(p, q) \times SO(1, 1) \subset \mathcal{C}$  consists of the pseudo-rotations and dilatations. The quotient



$C/\mathcal{W}$  is a  $2n$ -dimensional homogeneous manifold from which we immediately have a principal fiber bundle with fiber symmetry  $\mathcal{W}$ . We take the local structure of this bundle as a model for a curved space à la Cartan, modifying the manifold (if desired) and altering the connection subject to two conditions:

1. The resulting curvature 2-forms must be horizontal.
2. The resulting Cartan structure equations satisfy their integrability conditions (generalized Bianchi identities).

Let the connection forms dual to the generators of the Lie algebra be written as  $\omega^a_b$  ( $SO(p, q)$  transformations),  $e^a$  (translations),  $f_a$  (special conformal transformations, called co-translations in the context of these biconformal geometries), and  $\omega$  (dilatations). Then the Cartan structure equations are:

$$d\omega^a_b = \omega^c_b \wedge \omega^a_c + 2\Delta^{ad}_{cb} f_d \wedge e^c + \Omega^a_b \tag{9}$$

$$de^a = e^b \wedge \omega^a_b + \omega \wedge e^b + T^a \tag{10}$$

$$df_a = \omega^b_a \wedge f_b + f_a \wedge \omega + S_a \tag{11}$$

$$d\omega = e^a \wedge f_a + \Omega \tag{12}$$

Horizontality requires the curvature to be expanded in the  $(e^a, f_b)$  basis, giving each of the components  $(\Omega^a_b, T^a, S_a, \Omega)$  the general form

$$\Omega^A = \frac{1}{2} \Omega^A_{cd} e^c \wedge e^d + \Omega^{Ac}_d f_c \wedge e^d + \frac{1}{2} \Omega^{Acd} f_c \wedge f_d \tag{13}$$

and integrability follows from the Poincaré lemma,  $d^2 \equiv 0$ .

The  $\frac{(n+1)(n+2)}{2}$  curvature components  $(\Omega^a_b, T^a, S_a, \Omega)$  together comprise a single conformal curvature tensor. However, the local symmetries of the homogeneous Weyl symmetry of the biconformal bundle do not mix these four separate parts. Therefore, we call the  $SO(p, q)$  part of the full conformal curvature  $\Omega^a_b$  the *curvature*, the translational part of the curvature  $T^a$  the *torsion*, the special conformal part of the curvature the *co-torsion*,  $S_a$ , and the dilatational portion  $\Omega$  the *dilatational curvature* or simply the *dilatation*.

Each of the curvatures each has three distinguishable parts, as seen in Eq. (13). We call the  $e^a \wedge e^b$  term the *spacetime term*, the  $f_a \wedge e^b$  term the *cross term*, and the  $f_a \wedge f_b$  term the *momentum term*. While it may be somewhat abusive to call a signature  $(p, q)$  space “spacetime”, for the gravitational applications we consider the name is ultimately appropriate. In the cases where the co-solder forms generate a nonabelian Lie group, the name “momentum” is not appropriate, and we will speak of the relevant group manifold.

To avoid introducing too many symbols, the symbols for the three parts of curvatures are distinguished purely by index position. Thus,  $\Omega^a_b{}^c_d$  denotes the cross-term of the  $SO(p, q)$  curvature and  $\Omega^a_{bcd}$  the spacetime term of the  $SO(p, q)$  curvature. *These are independent functions*. We therefore do not raise or lower indices unless, on some submanifold, there is no chance for ambiguity. Note also that the raised and lowered index positions indicate the conformal weights,  $+1$  and  $-1$  respectively, of all definite weight objects. Therefore, the torsion cross-term  $T^{ab}_c$  has net conformal weight  $+1$ , the spacetime term of the co-torsion  $S_{abc}$  has conformal weight  $-3$ , and the full torsion 2-form  $T^a$  has conformal weight  $+1$ .

Note the similarity between Eqs. (10) and (11). This occurs because, by taking the quotient of the conformal group by its homogeneous Weyl subgroup instead of the more common inhomogeneous

geneous Weyl group, symmetry is maintained between the translations and the special conformal transformations. Indeed, in their action on the defining compactified  $(p, q)$  space, the special conformal transformations are simply translations in inverse coordinates,  $y_\mu = \frac{x_\mu}{x^2}$ . As a result, they behave near infinity exactly as translations do at the origin; correspondingly, the effect of a simple translation expressed in inverse coordinates is the same as that of a special conformal transformation at the origin. In the biconformal space, the resulting gauge field of translations,  $\mathbf{e}^a$ , and the gauge field of special conformal transformations,  $\mathbf{f}_a$ , form a cotangent basis. Each locally spans an  $n$ -dimensional subspace of the full biconformal cotangent space, which we ultimately show to be submanifolds. In parallel to calling  $\mathbf{e}^a$  the solder form, we call  $\mathbf{f}_a$  the *co-solder form*. Similarly, just as the field strength of the solder form is called the torsion,  $\mathbf{T}^a$ , we refer to the field strength of the co-solder form as the *co-torsion*,  $\mathbf{S}_a$ .

## 2.2. Bianchi identities

The generalized Bianchi identities are the integrability conditions for the Cartan equations. They are found by applying the Poincaré lemma,  $\mathbf{d}^2 \equiv 0$ , to each structure equation, then using the structure equations again to eliminate all but curvature terms. They always give covariant expressions – we are guaranteed that all purely connection terms must cancel because when all curvatures vanish the Cartan equations reduce to the Maurer-Cartan equations, for which the integrability conditions are the Jacobi identities, and therefore are automatically satisfied.

Knowing that all connection terms must cancel when we replace exterior derivatives with the corresponding curvatures makes it easier to derive the identities. Furthermore, every exterior derivative of a curvature becomes a covariant derivative. Using this knowledge, we may quickly find the identities. Thus, for the  $SO(p, q)$  curvature, we take the exterior derivative of Eq. (9),

$$\begin{aligned} 0 &\equiv \mathbf{d}^2 \omega^a_b \\ &= \mathbf{d}\omega^c_b \wedge \omega^a_c - \omega^c_b \wedge \mathbf{d}\omega^a_c + 2\Delta_{db}^{ac} \mathbf{d}\mathbf{f}_c \wedge \mathbf{e}^d - 2\Delta_{db}^{ac} \mathbf{f}_c \wedge \mathbf{d}\mathbf{e}^d + \mathbf{d}\Omega^a_b \\ 0 &= \Omega^c_b \wedge \omega^a_c - \omega^c_b \wedge \Omega^a_c + 2\Delta_{db}^{ac} \mathbf{S}_c \wedge \mathbf{e}^d - 2\Delta_{db}^{ac} \mathbf{f}_c \wedge \mathbf{T}^d + \mathbf{d}\Omega^a_b \\ 0 &= \mathbf{D}\Omega^a_b + 2\Delta_{db}^{ac} \mathbf{S}_c \wedge \mathbf{e}^d - 2\Delta_{db}^{ac} \mathbf{f}_c \wedge \mathbf{T}^d \end{aligned}$$

where we have identified the covariant exterior derivative,  $\mathbf{D}\Omega^a_b = \mathbf{d}\Omega^a_b + \Omega^c_b \wedge \omega^a_c - \omega^c_b \wedge \Omega^a_c$ . Proceeding through Eqs. (9)–(12), we find the full set of integrability conditions,

$$\mathbf{D}\Omega^a_b + 2\Delta_{cb}^{ad} (\mathbf{S}_d \wedge \mathbf{e}^c - \mathbf{f}_d \wedge \mathbf{T}^c) = 0 \quad (14)$$

$$\mathbf{D}\mathbf{T}^a - \mathbf{e}^b \wedge \Omega^a_b + \Omega \wedge \mathbf{e}^a = 0 \quad (15)$$

$$\mathbf{D}\mathbf{S}_a + \Omega^b_a \wedge \mathbf{f}_b - \mathbf{f}_a \wedge \Omega = 0 \quad (16)$$

$$\mathbf{D}\Omega + \mathbf{T}^a \wedge \mathbf{f}_a - \mathbf{e}^a \wedge \mathbf{S}_a = 0 \quad (17)$$

where the covariant derivatives are given by

$$\begin{aligned} \mathbf{D}\Omega^a_b &= \mathbf{d}\Omega^a_b + \Omega^c_b \wedge \omega^a_c - \omega^c_b \wedge \Omega^a_c \\ \mathbf{D}\mathbf{T}^a &= \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \omega^a_b - \omega \wedge \mathbf{T}^a \\ \mathbf{D}\mathbf{S}_a &= \mathbf{d}\mathbf{S}_a - \omega^b_a \wedge \mathbf{S}_b + \mathbf{S}_a \wedge \omega \\ \mathbf{D}\Omega &= \mathbf{d}\Omega \end{aligned} \quad (18)$$

Since each Bianchi identity contains the covariant derivative of a curvature, it is typically difficult to use them to help find solutions to the field equations. They are simply the conditions on

the curvatures that guarantee that a solution exists, and if we find a solution to the field equations, the Bianchi identities are necessarily satisfied. However, if one of the curvatures vanishes the relations become algebraic and can be extremely helpful.

### 2.3. The action functional

#### 2.3.1. Notational conventions

The metric is  $\eta_{ab}$  with pseudo-rotational invariance under  $SO(p, q)$ ,  $p + q = n$ . Lower case Latin indices run  $a, b \dots = 1, 2 \dots, n$ , and refer to orthonormal frames  $(\mathbf{e}^a, \mathbf{f}_a)$ . When coordinates are introduced they are given Greek indices. Thus, we may write

$$\mathbf{e}^a = e_\mu^a \mathbf{d}x^\mu + e^{\mu a} \mathbf{d}y_\mu$$

Until we have established appropriate submanifolds, we cannot use the components of the solder form,  $e_\mu^a$ , to change basis.

An antisymmetric projection operator on type  $\binom{0}{2}$  tensors may be written as

$$P_{fb}{}^{ed} = \frac{1}{2}(\delta_f^e \delta_b^d - \delta_f^d \delta_b^e)$$

If we raise the  $f$  index and lower  $e$ , this becomes

$$\begin{aligned} \Delta_{db}^{ac} &\equiv \eta_{de} \eta^{af} \frac{1}{2}(\delta_f^e \delta_b^c - \delta_f^c \delta_b^e) \\ &= \frac{1}{2}(\delta_d^a \delta_b^c - \eta^{ac} \eta_{bd}), \end{aligned}$$

the antisymmetric projection operator on type  $\binom{1}{1}$  tensors. This symbol occurs frequently.

#### 2.3.2. The volume form

The volume form is unusual, having two types of index. Since we can distinguish the conformal weight  $+1$  solder forms  $\mathbf{e}^a$  from the conformal weight  $-1$  co-solder forms,  $\mathbf{f}_a$ , we can always partially re-arrange. Thus, while the  $2n$ -dim volume form may be written as

$${}^e \binom{a}{\cdot} \binom{\cdot}{d} \dots \binom{b}{\cdot} \binom{\cdot}{e} \binom{c}{\cdot} \binom{\cdot}{f} = {}^e \left[ \binom{a}{\cdot} \binom{\cdot}{d} \dots \binom{b}{\cdot} \binom{\cdot}{e} \binom{c}{\cdot} \binom{\cdot}{f} \right]$$

where  $\binom{\cdot}{a}$  represents an index that contracts with  $\mathbf{e}^a$  and  $\binom{a}{\cdot}$  represents an index that contracts with  $\mathbf{f}_a$ , we can always insist that the weight  $+1$  indices go first and the weight  $-1$  go last,

$${}^e \left[ \binom{a}{\cdot} \binom{b}{\cdot} \dots \binom{c}{\cdot} \binom{\cdot}{d} \binom{\cdot}{e} \dots \binom{\cdot}{f} \right] \equiv e^{ab\dots c}{}_{de\dots f}$$

thereby reducing the  $(2n)!$  permutations to  $n!n!$ . Locally (and globally once submanifolds are established), there exist distinguishable subspaces on which we may write  $e^{ab\dots c}{}_{de\dots f}$  as a pair of  $n$ -dim Levi-Civita tensors,

$$e^{ab\dots c}{}_{de\dots f} = e^{ab\dots c} e_{de\dots f}$$

This convention means that a contraction,  $e^{ab\dots c}{}_{ae\dots f}$  is meaningful, when it would vanish immediately with the full antisymmetrization. This is, nonetheless, correct since there exists an unambiguous local separation by conformal weight, each with its own induced volume form. Since variation of the action is local, we may use this to find the field equations. A single contraction gives  $e^{ab\dots c}{}_{ae\dots f} = \delta_{e\dots f}^{b\dots c}$  and in general, contracting all but  $k$  pairs of indices,

$$\varepsilon^{c\dots de\dots f} \varepsilon_{a\dots bg\dots h} = (-1)^q (k-1)! (n-k+1)! \delta_{a\dots b}^{c\dots d} \tag{19}$$

The presence of this conformal separation also allows the dilatational curvature to be included as the  $\beta$  term in the action. It may be argued that this is not allowed if the subspaces are not integrable. We find that the subspaces *are* integrable, but have checked that setting  $\beta = 0$  throughout does not alter any of our conclusions.

There exist conditions that guarantee that such a splitting into subspaces is integrable across the full biconformal space. For example, the  $\mathbf{e}^a$  subspace is certainly integrable to a submanifold if the basis structure equation,

$$d\mathbf{e}^a = \mathbf{e}^b \wedge \omega^a_b + \omega \wedge \mathbf{e}^a + \mathbf{T}^a$$

is in involution, and this is true if the torsion  $\mathbf{T}^a$  is suitably restricted. Specifically, if the momentum term of the torsion vanishes,  $T^{acd} = 0$ , then the Eq. (13) for the torsion reduces to

$$\mathbf{T}^a = \frac{1}{2} T^a_{cd} \mathbf{e}^c \wedge \mathbf{e}^d + T^a{}_d \mathbf{f}_c \wedge \mathbf{e}^d \tag{20}$$

and  $\mathbf{e}^a$  is in involution. Similarly, the co-solder equation will be involute provided  $S_{acd} = 0$ .

We make no assumptions about the torsion or co-torsion in deriving the field equations. Though neither occurs explicitly in the curvature-linear action, integrations by parts after variation nonetheless introduce them into the field equations.

We define the Hodge dual of unity as a convenient volume form,

$$\begin{aligned} \Phi &\equiv *1 \\ &= \frac{1}{n!n!} e^{ab\dots c}{}_{de\dots f} \mathbf{e}^d \wedge \mathbf{e}^e \wedge \dots \wedge \mathbf{e}^f \wedge \mathbf{f}_a \wedge \mathbf{f}_b \wedge \dots \wedge \mathbf{f}_c \end{aligned} \tag{21}$$

It follows that

$$\mathbf{e}^d \wedge \mathbf{e}^e \wedge \dots \wedge \mathbf{e}^f \wedge \mathbf{f}_a \wedge \mathbf{f}_b \wedge \dots \wedge \mathbf{f}_c = e^{de\dots f}{}_{ab\dots c} \Phi \tag{22}$$

Eq. (22) is useful for finding the field equations. Taking a second dual,

$$\begin{aligned} {}^* \Phi &\equiv {}^{**}1 \\ &= * \left( \frac{1}{n!n!} e^{ab\dots c}{}_{de\dots f} \mathbf{e}^d \wedge \mathbf{e}^e \wedge \dots \wedge \mathbf{e}^f \wedge \mathbf{f}_a \wedge \mathbf{f}_b \wedge \dots \wedge \mathbf{f}_c \right) \\ &= \frac{1}{n!n!} e^{ab\dots c}{}_{de\dots f} \eta_{aa'} \eta_{bb'}; \dots \eta_{cc'} \eta^{dd'} \eta^{ee'} \dots \eta^{ff'} e^{a'b' \dots c'}{}_{d'e' \dots f'} \\ &= (-1)^{2q} \frac{1}{n!n!} n!n! \\ &= 1 \end{aligned}$$

regardless of the dimension or signature.

### 2.4. The action functional

Eq. (1) is the most general action linear in biconformal curvatures. It is defined on the  $2n$ -dimensional base manifold of the bundle, spanned by  $(\mathbf{e}^a, \mathbf{f}_a)$ . The initial conformally symmetric space has metric  $\eta_{ab}$  of any dimension  $n > 2$  and any signature  $(p, q)$ .

We find the field equations by varying the full set of connection 1-forms,  $\{\omega^a_b, \mathbf{e}^a, \mathbf{f}_a, \omega\}$ . Each variation has two parts when we expand in the  $(\mathbf{e}^a, \mathbf{f}_b)$  basis, for example,

$$\delta\omega^a_b = \delta A^a_{bc} \mathbf{e}^c + \delta B^a_b{}^c \mathbf{f}_c$$

with  $\delta A^a_{bc}$  and  $\delta B^a_b{}^c$  independent, arbitrary variations. We therefore find eight sets of field equations. To illustrate details of the variation technique, the variation of the spin connection  $\omega^a_b$  is given in Appendix B.

Carrying out each of the connection variations, we arrive at the final field equations:

$$T^ae{}_e - T^ea{}_e - S_e{}^{ae} = 0 \tag{23}$$

$$T^a{}_{ca} + S_c{}^a{}_a - S_a{}^a{}_c = 0 \tag{24}$$

$$\alpha \Delta_{sb}^{ar} (T^mb{}_a - \delta_a^m T^eb{}_e - \delta_a^m S_c{}^{bc}) = 0 \tag{25}$$

$$\alpha \Delta_{sb}^{ar} (\delta_c^b T^d{}_{ad} + S_c{}^b{}_a - \delta_c^b S_d{}^d{}_a) = 0 \tag{26}$$

$$\alpha (\Omega^a{}_e{}^e{}_b - \Omega^c{}_d{}^d{}_c \delta^a_b) + \beta (\Omega^a{}_b - \Omega^c{}_c \delta^a_b) + \Lambda \delta^a_b = 0 \tag{27}$$

$$\alpha \Omega^c{}_{acb} + \beta \Omega_{ab} = 0 \tag{28}$$

$$\alpha (\Omega^c{}_b{}^a{}_c - \Omega^c{}_e{}^e{}_c \delta^a_b) + \beta (\Omega^a{}_b - \Omega^c{}_c \delta^a_b) + \Lambda \delta^a_b = 0 \tag{29}$$

$$\alpha \Omega^a{}_c{}^{cb} + \beta \Omega^{ab} = 0 \tag{30}$$

where the constant  $\Lambda$  is defined to be  $\Lambda \equiv ((n - 1)\alpha - \beta + n^2\gamma)$ . Ultimately, all our results depend on a single parameter,  $\chi = \frac{1}{n-1} \frac{\Lambda}{(n-1)\alpha - \beta}$ .

### 2.5. Biconformal spaces

The system we wish to study consists of the structure equations Eqs. (9)–(12), their associated Bianchi identities Eqs. (14)–(17), and the field equations Eqs. (23)–(30). These have been written above with no additional conditions, and they apply to the biconformal geometry constructed from the conformal group in any dimension  $n$  and any signature  $(p, q)$ .

Our goal is to show how the full set of biconformal curvatures in  $2n$ -dimensions reduces to only those required to describe  $n$ -dimensional general relativity. Assuming only vanishing torsion and the field equations, we show in the next Section that the curvatures, each initially in the general form given in Eq. (13), reduce to

$$\Omega^a_b = \frac{1}{2} \Omega^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d + 2\chi \Delta_{db}^{ac} \mathbf{f}_c \wedge \mathbf{e}^d$$

$$\mathbf{T}^a = 0$$

$$\mathbf{S}_a = \frac{1}{2} S_{acd} \mathbf{e}^c \wedge \mathbf{e}^d + S_a{}^c{}_d \mathbf{f}_c \wedge \mathbf{e}^d + S_a{}^{cd} \mathbf{f}_c \wedge \mathbf{f}_d$$

$$\Omega = \chi \mathbf{e}^c \wedge \mathbf{f}_c$$

In Sections 4 and 5, we use the structure equations to reduce the coordinate dependence to  $x^\alpha$  only, with the exception of a few explicit terms linear in  $y_\alpha$ . There is also further reduction of the curvatures.

### 3. Reducing the curvatures of torsion-free biconformal spaces

We seek to reduce the field equations as far as possible. In particular, we will show that scale-invariant general relativity emerges from the vanishing torsion field equations. As in Riemannian geometry, vanishing torsion is a natural constraint on the full generality of a biconformal space. This has three definite consequences corresponding to the three parts in the expansion given by Eq. (13),

$$\mathbf{T}^a = \frac{1}{2}T^a_{cd}\mathbf{e}^c \wedge \mathbf{e}^d + T^{ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}T^{acd}\mathbf{f}_c \wedge \mathbf{f}_d$$

We expect the first term in this expansion to give the spacetime torsion, which is zero in general relativity. The cross term,  $T^{ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d$ , gives the extrinsic curvature of the spacetime submanifold in the full space, while the final term measures the non-involution – the degree to which the solder form fails to be in involution [23]. Taking the full torsion to vanish therefore has clear geometric consequences: it guarantees the existence of a spacetime submanifold with vanishing spacetime torsion, embedded with no extrinsic curvature in the larger biconformal space.

Note that it is important that we do *not* constrain the co-torsion. Indeed, we show at the end of this Section that setting both torsion and co-torsion to zero is overly restrictive, forcing the full space to have at most constant curvature and dilatation. Naturally, setting the torsion but not the co-torsion to zero breaks some of the symmetry between the solder form and the co-solder form. It would be equivalent to break the symmetry the other way, setting the co-torsion to zero and not the torsion.

We begin with the consequences of vanishing torsion in the Bianchi identities.

#### 3.1. Consequences of the Bianchi identities

If the torsion vanishes,  $\mathbf{T}^a = 0$ , then the second Bianchi identity, eq. (15), becomes an algebraic condition on the curvature and dilatation:

$$\mathbf{e}^b \wedge \Omega^a{}_b = \Omega \wedge \mathbf{e}^a$$

Expanding each of the curvatures in components,

$$\begin{aligned} \mathbf{e}^b \wedge \left( \frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d + \Omega^a{}_b{}^c{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}\Omega^a{}_b{}^{cd}\mathbf{f}_c \wedge \mathbf{f}_d \right) \\ = \left( \frac{1}{2}\Omega_{cd}\mathbf{e}^c \wedge \mathbf{e}^d + \Omega^c{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}\Omega^{cd}\mathbf{f}_c \wedge \mathbf{f}_d \right) \wedge \mathbf{e}^a \end{aligned}$$

This breaks into three independent equations, with components related by

$$\Omega^a{}_{[bcd]} = \delta^a_{[b}\Omega_{cd]} \quad (31)$$

$$\Omega^a{}_{[b}{}^c{}_d]} = \delta^a_{[b}\Omega^c{}_d]} \quad (32)$$

$$\Omega^a{}_b{}^{cd} = \delta^a_b\Omega^{cd} \quad (33)$$

Note in particular that since  $\Omega^a{}_b{}^{cd}$  is antisymmetric on  $ab$ , the trace gives  $\Omega^a{}_a{}^{cd} = 0$ , and the final condition requires

$$\Omega^{cd} = 0 \quad (34)$$

and therefore, the momentum space component of the  $SO(p, q)$  curvature vanishes,

$$\Omega^a{}_b{}^{cd} = 0 \tag{35}$$

The cross-term Bianchi may be used to express the cross curvature in terms of the cross dilatation. Expanding the antisymmetry in Eq. (32),

$$\Omega^a{}_b{}^c{}_d - \Omega^a{}_d{}^c{}_b = \delta^a{}_b \Omega^c{}_d - \delta^a{}_d \Omega^c{}_b$$

we formally lower the  $a$  index to  $e$ ,

$$0 = \eta_{ea} \Omega^a{}_b{}^c{}_d - \eta_{ea} \Omega^a{}_d{}^c{}_b - \eta_{eb} \Omega^c{}_d + \eta_{ed} \Omega^c{}_b$$

and cycle the  $b, e, d$  indices. Then adding the first two permutations and subtracting the third, we get

$$0 = \eta_{ea} \Omega^a{}_b{}^c{}_d - \eta_{ea} \Omega^a{}_d{}^c{}_b + \eta_{ba} \Omega^a{}_d{}^c{}_e - \eta_{ba} \Omega^a{}_e{}^c{}_d - \eta_{da} \Omega^a{}_e{}^c{}_b + \eta_{da} \Omega^a{}_b{}^c{}_e - \eta_{eb} \Omega^c{}_d + \eta_{ed} \Omega^c{}_b - \eta_{bd} \Omega^c{}_e + \eta_{be} \Omega^c{}_d + \eta_{de} \Omega^c{}_b - \eta_{db} \Omega^c{}_e$$

Now use the antisymmetry  $\eta_{ea} \Omega^a{}_b{}^c{}_d = -\eta_{ba} \Omega^a{}_e{}^c{}_d$  to solve,

$$\Omega^a{}_b{}^c{}_d = -2\Delta_{db}^{ae} \Omega^c{}_e \tag{36}$$

Vanishing torsion also affects the remaining Bianchi identities, but the effects are most pronounced when those are combined with the field equations. Therefore, we turn next to the simplification of the field equations.

