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$K \to \mu^+\mu^-$ as a clean probe of short-distance physics

Avital Dery,[a] Mitrajyoti Ghosh,[a] Yuval Grossman[a] and Stefan Schacht[b]

[a] *Department of Physics, LEPP, Cornell University, Ithaca, NY 14853, U.S.A.*

[b] *Department of Physics and Astronomy, University of Manchester, Manchester, M13 9PL, U.K.*

E-mail: avital.dery@cornell.edu, mg2338@cornell.edu, yg73@cornell.edu, stefan.schacht@manchester.ac.uk

Abstract: The $K \to \mu^+\mu^-$ decay is often considered to be uninformative of fundamental theory parameters since the decay is polluted by long-distance hadronic effects. We demonstrate that, using very mild assumptions and utilizing time-dependent interference effects, $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ can be experimentally determined without the need to separate the $\ell = 0$ and $\ell = 1$ final states. This quantity is very clean theoretically and can be used to test the Standard Model. In particular, it can be used to extract the CKM matrix element combination $|V_{ts}V_{td}\sin(\beta+\beta_s)| \approx |A^2\lambda^5\bar{\eta}|$ with hadronic uncertainties below 1%.

Keywords: CP violation, Kaon Physics, Quark Masses and SM Parameters

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1 Introduction

Rare flavor changing neutral current (FCNC) kaon decays [1–7] provide a unique way to probe the flavor sector of the Standard Model (SM) and, in particular, CP-violating effects. The program to measure the decay rates of $K^+ \to \pi^+\nu\bar{\nu}$ [8] and $K_L \to \pi^0\nu\bar{\nu}$ [9] is aimed at determining the CKM parameters with very high theoretical precision. In particular, the $K_L \to \pi^0\nu\bar{\nu}$ decay rate can be used to extract [10–12]

$$|V_{ts}V_{td}\sin(\beta+\beta_s)| \approx |A^2\lambda^5\bar{\eta}|\,, \tag{1.1}$$

where A, λ, and $\bar{\eta}$ are the Wolfenstein parameters and $\beta+\beta_s$ is one of the angles in the ds unitarity triangle such that [13]

$$\beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \quad \beta_s = \arg\left(-\frac{V_{ts}V_{tb}^*}{V_{cs}V_{cb}^*}\right), \quad \beta+\beta_s-\pi = \arg\left(-\frac{V_{ts}V_{td}^*}{V_{cs}V_{cd}^*}\right). \tag{1.2}$$

Experimentally, working with decays that involve charged leptons is much simpler than the above-mentioned neutrino modes. Nonetheless, the focus of the current kaon program is on the neutrino final states, primarily because decays to charged leptons are believed not to be theoretically clean. There are so-called long-distance effects that introduce hadronic uncertainties, making extractions of clean theory parameters challenging.

In this paper, we show that we can get very clean theoretical information from decays of kaons into charged leptons. This can be done only for the neutral kaons, by exploiting the interference effects between K_S and K_L. We focus on $K \to \mu^+\mu^-$, for which the relevant CKM observable is that of eq. (1.1). The theoretical precision in this case is superb, with hadronic uncertainties below the 1% level.

The importance of the interference terms in $K \to \mu^+\mu^-$ was emphasized in ref. [14]. In this paper, we generalize their results and demonstrate that one can get a very clean determination of the parameter combination in eq. (1.1) by studying the interference terms.

Before we get into the details, below we explain the main idea. We first recall the situation with $K_L \to \pi^0\nu\bar{\nu}$. The reason that this decay mode is theoretically clean is that it is to a very good approximation pure CP-violating. As such, it is all calculable using perturbation theory and we do not have to worry about non-calculable long-distance effects, as they are to a very good approximation CP conserving.

The issue with $K \to \mu^+\mu^-$ is that the final state is a mixture of $\ell = 0$ and $\ell = 1$ partial wave configurations. Thus, both K_S and K_L decays are not pure CP-violating, and both decays have non-calculable long distance effects. Yet, if we could experimentally distinguish between the $\ell = 0$ and $\ell = 1$ final states, the situation would be similar to $K_L \to \pi^0\nu\bar{\nu}$, as we could separate the CP-violating part that we can calculate. In particular, the $\ell = 0$ amplitude has significant CP violation effects in the SM, and the decay mode $K_S \to (\mu^+\mu^-)_{\ell=0}$ is very clean theoretically. What we show in this work is that under some mild assumptions we can extract the rate, that is, $\mathcal{B}(K_S \to (\mu^+\mu^-)_{\ell=0})$ without separating the $\ell = 0$ and $\ell = 1$ final states. This can be done by isolating the interference terms.

