

# Wilson loops and free energies in 3D $\mathcal{N} = 4$ SYM: exact results, exponential asymptotics, and duality

Miguel Tierz\*

*Departamento de Matemática, Grupo de Física Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal*

\*E-mail: tierz@fc.ul.pt

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 We show that  $U(N)$  3D  $\mathcal{N} = 4$  supersymmetric gauge theories on  $S^3$  with  $N_f$  massive fundamental hypermultiplets and with a Fayet–Iliopoulos (FI) term are solvable in terms of generalized Selberg integrals. Finite- $N$  expressions for the partition function and Wilson loop in arbitrary representations are given. We obtain explicit analytical expressions for Wilson loops with symmetric, antisymmetric, rectangular, and hook representations, in terms of Gamma functions of complex arguments. The free energy for the orthogonal and symplectic gauge groups is also given. The asymptotic expansion of the free energy is also presented, including a discussion of the appearance of exponentially small contributions. Duality checks of the analytical expressions for the partition functions are also carried out explicitly.  
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## 1. Introduction

The study of supersymmetric gauge theories in compact manifolds has been considerably pushed forward in recent years, after the development of the localization method [1], which reduces the original functional integral describing the quantum field theory into a much simpler finite-dimensional integral. The task of computing observables in a supersymmetric gauge theory then typically consists in analyzing the resulting integral representation, which is normally of the matrix model type. There is a large number of tools available in the study of matrix models. Here we will use results from random matrix theory and the theory of the Selberg integral [2].

We study in particular a 3D  $\mathcal{N} = 4$  gauge theory on  $S^3$ , which consists of a  $U(N)$  vector multiplet coupled to  $N_f$  massive hypermultiplets in the fundamental representation, together with a Fayet–Iliopoulos (FI) term. The rules of localization in three dimensions [3,4] immediately give the corresponding matrix model expression for the partition function:

$$Z_{N_f}^{U(N)} = \frac{1}{N!} \int d^N \mu \frac{e^{i\eta \sum_i \mu_i}}{\prod_i (2 \cosh(\frac{1}{2}(\mu_i + m)))^{N_f}} \prod_{i < j} 4 \sinh^2 \left( \frac{1}{2}(\mu_i - \mu_j) \right). \quad (1.1)$$

A more general case, also including adjoint hypermultiplets, has also been studied, e.g., in Refs. [5,6] and, with different methods, in Ref. [7]. In the case of one adjoint hypermultiplet, the theory is known to be dual to M-theory on  $AdS_4 \times S^7/N_f$ , where the quotient by  $N_f$  leads to an  $A_{N_f-1}$  singularity. While our results could be extended to this more complicated model, we focus here on Eq. (1.1). In what follows, we do not include the customary  $(1/N!)$  factor in Eq. (1.1) and simply

call the partition function  $Z_N$ . We will comment on this term at the end of the paper, when discussing dualities.

Other recent exact analytical evaluations of free energies of 3D supersymmetric gauge theories can be found in Refs. [8,9]. The models studied there are more general, but, due to remarkable identities satisfied by the double sine functions that appear in their matrix models, can also be analyzed analytically. However, we still find it valuable to focus on the simpler Eq. (1.1), relate it to an exact solvable model, and extend the study to Wilson loops. In turn, this leads to nontrivial aspects of the role of the FI parameter term and an ensuing discussion of exponentially small contributions in asymptotic expansions, in relation to the classification of these 3D theories as *good*, *ugly*, or *bad* [10–12].

In addition, notice that, in spite of the apparent simplicity of this model, even simpler models in 3D, like the Abelian gauge theory studied in Ref. [13], exhibit rich behavior such as large- $N_f$  phase transitions. This nontriviality of the models is in large part due to the presence of the FI parameter, which always implies an oscillatory Fourier kernel in the matrix model representation. Concerning Eq. (1.1), we will give explicit expressions for the free energy at finite and large  $N$ , and likewise for the average of a Wilson loop in arbitrary representation, which is given by the matrix integral

$$\langle W_\lambda(N) \rangle = \frac{1}{Z} \int d^N \mu \frac{s_\lambda(e^{\mu_1}, \dots, e^{\mu_N}) e^{i\eta \sum_i \mu_i}}{\prod_i (2 \cosh(\frac{1}{2}(\mu_i + m)))^{N_f}} \prod_{i < j} 4 \sinh^2 \left( \frac{1}{2}(\mu_i - \mu_j) \right), \quad (1.2)$$

where  $Z$  refers to Eq. (1.1) and  $s_\lambda$  denotes a Schur polynomial. Aspects of the analysis of Wilson loops in this theory, in particular the mirror symmetry between Wilson loops and vortex loops, have recently appeared in Ref. [6]. The matrix models have been considered without FI term in Refs. [5,7] and with one in Ref. [6]. Localization results for a large class of correlation functions of local operators in 3D  $\mathcal{N} = 4$  gauge theory on  $S^3$  are given in Refs. [14,15].

The case where all the  $N_f$  masses are different can also be solved exactly for the partition function [8] and has been studied and used in a number of works [16,17]. The solution in that case is based on a Cauchy determinant formula. Thus, the limiting case, when all masses become equal, is not immediate and it seems that even the partition function of the model has not been analyzed as the confluent limit of the expression when all masses are distinct. We show below how the large- $N$  behavior of the free energy has a different leading behavior when all masses are equal, since the usual leading term  $(N^2/2) \log N$  [16–18] cancels out. Other, in principle more general versions of the matrix model, dealing, e.g., with the case where the matter content consists of  $\mathcal{N} = 2$  mass deformation of  $\mathcal{N} = 3$  hypermultiplets, have been studied in Ref. [19], using the Cauchy theorem and in the context of factorization of 3D partition functions [20]. In this work, infinite series expressions are obtained and hence asymptotics could not be obtained with such formulas. In contrast to the previous works, our explicit analytical expressions will all be in terms of special functions, Barnes  $G$ -functions and Gamma functions, of complex arguments.

Notice that in the Abelian case, the theory is exactly solvable, since the observables then reduce to the evaluation of a very well-known Fourier transform

$$Z_t(\eta) = \int_{-\infty}^{\infty} \frac{e^{i\eta\mu} d\mu}{(\cosh(\mu + m))^t} = \frac{\Gamma((t + i\eta)/2) \Gamma((t - i\eta)/2)}{\Gamma(t)}, \quad (1.3)$$

which follows by immediate identification with Euler's Beta integral

$$\int_{-\infty}^{\infty} \frac{ce^{-cy} dy}{(1 + e^{-cy})^{p+q}} = \int_0^1 t^{p-1} (1-t)^{q-1} dt := B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}, \quad (1.4)$$

with  $\text{Re} c > 0$ ,  $\text{Re} p, \text{Re} q \geq 0$ , and  $\Gamma(p)$  the Gamma function.

The following exact evaluations, given in terms of a number of Barnes  $G$ -functions, admit the analysis of the observables for different large values of the parameters, such as  $N$ ,  $N_f$ , and  $\eta$ , leading to asymptotic series. The presence of the FI parameter makes the analysis richer, since it then involves complex arguments  $z$  for the Barnes  $G$ -functions and the Gamma functions below.

Therefore, the large- $z$  behavior of the special functions is required, involving naturally the whole complex plane, which leads to consideration of Stokes and anti-Stokes lines and the appearance of exponentially small contributions when crossing them. We can analyze the observables in these terms by using the exponential asymptotics of Barnes and Gamma functions [21–23]. Therefore, this theory is suitable to further explore ideas of resurgence and resummation, which have become a subject of considerable interest in the study of gauge theories in recent years (see Refs. [23,24]; e.g., Refs. [25–27] for supersymmetric gauge theories and localization; and Ref. [28] for a review).

