



# Four-dimensional Brane–Chern–Simons Gravity and Cosmology

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**Abstract** From the field equations corresponding to a four-dimensional brane embedded in the five-dimensional spacetime of the Einstein–Chern–Simons theory for gravity, we find cosmological solutions that describe an accelerated expansion for a flat universe. Apart from a quintessence-type evolution scheme, we obtain a transient phantom evolution, which is not ruled out by the current observational data. Additionally, a bouncing solution is shown. The introduction of a kinetic term in the action shows a de Sitter behavior although the energy density is not constant. A quintessence behavior is also found. We conjecture on a possible geometric origin of dark energy coming from this action.

## 1 Introduction

The Poincaré algebra and Poincaré group describe the symmetries of empty Minkowski spacetime. It is known since 1970 [1], that the presence of a constant electromagnetic field in Minkowski spacetime leads to the modification of Poincaré symmetries.

The presence of a constant classical electromagnetic field in Minkowski spacetime modifies the Poincaré algebra into the so-called Maxwell algebra [2–8]. This algebra can also be obtained from the anti-de Sitter (AdS) algebra and a particular semigroup  $S$  by means of the  $S$ -expansion procedure introduced in Refs. [9–12]. Using this method it is possible to obtain more general modifications to the Poincaré algebra (see, for example, [13, 14]). An interesting modification to the Poincaré symmetries, obtained by the aforementioned expansion procedure, is given by the so-called Lie  $\mathfrak{B}$  algebra also known as generalized Poincaré algebra, whose genera-

tors satisfy the commutation relation shown in Eq. (7) of Ref. [15].

The Einstein–Chern–Simons (EChS) gravity [13] is a gauge theory whose Lagrangian density is given by a five-dimensional Chern–Simons form for the  $\mathfrak{B}$  algebra. The field content induced by the  $\mathfrak{B}$  algebra includes the vielbein  $e^a$ , the spin connection  $\omega^{ab}$ , and two extra bosonic fields  $h^a$  and  $k^{ab}$ . The EChS gravity has the interesting property that the five-dimensional Chern–Simons Lagrangian for the  $\mathfrak{B}$  algebra, given by [13]

$$L_{ChS}^{(5)}[e, \omega, h, k] = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right), \quad (1)$$

where  $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$  and  $T^a = de^a + \omega_c^a e^c$ , leads to the standard general relativity without cosmological constant in the limit where the coupling constant  $l$  tends to zero while keeping the Newton's constant fixed. It should be noted that there is an absence of kinetic terms for the fields  $h^a$  and  $k^{ab}$  in the Lagrangian  $L_{ChS}^{(5)}$  (for details see Ref. [16]).

Recently was shown in Ref. [17] that the five-dimensional EChS gravity can be consistent with the idea of a four-dimensional spacetime. In this Reference was replaced a Randall–Sundrum type metric [18, 19] in the EChS gravity Lagrangian (1) to get (see Appendix)

$$\tilde{S}[\tilde{e}, \tilde{h}] = \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \left( -\frac{1}{2} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + C \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q - \frac{C}{4r^2} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right), \quad (2)$$

which is an gravity action with a cosmological constant for a four-dimensional brane embedded in the five-dimensional spacetime of the EChS theory of gravity.  $\tilde{\varepsilon}_{mnpq}$ ,  $\tilde{e}^m$ ,  $\tilde{R}^{mn}$  and  $\tilde{h}^m$  represent, respectively, the 4-dimensional versions of the Levi-Civita symbol, the vielbein, the curvature form

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and a matter field. It is of interest to note that the field  $h^a$ , a bosonic gauge field from the Chern–Simons gravity action, which gives rise to a form of positive cosmological constant, appears as a consequence of modification of the Poincaré symmetries, carried out through the expansion procedure.

On the other hand,  $C$  and  $r_c$  (the “compactification radius”) are constants. The corresponding version in tensor language (see Appendix) is given by

$$\tilde{S}[\tilde{g}, \tilde{h}] = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[ \tilde{R} + 2C \left( \tilde{R}\tilde{h} - 2\tilde{R}^\mu_\nu \tilde{h}^\nu_\mu \right) - \frac{3C}{2r_c^2} \tilde{h} \right], \tag{3}$$

where we can see that when  $l \rightarrow 0$  then  $C \rightarrow 0$  and hence (99) becomes the four-dimensional Einstein–Hilbert action.

In this paper we introduce the geometric framework obtained by gauging of the so called  $\mathfrak{B}$  algebra. Besides the vierbein  $e^a_\mu$  and the spin connection  $\omega^{ab}_\mu$ , our scheme includes the fields  $k^{ab}_\mu$  and  $h^a_\mu$  whose dynamic is described by the field equation obtained from the corresponding actions. The application of the cosmological principle shows that the field  $h^a$  has a similar behavior to that of a cosmological constant, which leads to the conjecture that the equations of motion and their accelerated solutions are compatible with the era of dark energy.

It might be of interest to note that, according to standard GR (Einstein framework in a FLRW background), a simple way to describe dark energy (also dark matter) is through an equation of state that relates density ( $\rho$ ) of a fluid and its pressure ( $p$ ) through the equation  $p = \omega\rho$ , where  $\omega$  is the parameter of the equation of state. Dark energy is characterized by  $-1 \leq \omega < -1/3$ ,  $\omega = -1$  represents the cosmological constant and  $\omega < -1$  corresponds to the so-called phantom dark energy. This means that in the context of general relativity the parameter  $\omega$  is “set by hand” and then contrasted with observational information.