Leptonic kaon decays have been studied for a long time [15–31]. Rare kaon decays have a lot of potential for the discovery of physics beyond the SM [32–50]. Also on the experimental side a lot of advances took place in the quest for rare kaon decays [8, 9, 51–57].

The SM predictions for $K \to \mu^+\mu^-$ [14, 58–60] and the corresponding long-distance contributions [58, 61–63] have been studied in great detail. The same goes for $K_S \to \gamma\gamma$ and $K_S \to \gamma l^+l^-$ [61] as well as kaon decays into four leptons [64]. See also the reviews refs. [65, 66].

2 Notation and formalism

We use the following standard notation [67], where the two neutral kaon mass eigenstates, $|K_S\rangle$ and $|K_L\rangle$, are linear combinations of the flavor eigenstates:

$$|K_S\rangle = p|K^0\rangle + q|\overline{K}^0\rangle, \qquad |K_L\rangle = p|K^0\rangle - q|\overline{K}^0\rangle. \tag{2.1}$$

The mass and width averages and differences are denoted by

$$\begin{aligned} m &= \frac{m_L + m_S}{2}, & \Gamma &= \frac{\Gamma_L + \Gamma_S}{2}, \\ \Delta m &= m_L - m_S, & \Delta\Gamma &= \Gamma_L - \Gamma_S. \end{aligned} \tag{2.2}$$

We define the decay amplitudes of $|K^0\rangle$ and $|\overline{K}^0\rangle$ to a final state f,

$$A_f = \langle f|\mathcal{H}|K^0\rangle, \qquad \overline{A}_f = \langle f|\mathcal{H}|\overline{K}^0\rangle, \tag{2.3}$$

and the parameter λ_f,

$$\lambda_f \equiv \frac{q}{p}\frac{\overline{A}_f}{A_f}\,. \tag{2.4}$$

We use an arbitrary normalization, such that A_f and $\overline{A}_f$ have the same normalization.

An amplitude is called *relatively real* if $\mathcal{I}m\lambda_f = 0$ and *relatively imaginary* if $\mathcal{R}e\lambda_f = 0$. Any amplitude can be written as a sum of a relatively real and a relatively imaginary part.

In any neutral meson system, the quantities A_f, $\overline{A}_f$, and q/p depend on the phase convention. However, $|A_f|$, $|\overline{A}_f|$, $|q/p|$, and λ_f are phase convention independent and are hence physical.

Consider a beam of neutral kaons. The time dependent decay rate as a function of proper time is given by [67]

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\right) = \mathcal{N}_f f(t), \tag{2.5}$$

where $\mathcal{N}_f$ is a time-independent normalization factor and the function $f(t)$ is given as a sum of four functions

$$f(t) = C_L e^{-\Gamma_L t} + C_S\, e^{-\Gamma_S t} + 2\left[C_{\sin}\sin(\Delta m t) + C_{\cos}\cos(\Delta m t)\right] e^{-\Gamma t}. \tag{2.6}$$

The form of eq. (2.6) is valid for any neutral kaon beam (that is, not only for a pure state) and also for a sum over several final states. We refer to the set of coefficients, $\{C_L, C_S, C_{\sin}, C_{\cos}\}$, as the *experimental parameters*. Note that C_L is the coefficient of the K_L decay term, C_S of the K_S decay term, while $C_{\sin}$ and $C_{\cos}$ come with the interference terms between K_L and K_S. For convenience we also define

$$C^2_{\text{Int.}} = C^2_{\cos} + C^2_{\sin}\,. \tag{2.7}$$

The C coefficients implicitly depend on the composition of the beam and on the relevant final states. The dependence on the final states enters via the parameters

$$\{|A_f|\,, \qquad |\overline{A}_f|\,, \qquad |q/p|\,, \qquad \arg(\lambda_f)\}. \tag{2.8}$$

We denote these as the *theory parameters*.