We advance that the issue of Stokes line crossing will not appear while moving the physical parameters (which is essentially here the FI parameter) of a given theory, unless we take  $\eta \rightarrow \infty$ . However, for large but finite  $\eta$ , we will see that the crossing occurs when the number of flavors is smaller than a certain value, in which case the theory becomes a *bad* theory.

The paper is organized as follows. In the next section, we show how Eqs. (1.1) and (1.2) can be evaluated by mapping the model into a random matrix ensemble and using the results in Ref. [29], which contains, among other results, a new extension of the Selberg integral. We show that the analytical continuation of the Beta function (1.4) extends the results to any number of flavors and to a nonzero FI parameter  $\eta$ . We use this to also obtain an analytical expression for the Wilson loop. Four specific sets of Wilson loops with FI parameter are studied in detail: symmetric, antisymmetric, rectangular, and hook representations, obtaining explicit expressions in terms of Gamma functions of complex arguments.

In Sect. 3, we focus exclusively on the partition function and show that the matrix model (1.1) and its counterparts for the orthogonal and symplectic gauge groups are all easily mapped into the Selberg integral, which gives an exact evaluation of the free energies in terms of Barnes  $G$ -functions. We also study the asymptotics for  $\eta = 0$  of the  $U(N)$  free energy for large  $N$  and constant Veneziano parameter  $\zeta = N_f/N$ , showing that the leading term is  $f(\zeta) N^2$  instead of  $N^2 \log N$  and we determine  $f(\zeta)$  and the subleading contributions to the free energy. An integral representation of the Mellin–Barnes type is given for the case of an  $SU(N)$  gauge group and the  $SU(2)$  case is computed explicitly in two ways.

In the last section we study the asymptotics of the free energy, with FI parameter, discussing the crossing of Stokes lines and the ensuing appearance of exponentially small contributions, coming from expansions of Gamma functions of complex arguments, in the asymptotic expansion of the partition functions. Explicit duality tests are carried out for the analytical expressions obtained for the partition functions. We conclude with some open directions for further work.

## 2. Exact evaluation of the Wilson loop average

We want the analytical evaluation (1.2). For this, notice that the usual change of variables  $e^\mu = y$ ,<sup>1</sup> useful when the matrix model contains the hyperbolic version of the Vandermonde determinant, immediately leads to the following matrix model:

$$\begin{aligned} \langle W_\lambda(N) \rangle &= \frac{1}{Z} \int_{[0,\infty)^N} d^N y \prod_{i=1}^N \frac{e^{\frac{mN_f}{2} i\eta + \frac{N_f}{2} - N} y_i}{(1 + e^m y_i)^{N_f}} s_\lambda(y_1, \dots, y_N) \prod_{i < j} (y_i - y_j)^2 \\ &= \frac{e^{-mN(i\eta + \frac{|\lambda|}{N})}}{Z} \int_{[0,\infty)^N} d^N x \prod_{i=1}^N \frac{x_i^{i\eta + \frac{N_f}{2} - N}}{(1 + x_i)^{N_f}} s_\lambda(x_1, \dots, x_N) \prod_{i < j} (x_i - x_j)^2. \end{aligned} \tag{2.1}$$

The mass dependence is accounted for in the prefactor

$$\exp \left[ -mN \left( i\eta + \frac{|\lambda|}{N} \right) \right],$$

and absent in the matrix model itself, as can also be simply seen directly from Eq. (1.1). The same holds for the normalizing partition function term  $Z$ , since it has the same matrix model representation (2.1) but without the Schur polynomial. Therefore, canceling the common mass-dependent prefactor, overall, only the term  $\exp(-m|\lambda|)$  will remain.

In contrast to the case of Chern–Simons theory, the absence of a Gaussian factor in the matrix model representation (1.1) implies that the  $x^{i\eta + N_f/2 - N}$  factor in the weight function is not removed with a shift of the above change of variables, such as in Ref. [30]. Without such a term the integrand in Eq. (2.1) is known to represent the joint probability distribution function of an ensemble of complex matrices [31]. The model can then be solved by using, e.g.,<sup>2</sup>  $\prod_{i=1}^N x_i^n s_\lambda(x_1, \dots, x_N) = s_{\lambda + (n^N)}(x_1, \dots, x_N)$ , together with the exact results in Lemma 3(a) in Ref. [29]. However, directly employing the later work [32] instead gives an immediate solution.

The implication of this solution for the gauge theories is what we discuss here. The central result in Ref. [32], for us, is the following: Let  $k$  and  $n$  be nonnegative integers,<sup>3</sup> then for any partition  $\lambda$  such that  $l(\lambda) \leq k$  and  $l(\lambda') \leq n$

$$\frac{s_\lambda(1_n) s_\lambda(1_k)}{s_{\lambda'}(1_a)} = \frac{1}{C_{n,k}^a} \int_0^\infty \dots \int_0^\infty s_\lambda(t_1, \dots, t_k) \Delta^2(t_1, \dots, t_k) \prod_{j=1}^k \frac{t_j^{n-k} dt_j}{(1 + t_j)^{a+n+k}}, \tag{2.2}$$

with

$$C_{n,k}^a = \int_0^\infty \dots \int_0^\infty \Delta^2(t_1, \dots, t_k) \prod_{j=1}^k \frac{t_j^{n-k} dt_j}{(1 + t_j)^{a+n+k}} = \prod_{j=0}^{k-1} \frac{\Gamma(j) \Gamma(j + n - k + 1) \Gamma(a + j + 1)}{\Gamma(a + n + j + 1)}. \tag{2.3}$$

The latter expression follows from Selberg’s integral (see Refs. [29] (Lemma 3(a)) and [32]) whereas Eq. (2.2) is a novel extension of the Selberg integral, obtained in Ref. [32] and previously, in a simpler

<sup>1</sup> Precisely, we use  $e^\mu = y$  and then  $ye^m = x$ .

<sup>2</sup> This leads to a discussion of the FI term, as this argument seemingly requires that the FI is either taken to be 0 (as is the case in Ref. [5]), or it is such that  $i\eta \in \mathbb{Z}$ . The latter choice has been discussed in order to avoid restriction on the charge (size of the partition) of the Wilson loop average [6] (where the choice is implemented through a modification of the contour integration).

<sup>3</sup> We change their notation slightly, since their  $m$  parameter could be confused with the mass here.

form, in Lemma 3(a) in Ref. [29]. Notice that the double constraint above on the partition  $\lambda$  of the Schur polynomial necessarily implies that<sup>4</sup>  $N_f > 2N$ , which is also the regime identified in Ref. [6]. Indeed, for these values the integrals are manifestly convergent. This regime corresponds to the *good* or *ugly* classification in Ref. [10]. In the last section, while studying the asymptotics of the observables, we will also be naturally led into considering the setting  $N < N_f < 2N$ , which corresponds to *bad* theories.