In the present work, the cosmological constant is not “set by hand” but rather arises from the framework that we present. An example is shown where a quintessence-type evolution as well as a phantom evolution are equally possible. This means that a possible geometric origin of dark energy can be conjectured in the context of the so-called Einstein–Chern–Simons gravity.

The article is organized as follows: in Section II, we rewrite the action (3) by introducing a scalar field associated to the field  $\tilde{h}_{\mu\nu}$ , we find the corresponding equations of motion, and then we discuss the cosmological consequences of this scheme. In Section III, a kinetic term is added in the action and its effects on cosmology are studied. Finally, Concluding Remarks are presented in Section IV. An Appendix is also included where we review the derivation of the action (3).

## 2 Cosmological consequences

In this Section we will study the cosmological consequences associated with the action (3). If we consider a maximally symmetric spacetime (for instance, the de Sitter’s space), the equation 13.4.6 of Ref. [20] allows us to write the field  $\tilde{h}_{\mu\nu}$  as

$$\tilde{h}_{\mu\nu} = \frac{\tilde{F}(\tilde{\varphi})}{4} \tilde{g}_{\mu\nu}, \tag{4}$$

where  $\tilde{F}$  is an arbitrary function of an 4-scalar field  $\tilde{\varphi} = \tilde{\varphi}(\tilde{x})$ . This means

$$\tilde{R}^\mu_\nu \tilde{h}^\nu_\mu = \frac{\tilde{F}(\tilde{\varphi})}{4} \tilde{R}, \quad \tilde{h} = \tilde{h}_{\mu\nu} \tilde{g}^{\mu\nu} = \tilde{F}(\tilde{\varphi}), \tag{5}$$

so that the action (3) takes the form (see Appendix)

$$\tilde{S}[\tilde{g}, \tilde{\varphi}] = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[ \tilde{R} + C\tilde{R}\tilde{F}(\tilde{\varphi}) - \frac{3C}{2r_c^2} \tilde{F}(\tilde{\varphi}) \right], \tag{6}$$

which corresponds to an action for the four-dimensional gravity coupled non-minimally to a scalar field. Note that this action has the form

$$\tilde{S}_B = \tilde{S}_g + \tilde{S}_{g\varphi} + \tilde{S}_\varphi, \tag{7}$$

where  $\tilde{S}_g$  is a pure gravitational action term,  $\tilde{S}_{g\varphi}$  is a non-minimal interaction term between gravity and a scalar field, and  $\tilde{S}_\varphi$  represents a kind of scalar field potential. In order to write down the action in the usual way, we define the constant  $\varepsilon$  and the potential  $V(\varphi)$  as (removing the symbols  $\sim$  in (6)). In fact

$$\varepsilon = \frac{4\kappa r_c^2}{3}, \quad V(\varphi) = \frac{3C}{4\kappa r_c^2} F(\varphi), \tag{8}$$

where  $\kappa$  is the gravitational constant. This permits to rewrite the action for a four-dimensional brane non-minimally coupled to a scalar field, immersed in a five-dimensional space-time as

$$S[g, \varphi] = \int d^4x \sqrt{-g} [R + \varepsilon R V(\varphi) - 2\kappa V(\varphi)]. \tag{9}$$

The corresponding field equations describing the behavior of the four-dimensional brane in the presence of the scalar field  $\varphi$  are given by

$$G_{\mu\nu} (1 + \varepsilon V) + \varepsilon H_{\mu\nu} = -\kappa g_{\mu\nu} V, \tag{10}$$

$$\frac{\partial V}{\partial \varphi} \left( 1 - \frac{\varepsilon R}{2\kappa} \right) = 0, \tag{11}$$

where

$$H_{\mu\nu} = g_{\mu\nu} \nabla^\lambda \nabla_\lambda V - \nabla_\mu \nabla_\nu V. \tag{12}$$

In order to construct a model of universe based on Eqs. (10, 11), we consider the Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\psi^2) \right), \tag{13}$$

where  $a(t)$  is the so called ‘‘cosmic scale factor’’ and  $k = 0, +1, -1$  describes flat, spherical and hyperbolic spatial geometries, respectively. Following the usual procedure, we find the following field equations

$$3 \left( H^2 + \frac{k}{a^2} \right) (1 + \varepsilon V) + 3\varepsilon H \dot{\varphi} \frac{\partial V}{\partial \varphi} = V, \tag{14}$$

$$\begin{aligned} \left( 2\dot{H} + 3H^2 + \frac{k}{a^2} \right) (1 + \varepsilon V) + \varepsilon \left( \dot{\varphi}^2 \frac{\partial^2 V}{\partial \varphi^2} \right. \\ \left. + (\ddot{\varphi} + 2H\dot{\varphi}) \frac{\partial V}{\partial \varphi} \right) = V, \end{aligned} \tag{15}$$

$$\frac{\partial V}{\partial \varphi} \left[ 1 - 3\varepsilon \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right) \right] = 0, \tag{16}$$

where  $H = \dot{a}/a$  is the Hubble parameter and we have used natural unities  $\kappa = 8\pi G = c = 1$ . Dot means derivative with respect to time.

From (14 to 16) we see that when  $\varepsilon = 0$  and  $V = const.$ , we have a de Sitter behavior governed by  $H = \sqrt{V/3}$ . On the another hand, from Eq. (8) we can see that  $\varepsilon$  is a positive quantity. This fact allows us to define an effective cosmological constant as

$$\Lambda_{eff} = \frac{1}{2\varepsilon}, \tag{17}$$

which will play an important role in the cosmological consequences that we will show below.