### 3.2. Simplifications of the torsion and co-torsion equations

Of the torsion and co-torsion equations, the first two relate various traces. Equations (23) and (24) identify two relationships between these traces. When the torsion vanishes, these become

$$S_a{}^{ca} = 0 \tag{37}$$

$$S_c{}^a{}_a = S_a{}^a{}_c \tag{38}$$

Using these in the next pair, Eqs. (25) is now identically satisfied while (26) determines the antisymmetric part of the cross-terms of the co-torsion, in terms of its trace,

$$\Delta_{sb}^{ar} S_c{}^b{}_a = \Delta_{sc}^{ar} S_a{}^e{}_e \tag{39}$$

The  $rc$  trace fixes the remaining possible contraction,

$$\eta^{ac} S_c{}^b{}_a = -(n-2)\eta^{bc} S_c{}^e{}_e$$

There is only one independent contraction of the cross-term, and it determines the antisymmetric part of the full cross-term.

### 3.3. Simplifications of the curvature and dilatation equations

Now consider the remaining four equations for the curvature and dilatation, Eqs. (27)–(30). Eq. (30) is already satisfied by the consequences of the torsion Bianchi identity, Eqs. (34) and (35). The difference of the two cross curvature equations, Eq. (27) and Eq. (29), shows the equality of the traces,

$$\Omega^a{}_e{}^e{}_b = \Omega^c{}_b{}^a{}_c$$

Eq. (27) together with Eq. (36) allows us to completely determine the cross terms of the curvature and dilatation. Starting with the trace of Eq. (27),

$$\alpha \Omega^c{}_d{}^d{}_c + \beta \Omega^c{}_c = \frac{n}{n-1} \Lambda$$

and substituting this back into Eq. (27), we find

$$\alpha \Omega^a{}_e{}^e{}_b + \beta \Omega^a{}_b = \frac{1}{n-1} \Lambda \delta^a{}_b \quad (40)$$

Now, using the  $(ad)$  trace of Eq. (36)

$$\Omega^a{}_b{}^c{}_a = -(n-1) \Omega^c{}_b$$

the equality of the cross curvature traces allows us to substitute into Eq. (40)

$$-(n-1) \alpha \Omega^a{}_b{}^c{}_a + \beta \Omega^a{}_b = \frac{1}{n-1} \Lambda \delta^a{}_b$$

to show that

$$\Omega^a{}_b = -\chi \delta^a{}_b \quad (41)$$

where we define

$$\chi \equiv \frac{1}{n-1} \frac{1}{((n-1)\alpha - \beta)} \Lambda$$

The cross-term of the curvature is now given by Eq. (36),

$$\Omega^a{}_b{}^c{}_d = 2\chi \Delta_{db}^{ac} \quad (42)$$

Next, we examine the remaining vanishing torsion Bianchi identity, Eq. (31). Expanding the antisymmetry and taking the  $ad$  trace,

$$\begin{aligned} \Omega^a{}_{bcd} + \Omega^a{}_{cdb} + \Omega^a{}_{dbc} &= \delta_b^a \Omega_{cd} + \delta_c^a \Omega_{db} + \delta_d^a \Omega_{bc} \\ \Omega^a{}_{cab} - \Omega^a{}_{bac} &= (n-2) \Omega_{bc} \end{aligned}$$

Combining this with the field equation, Eq. (28), for the corresponding components,  $\Omega^a{}_{bac} = -\frac{\beta}{\alpha} \Omega_{bc}$ , we have

$$((n-2)\alpha - 2\beta) \Omega_{bc} = 0$$

so the spacetime dilatation generically vanishes. The field equation then implies

$$\Omega_{ab} = 0 \quad (43)$$

$$\Omega^c{}_{acb} = 0 \quad (44)$$

The special case when  $((n-2)\alpha - 2\beta) = 0$  allows a non-integrable Weyl geometry and, likely being unphysical, will not concern us further.

Because of the constant form of the components of the dilatation, Eq. (41), the dilatation Bianchi identity gives constraints on the co-torsion. Starting with Eq. (17) with vanishing torsion and the complete dilatation now given by  $\Omega = \chi e^a \mathbf{f}_a$ , Eq. (17) gives

$$\begin{aligned} 0 &= \mathbf{D}(\chi e^a \wedge \mathbf{f}_a) - e^a \wedge \mathbf{S}_a \\ &= -(1 + \chi) e^a \wedge \mathbf{S}_a \end{aligned}$$



with components

$$\begin{aligned}
 (1 + \chi)S_{[abc]} &= 0 \\
 (1 + \chi)(S_a{}^b{}_c - S_c{}^b{}_a) &= 0 \\
 (1 + \chi)S_a{}^{bc} &= 0
 \end{aligned}
 \tag{45}$$

For generic constants in the action we may cancel the  $1 + \chi$  factor, but the  $\chi = -1$  case permits the presence of a non-abelian internal symmetry.

### 3.4. A theorem: vanishing torsion and co-torsion

We digress briefly to prove a useful result. From our results so far, we can easily prove the following theorem. We start with the definition of a *flat* and *trivial biconformal space*. Because of the “cosmological constant” term  $\Lambda$  in Eqs. (27) and (29), we cannot, in general, set all curvatures to zero unless  $\Lambda = 0$  as well. We therefore define a *flat biconformal space* [9] to have vanishing curvatures and  $\Lambda = 0$ , and a *trivial biconformal space* to have vanishing curvatures except for constant curvature and dilatation cross-terms, which then have the  $\Lambda$ -dependent forms given in Eqs. (41) and (42). That these constant values of the curvatures yield solutions to the field equations follows as a special case of the generic torsion free solution below.

**Triviality Theorem.** *Biconformal spaces in which both the torsion and the co-torsion vanish are trivial biconformal spaces.*

**Proof.** With vanishing torsion, we have already seen that the momentum curvature and dilatation vanish. By the symmetry of biconformal spaces, zero co-torsion requires the spacetime curvature and dilatation to vanish as well. Since, by assumption we have both  $\mathbf{T}^a = 0$  and  $\mathbf{S}_a = 0$ , the only nonvanishing curvature components are the dilatation and curvature cross-terms, shown above to necessarily have the forms given in Eqs. (41) and (42),

$$\begin{aligned}
 \Omega^a{}_b &= -\chi \delta^a{}_b \\
 \Omega^a{}_b{}^c{}_d &= 2\chi \Delta_{db}^{ac}
 \end{aligned}$$

vanishing if and only if  $\chi \equiv \frac{1}{n-1} \frac{1}{\alpha(n-1)-\beta} \Lambda$  is zero. The biconformal space is therefore trivial.  $\square$

There are interesting properties to trivial biconformal spaces. These homogeneous manifolds have been shown to be Kähler [22], and allow time to emerge as part of the solution from the properties of the underlying conformal group [31,22].

Still, there can be no spacetime or momentum space curvature if both the torsion and the co-torsion vanish completely, and therefore no local gravity. To achieve a meaningful gravity theory it is necessary that at least part of either the torsion or the co-torsion remains nonzero.

### 3.5. Summary of curvatures and remaining field equations

Initially, the four curvatures (“curvature”, torsion, co-torsion, and dilatation) have the three independent terms displayed in eq. (13). Using the assumption of vanishing torsion, we have now reduced these to

$$\begin{aligned}
 \Omega^a_b &= \frac{1}{2} \Omega^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d + 2\chi \Delta^a_{db} \mathbf{f}_c \wedge \mathbf{e}^d \\
 \mathbf{T}^a &= 0 \\
 \mathbf{S}_a &= \frac{1}{2} S_{acd} \mathbf{e}^c \wedge \mathbf{e}^d + S_a^c{}_d \mathbf{f}_c \wedge \mathbf{e}^d + S_a^{cd} \mathbf{f}_c \wedge \mathbf{f}_d \\
 \Omega &= \chi \mathbf{e}^c \wedge \mathbf{f}_c
 \end{aligned} \tag{46}$$

together with the remaining field equations

$$\begin{aligned}
 s_c &\equiv S_c^a{}_a = S_a^a{}_c \\
 \Delta_{sb}^{ar} S_c^b{}_a &= \Delta_{sc}^{ar} S_a \\
 S_c^{ac} &= 0 \\
 \Omega^c{}_{acb} &= 0
 \end{aligned}$$

and remaining Bianchi conditions,

$$\begin{aligned}
 (1 + \chi) S_{[abc]} &= 0 \\
 (1 + \chi) (S_a^b{}_c - S_c^b{}_a) &= 0 \\
 (1 + \chi) S_a^{bc} &= 0 \\
 \Omega^a{}_{[bcd]} &= 0
 \end{aligned}$$

Even when  $1 + \chi \neq 0$ , the equations involving the co-torsion cross-term do not determine the co-torsion further; we must turn to the structure equations to proceed.

While the severe restrictions evident in Eqs. (46) reduce the space considerably toward an  $n$ -dimensional theory, the remaining fields are still functions of all  $2n$  coordinates. It is only by using the structure equations that we fully reduce the theory to  $n$ -dimensional scale-covariant general relativity.

#### 4. The meaning of the doubled dimension

With  $\mathbf{T}^a = 0$ , the torsion Eq. (10) is in involution. This lets us first solve the structure equations on a submanifold and results in a substantial restriction of the connection forms. Extending back to the full space, we then work through the full structure equations to determine the final form of each connection form.

##### 4.1. The involution

The involution of the solder form,

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \omega^a_b + \omega \wedge \mathbf{e}^a$$

allows us to apply the Frobenius theorem, which tells us that there exist  $n$  functions on the manifold,  $x^\mu$ , such that

$$\mathbf{e}^a = e_\mu^a \mathbf{d}x^\mu$$

Furthermore, holding those functions constant,  $x^\mu = x_0^\mu$ , so that  $\mathbf{d}x^\mu = 0$  and  $\mathbf{e}^a = 0$ , the remaining structure equations describe submanifolds of a foliation of the full space. These remaining equations are

$$\begin{aligned}
 \mathbf{d}\tilde{\omega}^a_b &= \tilde{\omega}^c_b \wedge \tilde{\omega}^a_c + \tilde{\Omega}^a_b \\
 \mathbf{d}\tilde{\mathbf{f}}_a &= \tilde{\omega}^b_a \wedge \tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_a \wedge \tilde{\omega} + \tilde{\mathbf{S}}_a \\
 \mathbf{d}\tilde{\omega} &= \tilde{\Omega}
 \end{aligned}
 \tag{47}$$

where the tilde indicates the restriction to vanishing solder form, e.g.,

$$\tilde{\omega}^a_b \equiv \omega^a_b|_{x^\mu=x^\mu_0}$$

We will also examine the restriction of the integrability (i.e., the Bianchi identity) of the co-solder equation,

$$\begin{aligned}
 0 &= \mathbf{d}^2\tilde{\mathbf{f}}_a \\
 &= \tilde{\Omega}^a_b \wedge \tilde{\mathbf{f}}_b - \tilde{\omega}^b_a \wedge \tilde{\mathbf{S}}_b + \tilde{\mathbf{S}}_a \wedge \tilde{\omega} - \tilde{\mathbf{f}}_a \wedge \tilde{\Omega} + \mathbf{d}\tilde{\mathbf{S}}_a \\
 \tilde{\mathbf{D}}\tilde{\mathbf{S}}_a &= \tilde{\mathbf{f}}_a \wedge \tilde{\Omega} - \tilde{\Omega}^a_b \wedge \tilde{\mathbf{f}}_b
 \end{aligned}
 \tag{48}$$

When  $\mathbf{e}^a = 0$ , the curvature, co-torsion, and dilatation simplify to  $\tilde{\Omega}^a_b = \frac{1}{2}\Omega^a_b{}^{cd}\tilde{\mathbf{f}}_c \wedge \tilde{\mathbf{f}}_d$ ,  $\tilde{\mathbf{S}}_a = \frac{1}{2}S_a{}^{cd}\tilde{\mathbf{f}}_c \wedge \tilde{\mathbf{f}}_d$ , and  $\tilde{\Omega} = \frac{1}{2}\Omega^{cd}\tilde{\mathbf{f}}_c \wedge \tilde{\mathbf{f}}_d$ . In the previous section we showed that these components of the curvature and dilatation,  $\Omega^a_b{}^{cd}$  and  $\Omega^{cd}$ , vanish. Therefore, the structure equations and basis integrability reduce to

$$\begin{aligned}
 \mathbf{d}\tilde{\omega}^a_b &= \tilde{\omega}^c_b \wedge \tilde{\omega}^a_c \\
 \mathbf{d}\tilde{\omega} &= 0 \\
 \mathbf{d}\tilde{\mathbf{f}}_a &= \tilde{\omega}^b_a \wedge \tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_a \wedge \tilde{\omega} + \frac{1}{2}S_a{}^{cd}\tilde{\mathbf{f}}_c \wedge \tilde{\mathbf{f}}_d \\
 \tilde{\mathbf{D}}\tilde{\mathbf{S}}_a &= 0
 \end{aligned}$$

Let this submanifold be spanned by coordinates  $y_\mu$ . The first two equations show that the spin connection,  $\tilde{\omega}^a_b$ , and Weyl vector,  $\tilde{\omega}$ , are pure gauge on the submanifold,

$$\begin{aligned}
 \tilde{\omega}^a_b(x_0, y) &= -\tilde{F}^a_c(x^\mu_0, y_\nu)\mathbf{d}F^c_b(x^\mu_0, y_\nu) \\
 \tilde{\omega}(x_0, y) &= \mathbf{d}f(x^\mu_0, y_\nu)
 \end{aligned}$$

where at each  $x^\mu_0$  we are free to choose a local  $SO(p, q)$  transformation  $\Lambda^a_c(y)$  and a local dilatation  $\phi(y)$ . This allows us to gauge both  $\tilde{\omega}^a_b(x_0, y)$  and  $\tilde{\omega}(x_0, y)$  to zero if desired. It proves convenient to rename the restriction of the basis,  $\mathbf{h}_a \equiv \tilde{\mathbf{f}}_a$  and the restriction of the spin connection as  $\xi^a_b \equiv \tilde{\omega}^a_b(x_0, y)$ , while gauging the Weyl vector to zero. The basis  $\mathbf{h}_a$  must span the co-tangent space to the submanifold, so it must be nondegenerate. The submanifold is then described by

$$\mathbf{d}\xi^a_b = \xi^c_b \wedge \xi^a_c \tag{49}$$

$$\mathbf{d}\mathbf{h}_a = \xi^b_a \wedge \mathbf{h}_b + \frac{1}{2}S_a{}^{cd}\mathbf{h}_c \wedge \mathbf{h}_d \tag{50}$$

$$\tilde{\mathbf{D}}\tilde{\mathbf{S}}_a = 0 \tag{51}$$

To continue, we examine a manifold with these conditions, Eqs. (49)–(51). Notice that Eqs. (49)–(51) describe a differentiable manifold with flat connection for which the momentum part of the co-torsion  $\frac{1}{2}S_a{}^{cd}\mathbf{h}_c \wedge \mathbf{h}_d$  is the torsion of the submanifold. This submanifold torsion is constrained by Eq. (45),

$$(1 + \chi)S_a{}^{bc} = 0 \tag{52}$$

The importance of these properties will be seen in this and the following Sections.

#### 4.2. Foliation by a Lie group

We quote a well-known theorem due to Auslander and Markus [48]:

**Theorem 5.** *Let  $M$  be a differentiable manifold with complete, flat, affine connection  $\Gamma$  and holonomy group  $H(\mathcal{M}; \Gamma) = 0$ . Then  $\mathcal{M}$  is a complete Riemann space with Christoffel connection  $\Gamma$  and  $\mathcal{M}$  is differentiably isometric with a torus space.*

F.W. Kamber and Ph. Tondeur generalize this theorem [49], introducing their proof with the following:

Consider a linear connection on a smooth manifold. The connection is flat, if the curvature tensor  $R$  is zero. If the torsion tensor  $T$  has vanishing covariant derivative, the torsion is said to be parallel. A linear connection is complete, if every geodesic can be defined for any real value of the affine parameter. In this note the following structure theorem for smooth manifolds admitting a complete flat connection with parallel torsion is proved: *Any such manifold is the orbit space of a simply connected Lie group  $\mathcal{G}$  under a properly discontinuous and fixed-point free action of a subgroup of the affine group of  $\mathcal{G}$ .* This Theorem includes the classical cases of flat Riemannian manifolds and flat affine manifolds (Auslander and Markus), where the torsion is assumed to be zero and  $\mathcal{G}$  turns out to be  $\mathbb{R}^n$ , and also generalizes a theorem of Hicks Theorem 6 for complete connections with trivial holonomy group and parallel torsion tensor, stating that a manifold with such a connection is homogeneous. We consider the case where the curvature vanishes, without requiring the holonomy group to be trivial.

As noted above, our equations, Eqs. (49)–(51), exactly describe a manifold with flat connection  $\xi^a{}_b$  but with torsion satisfying only  $\mathbf{D}\tilde{S}_a = 0$ . This torsion conditions is weaker than those of the theorems above. Moreover, the additional conditions may or may not hold. Spacetime, and the more general  $SO(p, q)$  spaces we consider may be pseudo-Riemannian rather than Riemannian. Further, we know that spacetimes are generically incomplete [45–47] and that our physical spacetime contains black hole singularities and initial time incompleteness; the corresponding properties of the momentum subspace depend on the manifold chosen during the quotient construction. Finally, with our general considerations we cannot be certain of the remaining specifications regarding holonomy present in both theorems. Therefore, we do not attempt to apply Auslander-Markus or Kamber-Tondeur theorems, but derive our results directly, making our assumptions explicit.

We consider the two possible solutions to Eq. (52):

- Case 1:**  $S_a{}^{bc} = 0$ . In Sec. 6 below, we show that with no further assumptions, generic biconformal spaces (i.e., those with  $\chi \neq -1$ ) are foliated by an abelian Lie group. They therefore describe either the co-tangent bundle or torus space foliations over  $SO(p, q)$  spaces. Generically, therefore, the conclusion of Theorem 5 of Auslander and Markus holds for the momentum submanifolds of biconformal space.
- Case 2:**  $\chi = -1$ . In Sec. 7 below, we show that the subclass of biconformal spaces with  $1 + \chi = 0$  allows the possibility of foliation by a nonabelian Lie group. The result

is consistent with the claim of Kamber and Tondeur. To make further progress, we too assume vanishing covariant derivative of the torsion rather than vanishing covariant exterior derivative.

In the remainder of this Section and in Sec. 5, we show results that hold for either Case 1 or Case 2 by assuming  $S_a{}^{bc}$  constant and placing no condition on  $\chi$ . This is sufficient to show foliation by a Lie group; we leave detailed topological discussion to subsequent studies. In Sec. 5 we extend back to the full biconformal space, substituting the form of the connection into the structure equations to continue the reduction of the system toward general relativity. The program is completed in two different ways for Case 1 and Case 2, in Sections 6 and 7 respectively.

#### 4.2.1. Co-torsion Bianchi

We have seen that the vanishing torsion,  $\mathbf{T}^a = 0$ , combined with the dilatation Bianchi identity gives Eqs. (52). For the remainder of this Section, we will place a weaker constraint on the momentum co-torsion and  $\chi$  consistent with both Cases above. Thus, the conclusions of this Section hold for the discussions of both Sec. 6 and Sec. 7.

The integrability condition for the submanifold co-torsion, Eq. (51) is

$$\begin{aligned} 0 &= \tilde{\mathbf{D}}\tilde{\mathbf{S}}_a \\ &= \frac{1}{2}S_a{}^{[\alpha\beta;\mu]} \mathbf{d}y_\alpha \wedge \mathbf{d}y_\beta \wedge \mathbf{d}y_\mu \end{aligned} \tag{53}$$

so the covariant exterior  $y$ -derivative of  $\frac{1}{2}S_a{}^{cd}\mathbf{h}_c \wedge \mathbf{h}_d$  vanishes. Choosing the  $y_\alpha$ -dependent part of the gauge so that the submanifold spin connection and Weyl vector vanish, the covariant derivative reduces to a partial derivative,

$$0 = \tilde{\mathbf{d}}\tilde{\mathbf{S}}_a$$

and therefore for some 1-form,  $\tilde{\xi}_a$

$$\tilde{\mathbf{S}}_a = \tilde{\mathbf{d}}\tilde{\xi}_a$$

In coordinate components,

$$S_a{}^{\alpha\beta} = \xi_a{}^{\alpha,\beta} - \xi_a{}^{\beta,\alpha}$$

However, instead of such a general potential  $\tilde{\xi}_a$ , we assume

$$\partial^\mu S_a{}^{\alpha\beta} = 0 \tag{54}$$

This is one of the assumptions of the Kamber-Tondeur Theorem.

With the momentum co-torsion independent of  $y_\mu$  the structure equation on the  $\mathbf{e}^a = 0$  submanifolds becomes

$$\mathbf{d}\mathbf{h}_a = \frac{1}{2}S_a{}^{cd}(x_0)\mathbf{h}_c \wedge \mathbf{h}_d \tag{55}$$

In terms of the basis  $\mathbf{h}_a$  the integrability condition for eq. (55) is

$$\begin{aligned} 0 &\equiv \mathbf{d}^2\mathbf{h}_a \\ &= \frac{1}{2}S_a{}^{cd}(x_0)\mathbf{h}_c \wedge \mathbf{h}_d \\ &= S_a{}^{cd}(x_0)\mathbf{d}\mathbf{h}_c \wedge \mathbf{h}_d \\ &= \frac{1}{2}S_a{}^{cd}(x_0)S_c{}^{ef}(x_0)\mathbf{h}_e \wedge \mathbf{h}_f \wedge \mathbf{h}_d \end{aligned}$$

and therefore,

$$S_a^{c[d} S_c^{ef]} = 0$$

With  $S_a^{bc}(x_0^\mu)$  constant, we set  $c_a^{bc} \equiv -S_a^{bc}$ , and observe that the pair

$$d\mathbf{h}_a = -\frac{1}{2} c_a^{bc} \mathbf{h}_b \wedge \mathbf{h}_c \tag{56}$$

$$c_a^{c[d} c_c^{ef]} = 0 \tag{57}$$

form the Maurer-Cartan equations and the Jacobi identity (in the adjoint representation) for an  $n$ -dimensional Lie algebra. The field equation for the momentum co-torsion, Eq. (37) shows that the adjoint generators are traceless, so when the adjoint representation is faithful the Lie group elements will have unit determinant. With the observation that  $\mathbf{h}_a$  has an  $n$ -dimensional  $SO(p, q)$  or  $Spin(p, q)$  index (depending on which representation we have chosen for the beginning group), we have therefore proved the following theorem:

**Theorem.** *In any  $2n$ -dimensional, torsion-free biconformal spaces with  $\partial^\mu S_a^{\alpha\beta} = 0$ , there exists an  $n$ -dimensional foliation by a Lie group. If the adjoint representation is faithful, the group is special.*

We conjecture that the theorem holds for all torsion free biconformal spaces.

This is one of our most important new results, giving a definitive interpretation to the doubled dimension of biconformal spaces.

Introducing vector fields  $G^a$  dual to the one forms  $\mathbf{h}_a$ , we have

$$[G^a, G^b] = c_c^{ab} G^c \tag{58}$$

and

$$[G^a, [G^b, G^c]] + [G^b, [G^c, G^a]] + [G^c, [G^a, G^b]] = 0 \tag{59}$$

Let  $\mathcal{G}$  be the Lie group generated by the  $G^a$ . Then the  $y$ -submanifold at each  $x_0$  is the group manifold.

Since  $\mathbf{h}_a$  transforms as a vector under  $SO(p, q)$ , there may be constraints between  $SO(p, q)$  and  $\mathcal{G}$ .

Acting with  $\Lambda^a_b \in SO(p, q)$  on the structure equation of  $\mathbf{h}_a$ ,

$$\tilde{\mathbf{h}}_a = \mathbf{h}_a \Lambda^a_b$$

invariance of the structure equation requires

$$\begin{aligned} d\tilde{\mathbf{h}}_a &= -\frac{1}{2} \tilde{c}_a^{bc} \tilde{\mathbf{h}}_b \wedge \tilde{\mathbf{h}}_c \\ d\mathbf{h}_a \Lambda^a_b &= -\frac{1}{2} \tilde{c}_a^{bc} \mathbf{h}_d \Lambda^d_b \wedge \mathbf{h}_e \Lambda^e_c \\ -\frac{1}{2} c_a^{de} \mathbf{h}_d \wedge \mathbf{h}_e \Lambda^a_b &= -\frac{1}{2} \tilde{c}_a^{bc} \Lambda^d_b \Lambda^e_c \mathbf{h}_d \wedge \mathbf{h}_e \\ c_f^{de} &= \bar{\Lambda}^a_f \tilde{c}_a^{bc} \Lambda^d_b \Lambda^e_c \end{aligned}$$

so the structure constants must transform as a  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  tensor, consistent with  $\mathbf{S}_a$  being a tensor. Since  $SO(p, q)$  acts on itself, any subgroup of  $SO(p, q)$  will be allowed, but it is clear that there are additional possibilities. For example, the vanishing structure constants of an abelian group will be preserved, as will partly abelian combinations. We develop a concrete example.