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For an initial $|K^0\rangle$ and $|\overline{K}^0\rangle$ beam, respectively, and a single final state, f, the coefficients are explicitly given by [67]

$$
\begin{aligned}
C_L^{K^0} &= \frac{1}{2}|A_f|^2\left(1+|\lambda_f|^2-2\mathcal{R}e\lambda_f\right), &\qquad C_L^{\overline{K}^0} &= \frac{1}{2}|\overline{A}_f|^2\left(1+|\lambda_f|^{-2}-2\mathcal{R}e\lambda_f^{-1}\right),\\
C_S^{K^0} &= \frac{1}{2}|A_f|^2\left(1+|\lambda_f|^2+2\mathcal{R}e\lambda_f\right), &\qquad C_S^{\overline{K}^0} &= \frac{1}{2}|\overline{A}_f|^2\left(1+|\lambda_f|^{-2}+2\mathcal{R}e\lambda_f^{-1}\right),\\
C_{\sin}^{K^0} &= -|A_f|^2\mathcal{I}m\lambda_f\,, &\qquad C_{\sin}^{\overline{K}^0} &= -|\overline{A}_f|^2\mathcal{I}m\lambda_f^{-1},\\
C_{\cos}^{K^0} &= \frac{1}{2}|A_f|^2\left(1-|\lambda_f|^2\right), &\qquad C_{\cos}^{\overline{K}^0} &= \frac{1}{2}|\overline{A}_f|^2\left(1-|\lambda_f|^{-2}\right).
\end{aligned}
\tag{2.9}
$$

In the following we focus on decays into CP-eigenstate final states. For a given final state, f, we define $\eta_f = 1$ if it is CP-even and $\eta_f = -1$ if it is CP-odd. We define the CP-even and CP-odd amplitudes

$$
A_f^{\text{CP-even}} \equiv \frac{1}{\sqrt{2}}A_f\left(1+\eta_f\lambda_f\right), \qquad A_f^{\text{CP-odd}} \equiv \frac{1}{\sqrt{2}}A_f\left(1-\eta_f\lambda_f\right). \tag{2.10}
$$

We make several assumptions and approximations as we go on. Our first approximation is

(i) CP violation (CPV) in mixing is negligible.

Although our main interest is CP violating physics, CPV in mixing is subdominant in the effects we consider. We therefore neglect it throughout the paper and work in the limit

$$
\left|\frac{q}{p}\right| = 1.
$$

This approximation is known to work to order $\epsilon_K \sim 10^{-3}$ which we neglect from this point on.

Under the above assumption, the full set of decay-mode-specific independent physical parameters can be taken to be

$$
\{|A_f|, \qquad |\overline{A}_f|, \qquad \arg\left(\lambda_f\right)\}. \tag{2.11}
$$

Furthermore, in the limit of no CPV in mixing, the CP amplitudes of eq. (2.10) correspond to the amplitudes for the decays of K_S and K_L. For example, for $f = \pi^+\pi^-$, $\eta_f = 1$ and to a very good approximation $\lambda_f = 1$ and thus $A_{\pi^+\pi^-}^{\text{CP-odd}} = 0$. In the case of $K \to \pi\nu\bar{\nu}$, $\eta_f = 1$ and λ_f is to a very good approximation a pure phase, so that the amplitude for $K_L \to \pi\nu\bar{\nu}$ gives sensitivity to the phase $\arg(\lambda_f)$ [32].

In the following it will be useful to replace the set of independent physical parameters of eq. (2.11) with the equivalent set of physical parameters:

$$
\{|A_f^{\text{CP-even}}|, \qquad |A_f^{\text{CP-odd}}|, \qquad \arg\left(A_f^{\text{CP-even}*}A_f^{\text{CP-odd}}\right)\}. \tag{2.12}
$$

In particular, the time dependence for a beam of initial $|K^0\rangle$ into a CP-even final state is given by the coefficients

$$
\begin{aligned}
C_L^{K^0} &= |A_f^{\text{CP-odd}}|^2, &\qquad C_S^{K^0} &= |A_f^{\text{CP-even}}|^2,\\
C_{\cos}^{K^0} &= \mathcal{R}e(A_f^{\text{CP-odd}*}A_f^{\text{CP-even}}), &\qquad C_{\sin}^{K^0} &= -\mathcal{I}m(A_f^{\text{CP-odd}*}A_f^{\text{CP-even}}),
\end{aligned}
\tag{2.13}
$$