In the flat case, the Eqs. (14–16) are given by

$$3H^2 = -\frac{1}{V + 2\Lambda_{eff}} \left( 3H \frac{dV}{dt} - 2\Lambda_{eff} V \right), \tag{18}$$

$$\begin{aligned} 2\dot{H} + 3H^2 = -\frac{1}{V + 2\Lambda_{eff}} \\ \times \left( \frac{d^2V}{dt^2} + 2H \frac{dV}{dt} - 2\Lambda_{eff} V \right), \end{aligned} \tag{19}$$

where the Eq. (16) was not considered because it is not an independent equation. In fact, subtracting (19) from (18) we obtain

$$2\dot{H} = -\frac{1}{V + 2\Lambda_{eff}} \left( \frac{d^2V}{dt^2} - H \frac{dV}{dt} \right). \tag{20}$$

Deriving (18) with respect to time and using (20) we find (16), when  $k = 0$ . Bear in mind that, at the end of this section, we will study an interesting consequence derived from this equation.

We write now the Eqs. (18, 19) in the ‘‘standard’’ form

$$3H^2 = \rho, \quad \rho = -\frac{1}{V + 2\Lambda_{eff}} \left( 3H \frac{dV}{dt} - 2\Lambda_{eff} V \right), \tag{21}$$

$$\begin{aligned} \dot{H} + H^2 = -qH^2 = -\frac{1}{6}(\rho + 3p), \\ \frac{1}{6}(\rho + 3p) = \frac{1}{4\Lambda_{eff}} \frac{1}{(1 + V/2\Lambda_{eff})} \\ \times \left( \frac{d^2V}{dt^2} + H \frac{dV}{dt} - \frac{4\Lambda_{eff}}{3} V \right), \end{aligned} \tag{22}$$

being  $q$  the deceleration parameter defined by  $q = -1 - \dot{H}/H^2$  and  $p$  the pressure associated to  $\rho$  given by

$$p = \frac{1}{V + 2\Lambda_{eff}} \left( \frac{d^2V}{dt^2} + 2H \frac{dV}{dt} - 2\Lambda_{eff} V \right), \tag{23}$$

which allows to write the barotropic equation  $p = \omega\rho$ , where

$$\omega = -\left( \frac{2\Lambda_{eff} V - 2H dV/dt - d^2V/dt^2}{2\Lambda_{eff} V - 3H dV/dt} \right), \tag{24}$$

and we note that  $V = const.$  leads to  $\omega = -1$ , i.e., a de Sitter evolution.

Considering again Eqs. (18, 19) and defining  $x = V/2\Lambda_{eff}$ , we write the field equations in the form

$$3 \left( H^2 + H \frac{d}{dt} \ln(1+x) \right) = 2\Lambda_{eff} \left( \frac{x}{1+x} \right), \tag{25}$$

$$\begin{aligned} 3 \left( qH^2 - H \frac{d}{dt} \ln(1+x) \right) = -2\Lambda_{eff} \left( \frac{x}{1+x} \right) \\ + \frac{3}{2} \left( \frac{1}{1+x} \right) \frac{d^2x}{dt^2}, \end{aligned} \tag{26}$$

and we discuss some examples:

(a)  $x = x_0 = const.$  If  $x$  behaves as a constant, the solution for the Hubble parameter is given by

$$H = \sqrt{\frac{2\Lambda_{eff}}{3} \frac{x_0}{1+x_0}}, \tag{27}$$

i.e., a de Sitter evolution for all time.

(b)  $x = t/t_0$ . In this case, the solution is

$$H(t) = \frac{1}{2t_0(1+t/t_0)} \left( \sqrt{1 + \frac{8\Lambda_{eff}t_0}{3} (1+t/t_0)t - 1} \right), \tag{28}$$

where we can see that

$$H(t \rightarrow \infty) \rightarrow \sqrt{\frac{2\Lambda_{eff}}{3}}, \quad \rho(t \rightarrow \infty) \rightarrow 2\Lambda_{eff} \quad \text{and} \\ q(t \rightarrow \infty) = -1, \tag{29}$$

which means that we have a late de Sitter evolution.

(c)  $x = \exp(t/t_0)$ . Here, the Hubble parameter turns out to be

$$H(t) = \frac{1}{2t_0 [1 + \exp(t/t_0)]} \\ \times \left( \sqrt{1 + \frac{8\Lambda_{eff}t_0^2}{3} \exp(t/t_0) [1 + \exp(t/t_0)]} - 1 \right), \tag{30}$$

and

$$H(t \rightarrow \infty) \rightarrow \sqrt{\frac{2\Lambda_{eff}}{3}}, \quad \rho(t \rightarrow \infty) \rightarrow 2\Lambda_{eff} \quad \text{and} \\ q(t \rightarrow \infty) \rightarrow -1, \tag{31}$$

and, as in the previous case, we have a late de Sitter evolution.