Starting with a 3-dim representation of  $SO(3)$ , we require a 3-dimensional Lie group with structure constants that transform as a tensor under  $SO(3)$ . Consider  $ISO(2)$ , the two translations and one rotation of the plane. The Lie algebra is

$$\begin{aligned} [R, T_1] &= -2T_2 \\ [R, T_2] &= 2T_1 \end{aligned}$$

where we may think of  $R$  as the generator of rotations about the  $z$ -axis and  $T_k$  as translations in the  $xy$ -plane. The nonvanishing structure constants (using the conventional index positions) are then  $c^1_{32} = c^1_{13} = -c^1_{23} = -c^2_{31} = 2$ .

While this 3-dim picture of the group is clearly rotationally invariant, we may make the proof explicit by defining three unit vectors

$$n^i_{(a)} \equiv \delta^i_a$$

Letting the generators in an arbitrary basis form a 3-vector  $\mathbf{G} = (G_1, G_2, G_3)$  we set

$$\begin{aligned} R &= \mathbf{n}_{(3)} \cdot \mathbf{G} \\ T_1 &= \mathbf{n}_{(1)} \cdot \mathbf{G} \\ T_2 &= \mathbf{n}_{(2)} \cdot \mathbf{G} \end{aligned}$$

Then the structure constants may be written in terms of the unit vectors as

$$c^i_{jk} = 2n^i_{(1)}(n_{(3)j}n_{(2)k} - n_{(2)j}n_{(3)k}) - 2n^i_{(2)}(n_{(3)j}n_{(1)k} - n_{(1)j}n_{(3)k})$$

which now manifestly transforms as a  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  tensor under rotations.

For  $n = 4$ , the electroweak group,  $SU(2) \times U(1)$ , naturally springs to mind. This, and other particular cases will be explored explicitly in subsequent work.

Now consider the effect on  $S_a^{bc}$  of allowing  $x^\mu$  to vary. At each value of  $x^\mu$ ,  $S_a^{cd}(x)$ , comprises the structure constants of a Lie group. However, while the structure constants depend on the choice of the basis of group generator, we are limited to differentiable changes. Since Lie algebras are classified by a discrete set of possible root diagrams, a continuous transformation as we vary  $x^\mu$  cannot change to structure constants with a different root diagram. Moreover, by choosing the basis of dual 1-forms appropriately at each point, we may bring the structure constants to a given standard form  $c_a^{bc}$ . With  $S_a^{bc}(x) = -c_a^{bc}$  at each value of  $x^\mu$ , there is no  $x$ -dependence. With any such choice of basis,  $\mathbf{d}S_a^{bc} = 0$ , and we may set

$$S_a^{bc}(x, y) = -c_a^{bc}$$

across the entire biconformal manifold.

If the group  $\mathcal{G}$  is abelian, the structure constants are zero and the momentum co-torsion vanishes. In this case,  $\mathbf{d}\mathbf{h}_a = 0$  and we have

$$\mathbf{h}_a = f_a^\mu(x)\mathbf{d}y_\mu$$

where the  $y$ -dependence of the coefficients must now vanish, though in this case it is useful to allow the dependence on  $x^\mu$ . This describes an exact, orthonormal frame and therefore a flat space. Since we evaluate at fixed  $x^\alpha = x_0^\alpha$ , the coefficients  $f_a{}^\mu(x_0)$  are constants on the submanifold, but may be functions of  $x^\mu$  when we extend back to the full biconformal space. This is equivalent to the abelian Lie algebra of  $n$  translations, and the  $\mathcal{G}$ -foliation of the biconformal space may be identified as the co-tangent bundle of the remaining  $SO(p, q)$  space. Alternatively, the abelian group may be taken as a compactification on a torus.

4.2.2. *Parameterization of the group elements as coordinates*

The integral of the structure equations,

$$\begin{aligned} \mathbf{d}\mathbf{h}_a &= -\frac{1}{2}c_a{}^{bc}\mathbf{h}_b \wedge \mathbf{h}_c \\ S_a{}^{c[d}S_c{}^{ef]} &= 0 \end{aligned}$$

gives the group manifold, which is most easily coordinatized by the group elements. We may find these by exponentiating the Lie algebra,  $V = \{y_a G^a | y_a \in \mathbb{R}^n\}$  where  $G^a$  satisfy the Lie algebra relations in Eqs. (58) and (59). The group elements may be parameterized by coordinates  $y_a$ , by exponentiating the Lie algebra,  $g(y) = e^{y_a G^a} \in \mathcal{G}$ .

The basis forms,  $\mathbf{h}_a$  may be explicitly turned into Lie algebra valued one forms using any desired linear representation of the generators,  $\xi^C{}_B \equiv -\mathbf{h}_a[G^a]_B{}^C$ . For example, using the adjoint representation we contract a copy of the structure constants with the Maurer-Cartan equations, eq. (56), and define  $\xi^c{}_b \equiv -\mathbf{h}_a[G^a]_b{}^c = -c_b{}^{ac}\mathbf{h}_a$ . Then, using the Jacobi identity,

$$\begin{aligned} \mathbf{d}(c_b{}^{ac}\mathbf{h}_a) &= -\frac{1}{2}c_b{}^{ac}c_a{}^{de}\mathbf{h}_d \wedge \mathbf{h}_e \\ -\mathbf{d}\xi^c{}_b &= \frac{1}{2}(c_b{}^{ad}c_a{}^{ec} + c_b{}^{ae}c_a{}^{cd})\mathbf{h}_d \wedge \mathbf{h}_e \\ &= \frac{1}{2}(-c_b{}^{da}\mathbf{h}_d \wedge c_a{}^{ec}\mathbf{h}_e - c_b{}^{ea}\mathbf{h}_e \wedge c_a{}^{dc}\mathbf{h}_d) \\ &= \frac{1}{2}(-\xi^a{}_b \wedge \xi^c{}_a - \xi^a{}_b \wedge \xi^c{}_a) \\ &= -\xi^a{}_b \wedge \xi^c{}_a \end{aligned}$$

or

$$\mathbf{d}\xi^c{}_b = \xi^a{}_b \wedge \xi^c{}_a$$

This is the structure equation of a connection on a flat manifold, so we may write  $\xi^a{}_b$  as a pure gauge connection,

$$\xi^a{}_b = -\bar{g}_b{}^c \mathbf{d}g^a{}_c$$

where, in the adjoint representation,  $g^a{}_b = \exp(y_c[G^c]_b{}^a) = \exp(y_c c_b{}^{ca})$ . We check that this solves the structure equation,

$$\begin{aligned} \mathbf{d}\xi^c{}_b &= \xi^a{}_b \wedge \xi^c{}_a \\ \mathbf{d}(-\bar{g}_b{}^e \mathbf{d}g^c{}_e) &= (-\bar{g}_b{}^e \mathbf{d}g^a{}_e) \wedge (-\bar{g}_f{}^c \mathbf{d}g^c{}_f) \\ -\mathbf{d}\bar{g}_b{}^e \wedge \mathbf{d}g^c{}_e &= -\mathbf{d}\bar{g}_b{}^e g^a{}_e \wedge \bar{g}_f{}^c \mathbf{d}g^c{}_f \\ &= -\mathbf{d}\bar{g}_b{}^e \wedge \mathbf{d}g^c{}_e \end{aligned}$$



as required. We note that while this construction gives an explicit form for  $\mathbf{h}_a$ , this is not the usual connection,  $\omega^c_b \equiv -\frac{1}{2}\mathbf{h}_a[G^a]_b{}^c$ , which gives constant curvature [50].

### 5. Returning to the full space

We have established a geometric breakdown of the  $2n$ -dim biconformal space into an  $n$ -dimensional foliation with a Lie group for leaves. However, the connection forms still retain their dependence on the full set of coordinates  $(x^\alpha, y_\beta)$ . In this Section, and for the generic  $\chi \neq -1$  case in the next Section, we show that the structure equations further restrict this dependence so that except for certain explicit linear  $y_\alpha$  dependence, all fields depend only on the  $x^\alpha$ , up to coordinate choices and gauge transformations. To this end, we turn to the full space and the Cartan structure equations.

When we restore the solder form, letting  $x^\mu$  vary again, the connection forms must be given by their  $\mathbf{e}^a = 0$  parts, Eqs. (49), (56), and the vanishing Weyl vector) plus additional parts proportional to the solder form. Therefore,

$$\omega^a_b = \omega^a_{bc}(x, y)\mathbf{e}^c \tag{60}$$

$$\mathbf{e}^a = e_\mu{}^a(x, y)\mathbf{d}x^\mu \tag{61}$$

$$\begin{aligned} \mathbf{f}_a &= h_a{}^\mu(x, y)\mathbf{d}y_\mu + c_{ab}(x, y)\mathbf{e}^b \\ &\equiv \mathbf{h}_a + c_{ab}\mathbf{e}^b \\ &\equiv \mathbf{h}_a + \mathbf{c}_a \end{aligned} \tag{62}$$

$$\omega = W_a(x, y)\mathbf{e}^a \tag{63}$$

Eqs. (60)–(63) hold as long as we perform only  $x$ -dependent fiber transformations. While this form is convenient for recognizing the content of the geometry, the biconformal space is unchanged by more general transformations. General  $(x^\alpha, y_\beta)$ -dependent transformations on biconformal space act similarly to canonical transformations on phase spaces. They do not change the underlying physics.

We substitute these forms into the structure equations, with the reduced curvatures as given in Eq. (46),

$$\mathbf{d}\omega^a_b = \omega^c_b \wedge \omega^a_c + 2(1 + \chi)\Delta_{cb}{}^{ad}\mathbf{f}_d \wedge \mathbf{e}^c + \frac{1}{2}\Omega^a_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d \tag{64}$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \omega^a_b + \omega \wedge \mathbf{e}^a \tag{65}$$

$$\mathbf{d}\mathbf{f}_a = \omega^b_a \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \omega + \frac{1}{2}S_{acd}\mathbf{e}^c \wedge \mathbf{e}^d + S_a{}^c{}_d\mathbf{f}_c \wedge \mathbf{e}^d - \frac{1}{2}c_a{}^{cd}\mathbf{f}_c \wedge \mathbf{f}_d \tag{66}$$

$$\mathbf{d}\omega = (1 + \chi)\mathbf{e}^c \wedge \mathbf{f}_c \tag{67}$$

where  $\chi \equiv \frac{1}{n-1} \frac{1}{(n-1)\alpha-\beta} \Lambda$  and  $(1 + \chi)$  factors appear where we have combined the dilatation and curvature cross terms with matching pieces of the connection.

#### 5.1. The basis structure equations

First consider the solder form equation, Eq. (65). Substituting Eqs. (60), (61) and (63) for the current form of the connection,

$$\mathbf{d}x^\mu \wedge \partial_\mu \mathbf{e}^a + \mathbf{d}y_\mu \wedge \partial^\mu \mathbf{e}^a = \omega^a_{bc}\mathbf{e}^b \wedge \mathbf{e}^c + W_b\mathbf{e}^b \wedge \mathbf{e}^a$$

The sole mixed term must vanish,  $\mathbf{d}y_\mu \wedge \partial^\mu \mathbf{e}^a = 0$ , and this requires the solder form to be independent of  $y_\alpha$ . Therefore,

$$e_\mu^a(x, y) = e_\mu^a(x) \quad (68)$$

### 5.1.1. Solving the solder form equation for the spin connection

The next step is to solve the solder form equation for the spin connection. In the remaining  $\mathbf{e}^a \wedge \mathbf{e}^b$  part of the reduced solder form equation, Eq. (65) we may separate the connection into the familiar metric compatible piece, and a Weyl vector piece. Let  $\alpha^a_b$  be chosen as the  $\mathbf{e}^a$ -compatible connection, so that

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \alpha^a_b \quad (69)$$

We note that as a consequence of Eqs. (68) and (69),  $\alpha^a_b = \alpha^a_b(x)$ . Then writing  $\omega^a_b = \alpha^a_b + \beta^a_b$  with antisymmetry on each piece,  $\alpha^a_b = -\eta^{ac}\eta_{bd}\alpha^d_c$  and  $\beta^a_b = -\eta^{ac}\eta_{bd}\beta^d_c$ , we must have

$$0 = \mathbf{e}^b \wedge \beta^a_b + \omega \wedge \mathbf{e}^a$$

Since the solution is unique up to local Weyl transformations, we need only find an expression that works. Using the antisymmetric  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  projection operator  $\Delta_{bd}^{ac}$  to impose antisymmetry, and requiring linearity in the Weyl vector and the solder form, we guess that

$$\beta^a_b = -2\Delta_{db}^{ac} W_c \mathbf{e}^d \quad (70)$$

and check

$$\begin{aligned} \mathbf{e}^b \wedge \beta^a_b + \omega \wedge \mathbf{e}^a &= -2\Delta_{db}^{ac} W_c \mathbf{e}^b \wedge \mathbf{e}^d + \omega \wedge \mathbf{e}^a \\ &= -(\delta_d^a \delta_b^c - \eta^{ac}\eta_{bd}) W_c \mathbf{e}^b \wedge \mathbf{e}^d + \omega \wedge \mathbf{e}^a \\ &= -W_b \mathbf{e}^b \wedge \mathbf{e}^a + \omega \wedge \mathbf{e}^a \\ &= 0 \end{aligned}$$

as required. Therefore, the spin connection is

$$\omega^a_b = \alpha^a_b + \beta^a_b = \alpha^a_b - 2\Delta_{db}^{ac} W_c \mathbf{e}^d \quad (71)$$

where  $\alpha^a_b$  is the connection compatible with  $\mathbf{e}^a$ . Any  $y$ -dependence must come from the Weyl vector.

### 5.1.2. Coordinate form of the connection

We can also find the coordinate form of the connection. Starting from the solder form equation we expand,

$$\begin{aligned} 0 &= \mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \omega^a_b - \omega \wedge \mathbf{e}^a \\ &= (\partial_\mu e_\nu^a + e_\nu^b \omega_{b\mu}^a - W_\mu e_\nu^a) \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \end{aligned}$$

As the antisymmetric part of the coefficient expression in parentheses must vanish, it must equal a symmetric object, i.e.,

$$\Sigma^a_{\nu\mu} \equiv \partial_\mu e_\nu^a + e_\nu^b \omega_{b\mu}^a - W_\mu e_\nu^a$$

where  $\Sigma^a_{\nu\mu} = \Sigma^a_{\mu\nu}$ . Writing  $\Sigma^a_{\nu\mu} = e_\alpha^a \Sigma^\alpha_{\nu\mu}$  the equation takes the form of a vanishing covariant derivative,

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - e_\alpha^a \Sigma^\alpha_{\nu\mu} + e_\nu^b \omega_{b\mu}^a - W_\mu e_\nu^a = 0$$

We easily check that  $\Sigma^\alpha_{\nu\mu}$  is indeed the expected connection. First, contract with  $\eta_{ab} e_\beta^b$ ,

$$0 = \eta_{ab} e_\beta^b \partial_\mu e_\nu^a - \eta_{ab} e_\beta^b e_\alpha^a \Sigma^\alpha_{\nu\mu} + \eta_{ab} e_\beta^b e_\nu^c \omega_{c\mu}^a - \eta_{ab} e_\beta^b W_\mu e_\nu^a$$

Now symmetrize on  $\beta\nu$  and use  $g_{\alpha\beta} = \eta_{ab} e_\alpha^a e_\beta^b$ ,

$$\begin{aligned} 0 &= \eta_{ab} e_\beta^b \partial_\mu e_\nu^a - g_{\alpha\beta} \Sigma^\alpha_{\nu\mu} + e_\beta^b e_\nu^c \eta_{ab} \omega_{c\mu}^a - \eta_{ab} e_\beta^b e_\nu^a W_\mu \\ &\quad + \eta_{ab} e_\nu^b \partial_\mu e_\beta^a - g_{\alpha\nu} \Sigma^\alpha_{\beta\mu} + e_\nu^b e_\beta^c \eta_{ab} \omega_{c\mu}^a - \eta_{ab} e_\nu^b e_\beta^a W_\mu \\ &= \partial_\mu g_{\nu\beta} - g_{\alpha\beta} \Sigma^\alpha_{\nu\mu} - g_{\nu\alpha} \Sigma^\alpha_{\beta\mu} - 2g_{\nu\beta} W_\mu \end{aligned}$$

This is precisely the conformal metric compatibility of  $g_{\nu\beta}$  with  $\Sigma^\alpha_{\beta\mu}$  in a Weyl geometry. Solving for the connection by the usual cyclic permutation of  $\mu\nu\beta$ , adding the first two permutations and subtracting the third, we recover the explicit form of the compatible connection of a Weyl geometry [39]:

$$\begin{aligned} \Sigma_{\nu\beta\mu} &= \frac{1}{2}(\partial_\mu g_{\nu\beta} + \partial_\beta g_{\mu\nu} - \partial_\nu g_{\beta\mu}) - (g_{\nu\beta} W_\mu + g_{\mu\nu} W_\beta - g_{\beta\mu} W_\nu) \\ &= \Gamma_{\nu\beta\mu} - (g_{\nu\beta} W_\mu + g_{\mu\nu} W_\beta - g_{\beta\mu} W_\nu) \end{aligned} \tag{72}$$

where  $\Gamma_{\nu\beta\mu}$  is the Christoffel connection. This connection is compatible with the conformal class of metrics,  $\{g_{\alpha\beta} e^{2\phi} \mid \text{all } \phi(x, y)\}$ .

### 5.1.3. The covariant derivative of the solder form

The Weyl covariant derivative is compatible with the component matrix of the solder form,

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - e_\alpha^a \Sigma^\alpha_{\nu\mu} + e_\nu^b \omega_{b\mu}^a - W_\mu e_\nu^a = 0 \tag{73}$$

For the inverse component matrix, we contract with  $e_a^\beta$  and use the product rule to express the result as  $e_\nu^a D_\mu e_a^\beta$ , recognizing the covariant derivative of the inverse,

$$D_\mu e_a^\beta = \partial_\mu e_a^\beta + e_a^\alpha \Sigma^\beta_{\alpha\mu} - e_b^\beta \omega_{a\mu}^b + e_a^\beta W_\mu = 0 \tag{74}$$

Knowing the coordinate form of the covariant derivative lets us compute the covariant derivative of  $y_a$ . Notice that multiplying by  $e_a^\mu$  changes the conformal weight.

$$\begin{aligned} \mathbf{D}y_a &= \mathbf{d}y_a - y_b \omega_a^b + y_a \omega \\ &= \mathbf{d}e_a^\beta y_\beta + e_a^\mu \mathbf{d}y_\mu - y_b \omega_a^b + y_a \omega \\ &= (-e_a^\alpha \Sigma^\beta_{\alpha\mu} \mathbf{d}x^\mu + e_b^\beta \omega_{a\mu}^b \mathbf{d}x^\mu - e_a^\beta W_\mu \mathbf{d}x^\mu) y_\beta + e_a^\mu \mathbf{d}y_\mu - y_b \omega_a^b + y_a \omega \\ &= y_b \omega_a^b - e_a^\alpha y_\beta \Sigma^\beta_{\alpha\mu} \mathbf{d}x^\mu - y_a \omega + e_a^\mu \mathbf{d}y_\mu - y_b \omega_a^b + y_a \omega \\ &= e_a^\alpha (\mathbf{d}y_\mu - y_\beta \Sigma^\beta_{\alpha\mu} \mathbf{d}x^\mu) \end{aligned}$$

This will be of use shortly.

5.2. Curvature equation

We next study the curvature equation, Eq. (64). We begin with its integrability condition, which places strong constraints on the co-torsion. Then, substituting the connection forms from Eqs. (60)–(62), we impose the curvature field equation,  $\Omega^c_{acb} = 0$ .

5.2.1. Curvature Bianchi

Expanding the curvature Bianchi identity, Eq. (14), and substituting the reduced curvatures, it becomes

$$0 = \mathbf{d} \left( \frac{1}{2} \Omega^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) + \frac{1}{2} \Omega^c_{bef} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \omega^a_c - \frac{1}{2} \Omega^a_{cef} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \omega^c_b + 2(1 + \chi) \Delta^{ad}_{cb} \left( \frac{1}{2} S_{def} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^c + S_d^e{}_f \mathbf{f}_e \wedge \mathbf{e}^f \wedge \mathbf{e}^c - \frac{1}{2} c_d{}^{ef} \mathbf{f}_e \wedge \mathbf{f}_f \wedge \mathbf{e}^c \right) \quad (75)$$

To find the independent parts, we must break the exterior derivative into  $\mathbf{d}_{(x)}$  and  $\mathbf{d}_{(y)}$ . This also requires separating the independent parts of the co-solder form,

$$\mathbf{f}_a = h_a{}^\mu(x, y) \mathbf{d}y_\mu + c_{ab}(x, y) \mathbf{e}^b = \mathbf{h}_a + \mathbf{c}_a$$

where the solder form already lies only in the  $x$ -sector,  $\mathbf{e}^b = e_a{}^b \mathbf{d}x^a$ . Then the only term proportional to  $\mathbf{h}_a \wedge \mathbf{h}_b \wedge \mathbf{e}^c$  is the final one, so  $0 = 2(1 + \chi) \Delta^{ad}_{cb} c_d{}^{ef} \mathbf{h}_e \wedge \mathbf{h}_f \wedge \mathbf{e}^c$ . Since the structure constants are already antisymmetric we may drop the basis forms. Then the  $ac$  contraction shows that

$$0 = (1 + \chi) c_a{}^{bc}$$

Dropping this factor from the remaining parts of Eq. (75), we move to the  $\mathbf{h}_a \wedge \mathbf{e}^b \wedge \mathbf{e}^c$  components, antisymmetrizing to remove the basis forms

$$0 = \partial^\alpha \Omega^a{}_{bfc} + 2(1 + \chi) \Delta^{ad}_{cb} S_d^e{}_f h_e{}^\alpha - 2(1 + \chi) \Delta^{ad}_{fb} S_d^e{}_c h_e{}^\alpha \quad (76)$$

Taking the  $af$  trace and using the field equation,  $\Omega^a{}_{bac} = 0$ ,

$$0 = (1 + \chi) ((n - 2) S_b^e{}_c + \eta_{bc} \eta^{ad} S_d^e{}_a)$$

A further contraction with  $\eta^{bc}$  shows that  $(1 + \chi)(n - 1) \eta^{ad} S_d^e{}_a = 0$  and therefore,  $0 = (1 + \chi) S_b^a{}_c$ . Substituting this back into Eq. (76) shows that the spacetime curvature is independent of  $y^\alpha$ ,  $\partial^\alpha \Omega^a{}_{bcd} = 0$ .

Finally, defining the  $x$ -covariant derivative of the spacetime component of the curvature,

$$\mathbf{D}_{(\omega, x)} \left( \frac{1}{2} \Omega^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) \equiv \mathbf{d}_{(x)} \left( \frac{1}{2} \Omega^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) + \left( \frac{1}{2} \Omega^c{}_{bef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge \omega^a_c - \left( \frac{1}{2} \Omega^a{}_{cef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge \omega^c_b$$

we conclude

$$0 = D_{[e}^{(\omega, x)} \Omega^a{}_{|b|cd]} + 2(1 + \chi) \Delta^{af}_{[e|b} S_{f|cd]} \quad (77)$$

$$0 = (1 + \chi) S_a{}^b{}_c \quad (78)$$

$$0 = (1 + \chi) c_a{}^{bc} \quad (79)$$

along with the further consequence of Eq. (78),

$$\partial^\mu \Omega^a{}_{bcd} = 0 \tag{80}$$

5.2.2. *The curvature structure equation*

We next find the components of the curvature,  $\Omega^a{}_{bcd}$ , in terms of the connection, and impose the field equation.

Substituting Eq. (62) for  $\mathbf{f}_a$  and expanding the exterior derivative  $\mathbf{d}\omega^a{}_b = \mathbf{d}_{(x)}\omega^a{}_b - 2\Delta^{ac}_{db}\mathbf{d}_{(y)}W_c\mathbf{e}^d$  in the structure equation, Eq. (64), allows separation of the  $\mathbf{e}^c \wedge \mathbf{e}^d$  and  $\mathbf{e}^c \wedge \mathbf{h}_d$  parts into two equations

$$\mathbf{d}_{(x)}\omega^a{}_b = \omega^c{}_b \wedge \omega^a{}_c + 2(1 + \chi)\Delta^{ac}_{db}c_{ce}\mathbf{e}^e \wedge \mathbf{e}^d + \frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d \tag{81}$$

$$0 = 2\Delta^{ac}_{db}(\partial^\mu W_c + (1 + \chi)h_c{}^\mu)\mathbf{d}y_\mu \wedge \mathbf{e}^d \tag{82}$$

The *ad* trace of Eq. (82) requires

$$0 = (n - 1)(\partial^\mu W_b + (1 + \chi)h_b{}^\mu)$$

which solves the full cross-term equation. This same condition is also required by the dilatation equation below. For the  $\mathbf{e}^c \mathbf{e}^d$  terms, notice that we may still have some  $y_\mu$  dependence.