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For a CP-odd final state it is given by

$$
\begin{aligned}
C_L^{K^0} &= |A_f^{\text{CP-even}}|^2, & C_S^{K^0} &= |A_f^{\text{CP-odd}}|^2, \\
C_{\cos}^{K^0} &= \mathcal{R}e(A_f^{\text{CP-odd}*} A_f^{\text{CP-even}}), & C_{\sin}^{K^0} &= \mathcal{I}m(A_f^{\text{CP-odd}*} A_f^{\text{CP-even}}).
\end{aligned} \tag{2.14}
$$

For an initial $|\overline{K}^0\rangle$ state the result is obtained by multiplying $C_{\cos}$ and $C_{\sin}$ by -1 in eqs. (2.13) and (2.14).

We also define

$$
\varphi_f = \arg(A_f^{\text{CP-odd}*} A_f^{\text{CP-even}}), \tag{2.15}
$$

such that we can write for a CP-even final state

$$
C_{\cos}^{K^0} = |A_f^{\text{CP-odd}*}\, A_f^{\text{CP-even}}| \cos\varphi_f\,, \qquad C_{\sin}^{K^0} = -|A_f^{\text{CP-odd}*}\, A_f^{\text{CP-even}}| \sin\varphi_f\,. \tag{2.16}
$$

For a CP-odd final state we have analogously

$$
C_{\cos}^{K^0} = |A_f^{\text{CP-odd}*}\, A_f^{\text{CP-even}}| \cos\varphi_f\,, \qquad C_{\sin}^{K^0} = |A_f^{\text{CP-odd}*}\, A_f^{\text{CP-even}}| \sin\varphi_f\,. \tag{2.17}
$$

3 The $K \to \mu^+\mu^-$ decay

In the decay of a neutral kaon into a pair of muons, there are two orthogonal final states that are allowed by conservation of angular momentum — muons with a symmetric wave function ($\ell = 0$) and muons with an anti-symmetric wave function ($\ell = 1$). Note that since the leptons are fermions, the state with $\ell = 0$ has negative parity and so it is CP odd, and the state with $\ell = 1$ is CP even. The four relevant amplitudes can be written in terms of the CP amplitudes of eq. (2.10) as

$$
A_\ell^{\text{CP-even}} = \frac{1}{\sqrt{2}} A_\ell \left(1 - (-1)^\ell \lambda_\ell\right), \qquad A_\ell^{\text{CP-odd}} = \frac{1}{\sqrt{2}} A_\ell \left(1 + (-1)^\ell \lambda_\ell\right), \tag{3.1}
$$

with $\ell = 0, 1$. Note that we keep the normalization arbitrary, but if we want to maintain the same normalization for both A_0 and A_1 then we require a relative phase space factor between them, β_μ^2, with

$$
\beta_\mu \equiv \left(1 - \frac{4m_\mu^2}{m_K^2}\right)^{\frac{1}{2}}, \tag{3.2}
$$

see for details appendix B.

Note that under the approximation $|q/p| = 1$, eq. (3.1) allows us to write the CP-even and -odd amplitudes as amplitudes for the decays of the mass eigenstates $|K_S\rangle$ and $|K_L\rangle$:

$$
\begin{aligned}
A_0^{\text{CP-odd}} &= A(K_S \to \mu^+\mu^-)_{\ell=0}\,, \\
A_0^{\text{CP-even}} &= A(K_L \to \mu^+\mu^-)_{\ell=0}\,, \\
A_1^{\text{CP-odd}} &= A(K_L \to \mu^+\mu^-)_{\ell=1}\,, \\
A_1^{\text{CP-even}} &= A(K_S \to \mu^+\mu^-)_{\ell=1}.
\end{aligned} \tag{3.3}
$$

When measuring the total time dependent decay rate for $K \to \mu^+\mu^-$, the two di-muon configurations, $\ell = 0, 1$ add incoherently. The form of the function $f(t)$ defined in eq. (2.6), is unchanged. Theoretically, each of the C's is given by an implicit sum over the relevant amplitude expressions for different ℓ's. Thus we have two sets of decay-mode-specific physical theory parameters,

$$\{|A_\ell^{\text{CP-even}}|, \qquad |A_\ell^{\text{CP-odd}}|, \qquad \varphi_\ell \equiv \arg\left(A_\ell^{\text{CP-odd}*} A_\ell^{\text{CP-even}}\right)\}, \tag{3.4}$$

with $\ell = 0, 1$, bringing us to a total of six unknown physical parameters.