From (24) the parameter of the equation of state takes the form

$$\omega(t) = - \left( \frac{2\Lambda_{eff}x - 2Hdx/dt - d^2x/dt^2}{2\Lambda_{eff}x - 3Hdx/dt} \right), \tag{32}$$

and if  $x = t/t_0$  one finds

$$\omega(t) = - \left( \frac{t - H(t)/\Lambda_{eff}}{t - 3H(t)/2\Lambda_{eff}} \right), \tag{33}$$

so that if we identify  $t_0$  as the current time, then

$$\omega(t_0) = - \left( \frac{t_0 - H(t_0)/\Lambda_{eff}}{t_0 - 3H(t_0)/2\Lambda_{eff}} \right) < -1 \quad \text{and} \\ \omega(t \rightarrow \infty) = -1, \tag{34}$$

and we have a transient phantom evolution (not ruled out by the current observational data). Theoretical frameworks where this type of evolution is discussed can be seen in [21–23].

The shown examples above have a common characteristic, namely they show a late de Sitter evolution like, for instance,  $\Lambda$ CDM at late times, but we do not know if this characteristic comes from the formalism that we are inspecting or from the choice (Ansatz) that we make for  $V(t)$ . Since we do not have something to guide us towards a form for  $V(t)$  from first principles, we are tied to playing with different Ansätze for that potential. At least those shown here, give us interesting results, in particular, that obtained from the Ansatz given in (b), a transient phantom evolution.

Previously, we have seen that Eq. (16) is not independent, and therefore it was not analyzed in the first instance. However, it reveals an interesting fact of our scheme, the presence of a cosmological bounce. In case that  $k = 0$ ,  $\Lambda_{eff} = 1/2\epsilon$  and  $\partial V/\partial\varphi \neq 0$ , the Eq. (16) takes the form

$$\frac{2}{3}\Lambda_{eff} - (\dot{H} + 2H^2) = 0, \tag{35}$$

which leads to the following solution for the Hubble parameter

$$H(t) = \sqrt{\Lambda_{eff}/3} \left( \frac{\exp[4\sqrt{\Lambda_{eff}/3}(t-t_0)] - 1/\Delta(t_0)}{\exp[4\sqrt{\Lambda_{eff}/3}(t-t_0)] + 1/\Delta(t_0)} \right), \tag{36}$$

where

$$\Delta(t_0) = \frac{\sqrt{\Lambda_{eff}/3} + H_0}{\sqrt{\Lambda_{eff}/3} - H_0}. \tag{37}$$

Note that the Eq. (36) can be written in terms of an hyperbolic tangent as

$$H(t) = \sqrt{\frac{\Lambda_{eff}}{3}} \tanh \left( 2\sqrt{\frac{\Lambda_{eff}}{3}}(t-t_0) + \frac{1}{2} \ln[\Delta(t_0)] \right), \tag{38}$$

which reveals a cosmological bounce in

$$t_b = t_0 - \frac{1}{4} \sqrt{\frac{3}{\Lambda_{eff}}} \ln \Delta(t_0) \implies H(t_b) = 0. \tag{39}$$

Moreover, the expression (38) is also showing that  $H < 0$  for  $t < t_b$ , and  $H > 0$  for  $t > t_b$ ; i.e., there is a contraction for  $t < t_b$  and an expansion for  $t > t_b$ . In fact, from Eq. (38) the cosmic scale factor is obtained

$$a(t) = a_b \cosh^{1/2} \left( 2\sqrt{\frac{\Lambda_{eff}}{3}}(t-t_b) \right), \tag{40}$$

where  $a_b$  is the minimum of the scale factor, which occurs at  $t_b$ , and it is given by

$$a_b = a(t_b) = a(0) \cosh^{-1/2} \left( 2\sqrt{\frac{\Lambda_{eff}}{3}}(t_0-t_b) \right). \tag{41}$$

Behaviors of this kind have been frequently studied in the literature in the context of cosmological bouncing; see e.g., [24–26]. Finally, we note that

$$H(t \rightarrow \infty) \rightarrow \sqrt{\frac{\Lambda_{eff}}{3}}, \tag{42}$$

i.e., a late de Sitter evolution.

### 3 Introduction of the kinetic term (1/2) $\dot{\phi}^2$

In Ref. [14] it was found that the surface term  $B_{EChS}^{(4)}$  in the Lagrangian (1) is given by

$$B_{EChS}^{(4)} = \alpha_1 l^2 \epsilon_{abcde} e^a \omega^{bc} \left( \frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) + \alpha_3 \epsilon_{abcde} \left[ l^2 \left( h^a \omega^{bc} + k^{ab} e^c \right) \left( \frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) + l^2 k^{ab} \omega^{cd} \left( \frac{2}{3} de^e + \frac{1}{2} \omega_f^d e^e \right) + \frac{1}{6} e^a e^b e^c \omega^{de} \right]. \tag{43}$$

From (1) and (43) we can see that kinetic terms corresponding to the fields  $h^a$  and  $k^{ab}$ , absent in the Lagrangian, are present in the surface term. This situation is common to all Chern–Simons theories. This has the consequence that the action (9) does not have the kinetic term for the scalar field  $\phi$ .