As we see from the expressions below, involving Schur polynomials, in principle it seems that we need to consider the case of an even number of flavors and take, at least for the moment, the restricted view, shown above, for the FI parameter. For definiteness, we now take  $\eta = 0$  but, below, we show how to lift this restriction, using the analytical continuation given by the Beta function (1.4). The Wilson loop is then

$$\langle W_\lambda(N) \rangle = e^{-m|\lambda|} \frac{s_\lambda(1_{N_f/2})s_\lambda(1_N)}{s_{\lambda'}(1_{N_f/2-N})}, \tag{2.4}$$

which, by using the Weyl dimension formula for the specialization of the Schur polynomials

$$s_\mu(1^N) = \frac{1}{G(N+1)} \prod_{1 \leq j < k \leq N} (\mu_j - \mu_k + k - j), \tag{2.5}$$

valid when  $l(\mu) \leq N$ , can also be written as

$$\langle W_\lambda(N) \rangle = \frac{e^{-m|\lambda|} G(N_f/2 - N + 1) \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k + k - j) \prod_{1 \leq j < k \leq N_f/2} (\lambda_j - \lambda_k + k - j)}{G(N+1)G(N_f/2+1) \prod_{1 \leq j < k \leq N_f/2-N} (\lambda'_j - \lambda'_k + k - j)} \tag{2.6}$$

and the partition function from Eq. (2.3) is

$$Z_N = e^{-imN\eta} \frac{G(N+2) G(\frac{N_f}{2} - i\eta + 1) G(\frac{N_f}{2} + i\eta + 1) G(N_f - N + 1)}{G(\frac{N_f}{2} - N - i\eta + 1) G(\frac{N_f}{2} - N + i\eta + 1) G(N_f + 1)}. \tag{2.7}$$

Looking for potential poles or zeroes of the expressions, recall that the Barnes  $G$ -function is an entire function and its zeroes are located at  $G(-n) = 0$  for  $n = 0$  and  $n \in \mathbb{N}$ . As expected, there is a drastic difference with regard to convergence, according to the FI parameter. For  $\eta = 0$ , the two Barnes  $G$ -functions in the denominator do give zeroes, precisely for  $N_f < 2N - 1$ , and we have the known, expected, divergence of the partition function in this case. This is the well-known divergence of the partition function [12].

For  $\eta \neq 0$ , there is no divergence of the partition function, since the Barnes  $G$ -functions of the denominators do not have a zero there, for a bad theory with  $N_f \leq 2(N - 1)$ . The case of complex  $\eta$  [33] will be briefly touched upon in the last section. Notice that the Barnes  $G$ -function part in Eq. (2.7) is symmetric under  $\eta \rightarrow -\eta$ . Therefore, due to the imaginary prefactor,  $Z_N(-\eta) = \overline{Z_N(\eta)}$ , as also happens when there is a Chern–Simons term [33,34]. Note also that Eq. (2.7) admits alternative equivalent expressions, e.g., involving Gamma functions. This will be relevant below when discussing the  $SU(N)$  case and also at the end, when studying duality.

Regarding Wilson loops, it seems that the required specialization of Schur polynomials puts some restriction on the parameters of our model but actually these admit an expression in terms of Beta functions and this provides an extension to the whole complex plane. We show this explicitly now.

<sup>4</sup> Recall that  $s_\lambda(t_1, \dots, t_m) = 0$  if  $l(\lambda) > m$ .

2.1. *Symmetric and antisymmetric representations*

Thus, we focus now on two interesting specific instances of Eq. (2.4), namely antisymmetric representation and symmetric representation. Recall how difficult these specific cases are to analyze in  $\mathcal{N} = 4 U(N)$  supersymmetric Yang-Mills theory (SYM) theory in four dimensions, even though the corresponding matrix model there is a Gaussian ensemble (see Ref. [35] for a recent review).

In contrast, in this case, the solvability of the model, which is in general that of a multidimensional Beta function (Selberg integral), reduces to that of the ordinary Beta integral. This leads to compact exact expressions valid for all  $N$ . Precisely, recalling the Euler integral identity (1.4), then the elementary and homogeneous symmetric polynomials are given in terms of the Beta function

$$\frac{1}{e_r(1_n)} = (n + 1) \int_0^\infty \frac{t^r}{(1 + t)^{n+2}} dt \quad \text{and} \quad \frac{1}{h_r(1_n)} = (n - 1) \int_0^1 t^r (1 - t)^{n-2} dt. \quad (2.8)$$

Then, with  $\lambda$  a partition of length 1,  $\lambda = (r)$ , we have that

$$s_\lambda(1_n) = h_r(1_n) = \binom{n + r - 1}{r} \quad \text{and} \quad s_{\lambda'}(1_n) = e_r(1_n) = \binom{n}{r}. \quad (2.9)$$

The general Wilson loop expression (2.4) adopts a very concrete expression in terms of Gamma functions, for a symmetric representation

$$\langle W_{(r)}(N) \rangle = e^{-mNr} \frac{\Gamma\left(\frac{N_f}{2} + r\right) \Gamma(N + r) \Gamma\left(\frac{N_f}{2} - N - r + 1\right)}{\Gamma(r + 1) \Gamma\left(\frac{N_f}{2}\right) \Gamma(N) \Gamma\left(\frac{N_f}{2} - N + 1\right)},$$

and the antisymmetric representation

$$\langle W_{(1)^r}(N) \rangle = \frac{e^{-mNr} \Gamma\left(\frac{N_f}{2} + 1\right) \Gamma(N + 1) \Gamma\left(\frac{N_f}{2} - N\right)}{\Gamma(r + 1) \Gamma\left(\frac{N_f}{2} - r + 1\right) \Gamma(N - r + 1) \Gamma\left(\frac{N_f}{2} - N + r\right)}.$$

Notice that these expressions do not restrict  $N_f$  to be even. Likewise, the expressions in terms of Gamma functions suggest the consideration of the FI parameter. In that case, we would have

$$\langle W_\lambda(N, \eta) \rangle = e^{-m|\lambda|} \frac{s_\lambda(\mathbf{1}_{\frac{N_f}{2} + i\eta}) s_\lambda(\mathbf{1}_N)}{s_{\lambda'}(\mathbf{1}_{\frac{N_f}{2} - N - i\eta})}. \quad (2.10)$$

Because of Eq. (2.8), each term in this expression is a Beta function; two of these have one of their two parameters complex, so one can use Eq. (1.4),<sup>5</sup> and the resulting expressions with a generic  $N_f$

<sup>5</sup> A way to prove Eq. (1.4) is by induction when  $p$  is an integer, and since the integral is bounded and analytical for  $\text{Re } p, \text{Re } q \geq 0$ —and so is the r.h.s. of the formula—then, by the Carlson theorem [36], the expression follows for complex  $p$  and  $q$ . However, to prove the uniqueness of the analytical extension of the Wilson loops, one also needs to guarantee that the factorized expression (2.10) also holds for complex values. Here, we have just extended each part in a unique way, by using the Beta function.

and a nonzero FI parameter are then given by

$$\langle W_{(r)}(N, \eta) \rangle = e^{-mr} \frac{\Gamma\left(i\eta + \frac{N_f}{2} + r\right) \Gamma(N+r) \Gamma\left(\frac{N_f}{2} - N - i\eta - r + 1\right)}{\Gamma(r+1) \Gamma\left(\frac{N_f}{2} + i\eta\right) \Gamma(N) \Gamma\left(\frac{N_f}{2} - N - i\eta + 1\right)}, \quad (2.11)$$

$$\langle W_{(1)r}(N, \eta) \rangle = \frac{e^{-mr} \Gamma\left(\frac{N_f}{2} + i\eta + 1\right) \Gamma(N+1) \Gamma\left(\frac{N_f}{2} - N - i\eta\right)}{\Gamma(r+1) \Gamma\left(\frac{N_f}{2} - r + i\eta + 1\right) \Gamma(N-r+1) \Gamma\left(\frac{N_f}{2} - N - i\eta + r\right)}. \quad (2.12)$$

Note that in these expressions the Gamma functions can be traded for Pochhammer symbols, using  $r - 1$  times  $\Gamma(z + 1) = z\Gamma(z)$ . Some simple checks can be quickly made. For example, for  $r = 0$  the expressions above give 1, as expected, and for  $r = 1$  (fundamental representation) both expressions coincide, giving

$$\langle W_{(1)}(N, \eta) \rangle = e^{-m} \frac{N\Gamma\left(i\eta + \frac{N_f}{2} + 1\right) \Gamma\left(\frac{N_f}{2} - N - i\eta\right)}{\Gamma\left(\frac{N_f}{2} + i\eta\right) \Gamma\left(\frac{N_f}{2} - N - i\eta + 1\right)} = Ne^{-m} \frac{i\eta + \frac{N_f}{2}}{\frac{N_f}{2} - N - i\eta}.$$

Likewise, as for the partition function (but this time due to the Gamma functions and not the exponential prefactor, which is real), we have that  $\langle W_{\mu}(N, -\eta) \rangle = \overline{\langle W_{\mu}(N, \eta) \rangle}$ .