5.2.3. *The spacetime equation*

It is convenient to define the curvature 2-form of the connection compatible with the solder form,

$$\mathbf{R}^a{}_b(x) \equiv \mathbf{d}_{(x)}\alpha^a{}_b - \alpha^c{}_b \wedge \alpha^a{}_c \tag{83}$$

This is the Riemann curvature built from  $\alpha^a{}_b(x)$ , not the full scale-invariant curvature of the biconformal space. Writing the spin connection as

$$\omega^a{}_b = \alpha^a{}_b + \beta^a{}_b$$

where  $\beta^a{}_b(x, y) = -2\Delta^{ac}_{db}W_c\mathbf{e}^d$ , we substitute into Eq. (81),

$$\begin{aligned} \mathbf{d}_{(x)}\alpha^a{}_b + \mathbf{d}_{(x)}\beta^a{}_b &= \alpha^c{}_b \wedge \alpha^a{}_c + \alpha^c{}_b \wedge \beta^a{}_c + \beta^c{}_b \wedge \alpha^a{}_c + \beta^c{}_b \wedge \beta^a{}_c \\ &\quad + 2(1 + \chi)\Delta^{ac}_{db}c_{ce}\mathbf{e}^e \wedge \mathbf{e}^d + \frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d \end{aligned}$$

Solving for  $\frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d$ , using Eq. (83), and recognizing the  $\alpha$ -covariant derivative of  $\beta^a{}_b$  as

$$\mathbf{D}_{(\alpha,x)}\beta^a{}_b \equiv \mathbf{d}_{(x)}\beta^a{}_b + \beta^a{}_c \wedge \alpha^c{}_b - \beta^c{}_b \wedge \alpha^a{}_c$$

this becomes

$$\frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{R}^a{}_b + \mathbf{D}_{(\alpha,x)}\beta^a{}_b - \beta^c{}_b \wedge \beta^a{}_c - 2(1 + \chi)\Delta^{ac}_{db}c_{ce}\mathbf{e}^e \wedge \mathbf{e}^d$$

Recalling that  $\mathbf{D}_{(\alpha,x)}\mathbf{e}^a = 0$  the covariant exterior derivative of  $\beta^a{}_b$  becomes

$$\begin{aligned} \mathbf{D}_{(\alpha,x)}\beta^a{}_b &= \mathbf{D}_{(\alpha,x)}(-2\Delta^{ac}_{db}W_c\mathbf{e}^d) \\ &= -2\Delta^{ac}_{db}(\mathbf{D}_{(\alpha,x)}W_c) \wedge \mathbf{e}^d \end{aligned}$$

The  $\beta^c{}_b \wedge \beta^a{}_c$  term may be simplified considerably. With  $W^2 \equiv \eta^{ab}W_aW_b$ ,

$$\begin{aligned}
\beta_b^c \wedge \beta_c^a &= 2\Delta_{db}^{ce} 2\Delta_{gc}^{af} W_e W_f \mathbf{e}^d \wedge \mathbf{e}^g \\
&= (\delta_g^a \delta_d^f \delta_b^e - \eta^{ef} \eta_{bd} \delta_g^a - \eta^{af} \eta_{gd} \delta_b^e + \delta_g^e \eta_{bd} \eta^{af}) W_e W_f \mathbf{e}^d \wedge \mathbf{e}^g \\
&= -(\delta_c^a W_b W_d + \eta_{bd} \eta^{af} W_c W_f - \eta_{bd} \delta_c^a \eta^{ef} W_e W_f) \mathbf{e}^c \wedge \mathbf{e}^d \\
&= -\left( \delta_c^a \delta_b^e W_e W_d - \frac{1}{2} \eta_{ed} \delta_b^e \delta_c^a W^2 + \eta_{bd} \eta^{ae} W_c W_e - \frac{1}{2} \eta_{bd} \eta^{ae} \eta_{ce} W^2 \right) \mathbf{e}^c \wedge \mathbf{e}^d \\
&= -\left( \delta_c^a \delta_b^e \left( W_e W_d - \frac{1}{2} \eta_{ed} W^2 \right) - \eta_{bc} \eta^{ae} \left( W_d W_e - \frac{1}{2} \eta_{de} W^2 \right) \right) \mathbf{e}^c \wedge \mathbf{e}^d \\
&= -2\Delta_{cb}^{ae} \left( W_e W_d - \frac{1}{2} \eta_{ed} W^2 \right) \mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned}$$

Therefore,

$$\frac{1}{2} \Omega_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{R}^a_b - 2\Delta_{db}^{ae} \left( D_c^{(\alpha,x)} W_e + W_e W_c - \frac{1}{2} \eta_{ec} W^2 + (1 + \chi) c_{ec} \right) \mathbf{e}^c \wedge \mathbf{e}^d$$

It is convenient to define the curvature 2-form of the full spin connection as well,

$$\begin{aligned}
\mathcal{R}^a_b &= \mathbf{d}_{(x)} \omega_b^a - \omega_b^c \wedge \omega_c^a \\
&= \mathbf{R}^a_b - 2\Delta_{db}^{ae} \left( D_c^{(\alpha,x)} W_e + W_e W_c - \frac{1}{2} \eta_{ec} W^2 \right) \mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned} \tag{84}$$

where  $\omega_b^a = \alpha_b^a - 2\Delta_{db}^{ac} W_c \mathbf{e}^d$ . This may be recognized as the curvature tensor of an  $n$ -dim Weyl geometry [39].

We also identify the Schouten tensor

$$\mathcal{R}_a = \mathcal{R}_{ab} \mathbf{e}^b \equiv \frac{1}{(n-2)} \left( R_{ab} - \frac{1}{2(n-1)} \eta_{ab} R \right) \mathbf{e}^b \tag{85}$$

In any dimension greater than two, knowing the Schouten tensor is equivalent to knowing the Ricci tensor, since we may always invert,  $R_{ab} = (n-2)\mathcal{R}_{ab} + \eta_{ab}\mathcal{R}$ . In terms of the Schouten tensor, the decomposition of the Riemann curvature into the traceless Weyl conformal tensor,  $\mathbf{C}^a_b$  and its Ricci parts, takes the simple form [39],

$$\mathbf{R}^a_b = \mathbf{C}^a_b - 2\Delta_{db}^{ae} \mathcal{R}_e \wedge \mathbf{e}^d \tag{86}$$

Using this decomposition, the Ricci parts of the curvature combine with the additional terms from the scale covariance,

$$\begin{aligned}
\frac{1}{2} \Omega_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d &= \mathbf{C}^a_b - 2\Delta_{db}^{ae} \left( \mathcal{R}_{ec} + D_c^{(\alpha,x)} W_e + W_e W_c \right. \\
&\quad \left. - \frac{1}{2} \eta_{ec} W^2 + (1 + \chi) c_{ec} \right) \mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned} \tag{87}$$

To impose the field equation, set  $P_{ec} \equiv \mathcal{R}_{ec} + D_c^{(\alpha,x)} W_e + W_e W_c - \frac{1}{2} \eta_{ec} W^2 + (1 + \chi) c_{ec}$ . Then substituting  $\Omega_{bcd}^a = \mathbf{C}^a_{bcd} - 2\Delta_{db}^{ae} P_{ec} + 2\Delta_{cb}^{ae} P_{ed}$  into Eq. (44),

$$\mathbf{C}^c_{bcd} - 2\Delta_{db}^{ae} P_{ec} + 2\Delta_{cb}^{ae} P_{ed} = 0$$

Since  $\mathbf{C}^c_{bcd} = 0$ , this has the component form, for any  $P_{ab}$ ,  $0 = \Delta_{db}^{ce} P_{ec} - \Delta_{cb}^{ce} P_{ed}$ , which, expanding the projections and combining result with the further contraction with  $\eta^{bd}$ , is seen to be true if and only if  $P_{ab} = 0$ .

Applying this general result to the field equation by replacing  $P_{ec}$ , we have

$$\mathcal{R}_{ec} + D_c^{(\alpha,x)} W_e + W_e W_c - \frac{1}{2} \eta_{ec} W^2 + (1 + \chi) c_{ec} = 0 \tag{88}$$

This determines  $c_{ab}$  unless  $1 + \chi = 0$ . The symmetric part of the first four terms on the left side is the Weyl-Schouten tensor,

$$\mathcal{R}_a \equiv \mathcal{R}_a + \left( W_{(a;b)} + W_a W_b - \frac{1}{2} W^2 \eta_{ab} \right) \mathbf{e}^b$$

and we see that there is an antisymmetric part to the trace of the Riemann-Weyl tensor,

$$\begin{aligned} \mathcal{R}_{bd} \equiv \mathcal{R}^c{}_{bcd} &= R_{bd} + (n - 2) \left( D_d^{(\alpha,x)} W_b + W_b W_d - \frac{1}{2} \eta_{bd} W^2 \right) \\ &\quad - \eta_{bd} \eta^{ce} \left( D_c^{(\alpha,x)} W_e + W_e W_c - \frac{1}{2} \eta_{ec} W^2 \right) \\ \mathcal{R}_{[bd]} &= (n - 2) W_{[b;d]} \end{aligned}$$

This agrees with the trace of the corresponding term of the torsion-free Bianchi identity arising from Eq. (65), and shows that  $c_{ab}$  may have both symmetric and antisymmetric parts.

Returning to the full spacetime curvature after satisfying the field equation,

$$\Omega^a{}_{bcd} = C^a{}_{bcd}(\alpha) \tag{89}$$

so the spacetime piece of the biconformal curvature reduces to the Weyl (conformal) curvature of the metric compatible connection. This part of the curvature is independent of  $y_\mu$  as required by Eq. (80), since the only  $y_\mu$ -dependence of the connection must arise from the Weyl vector, and as seen in Eq. (87) the Weyl vector is only present in the trace terms of the curvature. The full  $SO(p, q)$  curvature may now be written as

$$\begin{aligned} \Omega^a{}_b &= \left( \frac{1}{2} C^a{}_{bcd} + 2\chi \Delta_{db}^{ae} c_{ec} \right) \mathbf{e}^c \wedge \mathbf{e}^d + 2\chi \Delta_{db}^{ac} \mathbf{h}_c \wedge \mathbf{e}^d \\ &= C^a{}_b - \frac{2\chi}{1 + \chi} \Delta_{db}^{ae} \mathcal{R}_e \wedge \mathbf{e}^d + 2\chi \Delta_{db}^{ac} \mathbf{h}_c \wedge \mathbf{e}^d \end{aligned}$$

with  $\mathbf{h}_a = h_a{}^\alpha \mathbf{d}y_\alpha$ . The second expression holds only when  $1 + \chi \neq 0$ .

#### 5.2.4. Form of the spacetime Bianchi identity

When we combine this solution for  $\Omega^a{}_b$  with the spacetime part of the curvature Bianchi identity, we have Eq. (77)

$$D_{[e}^{(\omega,x)} \Omega^a{}_{|b|cd]} + 2(1 + \chi) \Delta_{[e|b}^{af} S_{f|cd]} = 0$$

With  $\frac{1}{2} \Omega^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d = \frac{1}{2} C^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{C}^a{}_b$ , the covariant exterior derivative of the Weyl curvature is

$$\begin{aligned} \mathbf{D}_{(\omega,x)} \mathbf{C}^a{}_b &\equiv \mathbf{d}_{(x)} \mathbf{C}^a{}_b + \mathbf{C}^a{}_b \wedge \omega^a{}_c - \mathbf{C}^a{}_b \wedge \omega^c{}_b \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{C}^a{}_b + \mathbf{C}^a{}_b \wedge \beta^a{}_c - \mathbf{C}^a{}_c \wedge \beta^c{}_b \end{aligned}$$

where we have expanded  $\omega^a{}_c = \alpha^a{}_c + \beta^a{}_c$ . But  $\mathbf{C}^a{}_b$  is the usual traceless part of the Riemann curvature, which satisfies

$$\begin{aligned} 0 &\equiv \mathbf{D}_{(\alpha,x)} \mathbf{R}^a_b \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{C}^a_b - 2\Delta_{db}^{ae} \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \wedge \mathbf{e}^d \end{aligned}$$

and we may rewrite the covariant exterior derivative of the Weyl curvature in terms of the derivative of the Schouten tensor. Making this replacement and setting  $\beta_c^a = -2\Delta_{dc}^{ae} W_e \mathbf{e}^d$ , the Bianchi identity may be written as

$$0 = 2\Delta_{df}^{ge} (\delta_b^f \delta_g^a \mathbf{D}_{(\alpha,x)} \mathcal{R}_e - \delta_g^a W_e \mathbf{C}^f_b + \delta_g^c \delta_b^f W_e \mathbf{C}^a_c + (1 + \chi) \delta_g^a \delta_b^f \mathbf{S}_e^{(ee)}) \wedge \mathbf{e}^d$$

where we set  $\mathbf{S}_b^{(ee)} \equiv \frac{1}{2} S_{emn} \mathbf{e}^m \wedge \mathbf{e}^n$ . Now we expand the  $\Delta$ -projections, distribute and re-collect terms, then use the first Riemannian Bianchi  $\mathbf{C}_{db} \wedge \mathbf{e}^d = 0$ , to show that

$$0 = \Delta_{db}^{ac} (\mathbf{D}_{(\alpha,x)} \mathcal{R}_c - W_e \mathbf{C}^e_c + 2(1 + \chi) \mathbf{S}_c^{(ee)}) \wedge \mathbf{e}^d \quad (90)$$

Expanding both the  $\Delta$ -projection and the triple antisymmetrization, we show that for all  $n > 3$ , Eq. (90) holds if and only if

$$2(1 + \chi) S_{bfg} = D_g^{(\alpha,x)} \mathcal{R}_{bf} - D_f^{(\alpha,x)} \mathcal{R}_{bg} + W_e \mathbf{C}^e_{bfg}$$

Restoring two basis forms, we may write this as

$$\mathbf{D}^{(\alpha,x)} \mathcal{R}_b - W_e \mathbf{C}^e_b + 2(1 + \chi) \mathbf{S}_b^{(ee)} = 0 \quad (91)$$

which solves the full Bianchi relation, Eq. (90). From Eq. (91) it follows that:

**Theorem.** *In any torsion-free biconformal space with integrable Weyl vector;  $W_\alpha = \partial_\alpha \phi$ , and  $1 + \chi \neq 0$ , the spacetime co-torsion is the obstruction to conformal Ricci flatness.*

Complete details of the algebra leading to Eq. (90) are given in Appendix C.

### 5.3. The dilatation and co-torsion structure equations

Expanding the dilatation equation, Eq. (67), using Eqs. (61)–(63), to display the independent parts

$$\mathbf{d}_{(x)} \omega + \mathbf{d}_{(y)} \omega = (1 + \chi) \mathbf{e}^c \wedge \mathbf{h}_c + (1 + \chi) \mathbf{e}^c \wedge \mathbf{c}_c$$

we find two independent equations,

$$\mathbf{d}_{(x)} \omega = (1 + \chi) \mathbf{c} \quad (92)$$

$$\mathbf{d}_{(y)} \omega = (1 + \chi) \mathbf{e}^c \wedge \mathbf{h}_c \quad (93)$$

where we have set  $\mathbf{c} \equiv \mathbf{e}^c \wedge \mathbf{c}_c$ . The Bianchi identity reduces to  $0 \equiv \mathbf{d}^2 \omega = (1 + \chi) \mathbf{d}(\mathbf{e}^a \wedge \mathbf{f}_a) = (1 + \chi) \mathbf{D}(\mathbf{e}^a \wedge \mathbf{f}_a) = -(1 + \chi) \mathbf{e}^a \wedge \mathbf{S}_a$ , which reproduces Eqs. (78) and (79) and shows that

$$(1 + \chi) S_{[abc]} = 0$$

The dilatation structure equations may be integrated exactly, but the result depends crucially on whether or not  $(1 + \chi) = 0$ . The two cases will be handled separately in the next two Sections.

The co-torsion structure equation also depends on the case considered. In addition to the structure equations (66), we still have the field equations



$$\begin{aligned}
 s_c &\equiv S_c^a{}_a = S_a^a{}_c \\
 \Delta_{sb}^{ar} S_c^b{}_a &= \Delta_{sc}^{ar} s_a \\
 S_c^{ac} &= 0
 \end{aligned}
 \tag{94}$$

and constraints from the curvature and dilatation Bianchi identities,

$$\begin{aligned}
 (1 + \chi) S_{[abc]} &= 0 \\
 D_{[e}^{(\omega, x)} \Omega^a{}_{|b|cd]} + 2(1 + \chi) \Delta_{[e|b}^{af} S_{f|cd]} &= 0 \\
 (1 + \chi) S_a^b{}_c &= 0 \\
 (1 + \chi) S_a^{bc} &= 0
 \end{aligned}
 \tag{95}$$

To complete the reduction of the biconformal space, we turn to the  $1 + \chi \neq 0$  and  $1 + \chi = 0$  cases.

### 6. Generic case: $1 + \chi \neq 0$

In this Section, we consider the final reduction to spacetime for generic values of the constants  $\alpha, \beta, \gamma$  in the original action, assuming

$$1 + \chi \neq 0
 \tag{96}$$

It follows that we must have  $S_a^{bc} = 0$  and therefore on the  $\mathbf{e}^a = 0$  submanifold,

$$\begin{aligned}
 \mathbf{d}_{(y)} \mathbf{h}_a &= 0 \\
 \mathbf{h}_a &= \mathbf{d}_{(y)} y_a
 \end{aligned}$$

where the functions  $y_a$  may be written as coordinates with an  $x_0$ -dependent linear transformation,

$$h_a(x_0^\mu, y_\mu) = h_a{}^\mu(x_0^\mu)(y_\mu + \beta_\mu(x_0^\mu))$$

When we return to the full biconformal space, the linear coefficients  $h_a{}^\mu(x)$  and  $\beta_\mu(x)$  remain as arbitrary coordinate choices. The full co-solder form, Eq. (62), then satisfies

$$\begin{aligned}
 \mathbf{f}_a &= h_a{}^\mu(x) \mathbf{d}y_\mu + c_{ab}(x, y) \mathbf{e}^b \\
 &= h_a{}^\mu(x) \mathbf{d}y_\mu + h_a{}^\mu(x) \mathbf{d}\beta_\mu(x) + c_{ab}(x, y) \mathbf{e}^b
 \end{aligned}$$

Thus, the coordinate choice of the origin for  $y_\mu$  at each  $x^\alpha$  changes  $c_{ab}$ . We continue to define the co-basis  $\mathbf{h}_a$  as only the  $\mathbf{d}y_\mu$  part,

$$\mathbf{h}_a \equiv h_a{}^\mu(x) \mathbf{d}y_\mu
 \tag{97}$$

The momentum space is therefore foliated by abelian group manifolds. The foliation may be identified as  $R^n$  or a toroidal compactification of all or part of  $R^n$ . Being principally interested in the underlying presence of general relativity, we take it to be  $R^n$  and eventually identify it with the cotangent space at each  $x^\alpha$ . However, it may also be taken as the torus  $T^n$  of double field theory for other applications to string theory.

Given the form Eq. (97) of the co-basis  $\mathbf{h}_a$ , it is useful to begin with the natural inner product arising from the conformal Killing form together with our freedom to choose the  $x$ -coordinates. The coordinate freedom allows us to conveniently choose the functions  $h_\mu^a$  to be the inverse solder form, enabling us to integrate the dilatation equation for the Weyl vector.

### 6.1. The Killing metric

The field  $h_a^\mu(x)$  lets us choose a convenient orthonormal basis for the  $y_\alpha$  space at each point of the  $x^\alpha$  space. Taking the restriction of the conformal Killing form to the biconformal manifold,  $\begin{pmatrix} 0 & \delta_b^a \\ \delta_d^a & 0 \end{pmatrix}$ , as the biconformal metric lets us usefully control this coordinate freedom. The Killing metric gives the orthonormal inner product of  $(\mathbf{e}^a, \mathbf{f}_b)$  basis,

$$\begin{aligned} \langle \mathbf{e}^a, \mathbf{e}^b \rangle &= 0 \\ \langle \mathbf{e}^a, \mathbf{f}_b \rangle &= \delta_b^a \\ \langle \mathbf{f}_a, \mathbf{f}_b \rangle &= 0 \end{aligned}$$

Choosing arbitrary coordinates,  $\tilde{x}^\mu$  as the complement to  $y_\mu$ , the first of these three relations,  $\langle \mathbf{e}^a, \mathbf{e}^b \rangle = 0$ , shows that  $\langle \mathbf{d}\tilde{x}^\mu, \mathbf{d}\tilde{x}^\nu \rangle = 0$ . Substituting  $\mathbf{f}_a = \mathbf{h}_a + c_{ab}\mathbf{e}^b$  into  $\langle \mathbf{e}^a, \mathbf{f}_b \rangle$  and expanding in coordinates,

$$\begin{aligned} \delta_b^a &= \langle \mathbf{e}^a, \mathbf{h}_b + c_{bc}\mathbf{e}^c \rangle \\ &= \langle \mathbf{e}^a, \mathbf{h}_b \rangle \\ &= h_b^\mu(\tilde{x})e_\nu^a(\tilde{x})\langle \mathbf{d}\tilde{x}^\nu, \mathbf{d}y_\mu \rangle \\ e_b^\nu(\tilde{x}) &= h_b^\mu(\tilde{x})\langle \mathbf{d}\tilde{x}^\nu, \mathbf{d}y_\mu \rangle \end{aligned} \quad (98)$$

Using  $\langle \mathbf{e}^a, \mathbf{h}_b \rangle = \delta_b^a$  in the  $\langle \mathbf{f}_a, \mathbf{f}_b \rangle$  inner product, we find  $\langle \mathbf{h}_a, \mathbf{h}_b \rangle$ ,

$$\begin{aligned} 0 &= \langle \mathbf{f}_a, \mathbf{f}_b \rangle \\ &= \langle \mathbf{h}_a + c_{ac}\mathbf{e}^c, \mathbf{h}_b + c_{bd}\mathbf{e}^d \rangle \\ \langle \mathbf{h}_a, \mathbf{h}_b \rangle &= -(c_{ab} + c_{ba}) = -2c_{(ab)} \end{aligned} \quad (99)$$

We see from Eq. (98) that the inner product of  $\mathbf{d}y_\mu$  with  $\mathbf{d}\tilde{x}^\nu$  cannot depend on  $y_\mu$ ,

$$\langle \mathbf{d}\tilde{x}^\nu, \mathbf{d}y_\mu \rangle = k^\nu{}_\mu(\tilde{x})$$

Moreover, like  $e_b^\nu(\tilde{x})$  and  $h_b^\mu(\tilde{x})$ ,  $k^\nu{}_\mu(\tilde{x})$  must be invertible. Let  $x^\alpha = x^\alpha(\tilde{x})$  be any coordinate transformation of  $\tilde{x}^\alpha$ . Then in the new  $x$ -coordinates the inner product becomes

$$\begin{aligned} \langle \mathbf{d}x^\alpha, \mathbf{d}y_\mu \rangle &= \left\langle \frac{\partial x^\nu}{\partial \tilde{x}^\alpha} \mathbf{d}\tilde{x}^\alpha, \mathbf{d}y_\mu \right\rangle \\ &= \frac{\partial x^\nu}{\partial \tilde{x}^\alpha} k^\alpha{}_\mu(\tilde{x}) \end{aligned}$$

Since  $\frac{\partial x^\nu}{\partial \tilde{x}^\alpha}$  is an arbitrary general linear transformation at each point and  $k^\nu{}_\mu$  is invertible, we may choose  $\frac{\partial x^\nu}{\partial \tilde{x}^\alpha}$  to be its inverse. Then

$$\langle \mathbf{d}x^\nu, \mathbf{d}y_\mu \rangle = \delta_\mu^\nu$$

Writing eq. (98) in these new coordinates, we have

$$h_b^\mu(x) = e_b^\mu(x)$$

showing that in these coordinates  $h_b^\mu(x)$  is just the inverse matrix to  $e_\mu^a(x)$ . This fixes

$$\mathbf{h}_a = e_b^\mu(x)\mathbf{d}y_\mu \quad (100)$$

### 6.2. The dilatation equation

With the change of  $x$ -coordinate, the basis forms are now given by

$$\mathbf{e}^a = e_\mu^a(x) \mathbf{d}x^\mu \tag{101}$$

$$\mathbf{f}_a = e_a^\alpha (\mathbf{d}y_\alpha + \mathbf{d}\beta_\alpha + c_{\alpha\nu} \mathbf{d}x^\nu) \tag{102}$$

with the spin connection and Weyl vector given by Eq. (60) and Eq. (63). The  $x$ -dependent translation  $\beta_\mu$  remains an arbitrary coordinate choice.