It is well known that the decay $K \to \mu^+\mu^-$ receives long-distance and short-distance contributions [59, 68–70]. The long-distance contribution is dominated by diagrams with two intermediate on-shell photons, while the short-distance contribution is defined as originating from the weak effective Hamiltonian. The distinction between long-distance and short-distance physics is somewhat ambiguous. It is clear that the short-distance physics is to a good approximation dispersive (real), since it is dominated by heavy particles in the loops. However, long-distance diagrams contribute both to the absorptive (imaginary) amplitude and, when taken off-shell, also to the dispersive amplitude.

In the following we make one extra simplifying assumption, which results in reducing the number of unknown parameters for $K \to \mu^+\mu^-$. We consider only models where

(ii) The only source of CP violation is in the $\ell = 0$ *amplitude.*

What we mean by this assumption is that only the $\ell = 0$ amplitude has $\mathcal{I}m(\lambda_\ell) \neq 0$.

As we discuss in section 5 and in appendix C, this assumption is fulfilled to a very good approximation within the SM and in any model in which the leading leptonic operator is vectorial.

We can then draw an important conclusion from the above assumption:

$$A_1^{\text{CP-odd}} = 0. \tag{3.5}$$

This implies that the number of unknown parameters is reduced by two, leaving a single parameter, $|A_1^{\text{CP-even}}|$, for the $\ell = 1$ final state. Thus, we are left with the following list of four unknown physical parameters,

$$|A_0^{\text{CP-odd}}|, \qquad |A_0^{\text{CP-even}}|, \qquad |A_1^{\text{CP-even}}|, \qquad \arg(A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}). \tag{3.6}$$

In the rest of the paper we demonstrate how it is possible to extract these parameters, and specifically $|A_0^{\text{CP-odd}}| = A(K_S \to \mu^+\mu^-)_{\ell=0}$, which, as we explain below, is a clean probe of the SM.

4 Extracting $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$

As portrayed in eq. (2.6), the time-dependent decay rate for an arbitrary neutral kaon initial state is given in general by the sum of four independent functions of time that depend on the experimentally extracted parameters

$$\{C_L, \qquad C_S, \qquad C_{\cos}, \qquad C_{\sin}\}. \tag{4.1}$$

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Within our assumptions, these coefficients depend on the following four theory parameters

$$\{|A_0^{\text{CP-odd}}|, \qquad |A_0^{\text{CP-even}}|, \qquad |A_1^{\text{CP-even}}|, \qquad \varphi_0 \equiv \arg(A_0^{\text{CP-odd}*} A_0^{\text{CP-even}})\}. \tag{4.2}$$

We consider a case of a beam that at $t = 0$ was a pure K^0 beam (that is, no $\overline{K}^0$). Using eq. (2.9) we obtain that the result for this case is given by

$$\begin{aligned} C_L &= |A_0^{\text{CP-even}}|^2, \\ C_S &= |A_0^{\text{CP-odd}}|^2 + \beta_\mu^2 |A_1^{\text{CP-even}}|^2, \\ C_{\cos} &= \mathcal{R}e(A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}) = |A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}| \cos\varphi_0, \\ C_{\sin} &= \mathcal{I}m(A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}) = |A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}| \sin\varphi_0. \end{aligned} \tag{4.3}$$

We see that the four experimental parameters can be used to extract the four theory parameters. In particular, we find

$$|A_0^{\text{CP-odd}}|^2 = \frac{C_{\cos}^2 + C_{\sin}^2}{C_L} = \frac{C_{\text{Int.}}^2}{C_L}, \tag{4.4}$$

where $C_{\text{Int.}}^2 = C_{\cos}^2 + C_{\sin}^2$ was defined in eq. (2.7). Having the magnitude of the amplitude we can deduce the branching ratio in terms of other observables,

$$\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} = \mathcal{B}(K_L \to \mu^+\mu^-) \times \frac{\tau_S}{\tau_L} \times \left(\frac{C_{\text{int}}}{C_L}\right)^2. \tag{4.5}$$

Eq. (4.5) is our main result. It demonstrates that we can extract $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ from the experimental time dependent decay rate.