The case of antisymmetric representation with  $N$  boxes can be seen as a partition function computation because  $s_{(1^N)}(x_1, \dots, x_N) = e_N(x_1, \dots, x_N) = \prod_{i=1}^N x_i$ ; see below for this point of view in the more general case of a rectangular representation. Here, using Eq. (2.12) we have that

$$\langle W_{(1^N)}(N, \eta) \rangle = e^{-mN} \frac{\Gamma\left(\frac{N_f}{2} + i\eta + 1\right) \Gamma\left(\frac{N_f}{2} - N - i\eta\right)}{\Gamma\left(\frac{N_f}{2} - N + i\eta + 1\right) \Gamma\left(\frac{N_f}{2} - i\eta\right)}. \quad (2.13)$$

Notice that this is the same as the result for the fundamental but with the complex conjugate in the denominator.

### 2.2. Rectangular partitions

There are more general representations that can also be studied very explicitly. In particular, the case of rectangular partitions is specially interesting. Recall the statement made above, before Eq. (2.2), on rectangular partitions: if we consider a partition of length  $N$ ,  $(l, l, \dots, l)$ , which we denote by  $l^N = (l^N)$  then, assuming that  $\lambda$  is a partition of length equal or lower than  $N$ , we have that

$$s_{\lambda+l^N}(x_1, \dots, x_N) = e_N^l(x_1, \dots, x_N) s_{\lambda}(x_1, \dots, x_N), \quad (2.14)$$

which follows by recalling that  $e_N(x_1, \dots, x_N) = \prod_{i=1}^N x_i$ . Thus, as a simple extension of the result above for the  $(1^N)$  representation, the case where the representation is described by a rectangular partition (with number of rows equal to the rank  $N$  of the gauge group), the matrix model giving the Wilson loop has the same form as the one for the partition function, with a shift of parameters. Thus,

$$\begin{aligned} & \langle W_{(l^N)}(N, N_f, \eta) \rangle \\ &= \frac{Z_N(N, N_f, \eta - il)}{Z_N(N, N_f, \eta)} = e^{-mNl} \frac{\prod_{j=1}^l \Gamma\left(\frac{N_f}{2} + i\eta + 1 + l - j\right) \Gamma\left(\frac{N_f}{2} - N - i\eta + 1 - j\right)}{\prod_{j=1}^l \Gamma\left(\frac{N_f}{2} - i\eta + 1 - j\right) \Gamma\left(\frac{N_f}{2} - N + i\eta + l + 1 - j\right)}, \end{aligned}$$

where, in addition to taking the quotient using Eq. (2.7), we have also iteratively applied  $G(z + 1) = \Gamma(z)G(z)$ , which will also be crucial below to check Seiberg duality. Notice that for  $l = 1$ , it coincides with Eq. (2.13), as it should.

Likewise, from Eq. (2.14) it follows that

$$\langle W_{\lambda+lN}(N, N_f, \eta) \rangle = \langle W_\lambda(N, N_f, \eta - il) \rangle$$

Thus, this case is equivalent to that of a partition function with a complex FI parameter. This setting will be briefly discussed again at the end, when considering the asymptotics of Barnes  $G$ -functions and the crossing of Stokes lines.

### 2.3. Hook representations

The last explicit case that we analyze is the one corresponding to partitions represented by hooks,  $\lambda = (r - s, 1^s)$ . As above,  $|\lambda| = r$ . Notice that this notation describes a Young tableau of a row of size  $r - s$  and a column of size  $s + 1$  as  $1^s$  contains  $s$  boxes below the upper-left box of the tableau. Therefore,  $\lambda' = (s + 1, 1^{r-s-1})$  and we have

$$s_\lambda(1_n) = \frac{\Gamma(n + r - s)}{r\Gamma(r - s)\Gamma(s + 1)\Gamma(n - s)} \quad \text{and} \quad s_{\lambda'}(1_n) = \frac{\Gamma(n + s + 1)}{r\Gamma(s + 1)\Gamma(r - s)\Gamma(n - r + s + 1)}; \tag{2.15}$$

notice that for  $s = 0$ , then  $\lambda = (r)$  and  $\lambda' = (1, 1^{r-1}) = (1^r)$  and the above expressions reduce to the ones for the homogeneous and elementary symmetric polynomials (2.9), respectively, as they should. Likewise, the same consistency check is done for the dual situation, given by  $s = r - 1$ . Then, again using Eq. (2.10), we obtain

$$\langle W_{(r-s, 1^s)}(N, \eta) \rangle = \frac{e^{-mr} \Gamma\left(\frac{N_f}{2} + i\eta + r - s\right) \Gamma(N + r - s) \Gamma\left(\frac{N_f}{2} - N - i\eta - r + s + 1\right)}{r\Gamma(r - s)\Gamma(s + 1)\Gamma\left(\frac{N_f}{2} + i\eta - s\right)\Gamma(N - s)\Gamma\left(\frac{N_f}{2} - N - i\eta + s + 1\right)}; \tag{2.16}$$

if  $s = 0$  this expression indeed reduces to Eq. (2.11) and if  $s = r - 1$  it then gives Eq. (2.12). Notice that, alternatively to Eq. (2.10), one could also use Eq. (2.16), together with the Giambelli determinant formula, to obtain a Wilson loop in other representations, out of the hook expressions.

The asymptotics of these Wilson loop expressions is particularly rich because of having complex arguments in the Gamma functions, then the crossing of Stokes lines (and related phenomena, like Berry smoothing transitions across lines [37], etc.) appears. We will discuss this at the end, focusing more on the free energy.

### 3. $O(2N)$ , $O(2N + 1)$ , and $Sp(2N)$ cases, $U(N)$ asymptotics and integral representation for $SU(N)$

In this section, we show that the partition function of the matrix models also follows, after some change of variables, from the famous evaluation of the Selberg integral [2]

$$\begin{aligned} S_N(\lambda_1, \lambda_2, \gamma) &: = \int_0^1 \cdots \int_0^1 \prod_{i=1}^N t_i^{\lambda_1} (1 - t_i)^{\lambda_2} dt_i \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1 + 1 + j\gamma)\Gamma(\lambda_2 + 1 + j\gamma)\Gamma(1 + (j + 1)\gamma)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (N + j - 1)\gamma)\Gamma(1 + \gamma)}. \end{aligned} \tag{3.1}$$



The evaluation of this integral is valid for complex parameters  $\lambda_1, \lambda_2, \gamma$  such that

$$\Re(\lambda_1) > 0, \Re(\lambda_2) > 0, \Re(\gamma) > -\min\{1/N, \Re(\lambda_1)/(N-1), \Re(\lambda_2)/(N-1)\}, \quad (3.2)$$

corresponding to the domain of convergence of the integral. This evaluation is useful not only for the  $U(N)$  case discussed above, but also when the gauge group is the symplectic or the orthogonal group. These latter cases were studied with the Fermi gas formalism in Ref. [7]. We conclude this section with a discussion on how to obtain the  $SU(N)$  case by integration of the  $U(N)$  result (2.7) over the FI parameter.