Using the coefficients  $e_\mu^a$  to change basis in the usual way to convert between coordinate and orthonormal indices, we expand  $\mathbf{c}_a = e_a^\alpha c_{\alpha\nu} \mathbf{d}x^\nu$  and the Weyl vector  $\boldsymbol{\omega} = W_a \mathbf{e}^a = W_\mu \mathbf{d}x^\mu$  in Eq. (67) in coordinates.

$$\begin{aligned} & \mathbf{d}x^\mu \wedge \partial_\mu (W_\nu \mathbf{d}x^\nu) + \mathbf{d}y_\mu \wedge \partial^\mu (W_\nu \mathbf{d}x^\nu) \\ &= (1 + \chi) (e_\mu^a \mathbf{d}x^\mu) \wedge (e_a^\alpha (\mathbf{d}y_\alpha + \beta_{\alpha,\nu} \mathbf{d}x^\nu + c_{\alpha\nu} \mathbf{d}x^\nu)) \end{aligned}$$

Equating independent parts,

$$W_{[\mu,\nu]} = -(1 + \chi) (\beta_{[\mu,\nu]} + c_{[\mu\nu]}) \tag{103}$$

$$\partial^\mu W_\nu = -(1 + \chi) \delta_\nu^\mu \tag{104}$$

Eq. (104) is integrated immediately,

$$W_\mu = (1 + \chi) (-y_\mu + \alpha_\mu(x)) \tag{105}$$

This must satisfy both equations, so substituting into Eq. (103),

$$c_{[\mu\nu]} = -(\alpha_{[\mu,\nu]} + \beta_{[\mu,\nu]}) \tag{106}$$

Before making the obvious coordinate choice,  $\beta_\mu = -\alpha_\mu$ , it is suggestive to comment on the form of Eq. (105). The integration “constant”  $\alpha_\mu(x)$  is a potential for the antisymmetric part of  $c_{\mu\nu}$  and the antisymmetric part is independent of  $y_\mu$ . Since an  $x$ -dependent rescaling does not affect the vanishing of the  $\mathbf{f}_a$  component of the Weyl vector, we may perform a dilatation to modify  $\alpha_\mu(x)$ . This is precisely the form of the gauge transformation of the electromagnetic potential, but as with the failed Weyl theory of electromagnetism, it may lead to unphysical size changes since the dilatational curvature,  $\Omega_{\mu\nu}$  does not necessarily vanish. However, notice that biconformal space has a symplectic form. Eq. (67) describes a manifestly nondegenerate 2-form,  $\mathbf{e}^a \wedge \mathbf{f}_a$ , which is exact and therefore closed. This means we may interpret the full biconformal space as a relativistic particle phase space with canonical coordinates  $(x^\alpha, y_\beta)$ . In this view,  $y_\mu$  is a momentum and the Weyl vector (105) has exactly the form and gauge properties of the electromagnetic conjugate momentum if  $\alpha_\mu$  is taken proportional to the vector potential. Moreover, the previous well-known conflict with observation is avoided. The transformation by  $\beta_\mu$  to remove  $\alpha_\mu$  is then the canonical transformation between the conjugate electromagnetic momentum,  $\pi_\mu = p_\mu - eA_\mu$  and the simple particle momentum  $p_\mu$ .

The original ill-fated attempt by Weyl to identify the Weyl vector of a Weyl geometry as the vector potential of electromagnetism,  $W_\mu = eA_\mu$ , leads to nonvanishing dilatation in the presence of electromagnetic fields,  $\Omega_{\mu\nu} = eF_{\mu\nu}$ . Einstein immediately observed that this conflicts with experiment, and it is easy to show, for example, that two hydrogen atoms moving to produce a closed path that encloses some electromagnetic flux would emerge with different sizes, and therefore very different spectra. The precision of atomic spectra therefore disproves the simplest

version of the theory. The situation is completely different in the biconformal setting. Because of the extra  $\mathbf{e}^a \wedge \mathbf{f}_a$  term in the dilatation equation, it is possible to have vanishing dilatational curvature and retain the interpretation of  $\alpha_\mu$  (rather than  $W_\mu$ ) as the vector potential. The idea has been explored to some extent in [9]. Here, the form of the dilatation is given by

$$\begin{aligned}\Omega &= \Omega^a{}_b \mathbf{f}_a \wedge \mathbf{e}^b \\ &= \chi \mathbf{e}^a \wedge \mathbf{f}_a \\ &= \chi \delta_\nu^\mu e_a{}^\nu e_\mu{}^a \mathbf{d}x^\mu \wedge \mathbf{d}y_\nu + \chi c_{\mu\nu} e_\mu{}^a \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu\end{aligned}$$

or since we may also write  $\Omega = \frac{1}{2} \Omega_{\mu\nu} \mathbf{d}x^\mu \mathbf{d}x^\nu + \Omega^\mu{}_\nu \mathbf{d}y_\mu \mathbf{d}x^\nu + \Omega^{\mu\nu} \mathbf{d}y_\mu \mathbf{d}y_\nu$ , we have coordinate components

$$\begin{aligned}\Omega_{\mu\nu} &= \chi (c_{\mu\nu} - c_{\nu\mu}) \\ \Omega^\mu{}_\nu &= -\chi \delta_\nu^\mu \\ \Omega^{\mu\nu} &= 0\end{aligned}$$

Therefore, while the space spanned by  $\mathbf{f}_a = 0$  shows no unphysical size changes,  $\Omega_{ab} = 0$ , the space defined by setting  $y_\mu = 0$  has  $\Omega_{\mu\nu}$  given by the antisymmetric part of  $c_{\mu\nu}$ .

It is possible to avoid dilatational curvature altogether by setting  $\chi = 0$ . In this case, the full dilatational curvature is identically zero. There is still a symplectic form in this subclass of theories, since we still have  $\mathbf{d}\omega = \mathbf{e}^a \wedge \mathbf{f}_a$ . This permits the consistent interpretation of the Weyl vector as the conjugate electromagnetic momentum according to Eq. (105).

Notice that setting  $\chi = 0$  is inconsistent with the  $1 + \chi = 0$  cases to be studied in the next Section. The possibility of a geometric graviweak theory with  $1 + \chi = 0$  is more appealing than this  $\chi = 0$  case, since the success of the standard model strongly suggests that the electromagnetic and weak interactions should arise together. We continue with the generic picture, but eventually choose the  $y_a$  coordinate to be offset by  $\beta_\mu(x) = -\alpha_\mu(x)$ . This makes  $\Omega_{\mu\nu} = \chi c_{[\mu\nu]}$  vanish without restricting the action, while it leaves the cross-dilatation nonzero and  $c_{\mu\nu}$  symmetric. There is no effect of this on spacetime, but, identifying  $y_\mu = \frac{i}{\hbar} p_\mu$  as argued in [40,41] it leads to a non-integrability in phase space of the form  $\frac{i}{\hbar} \oint p_\mu dx^\mu \neq 0$  arising from the interesting conjunction of the dilatational curvature with the symplectic form. The result might be consistent with a quantum interpretation. This idea has been explored in [41,40,30].

Without further conjecture on the interpretation of the geometry, we continue with the generic case of the reduction toward general relativity. Without loss of generality, we choose the  $y_\mu$  coordinate so that  $\alpha_\mu = \tilde{\alpha}(x) + \beta_\mu(x) = 0$ , but this is merely a convenient coordinate choice. The solution retains full coordinate covariance.

Collecting the forms for the connection and basis established in Eqs. (71) and (101), and writing the gauged form of the Weyl vector co-solder form, we now have

$$\omega_b^a = \alpha_b^a(x) - 2\Delta_{db}^{ac} W_c \mathbf{e}^d \quad (107)$$

$$W_\alpha = -(1 + \chi) y_\alpha \quad (108)$$

$$\mathbf{e}^a = e_\alpha^a(x) \mathbf{d}x^\alpha \quad (109)$$

$$\mathbf{f}_a = e_a^\alpha \mathbf{d}y_\alpha + c_{ab}(x, y) \mathbf{e}^b = \mathbf{h}_a + \mathbf{c}_a \quad (110)$$

where

$$\begin{aligned} \mathbf{c}_a &= -\frac{1}{1+\chi} \left( \mathcal{R}_a + \mathbf{D}_{(\alpha,x)} W_a + W_a \boldsymbol{\omega} - \frac{1}{2} \eta_{ab} W^2 \mathbf{e}^b \right) \\ \mathbf{c} &= -\mathbf{d}\boldsymbol{\alpha} = 0 \end{aligned} \tag{111}$$

The dilatation may now be written as

$$\boldsymbol{\Omega} = \chi \mathbf{e}^a \wedge \mathbf{f}_a = \chi \mathbf{e}^a \wedge \mathbf{h}_a = \chi \mathbf{d}x^\mu \wedge \mathbf{d}y_\mu \tag{112}$$

### 6.3. The co-solder equation

Now consider the co-solder equation, Eq. (66), with the co-torsion constrained by the Bianchi identities, Eqs. (95)

$$\mathbf{d}\mathbf{f}_a = \boldsymbol{\omega}_a^b \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \boldsymbol{\omega} + \frac{1}{2} S_{acd} \mathbf{e}^c \wedge \mathbf{e}^d \tag{113}$$

First, note that the Bianchi identity  $S_{[abc]} = 0$  is identically satisfied, since contraction with the solder form vanishes identically:

$$\begin{aligned} \frac{1}{2} S_{[acd]} \mathbf{e}^a \wedge \mathbf{e}^c \wedge \mathbf{e}^d &= \mathbf{e}^a \wedge \mathbf{d}\mathbf{f}_a - \mathbf{e}^a \wedge \boldsymbol{\omega}_a^b \wedge \mathbf{f}_b - \mathbf{e}^a \wedge \mathbf{f}_a \wedge \boldsymbol{\omega} \\ &= -\mathbf{d}(\mathbf{e}^a \wedge \mathbf{f}_a) + (\mathbf{d}\mathbf{e}^b - \mathbf{e}^a \wedge \boldsymbol{\omega}_a^b - \boldsymbol{\omega} \wedge \mathbf{e}^b) \wedge \mathbf{f}_b \\ &= 0 \end{aligned}$$

Now, solving Eq. (113) for the co-torsion and substituting for the connection forms

$$\begin{aligned} \mathbf{S}_a &= \mathbf{d}\mathbf{f}_a - \boldsymbol{\omega}_a^b \wedge \mathbf{f}_b - \mathbf{f}_a \wedge \boldsymbol{\omega} \\ &= \mathbf{D}_{(\omega)} \mathbf{f}_a \\ &= \mathbf{D}_{(\omega)} (e_a^\alpha \mathbf{d}y_\alpha + c_{ab}(x, y) \mathbf{e}^b) \\ &= e_a^\alpha \mathbf{D}_{(\omega)} (\mathbf{d}y_\alpha) + \mathbf{D}_{(\omega)} \mathbf{c}_a \\ &= e_a^\alpha (\mathbf{d}(\mathbf{d}y_\alpha) + \mathbf{d}y_\beta \wedge \Sigma_{\alpha\mu}^\beta \mathbf{d}x^\mu) + (\mathbf{d}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\omega}_a^b - \boldsymbol{\omega} \wedge \mathbf{c}_a) \\ &= e_a^\alpha \Sigma_{\alpha\mu}^\beta \mathbf{d}y_\beta \wedge \mathbf{d}x^\mu + (\mathbf{d}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\omega}_a^b + \boldsymbol{\omega} \wedge \mathbf{c}_a) \end{aligned}$$

We first need the  $\mathbf{d}y_\beta$  dependent pieces of  $\mathbf{d}\mathbf{c}_a$ , where  $\mathbf{c}_a$  is given by Eq. (111). Since the only  $y$ -dependence is in the Weyl vector,

$$\mathbf{d}\mathbf{c}_a = \mathbf{d}_{(x)} \mathbf{c}_a - \frac{1}{1+\chi} \mathbf{d}_{(y)} \left( \mathbf{D}_{(\alpha,x)} W_a + W_a \boldsymbol{\omega} - \frac{1}{2} \eta_{ab} W^2 \mathbf{e}^b \right)$$

Expanding the  $x$ -dependent,  $\alpha$ -covariant derivative of the Weyl vector,

$$\begin{aligned} \mathbf{D}_{(\alpha,x)} W_b &= \mathbf{e}^a e_a^\mu D_\mu^{(\alpha,x)} (e_b^\nu W_\nu) \\ &= (1+\chi) \mathbf{e}^a e_a^\mu e_b^\nu D_\mu^{(\alpha,x)} (-y_\nu) \\ &= (1+\chi) \mathbf{e}^a e_a^\mu e_b^\nu (y_\alpha \Gamma_{\nu\mu}^\alpha(x)) \end{aligned}$$

the  $y$ -derivatives become

$$\begin{aligned}
\mathbf{d}_{(y)}\mathbf{c}_a &= -\frac{1}{1+\chi}\mathbf{d}_{(y)}\left(\mathbf{D}_{(\alpha,x)}W_a + W_a\boldsymbol{\omega} - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b\right) \\
&= -\frac{1}{1+\chi}e_b^\mu e_a^\nu \mathbf{d}_{(y)}\left((1+\chi)y_\alpha \Gamma_{\nu\mu}^\alpha + W_\mu W_\nu - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}W_\alpha W_\beta\right) \wedge \mathbf{e}^b \\
&= -e_b^\mu e_a^\nu (\mathbf{d}y_\alpha \Gamma_{\nu\mu}^\alpha - \mathbf{d}y_\mu W_\nu - W_\mu \mathbf{d}y_\nu + g_{\mu\nu}g^{\alpha\beta}W_\alpha \mathbf{d}y_\beta) \wedge \mathbf{e}^b \\
&= -e_b^\mu e_a^\nu (\Gamma_{\nu\mu}^\beta - \delta_\mu^\beta W_\nu - \delta_\nu^\beta W_\mu + g_{\mu\nu}g^{\alpha\beta}W_\alpha) \mathbf{d}y_\beta \wedge \mathbf{e}^b \\
&= -e_a^\nu \Sigma_{\nu\mu}^\beta \mathbf{d}y_\beta \wedge \mathbf{d}x^\mu
\end{aligned}$$

Substituting, the  $\mathbf{d}y_\alpha$  terms cancel identically, leaving

$$\begin{aligned}
\mathbf{S}_a &= e_a^\alpha \Sigma_{\alpha\mu}^\beta \mathbf{d}y_\beta \wedge \mathbf{d}x^\mu + (\mathbf{d}_{(x)}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\omega}_a^b + \boldsymbol{\omega} \wedge \mathbf{c}_a) - e_a^\nu \Sigma_{\nu\mu}^\beta \mathbf{d}y_\beta \wedge \mathbf{d}x^\mu \\
&= \mathbf{d}_{(x)}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\omega}_a^b + \boldsymbol{\omega} \wedge \mathbf{c}_a
\end{aligned}$$

This shows once again that the cross-term of the co-torsion vanishes,  $S_a{}^c{}_b = 0$ . Now we expand the spin connection, rewriting all of the derivatives as  $x$ -dependent,  $\alpha$ -covariant,  $\mathbf{D}_{(\alpha,x)}$ .

$$\begin{aligned}
\mathbf{S}_a &= \mathbf{d}_{(x)}\mathbf{c}_a + \mathbf{c}_b \wedge (\boldsymbol{\alpha}_a^b + \boldsymbol{\beta}_a^b) + \boldsymbol{\omega} \wedge \mathbf{c}_a \\
&= \mathbf{D}_{(\alpha,x)}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\beta}_a^b + \boldsymbol{\omega} \wedge \mathbf{c}_a \\
&= -\frac{1}{1+\chi}\mathbf{D}_{(\alpha,x)}\left(\mathcal{R}_a + \mathbf{D}_{(\alpha,x)}W_a + W_a\boldsymbol{\omega} - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b\right) \\
&\quad -\frac{1}{1+\chi}\left(\mathcal{R}_b + \mathbf{D}_{(\alpha,x)}W_b + W_b\boldsymbol{\omega} - \frac{1}{2}\eta_{bc}W^2\mathbf{e}^c\right) \wedge \boldsymbol{\beta}_a^b \\
&\quad -\frac{1}{1+\chi}\left(\boldsymbol{\omega} \wedge \mathcal{R}_a + \boldsymbol{\omega} \wedge \mathbf{D}_{(\alpha,x)}W_a - \frac{1}{2}\eta_{ab}W^2\boldsymbol{\omega} \wedge \mathbf{e}^b\right)
\end{aligned}$$

After distributing the covariant derivative and expanding  $\boldsymbol{\beta}_a^b$ , we separate curvature terms and simplify,

$$\begin{aligned}
\mathbf{S}_a &= -\frac{1}{1+\chi}\left(\mathbf{D}_{(\alpha,x)}\mathcal{R}_a + \mathbf{D}_{(\alpha,x)} \wedge \mathbf{D}_{(\alpha,x)}W_a - W_b \mathcal{R}_a \wedge \mathbf{e}^b + \eta^{bd}\eta_{ea}W_d \mathcal{R}_b \wedge \mathbf{e}^e\right) \\
&\quad -\frac{1}{1+\chi}\left(\mathbf{D}_{(\alpha,x)}W_a \wedge \boldsymbol{\omega} + \boldsymbol{\omega} \mathbf{D}_{(\alpha,x)}W_a + \eta^{bd}\eta_{ea}W_d (\mathbf{D}_{(\alpha,x)}W_b) \wedge \mathbf{e}^e\right) \\
&\quad -\eta_{ab}\eta^{cd}W_c (\mathbf{D}_{(\alpha,x)}W_d) \wedge \mathbf{e}^b \\
&\quad -\frac{1}{1+\chi}\left(\eta_{ea}W^2\boldsymbol{\omega} \wedge \mathbf{e}^e - \frac{1}{2}\eta_{ea}W^2\boldsymbol{\omega} \wedge \mathbf{e}^e - \frac{1}{2}\eta_{ab}W^2\boldsymbol{\omega} \wedge \mathbf{e}^b\right) \\
&= -\frac{1}{1+\chi}\left(\mathbf{D}_{(\alpha,x)}\mathcal{R}_a + \mathbf{D}_{(\alpha,x)} \wedge \mathbf{D}_{(\alpha,x)}W_a - W_b(\delta_a^d \delta_e^b - \eta^{bd}\eta_{ea})\mathcal{R}_d \wedge \mathbf{e}^e\right)
\end{aligned}$$

Using the partition of the Riemann tensor, Eq. (86), and the Ricci identity,

$$\mathbf{D}_{(\alpha,x)} \wedge \mathbf{D}_{(\alpha,x)}W_a = -W_b \mathbf{R}^b{}_a$$

we see that  $\mathbf{S}_a$  is

$$\mathbf{S}_a = -\frac{1}{1+\chi}\left(\mathbf{D}_{(\alpha,x)}\mathcal{R}_a - W_b \mathbf{C}^b{}_a\right) \quad (114)$$

If  $W_a$  were the gradient of a function of  $x^\mu$ , then Eq. (114) would be the condition for the spacetime to be conformal to a Ricci-flat spacetime. Since

$$\begin{aligned} \mathbf{d}_{(x)}\omega &= \mathbf{d}_{(x)}(- (1 + \chi)y_\alpha \mathbf{d}x^\alpha) \\ &= 0 \end{aligned}$$

this is the case, but only at constant  $y_\alpha$ .

$$\mathbf{S}_{\mu\alpha\beta} = -\frac{1}{1 + \chi} D_\alpha^{(\alpha,x)} \mathcal{R}_a + y_b \mathbf{C}^b_a$$

This is in agreement with our conclusion, Eq. (91), from the spacetime Bianchi equation combined with the usual Riemannian Bianchi identity. The result means that for vanishing co-torsion and *constant*  $y_\alpha$  there exists an  $x$ -dependent gauge transformation to a Ricci flat spacetime. This form is not unfamiliar, the same expression having been noted in another context in [17]. The remaining  $y$ -dependence is the only obstruction to the Triviality Theorem: if a conformal transformation could make  $S_{abc}$  vanish for *all*  $y_\alpha$ , then the biconformal space would necessarily be trivial.

Note that the field equations Eqs. (94) and the Bianchi identities Eqs. (95) for the co-torsion are now all satisfied for any allowed  $\mathbf{S}_a$ .

#### 6.4. Collecting the results $(1 + \chi) \neq 0$

We have now solved for the full connection and satisfied all of the field equations.

$$\begin{aligned} \omega^a_b &= \alpha^a_b(x) - 2\Delta^{ac}_{db} W_c \mathbf{e}^d \\ \mathbf{e}^a &= e_a^a(x) \mathbf{d}x^\alpha \\ \mathbf{f}_a &= e_a^\alpha \mathbf{d}y_\alpha - \frac{1}{1 + \chi} \left( \mathcal{R}_a + \mathbf{D}_{(\alpha,x)} W_a + W_a \omega - \frac{1}{2} \eta_{ab} W^2 \mathbf{e}^b \right) \\ &= \mathbf{h}_a - \frac{1}{1 + \chi} \mathcal{R}_a \\ \omega &= - (1 + \chi) y_\alpha \mathbf{d}x^\alpha \end{aligned}$$

where  $\chi \equiv \frac{1}{n-1} \frac{1}{(n-1)\alpha-\beta} \Lambda$ . Notice that  $\mathbf{d}\omega = (1 + \chi) \mathbf{e}^c \wedge \mathbf{f}_c$  defines a symplectic form.

The curvatures follow from the structure equations as

$$\mathbf{\Omega}^a_b = \mathbf{C}^a_b + 2(1 + \chi) \Delta^{ac}_{db} \mathbf{h}_c \wedge \mathbf{e}^d \tag{115}$$

$$\mathbf{T}^a = 0 \tag{116}$$

$$\mathbf{S}_a = -\frac{1}{1 + \chi} \mathbf{D}_{(\alpha,x)} \mathcal{R}_a + y_b \mathbf{C}^b_a \tag{117}$$

$$\mathbf{\Omega} = \chi \mathbf{e}^a \wedge \mathbf{h}_a \tag{118}$$

The combination  $\chi \mathbf{e}^a \wedge \mathbf{h}_a = \chi \mathbf{d}x^\alpha \wedge \mathbf{d}y_\alpha$  is also non-degenerate and closed, and therefore symplectic.

#### 6.5. The Lagrangian submanifold of spacetime

The basis forms  $\mathbf{h}_a = e_a^\mu \mathbf{d}y_\mu$  are manifestly involute. Holding  $y_\mu$  constant so that  $\mathbf{h}_a = 0$ , the resulting vanishing of the symplectic form shows that the  $\mathbf{h}_a = 0$  submanifold is Lagrangian. The structure equations for the resulting Lagrangian submanifold are

$$\mathbf{d}\omega_b^a = \omega_b^c \wedge \omega_c^a + 2(1 + \chi)\Delta_{db}^{ac}\mathbf{c}_c \wedge \mathbf{e}^d + \mathbf{C}^a_b(\alpha) \quad (119)$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \omega_b^a + \omega \wedge \mathbf{e}^a \quad (120)$$

$$\mathbf{d}\omega = 0$$

and the remaining part of the co-solder equation is,

$$\mathbf{S}_a = \mathbf{d}\mathbf{c}_a - \omega_b^a \wedge \mathbf{c}_b - \mathbf{c}_a \wedge \omega = \mathbf{D}_{(\omega,x)}\mathbf{c}_a \quad (121)$$

With  $y_\mu = y_\mu^0$  constant, the form of the connection is

$$\omega_b^a = \alpha_b^a(x) + 2(1 + \chi)\Delta_{db}^{ac}e_c^\alpha y_\alpha^0 \mathbf{e}^d$$

$$\mathbf{e}^a = e_\alpha^a(x)\mathbf{d}x^\alpha$$

$$W_\alpha = -(1 + \chi)y_\alpha^0$$

Notice that the Weyl vector is now the gradient of  $-(1 + \chi)y_\alpha^0 x^\alpha$  with respect to  $x^\alpha$ . There is one further consequence of the curvature field equation,

$$\mathbf{c}_a = -\frac{1}{1 + \chi}\left(\mathcal{R}_a + \mathbf{D}_{(\alpha,x)}W_a + W_a\omega - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b\right) = -\frac{1}{1 + \chi}\mathcal{R}_a$$

and the curvatures are as given in Eqs. (115)–(117) with  $y_\mu = y_\mu^0$ , together with  $\Omega = 0$ .