It could be interesting to add a kinetic term to the four-dimensional brane action. In this case, the action (9) takes the form

$$S[g, \phi] = \int d^4x \sqrt{-g} \left[ R + \epsilon R V(\phi) - 2\kappa \left[ \frac{1}{2} (\nabla_\mu \phi) (\nabla^\mu \phi) + V(\phi) \right] \right]. \tag{44}$$

The corresponding field equations are given by

$$G_{\mu\nu} (1 + \epsilon V) + \epsilon H_{\mu\nu} = \kappa T_{\mu\nu}^\phi, \tag{45}$$

$$\nabla_\mu \nabla^\mu \phi - \frac{\partial V}{\partial \phi} \left( 1 - \frac{\epsilon R}{2\kappa} \right) = 0, \tag{46}$$

where  $T_{\mu\nu}^\phi$  is the energy-momentum tensor of the scalar field

$$T_{\mu\nu}^\phi = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} \nabla^\lambda \phi \nabla_\lambda \phi + V \right), \tag{47}$$

and the rank-2 tensor  $H_{\mu\nu}$  is defined as

$$H_{\mu\nu} = g_{\mu\nu} \nabla^\lambda \nabla_\lambda V - \nabla_\mu \nabla_\nu V. \tag{48}$$

Following the usual procedure, we find that the FLRW type equations are given by

$$3 \left( \frac{\dot{a}^2 + k}{a^2} \right) (1 + \epsilon V) + 3\epsilon \frac{\dot{a}}{a} \dot{\phi} \frac{\partial V}{\partial \phi} = \kappa \left( \frac{1}{2} \dot{\phi}^2 + V \right), \tag{49}$$

$$\left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right) (1 + \epsilon V) + \epsilon \left[ \dot{\phi}^2 \frac{\partial^2 V}{\partial \phi^2} + \left( \ddot{\phi} + 2 \frac{\dot{a}}{a} \dot{\phi} \right) \frac{\partial V}{\partial \phi} \right] = -\kappa \left( \frac{1}{2} \dot{\phi}^2 - V \right), \tag{50}$$

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \frac{\partial V}{\partial \phi} \left[ 1 - \frac{3\epsilon}{\kappa} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right) \right] = 0. \tag{51}$$

In the case  $k = 0$ , and using  $\kappa = 1$ , Eqs. (49–51) takes the form

$$3H^2 (1 + \epsilon V) + 3\epsilon H \frac{dV}{dt} = \frac{1}{2} \dot{\phi}^2 + V, \tag{52}$$

$$(2\dot{H} + 3H^2) (1 + \epsilon V) + \epsilon \left( \frac{d^2V}{dt^2} + 2H \frac{dV}{dt} \right) = - \left( \frac{1}{2} \dot{\phi}^2 - V \right), \tag{53}$$

$$(\ddot{\phi} + 3H\dot{\phi})\dot{\phi} + \frac{dV}{dt} [1 - 3\epsilon(\dot{H} + 2H^2)] = 0, \tag{54}$$

and here, the Eq. (54) it is not an independent equation. In fact, subtracting the Eq. (53) from the Eq. (52) we obtain

$$2\dot{H} (1 + \epsilon) = \epsilon H \frac{dV}{dt} - \epsilon \frac{d^2V}{dt^2} + \dot{\phi}^2. \tag{55}$$

Deriving the Eq. (52) with respect to time and using the Eq. (55) we find the Eq. (54).

The combination  $(1/2) \dot{\phi}^2 + V(\phi)$ , together with the combination  $(1/2) \dot{\phi}^2 - V(\phi)$ , reminds us that in a standard scalar field theory

$$\rho = \frac{1}{2} \dot{\phi}^2 + V, \quad p = \frac{1}{2} \dot{\phi}^2 - V, \tag{56}$$

which is recovered at the limit  $\epsilon \rightarrow 0$ . In fact, when  $\epsilon \rightarrow 0$  the Eqs. (52–54) takes the form

$$3H^2 = \rho, \tag{57}$$

$$2\dot{H} + 3H^2 = -p, \tag{58}$$

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{59}$$

where we have used (56) to obtain (59) from (54) when  $\epsilon \rightarrow 0$ .

The Eq. (56) allows us to write (52, 53) in the form

$$3H^2 = \frac{1}{1 + \epsilon V} \left( \rho - 3\epsilon H \dot{\phi} \frac{\partial V}{\partial \phi} \right), \tag{60}$$

$$2\dot{H} + 3H^2 = -\frac{1}{1 + \epsilon V} \left( p - \epsilon \left[ (\rho + p) \frac{\partial^2 V}{\partial \phi^2} + \left( \ddot{\phi} + 2H\dot{\phi} \right) \frac{\partial V}{\partial \phi} \right] \right), \tag{61}$$

and if we choose  $p = -\rho$ , by thinking in a de Sitter evolution, we obtain

$$3H^2 = \frac{\rho}{1 + \epsilon V} - 3H \frac{d}{dt} \ln(1 + \epsilon V), \tag{62}$$

$$\dot{H} = \frac{1}{2} \left( \frac{\ddot{\phi}}{\dot{\phi}} + 5H \right) \frac{d}{dt} \ln(1 + \epsilon V). \tag{63}$$



According to (63),  $\dot{H} = 0$  say us

$$\frac{\ddot{\phi}}{\dot{\phi}} + 5H = 0 \text{ or } V(t) = \text{const.}, \tag{64}$$

and, according to (62),  $V(t) = \text{const.}$  implies  $\rho = \text{const.}$  i.e.,  $H = \text{const.}$ , i.e., an usual de Sitter evolution. But, if  $\ddot{\phi}/\dot{\phi} + 5H = 0$  and  $V(t) \neq \text{const.}$ , after to see (62) we have, with  $H = H_0 = \text{const.}$ ,

$$\rho(t) = 3H_0 \left[ H_0 + \frac{d}{dt} \ln(1 + \varepsilon V(t)) \right] (1 + \varepsilon V(t)), \tag{65}$$

and we have a de Sitter evolution although  $\rho \neq \text{const.}$  One more detail, the equation  $\ddot{\phi}/\dot{\phi} + 5H = 0$  has the solution  $\dot{\phi} \sim a^{-5}$  and so, the kinetic term  $(1/2)\dot{\phi}^2$  dissolves very quickly with evolution leading us to  $\rho \sim V(t)$  and  $p \sim -V(t)$  at late times, i.e., a de Sitter evolution.