### 3.1. $U(N)$ case revisited and free energy behavior for large $N$

Notice that, with only a very minor modification of the change of variables proposed in Ex. 4.1.3 in Ref. [2],

$$t_l = \frac{1}{e^{(s_l+m)/2} + 1},$$

the Selberg integral can be written as

$$S_N(\lambda_1, \lambda_2, \lambda) = e^{-(\lambda_1-\lambda_2)mN/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=1}^N \frac{e^{-(\lambda_1-\lambda_2)s_i/2} ds_i}{(2 \cosh(\frac{1}{2}(s_i + m)))^{\lambda_1+\lambda_2+2+2\lambda(N-1)}} \\ \times \prod_{i<j} \left(2 \sinh(\frac{1}{2}(s_i - s_j))\right)^{2\lambda},$$

and hence, choosing  $\lambda = 1, \lambda_1 = (2i\eta - 2N + N_f)/2$ , and  $\lambda_2 = (N_f - 2N - 2i\eta)/2$ , we have that

$$Z_N = e^{-i\eta mN} S_N \left(1, \frac{2i\eta - 2N + N_f}{2}, \frac{N_f - 2N - 2i\eta}{2}\right) \\ = e^{-i\eta mN} \prod_{j=0}^{N-1} \frac{\Gamma(i\eta + 1 + N_f/2 - N + j)\Gamma(-i\eta + 1 + N_f/2 - N + j)\Gamma(j + 2)}{\Gamma(-N + N_f + 1 + j)}. \quad (3.3)$$

Once again, the partition function can be written in terms of Barnes  $G$ -functions

$$Z_N = e^{-i\eta mN} \frac{G(i\eta + 1 + N_f/2)G(-i\eta + 1 + N_f/2)G(-N + N_f + 1)G(N + 2)}{G(i\eta + 1 + N_f/2 - N)G(-i\eta + 1 + N_f/2 - N)G(N_f + 1)G(2)}, \quad (3.4)$$

which, as expected, coincides with Eq. (2.7). The expression simplifies when  $\eta = 0$ , giving

$$Z_{N_f}^{U(N)}(\eta = 0) = \frac{G^2(1 + N_f/2)G(-N + N_f + 1)G(N + 2)}{G^2(1 + N_f/2 - N)G(N_f + 1)}. \quad (3.5)$$

In the  $\eta = 0$  case, as also seen from the condition (3.2), the result holds for  $N_f \geq 2N - 1$ , a well-known result [12]. In the large- $N$  and  $N_f$  limits, therefore, this implies a lower bound for the Veneziano parameter  $\zeta = N_f/N > 2$ . Now, one can immediately study the large  $N_f \rightarrow \infty$  and  $N \rightarrow \infty$  behavior in Eq. (3.5), directly using the original expansion by Barnes:

$$\log G(z + 1) = \frac{1}{2} - \log A + \frac{z}{2} \log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z - \frac{3z^2}{4} + \sum_{k=1}^N \frac{B_{2k+2}}{4k(k+1)z^{2k}} + O\left(\frac{1}{z^{2N+2}}\right), \quad (3.6)$$

where the constant  $A$  is the Glaisher–Kinkelin constant. The solution can be given in terms of  $N$  and the Veneziano parameter, in which case the partition function reads

$$Z(N, \zeta, \eta) = \frac{G(1 + i\eta + \frac{\zeta N}{2})G(1 - i\eta + \frac{\zeta N}{2})G((\zeta - 1)N + 1)G(N + 2)}{G(i\eta + (\zeta/2 - 1)N + 1)G(-i\eta + (\zeta/2 - 1)N + 1)G(\zeta N + 1)}. \quad (3.7)$$

For  $\eta = 0$  we can also study, e.g., the first convergent case, given by  $N_f = 2N + 1$  or, more generally,  $N_f = 2N + k$  with  $k \geq 1$  and finite, which gives

$$Z_{2N+1}^{U(N)}(\eta = 0) = \frac{G^2(1 + N + \frac{k}{2})G(N + k + 1)G(N + 2)}{G^2(1 + k/2)G(2N + 1 + k)};$$

this covers an arbitrarily large number of cases (indexed by  $k$ ), and the Veneziano parameter is always 2. Now, for  $\eta = 0$  and taking into account the asymptotics of the Barnes  $G$ -function, it follows that, for the double scaling limit  $N \rightarrow \infty$  and  $N_f \rightarrow \infty$  with  $\zeta = N_f/N = \text{cte}$ , if we write the free energy as  $F_N \equiv \ln Z_N$ , what would be the leading term in the free energy,

$$\frac{N^2}{2} \log N, \quad (3.8)$$

always cancels out and is not present. This holds because, if we denote by  $z_i$  the terms  $1 + z_i$  in the arguments of the different Barnes  $G$ -functions in Eq. (3.7) with  $\eta = 0$ ,<sup>6</sup> then it holds that

$$\sum_{i=1}^4 z_i^2 - \sum_{i=5}^7 z_i^2 = \sum_{i=1}^4 z_i - \sum_{i=5}^7 z_i = 2N + 1. \quad (3.9)$$

The final term that comes from the  $-3z^2/4$  part in Eq. (3.6) cancels completely due to Eq. (3.9) and the same happens, partially, for the term that results from the  $(z^2/2) \log z$  part in Eq. (3.6). The Eq. (3.8) behavior that would follow from this part cancels, and it only remains an  $N^2$  leading term that is multiplied by a Veneziano parameter-dependent factor. All together we have, for  $N \rightarrow \infty$ ,

$$F_N \sim N^2 f(\zeta) + N \log N + N \left( \log 2\pi - \frac{3}{2} \right) + \frac{5}{12} \log N,$$

where

$$f(\zeta) = \frac{\zeta^2}{2} \log \frac{\zeta}{2} + (\zeta - 1)^2 \log(\zeta - 1) - 2 \left( \frac{\zeta}{2} - 1 \right)^2 \log \left( \frac{\zeta}{2} - 1 \right) - \zeta^2 \log \zeta. \quad (3.10)$$

Note that for the particular case of  $\zeta = 2$  all the terms in Eq. (3.10) cancel, with the exception of the last one, and then  $f(\zeta) = -4 \log 2$ . In general, while naively at first it may seem that  $f(\zeta)$  vanishes for large  $\zeta$ , it turns out that it is well approximated by  $f(\zeta) \simeq -2\zeta \log 2$  for large  $\zeta$ .

Recall that the free energy of the larger family of 3D  $T_\rho^{\rho'}[SU(N)]$  theories, at leading order, has been studied in Refs. [16,17] and, more recently, from the point of view of gravitational duals, in Ref. [18], where it is shown that the leading term is of the form (3.8). This property also holds for the simpler  $T[SU(N)]$  theory, namely the  $SU(N)$  theory with  $N_f$  flavors of different masses  $m_l$  with  $l = 1, \dots, N_f$  [16,17], and we have just studied the case when all masses are equal, which leads to the cancellation of such a leading term. For a very recent discussion of other 3D SYM theories with similar behavior to the one obtained here, see Ref. [38].

<sup>6</sup>  $i = 1-4$  denotes the arguments in the numerator, 5–7 those in the denominator.