### 6.5.1. Interpreting $\mathbf{c}_a$

Combining Eq. (117) at constant  $y_\mu$  with Eq. (121), we expand the derivatives and replace the Weyl curvature using the partition of the Riemann tensor, Eq. (86),

$$\mathbf{S}_a = -\frac{1}{1 + \chi}\left(\mathbf{D}_{(\alpha,x)}\mathcal{R}_a + (1 + \chi)y_b^0\mathbf{C}^b_a\right)$$

$$\mathbf{D}_{(\alpha,x)}\mathbf{c}_a - W_b2\Delta_{da}^{bc}\mathbf{c}_c \wedge \mathbf{e}^d = -\frac{1}{1 + \chi}\left(\mathbf{D}_{(\alpha,x)}\mathcal{R}_a - W_b\mathbf{R}^b_a - W_b2\Delta_{da}^{bc}\mathcal{R}_c \wedge \mathbf{e}^d\right)$$

$$-(1 + \chi)\mathbf{D}_{(\alpha,x)}\mathbf{c}_a + 2(1 + \chi)W_b\Delta_{da}^{bc}\mathbf{c}_c \wedge \mathbf{e}^d = \mathbf{D}_{(\alpha,x)}\mathcal{R}_a - W_b\mathbf{R}^b_a - W_b2\Delta_{da}^{bc}\mathcal{R}_c \wedge \mathbf{e}^d$$

and therefore

$$0 = \mathbf{D}_{(\alpha,x)}(\mathcal{R}_a + (1 + \chi)\mathbf{c}_a) - W_b\mathbf{R}^b_a - W_b2\Delta_{da}^{bc}(\mathcal{R}_c + (1 + \chi)\mathbf{c}_c) \wedge \mathbf{e}^d \quad (122)$$

This is exactly the condition for the existence of a conformal transformation to a spacetime satisfying the Einstein equation with matter sources, found in [39], where the matter source  $(1 + \chi)c_{ab}$  is given in terms of the energy-momentum tensor  $T_{ab}$  by

$$(1 + \chi)c_{ab} = -\frac{1}{n - 2}\left(T_{ab} - \frac{1}{n - 1}\eta_{ab}T\right) \quad (123)$$

Therefore, there exists an  $x^\alpha$ -dependent rescaling of the solder form – the Riemannian gauge – such that the co-torsion equation becomes

$$\mathcal{R}_a = -(1 + \chi)\mathbf{c}_a \quad (124)$$

which, in turn, is the Einstein equation with source given by  $T_{ab}$ . Explicitly, substituting the definition of the Schouten tensor from Eq. (85) and the form of  $c_{ab}$  from Eq. (123),

$$\frac{1}{(n - 2)}\left(R_{ab} - \frac{1}{2(n - 1)}\eta_{ab}R\right) = \frac{1}{n - 2}\left(T_{ab} - \frac{1}{n - 1}\eta_{ab}T\right)$$



we substitute for the trace of the energy momentum,  $T = -\frac{1}{2}(n - 2)R$ . Solving for the energy-momentum tensor, we find the usual form of the Einstein equation,

$$R_{ab} - \frac{1}{2}\eta_{ab}R = T_{ab}$$

The  $\mathbf{h}_a = 0$  submanifold is therefore a spacetime satisfying the locally scale-covariant Einstein equation, including phenomenological matter sources. Study is underway to determine whether the energy-momentum of fundamental source fields automatically enter in this way in place of  $c_{ab}$ , or if special couplings to matter are required in the Lagrangian. Notice that in the Riemannian gauge Eq. (122) reduces to

$$W_b \mathbf{R}^b{}_a = 0$$

For a single vector to annihilate the curvature tensor can happen only in the simplest Petrov type spacetimes ( $O$  and  $N$ , and these are already conformally Ricci flat; see [17]) and we conclude that, generically, the gauge transformation that makes the spacetime Riemannian is simultaneously the one which makes the Weyl vector vanish.

### 6.5.2. Contractions of the Bianchi identity for the curvature on the Lagrangian submanifold

In components, the Bianchi identity for the Riemann-Weyl curvature on the spacetime submanifold, Eq. (91) becomes

$$D_a^{(\alpha,x)} \mathcal{R}_{bc} - D_c^{(\alpha,x)} \mathcal{R}_{ba} - W_e C^e{}_{bac} + (1 + \chi) S_{bac} = 0$$

Taking the  $bc$  contraction,

$$D_a^{(\alpha,x)} \mathcal{R} - D_{(\alpha,x)}^c \mathcal{R}_{ca} + (1 + \chi) \eta^{bc} S_{bac} = 0$$

In the Riemannian gauge, and therefore any gauge, the first two terms cancel since

$$D_a^{(\alpha,x)} \mathcal{R} = D_{(\alpha,x)}^c \mathcal{R}_{ca}$$

follows from the Bianchi identity for the Riemannian curvature. Therefore,

$$\eta^{bc} S_{bac} = 0$$

Now, expanding the co-torsion using Eq. (121)

$$\begin{aligned} \mathbf{S}_a &= \mathbf{D}_{(\omega,x)} \mathbf{c}_a \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{c}_a - 2\Delta_{da}^{bc} W_b \mathbf{c}_c \wedge \mathbf{e}^d \end{aligned}$$

In components, this is  $S_{abc} = D_b^{(\alpha,x)} c_{ac} - D_c^{(\alpha,x)} c_{ab}$ , so that

$$0 = \eta^{bc} S_{bac} = c_{;b} - c^a{}_{b;a}$$

and substituting for  $c_{ab}$  from Eq. (123) in

$$\begin{aligned} 0 &= \frac{1}{(n-1)(n-2)} T_{;b} + \frac{1}{n-2} \left( T^a{}_{b;a} - \frac{1}{n-1} T_{;a} \right) \\ &= \frac{1}{n-2} T^a{}_{b;a} \end{aligned}$$

and we have established  $c_{ab}$  to be both symmetric and conserved on spacetime, in agreement with the requirements for the energy momentum tensor. Naturally, this last condition also follows directly from the vanishing divergence of the Einstein tensor.

### 6.5.3. Metric on the Lagrangian submanifold

While we have used the Killing form to motivate the choice of  $h_a{}^\mu(x)$  in the basis form on the cotangent spaces, Eq. (100), the use of the Killing form is not necessary. Indeed, when general relativity is developed as a gauge theory from the Poincaré group, the Killing form vanishes when restricted to the base manifold. Instead, the spacetime metric may be motivated by the spin connection, which is compatible with Lorentzian signature. Ultimately, there is no inherent group structure that requires the choice except this compatibility. Similarly, in the biconformal gauging, we may introduce an  $SO(p, q)$  compatible metric by hand on the Lagrangian submanifolds where the restriction of the Killing form vanishes. For Lorentzian cases, with an original  $SO(n - 1, 1)$  spin connection, it is natural to introduce the corresponding Minkowski metric on each Lagrangian submanifold.

This choice is sufficient for the generic case, but since the restriction of the conformal killing form to biconformal space is non-degenerate, there are alternatives that trace their origin to the conformal group. These have been explored in a variety of ways ([40,31,22,43,32]) but these considerations take us too far beyond the scope of this class of solutions.

## 7. Non-abelian case: $1 + \chi = 0$

We note that the condition  $1 + \chi = 0$  becomes, in terms of the parameters of the original action,

$$0 = n\gamma - ((n - 1)\alpha - \beta)$$

and this does not coincide with any other special conditions.

We now return to the form of the connection, structure equations, and curvatures established at the end of Sec. 5, and set  $1 + \chi = 0$ . The connection forms still take the form,

$$\begin{aligned}\omega^a{}_b &= \alpha^a{}_b - 2\Delta_{db}^{ac} W_c \mathbf{e}^d \\ \mathbf{e}^a &= e_\mu{}^a(x) \mathbf{d}x^\mu \\ \mathbf{f}_a &= h_a{}^\mu(x, y) \mathbf{d}y_\mu + c_{ab}(x, y) \mathbf{e}^b \\ &\equiv \mathbf{h}_a + \mathbf{c}_a \\ \omega &= W_a(x, y) \mathbf{e}^a\end{aligned}$$

The form of the spin connection immediately gives the solution for  $\Omega^a{}_{bcd}$  as the Riemann-Weyl curvature tensor of an integrable Weyl geometry, given by Eq. (84). The field equation is the vanishing of the Weyl-Schouten tensor,

$$\begin{aligned}0 &= \Omega^a{}_{bac} \\ &= \mathcal{R}_{bc} \\ &= \mathcal{R}_{bc} + \phi_{(b;c)} + \phi_b \phi_c - \frac{1}{2} \eta_{bc} \eta^{ad} \phi_a \phi_d\end{aligned}$$

and therefore vanishing Weyl-Ricci tensor. The field equation reduces the full spacetime curvature to the Weyl curvature

$$\Omega^a{}_{bcd} = C^a{}_{bcd}(\alpha)$$

with  $\alpha^a{}_b$  the metric compatible connection.

The dilatation structure equation is now simply

$$\mathbf{d}\omega = 0$$

so to simplify the form of the field equations, we may gauge to  $\omega = 0$ . In the  $W_a = 0$  gauge, the Weyl connection becomes Riemannian,  $\omega^a{}_b = \alpha^a{}_b$  and the curvature is

$$\frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{d}\alpha^a{}_b - \alpha^c{}_b \wedge \alpha^a{}_c = \mathbf{R}^a{}_b \tag{125}$$

where  $\alpha^a{}_b$  is the metric compatible connection,  $\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \alpha^a{}_b$ . The curvature field equation is simply the vacuum Einstein equation. The dilatation and curvature cross terms are now of unit magnitude,

$$\begin{aligned} \Omega^a{}_b{}^c{}_d &= 2\Delta^a{}_b{}^c{}_d \\ \Omega^a{}_b &= \delta^a{}_b \end{aligned}$$

The only remaining field equations are those describing the co-torsion, and the only remaining structure equation is the co-solder equation,

$$\mathbf{d}\mathbf{f}_a = \alpha^b{}_a \wedge \mathbf{f}_b + \frac{1}{2}S_{acd}\mathbf{e}^c \wedge \mathbf{e}^d + S_a{}^c{}_d\mathbf{f}_c \wedge \mathbf{e}^d - \frac{1}{2}c_a{}^{cd}\mathbf{f}_c \wedge \mathbf{f}_d$$

When  $1 + \chi = 0$ , the remaining structure equation on the  $x^\mu = \text{constant}$ ,  $\mathbf{e}^a = 0$  submanifold is given by Eq. (56),

$$\mathbf{d}\mathbf{h}_a = -\frac{1}{2}c_a{}^{cd}\mathbf{h}_c \wedge \mathbf{h}_d \tag{126}$$

which is precisely the Maurer-Cartan equation for a Lie group  $\mathcal{G}$  with structure constants  $c_a{}^{cd}$ .

Here we see the realization of one of the motivations for the use of the conformal group as the starting point for Poincaré gravity, and the subsequent motivation for the biconformal gauging. One anticipates that by starting with the larger conformal group and taking the quotient by the inhomogeneous Weyl group  $\mathcal{C}/\mathcal{LW}$  that the resulting additional symmetry might account for some known or new fundamental interaction beyond gravity. This hope is frustrated by the finding that the additional special conformal gauge fields  $\mathbf{f}_a$  are always auxiliary, and determined by the Ricci tensor [42]. When these auxiliary gauge fields are substituted back into the rest of the model, they serve to turn the Riemann curvature tensor into the Weyl curvature tensor. As a result, though they enforce conformal symmetry, they never provide an additional interaction. As a way to avoid the elimination of  $\mathbf{f}_a$ , we are led to the biconformal gauging,  $\mathcal{C}/\mathcal{W}$ , the idea being that if both  $\mathbf{e}^a$  and  $\mathbf{f}_b$  together are required to span the base manifold, then  $\mathbf{f}_b$  cannot possibly be removed as auxiliary [8,9]. Although considerable subsequent work continues to find  $\mathbf{f}_a$  serving to turn  $\mathbf{R}^a{}_b$  to  $\mathbf{C}^a{}_b$  as in subsection 5.2.3, the emergence of an additional symmetry group is now realized in the  $1 + \chi$  subclass of cases.

The biconformal space comes equipped with the  $SO(p, q)$  pseudo-rotation group of an  $n$ -dimensional space, but these rotations and boosts act on a  $2n$ -dimensional manifold. This is much less than the  $SO(n, n)$  one might expect. Indeed, as seen above, the generic torsion-free solution dictates that half the space is flat, so there is no curvature corresponding to the  $\mathbf{d}y_\mu$  part of the spin connection. The spin connection reduces, essentially, to the metric compatible connection of  $\mathbf{e}^a$ ,

$$\omega^a{}_b = \omega^a{}_{b\mu}(x, y)\mathbf{d}x^\mu + \omega^a{}_b{}^\mu{}_\nu(x, y)\mathbf{d}y_\mu \Rightarrow (\alpha^a{}_{b\mu}(x) + 2\Delta^a{}_{\mu b}y_c)\mathbf{d}x^\mu$$

which is fully expressed on the  $n$ -dimensional, constant  $y_\mu$  Lagrangian submanifolds. This reduction of the spin connection reduces the number of physical fields, but when  $1 + \chi = 0$  the extra translational gauge fields – the co-solder form – make up for it by providing a new connection and field strength: there is necessarily an  $n$ -dimensional Lie group  $\mathcal{G}$  acting at each  $x^\mu$ . Since the  $n$ -dimensions of this group are labeled by an  $SO(p, q)$  index,  $SO(p, q)$  must act on  $\mathcal{G}$ . We show in this section that this internal symmetry group is gauged.

For a particularly pertinent example, suppose we have started with Euclidean 4-space. This does *not* preclude a spacetime Lagrangian submanifold, for it has been shown in [31] that time emerges uniquely from a Euclidean starting point, while [22] shows that this emergence arises purely from properties of the conformal group. With the 4-dim Euclidean starting point, the spin connection has symmetry  $SO(4) = SU(2) \times SU(2)$ . The obvious 4-dimensional subgroup is the electroweak symmetry,  $SU(2) \times U(1)$ . In this case, the symmetry breaking from a left-right symmetric electroweak theory to left-handed representations of  $SU(2)$  is forced by the requirement of a 4-dimensional subgroup. In a spinor representation of the conformal group, the  $P_a$  and  $K^a$  ( $x^\mu$  and  $y_\mu$  submanifolds) are left- and right-handed, respectively. Details of this case are currently under investigation.

It is important to note that although this symmetry  $\mathcal{G}$  is restricted to be acted on by  $SO(p, q)$  the connection gauge field, structure constants and field strength arise completely independently. The biconformal gauging has the additional fields required for this further symmetry.

While a fiber bundle gives us a foliation, the converse is not always the case. The central requirement to have a principal fiber bundle is the existence of a projection from the bundle to the base manifold. We establish this by showing a second involution. Separating the co-solder equation into independent parts,

$$\begin{aligned} \mathbf{d}_{(x)}\mathbf{c}_a &= \alpha_a^b \wedge \mathbf{c}_b + \frac{1}{2}S_{acd}\mathbf{e}^c \wedge \mathbf{e}^d + S_a^c{}_d\mathbf{c}_c \wedge \mathbf{e}^d - \frac{1}{2}c_a^{cd}\mathbf{c}_c \wedge \mathbf{c}_d \\ \mathbf{d}_{(y)}\mathbf{c}_a + \mathbf{d}_{(x)}\mathbf{h}_a &= \alpha_a^b \wedge \mathbf{h}_b + S_a^c{}_d\mathbf{h}_c \wedge \mathbf{e}^d - c_a^{cd}\mathbf{h}_c \wedge \mathbf{c}_d \\ \mathbf{d}_{(y)}\mathbf{h}_a &= -\frac{1}{2}c_a^{cd}\mathbf{h}_c \wedge \mathbf{h}_d \end{aligned}$$

we observe that the exterior  $y_\alpha$ -derivative of  $\mathbf{c}_a$  must be linear in  $\mathbf{d}y_\alpha$ , and so linear in  $\mathbf{h}_a$ . We may therefore write  $\mathbf{d}_{(y)}\mathbf{c}_a = C_a^b\mathbf{h}_b \wedge \mathbf{e}^a$  and solve for  $\mathbf{d}\mathbf{h}_a$  on the full biconformal space,

$$\begin{aligned} \mathbf{d}\mathbf{h}_a &= \mathbf{d}_{(x)}\mathbf{h}_a + \mathbf{d}_{(y)}\mathbf{h}_a = \alpha_a^b \wedge \mathbf{h}_b + S_a^c{}_d\mathbf{h}_c \wedge \mathbf{e}^d - c_a^{cd}\mathbf{h}_c \wedge \mathbf{c}_d - C_a^b\mathbf{h}_b \wedge \mathbf{e}^a \\ &\quad - \frac{1}{2}c_a^{cd}\mathbf{h}_c \wedge \mathbf{h}_d \end{aligned}$$

This shows that  $\mathbf{h}_a$  is in involution. Setting  $\mathbf{h}_a = 0$  constitutes a projection to an  $n$ -dimensional submanifold spanned by  $\mathbf{e}^a$ .

### 7.1. Gauging $\mathcal{G}$

We compare our usual gauging of  $\mathcal{G}$  with the structures already present in the biconformal geometry.

Our usual gauging of a symmetry is to take the Cartan generalization of the Maurer-Cartan equation for  $\mathcal{G}$ , eq. (126). For this we replace the Maurer-Cartan connection  $\mathbf{h}_a$  with a general connection, leading to the introduction of a field strength. Taking  $\mathbf{A}_a$  to be the generalization of  $\mathbf{h}_a$ , the Maurer-Cartan equation becomes the Cartan equation,

$$d\mathbf{A}_a = -\frac{1}{2}c_a^{cd}\mathbf{A}_c \wedge \mathbf{A}_d + \mathbf{F}_a \tag{127}$$

where the field strength  $\mathbf{F}_a$  is required to be horizontal and the equation integrable. Horizontality demands

$$\mathbf{F}_a = \frac{1}{2}F_{acd}\mathbf{e}^c \wedge \mathbf{e}^d \tag{128}$$

while integrability requires

$$\begin{aligned} 0 &\equiv d^2\mathbf{A}_a \\ &= c_a^{cd}\mathbf{A}_c \wedge d\mathbf{A}_d + d\mathbf{F}_a \\ &= -\frac{1}{2}c_d^{[ef}c_a^{c]d}\mathbf{A}_c \wedge \mathbf{A}_e \wedge \mathbf{A}_f + c_a^{cd}\mathbf{A}_c \wedge \mathbf{F}_d + d\mathbf{F}_a \\ &= \mathbf{D}_{(\mathcal{G})}\mathbf{F}_a \end{aligned}$$

since  $c_d^{[ef}c_a^{c]d} = 0$  by the Jacobi identity for  $\mathcal{G}$ .

Within the biconformal solution, we interpret the co-solder forms  $\mathbf{f}_a$  as these generalized potentials  $\mathbf{A}_a$  for the  $\mathcal{G}$ -connection. The full structure equation for  $\mathbf{f}_a$ , however, is not exactly what we expect for a typical gauging. Instead we have

$$d\mathbf{f}_a = \omega_a^b \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \omega + \frac{1}{2}S_{acd}\mathbf{e}^c \wedge \mathbf{e}^d + S_a^c{}_d\mathbf{f}_c \wedge \mathbf{e}^d - \frac{1}{2}c_a^{cd}\mathbf{f}_c \wedge \mathbf{f}_d$$

The situation appears to be similar to what Kibble encountered in writing general relativity as a gauge theory of the Poincaré group [2]. Kibble introduced Poincaré fibers over spacetime, then “soldered” the translational gauge fields of the fiber symmetry to the cotangent basis of the bundle. This identification avoided double counting the translations. With the quotient method, such identification is no longer needed since the quotient of the Poincaré group by the Lorentz group automatically changes the translational symmetry into the base manifold.

Here, we already have an  $SO(p, q)$  symmetry on the fibers and find that the base manifold has a similar, but restricted symmetry. It is this latter, emergent symmetry we would like to use. The present situation differs from the Poincaré case since it is the connection and not the frame field that is doubled, with  $\omega_a^d \wedge \mathbf{f}_d$  and  $(-\frac{1}{2}c_a^{cd}\mathbf{f}_c) \wedge \mathbf{f}_d$  both acting on the same index of  $\mathbf{f}_d$ . We cannot simply solder the two together because they produce covariance with respect to different symmetries. Moreover, we still need the original symmetry to act on the remaining gauge fields in the usual way. There are a number of possible resolutions to the difficulty. First, it might be possible to build the gauging of the co-solder form into the original quotient. However, though closer examination may reveal a natural way to do this, it would seem to lose the appealing feature of an emergent internal symmetry. A second approach is suggested by [22,23,43], in which an initially  $SO(n)$  connection is written as a Lorentz connection plus additional terms, which then introduce physical fields. If a similar technique is applied here, perhaps along with the transformation of [22], it is possible that the restriction of the connection can occur directly.

A third approach is to keep both connections but keep careful track of which fields transform under which symmetry. This actually causes no problem for additional fields we might wish to introduce, since these will enter as representations of either the  $SO(p, q)$  transformations, the  $\mathcal{G}$  transformations, or both, leaving no ambiguity about their transformations. The only potential conflict arises with  $\mathbf{f}_a \Leftrightarrow \mathbf{f}_A$  itself. Thinking of  $\mathcal{G}$  as a subgroup of  $SO(p, q)$ , we must wonder whether a full  $SO(p, q)$  transformation would introduce sensible but distinct copies of the gauge potential. In the case of electroweak symmetry, we could construct a theory with  $SO(4)$  breaking

to  $SU(2) \times U(1)$  on the fibers while the full  $SO(4)$  is written following [22], as a Lorentz connection plus additional scalar field and cosmological constant. It is not clear whether the resulting weak fields would violate the causal structure of the Lorentz sector.

However, these conjectures will require—and are the subject of—further study.

For the present we content ourselves with the following. First, we satisfy the final field equation for the cross-term of the co-torsion,

$$\Delta_{sb}^{ar} S_c^b{}_a = \Delta_{sc}^{ar} S_a^b{}_b$$

by the sufficient condition  $S_c^b{}_a = 0$ . The only remaining field equation is  $S_a^{ba} = -c_a^{ba} = 0$ , the tracelessness of the structure constants. This leads to unit determinant for elements of  $\mathcal{G}$ .

Having solved the field equations, we identify  $\mathbf{f}_a$  as the gauge field  $\mathbf{A}_a$ , modifying the indices to make it clear which gauge group applies. For fields that transform under  $SO(p, q)$ , we retain the lower case Latin indices, while for fields that transform under  $\mathcal{G}$  we replace the relevant indices by upper case Latin. Both sets range from 1 to  $n$ . The connection becomes a pair,

$$\alpha^a{}_b \Rightarrow \begin{pmatrix} \alpha^a{}_b & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}c_a{}^{cd}\mathbf{f}_c \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}c_B{}^{CA}\mathbf{f}_C \end{pmatrix}$$

so in the new notation,  $\alpha^A{}_B = 0$  and  $\frac{1}{2}c_a{}^{cd}\mathbf{f}_c = 0$ .

The curvature and solder form equations are unchanged, but the co-torsion equation becomes

$$d\mathbf{f}_A = \alpha^B{}_A \wedge \mathbf{f}_B + \frac{1}{2}S_{Acd}\mathbf{e}^c \wedge \mathbf{e}^d - \frac{1}{2}c_B{}^{CA}\mathbf{f}_C \wedge \mathbf{f}_D$$

or, since  $\alpha^B{}_A = 0$ ,

$$\frac{1}{2}S_{Acd}\mathbf{e}^c \wedge \mathbf{e}^d = d\mathbf{f}_A + \frac{1}{2}c_B{}^{CA}\mathbf{f}_C \wedge \mathbf{f}_D \tag{129}$$

This exactly reproduces the form of a Yang-Mills field, Eqs. (127) and (128).

Consistency of this restriction is now immediate because in the full set of structure equations,

$$d\alpha^a{}_b = \alpha^c{}_b \wedge \alpha^a{}_c + \frac{1}{2}R^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d$$

$$d\mathbf{e}^a = \mathbf{e}^b \wedge \alpha^a{}_b$$

$$d\mathbf{f}_A + \frac{1}{2}c_A{}^{CD}\mathbf{f}_C \wedge \mathbf{f}_D = \frac{1}{2}S_{Acd}\mathbf{e}^c \wedge \mathbf{e}^d$$

the internal symmetry  $\mathcal{G}$  has fully decoupled from the spacetime geometry.

### 7.1.1. Metric on the submanifolds in the nonabelian case

The nonabelian case still allows us to simply choose the spacetime metric as is done in general relativity and was used in the generic case above. For the group submanifold, it is natural to choose the Killing form of  $\mathcal{G}$  if it is nondegenerate. While this assignment is certainly possible for semisimple  $\mathcal{G}$ , the details depend on the particular group and will be discussed elsewhere. As with the generic case, the considerations of [40,31,22,43,32] may be relevant.

### 7.2. Remaining issues

While we have arrived at a satisfactory separation of a new internal symmetry, we still lack both the field equation for  $\mathbf{S}_A$  and the contribution of this additional field to the Einstein equation. We consider three possible resolutions:

1. The absence of a source for gravity arising from  $\mathcal{G}$  is not surprising given the restriction of the action to linear curvature terms. Including up to quadratic curvature terms in the original action could provide both field equation and gravitational coupling.
2. Allow nonvanishing cross-term to the torsion. This preserves the involution of the solder form and avoids spacetime torsion. The cross-term gives extrinsic curvature to the embedding of the submanifolds into the full biconformal space, and allows curvature of the group manifold as in [50]. The cross-term of the torsion, driven by the internal symmetry, then enters the spacetime curvature quadratically and might supply the required gravitational source.
3. The gravitational instanton has been shown to introduce both field equations and gravitational source in the usual form [51,52].