Writing (60, 61) in the form

$$3H^2 = \frac{1}{1 + \varepsilon V} \left( \rho - 3\varepsilon H \frac{dV}{dt} \right), \tag{66}$$

$$\dot{H} + H^2 = -\frac{1}{6(1 + \varepsilon V)} \left[ (\rho + 3p) + 3\varepsilon \left( \frac{d^2V}{dt^2} + H \frac{dV}{dt} \right) \right], \tag{67}$$

we see that when  $\varepsilon = 0$  we recover the results of General Relativity, i.e.,  $3H^2 = \rho$  and  $\dot{H} + H^2 = -(1/6)(1 + 3\omega)\rho$ . By following this reminder, we write (66, 67) in the standard form

$$3H^2 = \rho_{tot}, \tag{68}$$

$$\dot{H} + H^2 = -\frac{1}{6}(\rho_{tot} + 3p_{tot}), \tag{69}$$

$$\rho_{tot} = \frac{1}{1 + \varepsilon V} \left( \rho - 3H \frac{d(\varepsilon V)}{dt} \right), \tag{70}$$

$$p_{tot} = \frac{1}{1 + \varepsilon V} \left( p + 2H \frac{d(\varepsilon V)}{dt} + \frac{d^2(\varepsilon V)}{dt^2} \right), \tag{71}$$

we can see that we can build a barotropic equation  $p_{tot} = \omega_{tot} \rho_{tot}$ , where

$$\omega_{tot} = \frac{p + 2Hd(\varepsilon V)/dt + d^2(\varepsilon V)/dt^2}{\rho - 3Hd(\varepsilon V)/dt} \text{ and } q = \frac{1}{2}(1 + 3\omega_{tot}). \tag{72}$$

By doing  $p = \omega\rho$ , we can write (72) as

$$\omega_{tot} = \omega + \frac{(2 + 3\omega) Hd(\varepsilon V)/dt + d^2(\varepsilon V)/dt^2}{\rho - 3Hd(\varepsilon V)/dt}. \tag{73}$$

Here we can see that if  $d^2(\varepsilon V)/dt^2 = 0$  and  $d(\varepsilon V)/dt \neq 0$ , then  $\omega = -2/3$ ,  $\omega_{tot} = -2/3$  and  $q = -1/2$ . This means that  $\omega_{tot}$  belongs to the quintessence zone. So, with  $(\alpha, \beta)$

constants,  $\varepsilon V(t) = \alpha(t/t_0) + \beta$  is an obvious choice for  $\varepsilon V(t)$ .

On the other hand, it is direct to show that

$$\dot{\rho}_{tot} + 3H(1 + \omega_{tot})\rho_{tot} = 0, \tag{74}$$

so that

$$\rho_{tot}(a) = \rho_{tot}(a_0) \left( \frac{a_0}{a} \right)^3 \exp \left( -3 \int_{t_0}^t \omega_{tot}(t) d \ln a \right),$$

$$\text{and } \omega_{tot} = -\frac{2}{3} \rightarrow \rho_{tot}(a) = \rho_{tot}(a_0) \left( \frac{a_0}{a} \right), \tag{75}$$

and the same is true for  $\rho(a)$ , that is,  $\rho(a) = \rho(0)(a_0/a)$ .

Note that, if  $V(t) = V_0 = \text{const.}$

$$\rho_{tot} = \frac{1}{1 + \varepsilon V_0} \rho, \quad p_{tot} = \frac{1}{1 + \varepsilon V_0} p \text{ and } \omega_{tot} = \omega, \tag{76}$$

and if  $\omega = 0$  then  $\omega_{tot} = 0$  and then  $p_{tot} = 0$ . This means that  $\omega_{tot} = 0$  plays the role of the usual dark matter ( $\omega = 0$ ), although  $\rho_{tot} \neq \rho$ .

Finally, we have been using the quantity  $\Lambda_{eff} = 1/2\varepsilon$ , where  $\varepsilon = 4\kappa r_c^2/3 = \text{const.}$  is a parameter derived from the mechanism of dimensional reduction under consideration, which depends on the gravitational constant  $\kappa$  and the compactification radius  $r_c$ . This parameter plays the role of an effective cosmological constant (its inverse) recalling that in the action  $S[g, \varphi] = \int d^4x \sqrt{-g} [R + (\varepsilon R - 2\kappa) V(\varphi)]$  there is no a ‘‘bare’’ cosmological constant. This fact could lead us to conjecture that the  $h$ -field (or  $\tilde{h}$ -field), in some way, manifests itself as dark energy. If so, the next step will be to submit the present outline to the verdict of observation.

#### 4 Concluding remarks

We have considered a modification of the Poincaré symmetries known as given  $\mathfrak{B}$  Lie algebra also known as generalized Poincaré algebra, whose generators satisfy the commutation relation shown in Eq. (7) of Ref. [15]. Besides the vierbein  $e^a_\mu$  and the spin connection  $\omega^{ab}_\mu$ , our scheme includes the fields  $k^{ab}_\mu$  and  $h^a_\mu$  whose dynamic is described by the field equation obtained from the corresponding actions.