### 3.2. $O(2N)$ , $O(2N + 1)$ , and $Sp(2N)$ cases

Now, setting the mass and the FI parameter  $\eta = m = 0$ , the explicit expression of the free energy when the gauge groups are the orthogonal and symplectic groups can be obtained, again with the same Selberg integral, just by considering another change of variables. Indeed, with the change of variables (see Ex. 4.1.3 in Ref. [2])

$$t_l = \frac{2}{\cosh(s_l/2) + 1}$$

we have that

$$S_N(\lambda_1, \lambda_2, \lambda) = 2^{-N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^N \frac{(\sinh^2(s_i/2))^{\lambda_2+1/2} ds_i}{(\cosh^2(s_i/2))^{\lambda_1+\lambda_2+2+2\lambda(N-1)}} \times \prod_{i<j} \left( \sinh\left(\frac{1}{2}(s_i - s_j)\right) \sinh\left(\frac{1}{2}(s_i + s_j)\right) \right)^{2\lambda}. \tag{3.11}$$

Recall that the Vandermonde determinant of the matrix model in the case of the orthogonal and symplectic gauge groups is the corresponding hyperbolic version of the Haar measure, explicitly given by:

$$\Delta_{O(2N)}^2(e^{is}) = \prod_{i<j} \left( 2 \sinh\left(\frac{1}{2}(s_i - s_j)\right) 2 \sinh\left(\frac{1}{2}(s_i + s_j)\right) \right)^2,$$

$$\Delta_{O(2N+1)}^2(e^{is}) = \prod_{i<j} \left( 2 \sinh\left(\frac{1}{2}(s_i - s_j)\right) 2 \sinh\left(\frac{1}{2}(s_i + s_j)\right) \right)^2 \prod_{i=1}^N \sinh^2(s_i/2),$$

$$\Delta_{Sp(2N)}^2(e^{is}) = \prod_{i<j} \left( 2 \sinh\left(\frac{1}{2}(s_i - s_j)\right) 2 \sinh\left(\frac{1}{2}(s_i + s_j)\right) \right)^2 \prod_{i=1}^N \sinh^2(s_i).$$

Thus, we find that, in terms of the  $S_N$  given by Eq. (3.1), the partition functions read:

$$Z_{N_f}^{O(2N)} = 2^{2N(N-(N_f-3)/2)} S_N((N_f + 1)/2 - 2N, -1/2, 1)$$

$$Z_{N_f}^{O(2N+1)} = 2^{2N(N-(N_f-3)/2)} S_N((N_f - 1)/2 - 2N, +1/2, 1)$$

$$Z_{N_f}^{Sp(2N)} = S_N((N_f - 3)/2 - 2N, +1/2, 1),$$

where for the symplectic case we have used a half-angle formula for the sinh function in Eq. (3.11). The partition functions in terms of the Barnes  $G$ -function are then given by

$$Z_{N_f}^{O(2N)} = 2^{2N(N-(N_f-3)/2)} \frac{G((N_f + 1)/2 - N + 1) G(N + 1/2) G(N + 2) G(N_f/2 - N)}{G((N_f + 1)/2 - 2N + 1) G(1/2) G(N_f/2)},$$

$$Z_{N_f}^{O(2N+1)} = 2^{2N(N-(N_f-3)/2)} \frac{G((N_f - 1)/2 - N + 1) G(N + 3/2) G(N + 2) G(N_f/2 - N + 1)}{G(N_f/2 + 1) G(3/2) G(N_f/2 + 1/2)},$$

$$Z_{N_f}^{Sp(2N)} = \frac{G((N_f - 3)/2 - N + 1) G(N + 3/2) G(N_f/2 - N)}{G(N_f/2) G((N_f - 3)/2 - 2N + 1) G(3/2)}.$$

### 3.3. Mellin–Barnes representation for $SU(N)$

To conclude this section, notice that the  $SU(N)$  case could also be computed by integrating the  $U(N)$  result over the FI parameter (as this is equivalent to introducing a Dirac delta in the matrix model). We start by rewriting the expression (2.7) in terms of Gamma functions for the part containing the FI parameter. That is,

$$Z_N = e^{-imN\eta} \prod_{j=1}^N \left| \Gamma \left( \frac{N_f}{2} - j + i\eta + 1 \right) \right|^2 \frac{G(N+2) G(N_f - N + 1)}{G(N_f + 1)}, \quad (3.12)$$

which follows immediately by repeatedly using  $G(z+1) = \Gamma(z)G(z)$ . First, we analyze the  $SU(2)$  case. In the massless case, we can directly use Barnes’ first lemma

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+x) \Gamma(b+x) \Gamma(c-x) \Gamma(d-x) dx = \frac{\Gamma(a+c) \Gamma(a+d) \Gamma(b+c) \Gamma(b+d)}{\Gamma(a+b+c+d)},$$

which is an extension of the Beta integral and equivalent to Gauss summation of the hypergeometric function. Then, normalizing by  $2\pi$  to account for the Dirac delta representation as a Fourier transform of the identity, we have

$$\begin{aligned} Z_{SU(2)} &= \frac{G(4) G(N_f - 1)}{(2\pi) G(N_f + 1)} \int_{-\infty}^{\infty} \left| \Gamma \left( \frac{N_f}{2} + i\eta \right) \right|^2 \left| \Gamma \left( \frac{N_f}{2} + i\eta - 1 \right) \right|^2 d\eta \\ &= \frac{G(4) G(N_f - 1) \Gamma(N_f) \Gamma(N_f - 1)^2 \Gamma(N_f - 2)}{G(N_f + 1) \Gamma(2N_f - 2)}. \end{aligned}$$

Notice that the expression diverges for  $N_f = 1$  and  $N_f = 2$  and indeed these two cases correspond to *bad* theories. Of course, for this case the direct computation of the matrix model (1.1) with the  $\delta(x_1 + x_2)$  insertion is direct as well, with the resulting 1D integral, giving:

$$\tilde{Z}_{SU(2)} = \int_{-\infty}^{\infty} \frac{4 \sinh^2 x}{(2 \cosh(x/2))^{2N_f}} dx, \quad (3.13)$$

and, using, e.g., Eq. (1.3), one checks that indeed  $\tilde{Z}_{SU(2)} = Z_{SU(2)}$ . The massive version of Eq. (3.13) is also solvable (e.g., using the results in Ref. [13]) and can be related to Wilson loops in a  $U(1)$  theory with  $N_f$  hypermultiplets of mass  $m$  and  $N_f$  hypermultiplets of mass  $-m$ , but this will be discussed elsewhere.

From the expression (3.12), it follows that the partition function for the  $SU(N)$  theory admits a 1D Mellin–Barnes type of integral representation:

$$\begin{aligned} Z_{SU(N)} &= \frac{G(N+2) G(N_f - N + 1)}{G(N_f + 1)} \int_{-\infty}^{\infty} e^{-imN\eta} \prod_{j=1}^N \Gamma \left( \frac{N_f}{2} - j + i\eta + 1 \right) \Gamma \left( \frac{N_f}{2} - j - i\eta + 1 \right) d\eta, \end{aligned} \quad (3.14)$$

with a Fourier kernel too, in the massive case. Both asymptotics and a full analytical solution are possible but not entirely immediate (a formula for generic values of all parameters ends up being quite involved), requiring an analysis in itself, and hence it will be presented elsewhere.

#### 4. On exponential asymptotics and duality

The presence of the FI parameter has an interesting implication: the expressions for the free energies and the Wilson loops, given above, are in terms of Barnes  $G$ - or Gamma functions of *complex* arguments. This leads naturally to consideration of the behavior of these functions in the whole complex plane, where a richer behavior, involving Stokes lines [21,37], is well known to emerge.

Let us remind ourselves first that  $G(z+1)$  and  $\Gamma(z)$  (we will directly look at the asymptotic expansion of their logarithm) have Stokes lines at  $z = \pm\pi/2$ . Thus, looking at the logarithm of the solution (2.7), it appears that, for certain values of the rank of the gauge group  $N$  and the number of flavors  $N_f$  (and, in the case of the Wilson loops, the size of the representation), we will have the appearance of exponentially small contributions in the asymptotic expansions of the observables. Notice that these are not phase transitions or crossovers within eventual different regimes of the theory, since the controlling parameters are the numbers of flavors and colors.