Leaving these considerations to further work, we make only the following observation. It would be natural to introduce a quadratic term such as  $\int \mathbf{T}^a \wedge * \mathbf{S}_a$  into the action. With vanishing torsion, only the variation  $\delta \mathbf{T}^a = \mathbf{D} \delta \mathbf{e}^a$  yields nonvanishing contributions

$$\delta_e \int \mathbf{T}^a \wedge * \mathbf{S}_a = \int \delta \mathbf{e}^a \wedge \mathbf{D} * \mathbf{S}_a + \text{surface term} + \text{terms proportional to torsion}$$

This term leads to a divergence of  $\mathbf{S}_a$  in the curvature equation, but it because of the presence of  $\mathbf{f}_a$  in the volume form also gives terms quadratic in the components of  $\mathbf{S}_a$ . However, with vanishing co-torsion cross-term the quadratic terms added to the spacetime curvature field equations involve only products of the spacetime and the momentum components,

$$\begin{aligned} \delta_{e2} S &\sim \frac{n(n-2)}{12} (S_{acd} S_b{}^{cd} + S_{bcd} S_a{}^{cd} + S_{abc} S_d{}^{cd} + S_{abc} S_d{}^{cd}) \\ &= -\frac{n(n-2)}{12} (S_{acd} c_b{}^{cd} + S_{bcd} c_a{}^{cd}) \end{aligned}$$

and these are not in the form of energy momentum tensor. This suggests that a combination of this quadratic term and nonvanishing cross-term for torsion may provide a solution.

## 8. Conclusions

We have shown how general relativity emerges from the torsion-free solutions to biconformal gravity. The derivations involve field equation driven dimensional reduction and may therefore have relevance to dimensional reduction of twistor string, or the reduction of Drinfeld doubles.

### 8.1. Results in biconformal gravity

We began with the conformal group  $\mathcal{C}_{p,q}$  of a compactified space with  $(p, q)$ -signature metric in  $n = p + q$  dimensions. The quotient of  $\mathcal{C}_{p,q}$  by its homogeneous Weyl subgroup  $\mathcal{W}$  gives a  $2n$ -dimensional Kahler manifold with local  $SO(p, q)$  and scale invariance. Generalizing this local structure leads to a curved geometry characterized by  $SO(p, q)$  curvature, torsion, co-torsion

and dilatational curvature corresponding to the generators of the conformal group. Throughout,  $SO(p, q)$  may be replaced by  $Spin(p, q)$  when a spinor representation is desired. This biconformal space admits a scale invariant action functional linear in the Cartan curvatures, Eq. (1). Varying the action yields a gravity theory in  $2n$  dimensions. All models with  $(n - 2)\alpha - 2\beta$  nonzero are considered. The special case when  $((n - 2)\alpha - 2\beta) = 0$  may include non-integrable Weyl geometries and, likely being unphysical, these are not considered in detail.

We established the following distinct results for this model.

1. **Triviality with vanishing torsion and co-torsion.** If, in addition to vanishing torsion, the co-torsion (the field strength of special conformal transformations) is taken to vanish, the biconformal space takes a trivial form, with the only nonvanishing components of the  $SO(p, q)$  and dilatational curvatures being constant cross-terms.
2. **Foliation by a Lie subgroup.** We prove that half of the biconformal space is foliated by an  $n$ -dimensional Lie group  $\mathcal{G}$  with structure constants lying in a representation of  $SO(p, q)$ . This result follows from the involution guaranteed by vanishing torsion, together with the field equations. When  $\alpha, \beta$  and  $\gamma$  are chosen such that  $\chi = \frac{1}{n-1}(1 + \frac{n^2\gamma}{(n-1)\alpha-\beta}) = -1$ ,  $\mathcal{G}$  may be non-abelian, otherwise it is abelian.
3. **Generic solution.** The generic solution assumes only that  $1 + \chi$  is nonzero, together with vanishing torsion,  $\mathbf{T}^a = 0$ . The field equations reduce the only nontrivial dependence of all the remaining curvatures – the  $SO(p, q)$  curvature, co-torsion, and dilatation – from  $2n$  independent variables  $(x^\alpha, y_\beta)$  down to  $n$  variables  $(x^\alpha)$  and reduce the number and form of the independent components. Explicitly, the form and dependency of each curvature  $\Omega^A \in \{\Omega^a{}_b, \mathbf{S}_a, \Omega\}$  begins as three distinct tensors dependent on  $2n$  coordinates,

$$\Omega^A = \frac{1}{2}\Omega^A{}_{cd}(x^\alpha, y_\beta) \mathbf{e}^c \wedge \mathbf{e}^d + \Omega^{Ac}{}_d(x^\alpha, y_\beta) \mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}\Omega^{Ac d}(x^\alpha, y_\beta) \mathbf{f}_c \wedge \mathbf{f}_d$$

and after application of the field equations, each is reduced as follows:

$$\begin{aligned} \Omega^a{}_b &= \frac{1}{2}C^a{}_{bcd}(x)\mathbf{e}^c \wedge \mathbf{e}^d + 2(1 + \chi)\Delta_{cb}^{ac}\mathbf{f}_c \wedge \mathbf{e}^d \\ \mathbf{S}_a &= -\frac{1}{1 + \chi}\left(D_c^{(x)}\mathcal{R}_{ad}(x) - \frac{1}{2}(1 + \chi)y_b C^b{}_{acd}(x)\right)\mathbf{e}^c \wedge \mathbf{e}^d \\ \Omega &= \chi \mathbf{e}^c \wedge \mathbf{f}_c \end{aligned}$$

where  $\chi = \frac{1}{n-1}(1 + \frac{n^2\gamma}{(n-1)\alpha-\beta})$ . The resultant curvatures,  $\frac{1}{2}C^b{}_{acd}(x)\mathbf{e}^c \wedge \mathbf{e}^d$  and  $\mathcal{R}_{ab}(x)$  are the Weyl and Schouten parts of the Riemann curvature tensor computed from the connection compatible with the  $n$ -dimensional solder form,  $\mathbf{e}^a(x)$ . Each of the 2-forms

$$\begin{aligned} \mathbf{d}\omega &= (1 + \chi)\mathbf{e}^c \wedge \mathbf{f}_c \\ \Omega &= \chi \mathbf{e}^a \wedge \mathbf{h}_a = \chi \mathbf{d}x^\alpha \wedge \mathbf{d}y_\alpha \end{aligned}$$

is closed and non-degenerate, hence symplectic on the full biconformal space. The basis forms  $\mathbf{e}^a, \mathbf{f}_b$  are given by

$$\begin{aligned} \mathbf{e}^a &= e_\alpha{}^a(x)\mathbf{d}x^\alpha \\ \mathbf{f}_a &= e_a{}^\alpha \mathbf{d}y_\alpha - \left(\frac{1}{1 + \chi}\mathcal{R}_{ab} - D_b^{(\alpha,x)}y_a + (1 + \chi)\left(y_a y_b - \frac{1}{2}\eta_{ab}\eta^{cd}y_c y_d\right)\right)\mathbf{e}^b \\ &= \mathbf{h}_a - \frac{1}{1 + \chi}\mathcal{R}_a \end{aligned}$$



and the manifest involution of  $\mathbf{h}_a = e_a^\alpha \mathbf{d}y_\alpha$  shows that setting  $y_\alpha = \text{constant}$  gives a Lagrangian submanifold for spacetime. Conversely, setting  $x^\alpha = \text{constant}$  gives conjugate Lagrangian submanifolds, each the leaf of a foliation by flat manifolds. The entire  $2n$ -dimensional biconformal spaces is therefore interpreted as the cotangent bundle of spacetime. The spin connection and Weyl vector are given by

$$\begin{aligned} \omega_b^a &= \alpha_b^a(x) + 2(1 + \chi)\Delta_{db}^{ac}y_c \mathbf{e}^d \\ W_\alpha &= -(1 + \chi)y_\alpha \end{aligned}$$

where  $\alpha_b^a(x)$  is the metric compatible connection,  $\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \alpha_b^a(x)$ . Here the orthonormal frame field and spin connection pair  $(\mathbf{e}^a, \alpha_b^a)$  is equivalent to the metric and Christoffel connection pair,  $(g_{\mu\nu}, \Gamma_{\mu\nu}^\alpha)$ .

The sole remaining constraint on the system is the locally scale covariant Einstein equation with source  $c_{ab}$ , Eq. (122):

$$0 = \mathbf{D}_{(\alpha,x)}(\mathcal{R}_\alpha + (1 + \chi)\mathbf{c}_\alpha) - W_b \mathbf{R}^b_a - W_b 2\Delta_{da}^{bc}(\mathcal{R}_c + (1 + \chi)\mathbf{c}_c) \wedge \mathbf{e}^d$$

which has been shown in [39] to be the condition for the existence of a conformal transformation to the sourced Einstein equation when the source  $\mathbf{c}_a = c_{ab}\mathbf{e}^b$  is written as

$$c_{ab} = -\frac{1}{n-2} \frac{1}{1+\chi} \left( T_{ab} - \frac{1}{n-1} \eta_{ab} T \right)$$

We showed from the properties of  $c_{ab}$  that  $T_{ab}$  is symmetric and divergence free, and in the Riemannian gauge (in which the Weyl vector vanishes),

$$R_{ab} - \frac{1}{2} \eta_{ab} R = T_{ab}$$

We conclude that the generic case describes the locally scale covariant  $n$ -dimensional Einstein equation sourced by a symmetric, divergence free tensor and formulated on the cotangent bundle of spacetime. The reduction to  $n$ -dimensions is accomplished using only the field equations with vanishing torsion.

**4. The non-abelian cases.** When  $\alpha, \beta$  and  $\gamma$  are chosen such that  $\chi = \frac{1}{n-1} (1 + \frac{n^2\gamma}{(n-1)\alpha-\beta}) = -1$ ,  $\mathcal{G}$  may be non-abelian, and there are substantial differences in the Cartan structure equations. For these cases, we again showed the reduction to dependence on the spacetime solder form,  $\mathbf{e}^a(x)$ , but now the final forms of the curvature and dilatation are

$$\begin{aligned} \Omega_b^a &= \frac{1}{2} C^a_{bcd}(\alpha) \mathbf{e}^c \wedge \mathbf{e}^d + 2\Delta_{db}^{ac} \mathbf{f}_c \wedge \mathbf{e}^d \\ \Omega &= \mathbf{e}^a \wedge \mathbf{f}_a \end{aligned}$$

The curvature is subject to the scale-covariant vacuum Einstein equation,

$$\mathcal{R}_{ec} + D_c^{(x)}\phi_e + \phi_e\phi_c - \frac{1}{2}\eta_{ec}\phi^d\phi_d = 0$$

For the co-torsion, Lorentz transformations are suppressed while the appearance of the  $\mathcal{G}$ -connection is automatic. This leads to the co-solder form becoming the  $\mathcal{G}$  gauge field and the spacetime co-torsion becoming the usual Yang-Mills field strength,

$$\frac{1}{2} S_{Acd} \mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{D}_{(W)} \mathbf{f}_A + \frac{1}{2} c_A^{CD} \mathbf{f}_C \wedge \mathbf{f}_D$$

where  $\mathbf{D}_{(W)}\mathbf{f}_A = \mathbf{d}\mathbf{f}_A - \mathbf{f}_A \wedge \boldsymbol{\omega}$  is the dilatation-covariant derivative. The biconformal space becomes a principal  $\mathcal{G}$ -bundle over spacetime.

Effectively, the total principal bundle has homogeneous Weyl and  $\mathcal{G}$  symmetry,  $\mathcal{W} \times \mathcal{G}$ . This is *not* what occurs if the original quotient is of the conformal group by inhomogeneous Weyl,  $C_{(p,q)}/\mathcal{I}\mathcal{W}$ , which essentially gives Poincaré fibers over spacetime [42,17]. The emergence of non-abelian symmetry from the degrees of freedom of the special conformal gauge fields of the conformal group is a new result. In 4-dimensions, with  $SO(p, q) = SO(4)$ , the maximal  $\mathcal{G}$  is the electroweak symmetry with necessary parity violation.

## 8.2. A note on the metric and signature change

To formulate general relativity as a gauge theory using the Cartan techniques, we take the quotient of the Poincaré group by the Lorentz group. The only guidance as to the metric is the presence of the Lorentzian connection, which will leave the usual orthonormal metric,  $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$  invariant. We then introduce the metric by hand using the orthonormal frame field,  $g_{\alpha\beta} = e_a^{\alpha} e_b^{\beta} \eta_{ab}$ . The situation is different in biconformal space, where there are natural metric structures present. Of course, there is the  $SO(p, q)$  connection, making it possible to introduce a  $(p, q)$  signature metric by hand, just as we do in general relativity. However, the Killing form of the conformal group has a non-degenerate restriction to biconformal space, and we may use this instead. The resulting metric on spacetime depends not only on the original signature, but also on what submanifold is taken as spacetime.

Using the restricted Killing form, the metric is

$$K_{AB} = \begin{pmatrix} 0 & \delta_b^a \\ \delta_a^b & 0 \end{pmatrix}$$

and the restriction to the  $\mathbf{e}^a$  or  $\mathbf{f}_b$  subspaces has vanishing restriction. It is shown in [31], however, that if we seek orthogonal Lagrangian submanifolds on which the Killing metric is non-degenerate, there are limited possibilities, with initial spaces of signature  $(n, 0)$ ,  $(\frac{n}{2}, \frac{n}{2})$  or  $(0, n)$  being the only consistent starting points, and the two Euclidean cases leading uniquely to Lorentzian signature,  $(n - 1, 1)$  or  $(1, n - 1)$ . This development of a time direction from an initially Euclidean space is appealing.

There are other possibilities. If we drop the orthogonality requirement from the theorem of [31], it becomes possible to have different signature on the two Lagrangian submanifolds. This too has its advantages, as we might arrange for Lorentzian signature on one submanifold and Euclidean on the other, enabling an additional compact internal symmetry.

Some of these avenues have been explored. The Euclidean starting point leading to Lorentzian signature on both Lagrangian submanifolds is studied in [22], in which all the results are seen to depend only on structures inherent in the conformal group. In [32] connections of both types are introduced, and some possibilities are explored in [44]. Work is currently underway to examine a  $4 + 4$  dimensional model with mixed signatures, to take advantage of the potential graviweak theory.

There is still another metric possibility, because the metric compatible with the Kähler structure is different, having signature  $(2p, 2q)$  while the Killing metric has signature  $(n, n)$ .

For the present results, it seems best to simply impose the metric we choose. If we let the original  $SO(p, q)$  be Lorentzian,  $SO(n - 1, 1)$  then the natural choice for spacetime is Lorentzian (but see [22]).

### 8.3. Discussion

Biconformal spaces with appropriate signature give rise to general relativity, generically formulated on the cotangent bundle of spacetime. In a subclass of cases there may be an additional non-abelian internal symmetry. While this internal symmetry ultimately arises from the special conformal transformations, no previous gauging of the conformal group has shown the direct possibility of a non-abelian symmetry. This opens the possibility of a graviweak unification, which, while still requiring additional structure for the strong force, holds out the hope of a deeper understanding of parity violation and the breaking of a left-right symmetric  $SU(2) \times SU(2)$  model. This possibility is under current investigation.

As described in the introduction, these gravity models may provide new insights into string theory. The existence of a conformal route to general relativity, as opposed to fourth-order Weyl gravity, allows for the consistent use of twistor string models. In addition, the doubled dimension makes possible a compactification from 10-dimensional superstring theory to an 8-dimensional biconformal space with an immediate interpretation as 4-dimensional general relativity. In this way, the myriad 6-dimensional compact spaces are avoided, to be replaced by compactification of only 2-dimensions and possibly uniquely to a torus if other structures are to be maintained.

Finally, this reduction of the biconformal gauging shares many features with Drinfeld doubles. The match between the Killing form and the symmetric form of the Drinfeld product may suggest systematic ways of reducing the doubles to their half-dimension.

### Acknowledgements

The author thanks Jeffrey Shafiq Hazboun for his careful reading and discussion of these results.

### Appendix A. Compactification of spacetimes

Consider a flat space  $\mathcal{S}$  of signature  $(p, q)$ , i.e., with indefinite metric

$$\eta_{ab} = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q)$$

This space is noncompact due to curves heading off to infinite distance. We complete the space by appending an inverse to every vector from the origin.

Let  $N$  be the set of null vectors from the origin,  $N = \{x^\alpha | x^\beta x_\beta = 0, x^\alpha \in \mathcal{V}\}$ , and  $\mathcal{CN}$  its complement,  $\mathcal{CN} = \{x^\alpha | x^\beta x_\beta \neq 0, x^\alpha \in \mathcal{V}\}$ .

For all points in  $\mathcal{CN}$ , we consider the set,

$$\mathcal{W}_0 = \left\{ w^\alpha \mid w^\alpha = -\frac{x^\alpha}{|x|^2} \right\}$$

where  $|x|^2 = \eta_{\alpha\beta} x^\alpha x^\beta$ . Notice that  $\mathcal{W}_0$  contains no null vectors since this would require  $|x|^2 = 0$ . Inverting the transformation we have

$$x^\alpha = -\frac{w^\alpha}{|w|^2}$$

so we see the mapping between  $x^\alpha$  in  $\mathcal{CN}$  and  $w^\alpha$  is a bijection.

We extend  $\mathcal{W}_0$  by taking the union with a new set  $\hat{\mathcal{W}}$  of points  $\hat{w}^\alpha$  satisfying

$$\hat{\mathcal{W}} = \{ w^\alpha \mid \eta_{\alpha\beta} \hat{w}^\alpha \hat{w}^\beta = 0 \}$$

Clearly  $\hat{\mathcal{W}} \cap \mathcal{W}_0 = \emptyset$ . We suggest that

$$\mathcal{W} \equiv \hat{\mathcal{W}} \cup \mathcal{W}_0$$

provides a compactification of the space.

If the signature is Euclidean,  $(p, q) = (n, 0)$ , then  $N$  consists of the origin alone, and  $\hat{\mathcal{W}}$  is the 1-point compactification of  $R^n$ .

More generally, a detailed proof of compactness must rely on the specification of a topology on indefinite spaces. For general spacetimes this relies on introducing ideal points [45–47]. While such methods should meet no obstruction in the flat, nonsingular spaces considered here, the definition of the conformal group requires only the existence of inverses. This much has already been accomplished with the definition of  $\mathcal{W}$ . We therefore content ourselves by defining a suitable extension and indicating compactness by studying the resulting extensions of spacetime curves.

First, consider a straight line that runs off to infinity in spacetime at half the speed of light,

$$\mathcal{C}(\lambda) = \left\{ x^\alpha = \left( \lambda, \frac{\lambda}{2}, 0, 0 \right) \right\}$$

This curve starts at the origin and reaches no endpoint in the original space, but in the compactification we may continue the curve to some finite value of  $\lambda = \lambda_0$  then change to  $w^\alpha$  coordinates,

$$\mathcal{C}(\lambda_0) = \left\{ x^\alpha = \left( \lambda_0, \frac{\lambda_0}{2}, 0, 0 \right) \right\} = \left\{ w^\alpha = \frac{4}{3} \left( \frac{1}{\lambda_0}, \frac{1}{2\lambda_0}, 0, 0 \right) \right\}$$

Define a new parameter,  $\xi = \frac{1}{\lambda}$  and continue the curve

$$\mathcal{C}(\xi) = \left\{ w^\alpha = \frac{4}{3} \left( \xi, \frac{1}{2}\xi, 0, 0 \right) \right\}$$

We continue the curve through  $w^\alpha(0)$  and continue  $\xi$  to negative values. At some finite value,  $\xi = -\xi_0 < 0$ , we are at the point

$$w^\alpha = -\frac{4}{3} \left( \xi_0, \frac{1}{2}\xi_0, 0, 0 \right)$$

corresponding to

$$\begin{aligned} x^\alpha &= -\left( \frac{1}{\left(\frac{16}{9}\right)\left(-\xi_0^2 + \frac{1}{4}\xi_0^2\right)} \right) - \frac{4}{3} \left( \xi_0, \frac{1}{2}\xi_0, 0, 0 \right) \\ &= -\frac{4}{3} \left( \frac{1}{\frac{4}{3}\xi_0^2} \right) \left( \xi_0, \frac{1}{2}\xi_0, 0, 0 \right) \\ &= -\left( \frac{1}{\xi_0}, \frac{1}{2}\frac{1}{\xi_0}, 0, 0 \right) \end{aligned}$$

This is a point in the past light cone of  $x^\alpha = 0$ . Letting  $\lambda' = \frac{1}{\xi}$ , we continue the path back to the origin as

$$x^\alpha = -\left(\lambda', \frac{1}{2}\lambda', 0, 0\right)$$

and the curve is closed.

As an example of a curve reaching null infinity (hence some nonzero null vector  $|w|^2 = 0$ ), consider the accelerating curve

$$\mathcal{C}(\lambda) = \{x^\alpha = (a \cosh \lambda, a \sinh \lambda, 0, 0)\}$$

with timelike norm

$$|x|^2 = -a^2$$

For this we choose a new parameter,

$$\sigma = \tan^{-1}(\sinh \lambda)$$

which approaches  $\frac{\pi}{2}$  as  $\lambda \rightarrow \infty$ . The curve takes the form

$$x^\alpha = a(\sqrt{1 + \tan^2 \sigma}, \tan \sigma, 0, 0)$$

At a value  $\sigma = \frac{\pi}{2}$  we now have a null vector. We need to reparameterize the curve before this happens, so set

$$w^\alpha = \frac{a}{\tan \sigma}(\sqrt{1 + \tan^2 \sigma}, \tan \sigma, 0, 0)$$

Now as  $\sigma$  approaches  $\frac{\pi}{2}$ ,  $w^\alpha$  is given by

$$\begin{aligned} w^\alpha(\sigma) &= a\left(\sqrt{\frac{1}{\tan^2 \sigma} + 1}, 1, 0, 0\right) \\ &\rightarrow a(1, 1, 0, 0) \end{aligned}$$

Let the parameterization of the curve in  $w^\alpha$  coordinates be  $\xi = \frac{1}{\tan \sigma}$  so that

$$w^\alpha(\xi) = a(\sqrt{1 + \xi^2}, 1, 0, 0)$$

As  $\xi \rightarrow 0$  this approaches a null vector,  $w^\alpha = (a, a, 0, 0)$ , so we may now identify the limit  $\sigma \rightarrow \infty$  with the null vector  $w^\alpha = (a, a, 0, 0)$ . This vector is in the space, and we may again continue the curve to negative  $\xi$ .

### Appendix B. Variation of the spin connection

Here we give details of the variation of the spin connection, since some of the steps are novel. Because many of the expressions are long, we introduce some notational conventions to make expressions more compact and transparent. Specifically, since all differential forms are rendered in boldface, there is no loss of information if we assume wedge products between all adjacent forms, dropping the explicit wedge. We further define a multi-index form,

$$\omega_{c\dots d} \equiv \omega_c \omega_{c_1} \dots \omega_d \equiv \omega_c \wedge \omega_{c_1} \wedge \dots \wedge \omega_d$$

for any number of basis 1-forms  $\omega_c$ . It is always possible to deduce the correct number of indices from the Levi-Civita tensor.

The spin connection occurs only in the  $SO(p, q)$  curvature,  $\Omega^a_b$ , so the spin connection variation affects only the  $\alpha$  term of the action,

$$\begin{aligned}
\delta\omega_b^a S &= \delta\omega_b^a \int (\alpha\Omega_b^a + \beta\delta_b^a\Omega + \gamma\mathbf{e}^a\mathbf{f}_b)\mathbf{e}^{e\dots f}\mathbf{f}_{c\dots d}e^{bc\dots d}{}_{ae\dots f} \\
&= \alpha \int \delta\omega_b^a (\mathbf{d}\omega_b^a - \omega_b^c\omega_c^a - 2\Delta_{cb}^{ad}\mathbf{f}_d\mathbf{e}^c)\mathbf{e}^{e\dots f}\mathbf{f}_{c\dots d}e^{bc\dots d}{}_{ae\dots f} \\
&= \alpha \int (\mathbf{d}\delta\omega_b^a - \delta\omega_b^c\omega_c^a - \omega_b^c\delta\omega_c^a)\mathbf{e}^{e\dots f}\mathbf{f}_{c\dots d}e^{bc\dots d}{}_{ae\dots f} \\
&= \alpha \int \mathbf{D}(\delta\omega_b^a)\mathbf{e}^{e\dots f}\mathbf{f}_{c\dots d}e^{bc\dots d}{}_{ae\dots f}
\end{aligned}$$

Integrating the covariant derivative by parts and discarding the surface term,

$$\begin{aligned}
0 &= \alpha \int \delta\omega_b^a \mathbf{D}(\mathbf{e}^{e\dots f}\mathbf{f}_{c\dots d}e^{bc\dots d}{}_{ae\dots f}) \\
0 &= \alpha \int \delta\omega_b^a (\mathbf{T}^e\mathbf{e}^{e_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f} + (-1)^{n-1}\mathbf{e}^{ee_1\dots f}\mathbf{S}_c\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}) \\
&= \alpha \int (\delta A_{bc}^a\mathbf{e}^c + \delta B_{b\ c}^a\mathbf{f}_c)(\mathbf{T}^e\mathbf{e}^{e_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f} \\
&\quad + (-1)^{n-1}\mathbf{S}_c\mathbf{e}^{ee_1\dots f}\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f})
\end{aligned}$$

The covariant derivatives of the basis forms divide naturally into two tensors, the torsion,  $\mathbf{T}^a = \mathbf{D}\mathbf{e}^a$  and the co-torsion,  $\mathbf{S}_a = \mathbf{D}\mathbf{f}_a$ . We show in the main text that if both of these vanish, the solution must be trivial (i.e., non-gravitating) so it is important to realize when considering torsion-free solutions that the co-torsion  $\mathbf{S}_a$  remains non-zero.