We have used the field equations for a four-dimensional brane embedded in the five-dimensional spacetime of [17] to study their cosmological consequences. The corresponding FLRW equations are found by means of the usual procedure and cosmological solutions are shown and discussed. We highlight two solutions, by choosing  $\partial V/\partial t = \text{const.}$ , a transient phantom evolution (not ruled out by the current observational data) is obtained and if  $\partial V/\partial t \neq \text{const.}$  we obtain a bouncing solution.

Since the kinetic terms corresponding to the fields  $h^a$  and  $k^{ab}$  are present in the surface term (see (1) and (43)) it was necessary to introduce a kinetic term to the four-dimensional action. As a consequence of this, in the corresponding cosmological framework we highlight a de Sitter evolution even when the energy density involved is not constant.

Whatever it is, and since we do not have something to guide us towards a form for  $V(t)$  from first principles, we are tied to playing with different Ansätze for that potential. At least in the cases that were considered give us interesting results. But, we must insist, we are completely dependent on the Ansätze for  $V(t)$ . If we are thinking on cosmology, the results shown here suffer from this “slavery”. The hope, a common feeling, is that what is shown can be a contribution that guides us towards a better understanding of the present formalism and its chance of being a possible alternative to General Relativity. It is evident that the observational information will be key when it comes to discriminating between both models. To extract information that leads us to  $V(t)$  in order to visualize if the scalar field philosophy has a viable chance of being real when it comes to doing cosmology is the challenge to face.

**Appendix: Derivation of the action for a four-dimensional brane embedded in the five-dimensional spacetime**

In this Appendix we briefly review the derivation of the action (3). In order to find it, we will first consider the following five-dimensional Randall–Sundrum [18, 19] type metric

$$\begin{aligned}
 ds^2 &= e^{2f(\phi)} \tilde{g}_{\mu\nu}(\tilde{x}) d\tilde{x}^\mu d\tilde{x}^\nu + r_c^2 d\phi^2, \\
 &= \eta_{ab} e^a e^b, \\
 &= e^{2f(\phi)} \tilde{\eta}_{mn} \tilde{e}^m \tilde{e}^n + r_c^2 d\phi^2,
 \end{aligned}
 \tag{77}$$

where  $e^{2f(\phi)}$  is the so-called “warp factor”, and  $r_c$  is the so-called “compactification radius” of the extra dimension, which is associated with the coordinate  $0 \leq \phi < 2\pi$ . The symbol  $\sim$  denotes four-dimensional quantities. We will use the usual notation

$$\begin{aligned}
 x^\alpha &= (\tilde{x}^\mu, \phi); \quad \alpha, \beta = 0, \dots, 4; \quad a, b = 0, \dots, 4; \\
 \mu, \nu &= 0, \dots, 3; \quad m, n = 0, \dots, 3; \\
 \eta_{ab} &= \text{diag}(-1, 1, 1, 1, 1); \quad \tilde{\eta}_{mn} = \text{diag}(-1, 1, 1, 1),
 \end{aligned}
 \tag{78}$$

which allows us to write the vielbein

$$e^m(\phi, \tilde{x}) = e^{f(\phi)} \tilde{e}^m(\tilde{x}) = e^{f(\phi)} \tilde{e}^m_\mu(\tilde{x}) d\tilde{x}^\mu; \quad e^4(\phi) = r_c d\phi,
 \tag{79}$$

where  $\tilde{e}^m$  is the vierbein.

From the vanishing torsion condition

$$T^a = de^a + \omega^a_b e^b = 0,
 \tag{80}$$

we obtain the connections

$$\omega^a_{b\alpha} = -e^\beta_b \left( \partial_\alpha e^a_\beta - \Gamma^\gamma_{\alpha\beta} e^a_\gamma \right),
 \tag{81}$$

where  $\Gamma^\gamma_{\alpha\beta}$  is the Christoffel symbol.

From Eqs. (79) and (80) we find

$$\omega^m_4 = \frac{e^f f'}{r_c} \tilde{e}^m,
 \tag{82}$$

and the four-dimensional vanishing torsion condition

$$\tilde{T}^m = \tilde{d}\tilde{e}^m + \tilde{\omega}^m_n \tilde{e}^n = 0,
 \tag{83}$$

where  $f' = \frac{\partial f}{\partial \phi}$ ,  $\tilde{\omega}^m_n = \omega^m_n$  and  $\tilde{d} = d\tilde{x}^\mu \frac{\partial}{\partial \tilde{x}^\mu}$ .

From (82), (83) and the Cartan’s second structural equation,  $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$ , we obtain the components of the 2-form curvature

$$\begin{aligned}
 R^{m4} &= \frac{e^f}{r_c} \left( f'^2 + f'' \right) d\phi \tilde{e}^m, \\
 R^{mn} &= \tilde{R}^{mn} - \left( \frac{e^f f'}{r_c} \right)^2 \tilde{e}^m \tilde{e}^n,
 \end{aligned}
 \tag{84}$$

where the four-dimensional 2-form curvature is given by

$$\tilde{R}^{mn} = \tilde{d}\tilde{\omega}^{mn} + \tilde{\omega}^m_p \tilde{\omega}^{pn}.
 \tag{85}$$