We focus on the  $U(N)$  free energy, given by the logarithm of Eq. (3.4). Its analysis immediately follows from considering the mathematical results on the Barnes  $G$ -function [21,23]. The exponentially improved asymptotic expansion of the Barnes  $G$ -function reads [21]

$$\begin{aligned} \log G(z+1) &\sim \frac{1}{4}z^2 + z \log \Gamma(z+1) - \left( \frac{1}{2}z(z+1) + \frac{1}{12} \right) \log z - \log A \\ &+ \sum_{k=1}^{\infty} S_k(\theta) e^{\pm 2\pi ikz} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}}, \end{aligned} \quad (4.1)$$

where

$$S_k(\theta) = \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \mp \frac{1}{22\pi ik^2} & \text{if } \theta = \pm \frac{\pi}{2} \\ \mp \frac{1}{2\pi ik^2} & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} \quad (4.2)$$

and  $\theta = \arg z$ . The upper or lower sign is taken according to  $z$  being in the upper or lower half-plane. The term with  $S_k(\theta)$  describes the Stokes singularities and the rest of Eq. (4.1) is the asymptotic expansion of the Barnes  $G$ -function, as given in Ref. [39]. Note that this is not the original asymptotic expansion by Barnes, used above, but, using the asymptotics for the  $\log \Gamma(z+1)$  in Eq. (4.1) (including exponentially small contributions), one can write down the analogous result for the Barnes form above Eq. (3.6). At any rate, as will be seen below, we crucially need both asymptotics for our analysis of the free energy.

The four  $G(1+z)$  functions of the free energy (taking the logarithm of Eq. (2.7)) with complex arguments have a  $z$  variable given by:

$$\begin{aligned} z_{1,\pm} &= \pm i\eta + N_f/2 - N, \\ z_{2,\pm} &= \pm i\eta + N_f/2. \end{aligned}$$

The Stokes lines are located at  $\pm\pi/2$ , hence, for large but finite  $\eta$ , we need to look when the real part of the arguments becomes 0 and/or negative. A null or negative real part argument can only physically happen for the  $z_{1,\pm}$  case above. The first case, which is right at the Stokes line, corresponds to

$N_f = 2N$ , which is a self-dual case in terms of dualities (see below). The first actual crossing of a Stokes line occurs for  $N_f = 2N - 1$ , the ugly case, and the rest is then for

$$N_f < 2N - 1,$$

which is well known to correspond to the so-called *bad* theories. Therefore, we also discuss below this eventual crossing of the Stokes line in the context of the analysis of *bad*, *good*, and *ugly* theories [11,12] and dualities below.

We need the asymptotics of the Gamma function too, whose Stokes phenomenon is similar to that of the Barnes  $G$ -function above, with the Stokes lines also at  $\pm\pi/2$ . One difference is the different decay of the  $S_k(\theta)$  coefficients (Stokes multipliers), where the quadratic decay in the Barnes  $G$ -function is a linear decay in the Gamma function case [22]. More crucially, there is a sign difference in the respective Stokes multipliers, as we show in what follows. For the Gamma function the following asymptotic expansion holds as  $z \rightarrow \infty$ :

$$\log \Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} - \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \frac{1}{2} \log(1 - e^{\pm 2\pi iz}) & \text{if } \theta = \pm \frac{\pi}{2} \\ \log(1 - e^{\pm 2\pi iz}) & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} .$$

The expansion of the logarithm gives the asymptotics in the same form as above:

$$\log \Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \sum_{k=1}^{\infty} \tilde{S}_k(\theta) e^{\pm 2\pi ikz}, \tag{4.3}$$

in the sector  $|\arg z| \leq \pi - \delta < \pi$  for any  $0 < \delta \leq \pi$  with<sup>7</sup>

$$\tilde{S}_k(\theta) = \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \frac{1}{2k} & \text{if } \theta = \pm \frac{\pi}{2} \\ \frac{1}{k} & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} , \tag{4.4}$$

where the usual definition

$$\Gamma^*(z) = \frac{\Gamma(z)}{\sqrt{\pi} z^{z-1/2} e^{-z}}$$

is used. In addition to the different decay, the sign difference between  $\tilde{S}_k(\theta)$  and  $S_k(\theta)$  is crucial with regard to the cancellation of exponentially small terms in the asymptotics of the observables. Notice also the difference in how the variable is written in the argument of the functions in Eq. (4.3). Thus, the relevant variable indicating a possible Stokes line crossing, in this case, is then

$$\tilde{z}_{1,\pm} = \pm i\eta + N_f/2 - N + 1. \tag{4.5}$$

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<sup>7</sup> Note that, with regard to the location of the Stokes lines, the asymptotics of the Gamma function is with variable  $z$  whereas that of the Barnes  $G$ -function is with  $z + 1$ .

### 4.1. Exponentially small contributions in the asymptotics

Taking into account the specific form of the Stokes multiplier parts in Eqs. (4.1) and (4.3), we discuss the appearance of exponentially small contributions. In all cases, it is due to the Gamma asymptotics because the part corresponding to the Stokes multipliers in the Barnes  $G$ -asymptotics will cancel out, as we shall see. Since we want to use the asymptotic result for  $z \rightarrow \infty$  we then need to take the  $\eta \rightarrow \infty$  limit.

There is a small difference between the cases corresponding to *good* and *ugly* theories, where we will be in the Stokes line if  $\eta \rightarrow \infty$ , in comparison to the case of *bad* theories, where we will also be in the Stokes line, or cross it, for large but finite  $\eta$ . Since we are considering the  $\eta \rightarrow \infty$  limit there are therefore exponentially small contributions in the asymptotics of all cases.

For example, in the *good* theory case that is self-dual,  $N_f = 2N$ , then  $z_{1,\pm} = \pm i\eta$ ; we are always on the two Stokes lines for the two Barnes  $G$ -functions in the denominator of Eq. (2.7). We focus only on the exponentially small contributions to the free energy, i.e., on the part with the Stokes multipliers in Eq. (4.1) for the two Barnes  $G$ -functions  $\log G(i\eta + 1) + \log G(-i\eta + 1)$ . We find that the two set of contributions cancel each other:

$$F_{N,\text{good}}^{\text{Stokes}} \sim -\frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta} - e^{-2\pi k\eta}}{2\pi i k^2} = 0.$$

Notice that this result and the one below also holds for finite  $N$ ; however, for the asymptotics of the rest of the Barnes  $G$ -functions in Eq. (2.7), one needs to take the large- $N$  limit. Regarding the Gamma asymptotics, using Eq. (4.5), we have that  $\tilde{z}_{1,\pm} = \pm i\eta + 1$ . Therefore, while for large but finite  $\eta$  we have not reached the Stokes line in this case, the  $\eta \rightarrow \infty$  limit effectively puts us on the Stokes line and we have exponentially small contributions in the asymptotics, which are those of the Gamma function (4.3). We write down one case explicitly below.