The variation must preserve the antisymmetry,  $\eta_{bc}\eta^{ad}\omega_c^d = -\omega_b^a$ , of the spin connection, so,  $\delta\omega_b^a\Delta_{sb}^{ar} = \delta\omega_s^r$ . Therefore, the coefficients of the variation,  $\delta A_{bc}^a$  and  $\delta B_{b\ c}^a$ , are antisymmetric on the first pair of indices. As a result, only the antisymmetric part of the rest of the integrand vanishes, so we require the projection operator,  $\Delta_{db}^{ac} \equiv \frac{1}{2}(\delta_d^a\delta_b^c - \eta^{ac}\eta_{bd}) = \frac{1}{2}\eta^{an}\eta_{md}(\delta_n^m\delta_b^c - \delta_n^c\delta_b^m)$  with  $\Delta_{sb}^{ar}\Delta_{nr}^{sm} = \Delta_{nb}^{am}$ . This acts to antisymmetrize  $\binom{1}{1}$  tensors,  $\Delta_{db}^{ac}T_c^d = \frac{1}{2}\eta^{an}(T_{nb} - T_{bn})$ .

With  $\delta A_{bc}^a$  and  $\delta B_{b\ c}^a$  independent we find two equations,

$$\begin{aligned}
0 &= \alpha\Delta_{sb}^{ar}\mathbf{e}^m(\mathbf{T}^e\mathbf{e}^{e_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f} + (-1)^{n-1}\mathbf{e}^{ee_1\dots f}\mathbf{S}_c\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}) \\
0 &= \alpha\Delta_{sb}^{ar}\mathbf{f}_m(\mathbf{T}^e\mathbf{e}^{e_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f} + (-1)^{n-1}\mathbf{e}^{ee_1\dots f}\mathbf{S}_c\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f})
\end{aligned}$$

Next, we substitute the expansion of the torsion Eq. (20) and use Eq. (13) to write the co-torsion, finding the one term of each which completes the volume form. Rearranging the basis forms in standard order, we use the volume form replacement of eq. (22):

$$\begin{aligned}
0 &= \alpha\Delta_{sb}^{ar}(\mathbf{T}^e\mathbf{e}^{me_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f} + (-1)^{n-1}\mathbf{e}^{mee_1\dots f}\mathbf{S}_c\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}) \\
&= \alpha\Delta_{sb}^{ar}\left(T^{eu}{}_{\nu}\mathbf{f}_u\mathbf{e}^{\nu}\mathbf{e}^{me_1\dots f}\mathbf{f}_{cc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}\right. \\
&\quad \left.+ (-1)^{n-1}\frac{1}{2}S_c{}^{uv}\mathbf{f}_u\mathbf{f}_\nu\mathbf{e}^{mee_1\dots f}\mathbf{f}_{c_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}\right) \\
&= \alpha\Delta_{sb}^{ar}\left((-1)^n T^{eu}{}_{\nu}\mathbf{e}^{\nu me_1\dots f}\mathbf{f}_{ucc_1\dots d}e^{bcc_1\dots d}{}_{aee_1\dots f}\right)
\end{aligned}$$

$$\begin{aligned}
 & + (-1)^{n-1} \frac{1}{2} S_c^{uv} \mathbf{e}^{mee_1 \dots f} \mathbf{f}_{uvc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \Big) \\
 = & \alpha \Delta_{sb}^{ar} \left( (-1)^n T^{eu}{}_v e^{vme_1 \dots f}_{ucc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right. \\
 & \left. + (-1)^{n-1} \frac{1}{2} S_c^{uv} e^{mee_1 \dots f}_{uvc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \Phi
 \end{aligned}$$

Then, taking the dual to eliminate the forms and replacing the resulting double Levi-Civita tensor using eq. (19) we arrive at the final field equation.

$$\begin{aligned}
 0 = & \alpha \Delta_{sb}^{ar} \left( T^{eu}{}_v e^{vme_1 \dots f}_{ucc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} - \frac{1}{2} S_c^{uv} e^{mee_1 \dots f}_{uvc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \\
 = & (n-2)!(n-1)! \alpha \Delta_{sb}^{ar} \left( T^{eu}{}_v (\delta_a^v \delta_e^m - \delta_a^m \delta_e^v) \delta_u^b - \frac{1}{2} S_c^{uv} \delta_a^m (\delta_u^b \delta_v^c - \delta_v^b \delta_u^c) \right) \\
 = & (n-2)!(n-1)! \alpha \Delta_{sb}^{ar} (T^{mb}{}_a - \delta_a^m T^{eb}{}_e - \delta_a^m S_c{}^bc)
 \end{aligned}$$

so that

$$0 = \alpha \Delta_{sb}^{ar} (T^{cb}{}_a - \delta_a^c T^{eb}{}_e - \delta_a^c S_e{}^be) \tag{130}$$

For the second equation, the same steps yield,

$$\begin{aligned}
 0 = & \alpha \Delta_{sb}^{ar} \left( (-1)^n \mathbf{T}^e \mathbf{e}^{e_1 \dots f} \mathbf{f}_{mcc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} + \mathbf{e}^{ee_1 \dots f} \mathbf{S}_c \mathbf{f}_{mc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \\
 = & \alpha \Delta_{sb}^{ar} \left( (-1)^n \frac{1}{2} T^e{}_{uv} \mathbf{e}^{uve_1 \dots f} \mathbf{f}_{mcc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right. \\
 & \left. + S_c{}^u{}_v \mathbf{f}_a \mathbf{e}^v \mathbf{e}^{ee_1 \dots f} \mathbf{f}_{mc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \\
 = & \alpha \Delta_{sb}^{ar} \left( (-1)^n \frac{1}{2} T^e{}_{uv} \mathbf{e}^{uve_1 \dots f} \mathbf{f}_{mcc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right. \\
 & \left. + (-1)^n S_c{}^u{}_v \mathbf{e}^{vee_1 \dots f} \mathbf{f}_{umc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \\
 = & (-1)^n \alpha \Delta_{sb}^{ar} \left( \frac{1}{2} T^e{}_{uv} e^{uve_1 \dots f}_{mcc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} + S_c{}^u{}_v e^{vee_1 \dots f}_{umc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \Phi
 \end{aligned}$$

Then

$$\begin{aligned}
 0 = & \alpha \Delta_{sb}^{ar} \left( \frac{1}{2} T^e{}_{uv} e^{uve_1 \dots f}_{mcc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} + S_c{}^u{}_v e^{vee_1 \dots f}_{umc_1 \dots d} e^{bcc_1 \dots d}_{aee_1 \dots f} \right) \\
 = & (n-2)!(n-1)! \alpha \Delta_{sb}^{ar} \left( \frac{1}{2} T^e{}_{uv} \delta_m^b (\delta_a^u \delta_e^v - \delta_a^v \delta_e^u) + S_c{}^u{}_v \delta_a^v (\delta_u^b \delta_m^c - \delta_u^c \delta_m^b) \right) \\
 = & (n-2)!(n-1)! \alpha \Delta_{sb}^{ar} (T^e{}_{ae} \delta_m^b + S_m{}^b{}_a - \delta_m^b S_u{}^u{}_a)
 \end{aligned}$$

so finally,

$$\alpha \Delta_{sb}^{ar} (\delta_c^b T^e{}_{ae} + S_c{}^b{}_a - \delta_c^b S_e{}^e{}_a) = 0 \tag{131}$$

### Appendix C. The curvature Bianchi identity

Here we give details of the development of the curvature Bianchi identity leading from Eq. (77) to Eq. (90).

When we combine the solution for  $\Omega^a_b$  with the spacetime part of the curvature Bianchi identity, we have Eq. (77),

$$\begin{aligned} D_{[e}^{(\omega,x)} \Omega^a_{|b|cd]} + 2(1 + \chi) \Delta_{[e|b}^{af} S_{f|cd]} &= 0 \\ D_e^{(\omega,x)} C^a_{bcd} + D_c^{(\omega,x)} C^a_{bde} + D_d^{(\omega,x)} C^a_{bec} \\ &= -2(1 + \chi) (\Delta_{eb}^{af} S_{fcd} + \Delta_{cb}^{af} S_{fde} + \Delta_{db}^{af} S_{fec}) \end{aligned}$$

With  $\frac{1}{2} \Omega^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d = \frac{1}{2} C^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d = \mathbf{C}^a_b$ , the covariant exterior derivative of the Weyl curvature is

$$\begin{aligned} \mathbf{D}_{(\omega,x)} \left( \frac{1}{2} C^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) &\equiv \mathbf{d}_{(x)} \left( \frac{1}{2} C^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) + \left( \frac{1}{2} C^c_{bef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge \omega^a_c \\ &\quad - \left( \frac{1}{2} C^a_{cef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge \omega^c_b \\ &= \mathbf{D}_{(\alpha,x)} \left( \frac{1}{2} C^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \right) + \left( \frac{1}{2} C^c_{bef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge (\alpha^a_c + \beta^a_c) \\ &\quad - \left( \frac{1}{2} C^a_{cef} \mathbf{e}^e \wedge \mathbf{e}^f \right) \wedge (\alpha^c_b + \beta^c_b) \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{C}^a_b + \mathbf{C}^a_b \wedge \beta^a_c - \mathbf{C}^a_c \wedge \beta^c_b \end{aligned}$$

But  $\mathbf{C}^a_b$  is the usual traceless part of the Riemann curvature, which satisfies

$$\begin{aligned} 0 &\equiv \mathbf{D}_{(\alpha,x)} \mathbf{R}^a_b \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{C}^a_b - 2\Delta_{db}^{ae} \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \mathbf{e}^d \end{aligned}$$

Restoring the basis, the Bianchi identity is

$$\begin{aligned} 0 &= \mathbf{D}_{(\omega,x)} \mathbf{C}^a_b + 2(1 + \chi) \Delta_{cb}^{ad} \mathbf{S}_d^{(ee)} \wedge \mathbf{e}^c \\ &= \mathbf{D}_{(\alpha,x)} \mathbf{C}^a_b + \mathbf{C}^c_b \wedge \beta^a_c - \mathbf{C}^a_c \wedge \beta^c_b + 2(1 + \chi) \Delta_{cb}^{ad} \mathbf{S}_d^{(ee)} \wedge \mathbf{e}^c \\ &= 2\Delta_{db}^{ae} \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \mathbf{e}^d + \mathbf{C}^c_b \wedge (-2\Delta_{dc}^{ae} W_e \mathbf{e}^d) - \mathbf{C}^a_c \wedge (-2\Delta_{db}^{ce} W_e \mathbf{e}^d) \\ &\quad + 2(1 + \chi) \Delta_{cb}^{ad} \mathbf{S}_d^{(ee)} \wedge \mathbf{e}^c \\ &= 2\Delta_{db}^{ae} \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \mathbf{e}^d - 2\Delta_{dc}^{ae} \mathbf{C}^c_b \wedge W_e \mathbf{e}^d + 2\Delta_{db}^{ce} \mathbf{C}^a_c \wedge W_e \mathbf{e}^d + 2(1 + \chi) \Delta_{db}^{ae} \mathbf{S}_e^{(ee)} \wedge \mathbf{e}^d \\ &= 2\Delta_{df}^{ge} (\delta_b^f \delta_g^a \mathbf{D}_{(\alpha,x)} \mathcal{R}_e - \delta_g^a W_e \mathbf{C}^f_b + \delta_g^c \delta_b^f W_e \mathbf{C}^a_c + (1 + \chi) \delta_g^a \delta_b^f \mathbf{S}_e^{(ee)}) \wedge \mathbf{e}^d \end{aligned}$$

Now expand  $\Delta$  projections and carry out the simplifications

$$\begin{aligned} 0 &= 2\Delta_{db}^{ae} \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \mathbf{e}^d - 2\Delta_{dc}^{ae} \mathbf{C}^c_b \wedge W_e \mathbf{e}^d + 2\Delta_{db}^{ce} \mathbf{C}^a_c \wedge W_e \mathbf{e}^d + 2(1 + \chi) \Delta_{db}^{ae} \mathbf{S}_e^{(ee)} \wedge \mathbf{e}^d \\ &= (\delta_d^a \delta_b^e - \eta^{ae} \eta_{bd}) \mathbf{D}_{(\alpha,x)} \mathcal{R}_e \mathbf{e}^d - (\delta_d^a \delta_c^e - \eta^{ae} \eta_{cd}) \mathbf{C}^c_b \wedge W_e \mathbf{e}^d \\ &\quad + (\delta_d^c \delta_b^e - \eta^{ce} \eta_{bd}) \mathbf{C}^a_c \wedge W_e \mathbf{e}^d + (1 + \chi) (\delta_d^a \delta_b^e - \eta^{ae} \eta_{bd}) \mathbf{S}_e^{(ee)} \wedge \mathbf{e}^d \\ &= \mathbf{D}_{(\alpha,x)} \mathcal{R}_b \wedge \mathbf{e}^a - \eta_{bd} \mathbf{D}_{(\alpha,x)} \mathcal{R}^a \wedge \mathbf{e}^d - W_e \mathbf{C}^e_b \wedge \mathbf{e}^a + \eta^{ae} W^a \mathbf{C}_{db} \wedge \mathbf{e}^d \\ &\quad + W_b \mathbf{C}^a_c \wedge \mathbf{e}^c + \eta^{ca} \eta_{bd} W_e \mathbf{C}^e_c \wedge \mathbf{e}^d + 2(1 + \chi) \mathbf{S}_b^{(ee)} \wedge \mathbf{e}^a - (1 + \chi) \eta^{ae} \eta_{bd} \mathbf{S}_e^{(ee)} \wedge \mathbf{e}^d \end{aligned}$$



$$\begin{aligned}
 &= (\mathbf{D}_{(\alpha,x)} \mathcal{R}_b - W_e \mathbf{C}^e_b + 2(1 + \chi) \mathbf{S}_b^{(ee)}) \wedge \mathbf{e}^a \\
 &\quad - \eta_{bd} \eta^{ac} (\mathbf{D}_{(\alpha,x)} \mathcal{R}_c - W_e \mathbf{C}^e_c + (1 + \chi) \mathbf{S}_c^{(ee)}) \wedge \mathbf{e}^d \\
 &= \delta_d^a \delta_b^c (\mathbf{D}_{(\alpha,x)} \mathcal{R}_c - W_e \mathbf{C}^e_c + 2(1 + \chi) \mathbf{S}_c^{(ee)}) \wedge \mathbf{e}^d \\
 &\quad - \eta_{bd} \eta^{ac} (\mathbf{D}_{(\alpha,x)} \mathcal{R}_c - W_e \mathbf{C}^e_c + (1 + \chi) \mathbf{S}_c^{(ee)}) \wedge \mathbf{e}^d
 \end{aligned}$$

since the first Bianchi gives  $\mathbf{C}_{db} \wedge \mathbf{e}^d = 0$ . Then we have

$$0 = \Delta_{db}^{ac} (\mathbf{D}_{(\alpha,x)} \mathcal{R}_c - W_e \mathbf{C}^e_c + 2(1 + \chi) \mathbf{S}_c^{(ee)}) \wedge \mathbf{e}^d \tag{132}$$

*Resolving the projection*

We now show that we can eliminate the  $\Delta$ -projection and the wedge product with the solder form  $\mathbf{e}^d$ . Extracting the basis forms, we antisymmetrize,

$$\begin{aligned}
 0 &= \Delta_{db}^{ac} \left( D_f^{(\alpha,x)} \mathcal{R}_{cg} - \frac{1}{2} W_e C^e_{cfg} + (1 + \chi) S_{cfg}^{(ee)} \right) \mathbf{e}^f \wedge \mathbf{e}^g \wedge \mathbf{e}^d \\
 0 &= \Delta_{db}^{ac} (D_f^{(\alpha,x)} \mathcal{R}_{cg} - W_e C^e_{cfg} + 2(1 + \chi) S_{cfg}^{(ee)}) \\
 &\quad + \Delta_{fb}^{ac} (D_g^{(\alpha,x)} \mathcal{R}_{cd} - W_e C^e_{cgd} + 2(1 + \chi) S_{cgd}^{(ee)}) \\
 &\quad + \Delta_{gb}^{ac} (D_d^{(\alpha,x)} \mathcal{R}_{cf} - W_e C^e_{cdf} + 2(1 + \chi) S_{cdf}^{(ee)}) \\
 &\quad - \Delta_{db}^{ac} D_g^{(\alpha,x)} \mathcal{R}_{cf} - \Delta_{fb}^{ac} D_d^{(\alpha,x)} \mathcal{R}_{cg} - \Delta_{gb}^{ac} D_f^{(\alpha,x)} \mathcal{R}_{cd}
 \end{aligned}$$

Now, contract  $ad$ ,

$$\begin{aligned}
 0 &= (n - 1) (D_f^{(\alpha,x)} \mathcal{R}_{bg} - W_e C^e_{bfg} + 2(1 + \chi) S_{bfg}^{(ee)}) \\
 &\quad + 2\Delta_{fb}^{ac} (D_g^{(\alpha,x)} \mathcal{R}_{ca} - W_e C^e_{cga} + 2(1 + \chi) S_{cga}^{(ee)}) \\
 &\quad + 2\Delta_{gb}^{ac} (D_a^{(\alpha,x)} \mathcal{R}_{cf} - W_e C^e_{caf} + 2(1 + \chi) S_{caf}^{(ee)}) - (n - 1) D_g^{(\alpha,x)} \mathcal{R}_{bf} \\
 &\quad - 2\Delta_{fb}^{ac} D_a^{(\alpha,x)} \mathcal{R}_{cg} - 2\Delta_{gb}^{ac} D_f^{(\alpha,x)} \mathcal{R}_{ca} \\
 &= (n - 1) D_f^{(\alpha,x)} \mathcal{R}_{bg} - (n - 1) W_e C^e_{bfg} + 2(n - 1) (1 + \chi) S_{bfg}^{(ee)} \\
 &\quad + (D_g^{(\alpha,x)} \mathcal{R}_{bf} - W_e C^e_{bgf} + 2(1 + \chi) S_{bgf}^{(ee)}) \\
 &\quad - \eta^{ac} \eta_{bf} D_g^{(\alpha,x)} \mathcal{R}_{ca} - 2(1 + \chi) \eta^{ac} \eta_{bf} S_{cga}^{(ee)} \\
 &\quad + D_g^{(\alpha,x)} \mathcal{R}_{bf} - W_e C^e_{bgf} + 2(1 + \chi) S_{bgf}^{(ee)} - \eta^{ac} \eta_{bg} D_a^{(\alpha,x)} \mathcal{R}_{cf} - 2(1 + \chi) \eta^{ac} \eta_{bg} S_{caf}^{(ee)} \\
 &\quad - (n - 1) D_g^{(\alpha,x)} \mathcal{R}_{bf} - D_f^{(\alpha,x)} \mathcal{R}_{bg} + \eta_{bf} \eta^{ac} D_a^{(\alpha,x)} \mathcal{R}_{cg} \\
 &\quad - D_f^{(\alpha,x)} \mathcal{R}_{bg} + \eta^{ac} \eta_{bg} D_f^{(\alpha,x)} \mathcal{R}_{ca} \\
 &= -(n - 3) W_e C^e_{bfg} + 2(1 + \chi) ((n - 3) S_{bfg}^{(ee)} - \eta_{bf} \eta^{ac} S_{cga}^{(ee)} + \eta_{bg} \eta^{ac} S_{cfa}^{(ee)}) \\
 &\quad + (n - 3) D_f^{(\alpha,x)} \mathcal{R}_{bg} - (n - 3) D_g^{(\alpha,x)} \mathcal{R}_{bf} + \eta_{bf} \eta^{ac} D_a^{(\alpha,x)} \mathcal{R}_{cg} - \eta_{bg} \eta^{ac} D_a^{(\alpha,x)} \mathcal{R}_{cf} \\
 &\quad + \eta_{bg} D_f^{(\alpha,x)} \mathcal{R} - \eta_{bf} D_g^{(\alpha,x)} \mathcal{R}
 \end{aligned}$$

Write this as

$$\begin{aligned}
 &(n - 3)(D_g^{(\alpha,x)}\mathcal{R}_{bf} - D_f^{(\alpha,x)}\mathcal{R}_{bg} + W_e C^e_{bfg}) \\
 &= 2(1 + \chi)((n - 3)S_{bfg}^{(ee)} - \eta_{bf}\eta^{ac}S_{cga}^{(ee)} + \eta_{bg}\eta^{ac}S_{cfa}^{(ee)}) \\
 &\quad + \eta_{bf}\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cg} - \eta_{bg}\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} + \eta_{bg}D_f^{(\alpha,x)}\mathcal{R} - \eta_{bf}D_g^{(\alpha,x)}\mathcal{R}
 \end{aligned}$$

If we trace  $bg$ ,

$$\begin{aligned}
 &(n - 3)\eta^{bg}(D_g^{(\alpha,x)}\mathcal{R}_{bf} - D_f^{(\alpha,x)}\mathcal{R}_{bg}) \\
 &= 2(1 + \chi)((n - 3)\eta^{bg}S_{bfg}^{(ee)} - \eta^{ac}S_{cfa}^{(ee)} + n\eta^{ac}S_{cfa}^{(ee)}) \\
 &\quad + \eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} - n\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} + nD_f^{(\alpha,x)}\mathcal{R} - D_f^{(\alpha,x)}\mathcal{R} \\
 4(1 + \chi)(n - 2)\eta^{bg}S_{bfg}^{(ee)} &= 2(n - 2)\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} - 2(n - 2)D_f^{(\alpha,x)}\mathcal{R} \\
 2(1 + \chi)\eta^{bg}S_{bfg}^{(ee)} &= \eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} - D_f^{(\alpha,x)}\mathcal{R}
 \end{aligned}$$

Substituting back into the original,

$$\begin{aligned}
 &(n - 3)(D_g^{(\alpha,x)}\mathcal{R}_{bf} - D_f^{(\alpha,x)}\mathcal{R}_{bg} + W_e C^e_{bfg}) \\
 &= 2(1 + \chi)((n - 3)S_{bfg}^{(ee)} - \eta_{bf}\eta^{ac}S_{cga}^{(ee)} + \eta_{bg}\eta^{ac}S_{cfa}^{(ee)}) \\
 &\quad - \eta_{bg}(\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cf} - D_f^{(\alpha,x)}\mathcal{R}) + \eta_{bf}(\eta^{ac}D_a^{(\alpha,x)}\mathcal{R}_{cg} - D_g^{(\alpha,x)}\mathcal{R}) \\
 &= 2(1 + \chi)((n - 3)S_{bfg}^{(ee)} - \eta_{bf}\eta^{ac}S_{cga}^{(ee)} + \eta_{bg}\eta^{ac}S_{cfa}^{(ee)}) \\
 &\quad - \eta_{bg}2(1 + \chi)\eta^{bg}S_{bfg}^{(ee)} + \eta_{bf}2(1 + \chi)\eta^{bc}S_{bgc}^{(ee)} \\
 &= 2(1 + \chi)(n - 3)S_{bfg}^{(ee)}
 \end{aligned}$$

Therefore, in dimensions greater than 3,

$$2(1 + \chi)S_{bfg}^{(ee)} = D_g^{(\alpha,x)}\mathcal{R}_{bf} - D_f^{(\alpha,x)}\mathcal{R}_{bg} + W_e C^e_{bfg}$$

Restoring two basis forms, we may write this as Eq. (90),

$$D^{(\alpha,x)}\mathcal{R}_b - W_e C^e_b + 2(1 + \chi)S_b^{(ee)} = 0$$

which solves the full Bianchi relation, Eq. (132) as desired. We note that this proves:

**Theorem.** *In any torsion-free biconformal space with integrable Weyl vector,  $W_\alpha = \partial_\alpha\phi$ , and  $1 + \chi \neq 0$ , the spacetime co-torsion is the obstruction to conformal Ricci flatness.*

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