The torsion-free condition implies that the third term in the EChS action, given in Eq. (1), vanishes. This means that the corresponding Lagrangian is no longer dependent on the field  $k^{ab}$ . So, the Lagrangian (1) has now two independent fields,  $e^a$  and  $h^a$ , and it is given by

$$\begin{aligned}
 L_{ChS}^{(5)}[e, h] &= \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e \\
 &+ \alpha_3 \varepsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + l^2 R^{ab} R^{cd} h^e \right).
 \end{aligned}
 \tag{86}$$

From Eq. (86) we can see that the Lagrangian contains the Gauss–Bonnet term  $L_{GB}$ , the Einstein–Hilbert term  $L_{EH}$  and a term  $L_H$  which couples geometry and matter. In fact, replacing (79) and (84) in (86) and using  $\tilde{\varepsilon}_{mnpq} = \varepsilon_{mnpq4}$ , we obtain

$$\begin{aligned}
 \tilde{S}[\tilde{e}, \tilde{h}] &= \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \left( A \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + B \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right. \\
 &\quad \left. + C \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q + E \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right),
 \end{aligned}
 \tag{87}$$

where

$$h^m(\phi, \tilde{x}) = e^{g(\phi)} \tilde{h}^m(\tilde{x}), \quad h^4 = 0,
 \tag{88}$$

and

$$A = 2r_c \int_0^{2\pi} d\phi e^{2f} \left[ \alpha_3 - \frac{\alpha_1 l^2}{r_c^2} (3f'^2 + 2f'') \right], \tag{89}$$

$$B = -\frac{1}{r_c} \int_0^{2\pi} d\phi e^{4f} \left[ \frac{2\alpha_3}{3} (5f'^2 + 2f'') - \frac{\alpha_1 l^2}{r_c^2} f'^2 (5f'^2 + 4f'') \right], \tag{90}$$

$$C = -\frac{4\alpha_3 l^2}{r_c} \int_0^{2\pi} d\phi e^f e^g (f'^2 + f''), \tag{91}$$

$$E = \frac{4\alpha_3 l^2}{r_c^3} \int_0^{2\pi} d\phi e^{3f} e^g f'^2 (f'^2 + f''), \tag{92}$$

with  $f(\phi)$  and  $g(\phi)$  representing functions that can be chosen (non-unique choice) as  $f(\phi) = g(\phi) = \ln(K + \sin\phi)$  with  $K = \text{constant} > 1$ ; and therefore we have

$$A = \frac{2\pi}{r_c} [\alpha_3 r_c^2 (2K^2 + 1) + \alpha_1 l^2], \tag{93}$$

$$B = \frac{\pi}{2r_c} \left[ \alpha_3 (4K^2 + 1) - \frac{\alpha_1 l^2}{2r_c^2} \right], \tag{94}$$

$$C = -4r_c^2 E = \frac{4\pi\alpha_3 l^2}{r_c}. \tag{95}$$

Taking into account that  $L_{ChS}^{(5)}[e, h]$  flows into  $L_{EH}^{(5)}$  when  $l \rightarrow 0$  [13], we have that action (87) should lead to the action of Einstein–Hilbert when  $l \rightarrow 0$ . From (87) it is direct to see that this occurs when  $A = -1/2$  and  $B = 0$ . In this case, from Eqs. (93) to (95), we can see that

$$\alpha_1 = -\frac{r_c(4K^2 + 1)}{2\pi l^2(10K^2 + 3)}, \tag{96}$$

$$\alpha_3 = -\frac{1}{4\pi r_c(10K^2 + 3)}, \tag{97}$$

$$C = -4r_c^2 E = -\frac{l^2}{r_c^2(10K^2 + 3)}, \tag{98}$$

and therefore the action (87) takes the form

$$\tilde{S}[\tilde{e}, \tilde{h}] = \int_{\Sigma_4} \tilde{\epsilon}_{mnpq} \left( -\frac{1}{2} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + C \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q - \frac{C}{4r_c^2} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right), \tag{99}$$

corresponding to a four-dimensional brane embedded in the five-dimensional spacetime of the EChS gravity. We can see that when  $l \rightarrow 0$  then  $C \rightarrow 0$  and hence (99) becomes the four-dimensional Einstein–Hilbert action.

Finally, it is convenient to express the action (99) in tensorial language. To achieve this, we write  $\tilde{e}^m(\tilde{x}) = \tilde{e}^m_\mu(\tilde{x}) d\tilde{x}^\mu$  and  $\tilde{h}^m = \tilde{h}^m_\mu d\tilde{x}^\mu$ , and then we compute the individual terms in (99) as

$$\tilde{\epsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q = -2\sqrt{-\tilde{g}} \tilde{R} d^4 \tilde{x}, \tag{100}$$

$$\tilde{\epsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q = 2\sqrt{-\tilde{g}} \left( \tilde{R} \tilde{h} - 2\tilde{R}^\mu_\nu \tilde{h}^\nu_\mu \right) d^4 \tilde{x}, \tag{101}$$

$$\tilde{\epsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q = 6\sqrt{-\tilde{g}} \tilde{h} d^4 \tilde{x}, \tag{102}$$

where it has been defined  $\tilde{h} \equiv \tilde{h}^\mu_\mu$ . So, the four-dimensional action for the brane immersed in the five-dimensional spacetime of the EChS gravitational theory is given by

$$\tilde{S}[\tilde{g}, \tilde{h}] = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ \tilde{R} + 2C \left( \tilde{R} \tilde{h} - 2\tilde{R}^\mu_\nu \tilde{h}^\nu_\mu \right) - \frac{3C}{2r_c^2} \tilde{h} \right]. \tag{103}$$

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