In the *ugly* case  $N_f = 2N - 1$  then  $z_{1,\pm} = \pm i\eta - 1/2$  and we have crossed the Stokes line in the Barnes  $G$ -function asymptotics. As above, we have cancellation, due to the symmetric way in which the Barnes  $G$ -function appears in the partition function. Focusing on the part with the Stokes multipliers in Eq. (4.1), for the two Barnes  $G$ -functions  $\log G(i\eta + 1/2) + \log G(-i\eta + 1/2)$  we have

$$F_{N,\text{ugly}}^{\text{Stokes}} \sim -\sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta + \pi i k} - e^{-2\pi k\eta - \pi i k}}{2\pi i k^2} = 0.$$

In addition, this is the last case where the Gamma function asymptotics would not contribute at large but finite  $\eta$ , since the Stokes line is not yet reached, as  $\tilde{z}_{1,\pm} = \pm i\eta + 1/2$ . Again, regardless of this, the  $\eta \rightarrow \infty$  limit situates us in the Stokes line, leading to exponentially small contributions to the asymptotics.

For bad theories, we have  $N_f = 2N - 2 - l$  with  $l = 0, 1, 2, \dots$  then  $z_{1,\pm} = \pm i\eta - 1 - l/2$  for the Barnes  $G$ -function and  $\tilde{z}_{1,\pm} = \pm i\eta - l/2$  for the corresponding Gamma function asymptotics, then

$$F_{N,\text{bad}}^{\text{Stokes}} = -\sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta} \sin(\pi k l)}{2\pi i k^2} - \sum_{k=1}^{\infty} \frac{2e^{-2\pi k\eta} \cos[2\pi k(1 + l/2)]}{k} = -2(-1)^l \sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta}}{k},$$

and therefore the sign of the contribution is opposite according to the parity of  $N_f$ . Therefore, we have seen that only the exponentially small terms in the asymptotics of the Gamma function eventually

contribute. Another way of obtaining this result would have been to directly invoke the equivalent expression (3.12) for the partition function, but it is illustrative to obtain it from the asymptotics of the Barnes  $G$ -function.

Now, localization on  $S^3$  [33] also permits consideration of the case  $\eta \in \mathbb{C}$ . If we write  $\eta = \eta_R + i\eta_I$ , then there are clearly many more possibilities of crossing Stokes lines and the four Barnes  $G$ -functions can now give corresponding exponentially small contributions, because the crossings are now determined by:

$$\begin{aligned} \pm\eta_I + N_f/2 - N &\leq 0 \\ \pm\eta_I + N_f/2 &\leq 0, \end{aligned}$$

for the Stokes multipliers in Eq. (4.1) and

$$\begin{aligned} \pm\eta_I + N_f/2 - N &\leq 0 \\ \pm\eta_I + N_f/2 &\leq 0, \end{aligned}$$

for the ones in Eq. (4.3), corresponding to the Gamma function.

Besides, we saw above that this case is equal to an unnormalized (not divided by the partition function) Wilson loop with a rectangular representation of the type  $(N^m)$ . The case of Wilson loops is somewhat richer since we also have the additional parameter  $r$ , characterizing the representation, but it is analyzed in the same way.

#### 4.2. Duality

Some simple tests of Seiberg duality can be quickly carried out. This is an additional test of the analytical formula (2.7). We consider the generic case given by  $N_f = 2N + k$  for a general integer value of  $k$ . We start first with  $k = -1$ . Then, theory  $U(N)$  with  $N_f = 2N - 1$  is in the *ugly* class, containing a decoupled free sector, generated by BPS monopole operators of dimension  $\frac{1}{2}$ . It is known that the rest is dual to the IR limit of the  $U(N - 1)$  gauge theory with  $N_f = 2N - 1$ , and thus a *good* theory. Using Eq. (2.7) and again  $G(z + 1) = \Gamma(z)G(z)$ , one quickly finds that

$$Z_N(N_f = 2N - 1) = e^{-im\eta} N \Gamma(-i\eta + 1/2) \Gamma(i\eta + 1/2) Z_{N-1}(N_f = 2N - 1); \quad (4.6)$$

the  $N$  arises due to the fact that we dropped the  $N!$  normalizing factor in Eq. (1.1) and denoted the resulting partition function by  $Z_N$ ; restoring it, we obtain that for Eq. (1.1) it holds that

$$Z_{N_f=2N-1}^{U(N)} = Z_{N_f=2N-1}^{U(N-1)} \frac{\pi e^{-im\eta}}{\cosh(\pi\eta)}, \quad (4.7)$$

and we have the expected duality between the  $U(N)$  and  $U(N - 1)$  theories, together with the expected appearance of a free hypermultiplet. This is the case where the duality is between the *good* and *ugly* theories, whereas the rest will be between the *good* and *bad* theories.

If we take  $k = 1$  instead of  $k = -1$  then we have to obtain the same duality, but starting from the *good* theory side. An immediate computation shows that this is indeed the case, giving again Eq. (4.7). The self-dual case  $N_f = 2N$  is evident and the rest is, naively, between *good* and *bad* theories, which corresponds to starting with a  $U(N)$  theory with  $N_f > 2N + 1$ . For example, for a  $U(N)$  theory with  $N_f = 2N + 2$ , we have

$$Z_N(N_f = 2N + 2) = \frac{e^{2im\eta}}{(N + 2)(N + 1)} |\Gamma(-i\eta + 1)|^{-2} |\Gamma(-i\eta)|^{-2} Z_{N+2}(N_f = 2N + 2), \quad (4.8)$$



then

$$Z_{N_f=2N+2}^{U(N+2)} = Z_{N_f=2N+2}^{U(N)} \frac{\pi^2 e^{-2im\eta}}{\sinh^2(\pi\eta)}.$$

Thus, this is the duality between a good theory on the l.h.s. of Eq. (4.8) and a bad theory on the r.h.s. This duality check can be extended to the generic cases of even and odd numbers of flavors  $N_f = 2N + 2k$  for  $k = 1, 2, 3, \dots$  and  $N_f = 2N + 2k + 1$  for  $k = 0, 1, 2, 3, \dots$  (the negative  $k$  gives the same duality as its positive counterpart). We obtain:

$$Z_{N_f=2N+2k}^{U(N+2k)} = Z_{N_f=2N+2k}^{U(N)} e^{-2ikm\eta} \prod_{j=1-k}^k |\Gamma(j+i\eta)|^2 = Z_{N_f=2N+2k}^{U(N)} \left( \frac{\pi e^{-im\eta}}{\sinh(\pi\eta)} \right)^{2k},$$

$$Z_{N_f=2N+2k+1}^{U(N+2k+1)} = Z_{N_f=2N+2k+1}^{U(N)} e^{-i(2k+1)m\eta} \prod_{j=-k}^k \left| \Gamma\left(j + \frac{1}{2} + i\eta\right) \right|^2 = Z_{N_f=2N+2k+1}^{U(N)} \left( \frac{\pi e^{-im\eta}}{\cosh(\pi\eta)} \right)^{2k+1}.$$

## 5. Outlook

We expect to study further the asymptotics together with the duality, including further discussion on the case of Wilson loops and the setting where a gauge-R Chern–Simons term is present, characterized by an additional imaginary part in the FI parameter. The asymptotics in this case will admit more possibilities and it should be possible to also look at it from the point of view of Borel transforms. The Mellin–Barnes type of integral given for the  $SU(N)$  theory can also be exploited for both a full analytical solution and for a study of asymptotics as well.

We conclude by briefly commenting on the fact that the matrix model studied here is not only related to an extended Selberg integral [29,32], but also has several equivalent representations [29,32], a well-known result in the context of Berezin quantization, whose original connections with matrix models precisely relied heavily on matrix integration identities [40]. In this way, the matrix model for the partition function without an FI term also appears in the (Berezin) quantization analysis characterizing the Hilbert space of a collective field theory of the singlet sector of the symplectic  $Sp(2N)$  sigma model, of importance in the study of dS/CFT correspondence [41,42]. In that setting, it computes the size of the Hilbert space. In principle, the Wilson loop studied here will also have an interpretation in this Hilbert space picture.

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