A few comments are in order regarding eq. (4.5):

1. Our ability to extract $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ comes from the interference terms. It cannot be extracted from pure K_L or K_S terms.

2. A measurement of the interference terms additionally amounts to a measurement of the phase φ_0, which is not calculable from short-distance physics.

3. In order to extract $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ we need only three of the four experimental parameters. The fourth parameter, C_S, can then be used to extract $|A_1^{\text{CP-even}}|$, or equivalently $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=1}$. Yet, this is not our main interest, as $|A_1^{\text{CP-even}}|$ is not calculable from short-distance physics.

4. For a pure $\overline{K}^0$ beam, C_S and C_L in eq. (4.3) are unchanged while $C_{\cos}$ and $C_{\sin}$ pick up a minus sign, and eq. (4.5) is unchanged.

While we have only discussed a pure K^0 beam in this section, as long as we have sensitivity to the interference terms, it is possible to determine $|A_0^{\text{CP-odd}}|$. In particular, as long as the kaon decays in vacuum, one can write the branching ratio $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ in terms of $\mathcal{B}(K_L \to \mu^+\mu^-)$ in the following way:

$$\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} = \mathcal{D}_F \times \mathcal{B}(K_L \to \mu^+\mu^-) \times \frac{\tau_S}{\tau_L} \times \left(\frac{C_{\text{int}}}{C_L}\right)^2. \tag{4.6}$$

where $\mathcal{D}_F$ is a dilution factor that takes into account the particular composition of the kaon beam. We discuss two cases, that of a mixed beam, and of a K_L beam with regeneration, in appendix A.

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5 Calculating $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$

We move to discuss the theoretical calculation of $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$.

5.1 General calculation

We define

$$A_\ell = A_\ell^{SD} + A_\ell^{LD}. \tag{5.1}$$

The short-distance (SD) amplitude, A_ℓ^{SD}, is the one that can be calculated perturbatively from the effective Hamiltonian of any model. Note that at leading order in the perturbative calculation it carries no strong phase. By definition, the long-distance (LD) amplitude, A_ℓ^{LD}, is the part that is not captured by that calculation. In general, it carries a strong phase. We further define

$$\lambda_\ell^{SD} = \frac{q}{p}\frac{\overline{A}_\ell^{SD}}{A_\ell^{SD}}, \qquad \lambda_\ell^{LD} = \frac{q}{p}\frac{\overline{A}_\ell^{LD}}{A_\ell^{LD}}. \tag{5.2}$$

Note that since we assume that the SD amplitude carries no strong phase, we have $|\lambda_\ell^{SD}| = 1$.

We now adopt one more working assumption, that is, we consider only models where:

> *(iii) The long-distance physics is CP conserving.*
>
> That is, we only consider cases where A_ℓ^{LD} is relatively real, that is, $\mathcal{I}m(\lambda_\ell^{LD}) = 0$.

In particular, this assumption implies that we can trust the perturbative calculation for the CP-violating amplitude, using specific operators described by quarks.

We are now ready to discuss the CP-odd amplitudes. Because of assumption (*ii*) we have $A_1^{\text{CP-odd}} = 0$. Thus we only need to consider the $\ell = 0$ CP-odd amplitude. Using eqs. (2.10) and (3.1) we write it as

$$A_0^{\text{CP-odd}} = \frac{1}{\sqrt{2}} A_0^{SD}\left(1 + \lambda_0^{SD}\right). \tag{5.3}$$

Then, using the fact that $|\lambda_0^{SD}| = 1$, we get

$$|A_0^{\text{CP-odd}}|^2 = |A_0^{SD}|^2\left[1 + \text{Re}(\lambda_0^{SD})\right] = |A_0^{SD}|^2\left[1 - \cos\left(2\phi_0^{SD}\right)\right] = 2|A_0^{SD}|^2 \sin^2\phi_0^{SD}\,, \tag{5.4}$$

where we define

$$\phi_0^{SD} = \frac{1}{2}\arg\left(-\lambda_0^{SD}\right). \tag{5.5}$$

Note that the result is independent of the way we choose to split the amplitude into long- and short-distance physics as long as the part we call "long-distance" is relatively real. Moreover, we can subtract from A_0^{SD} any part that is relatively real without affecting the result. We use this freedom below when we discuss the SM prediction.

We conclude that in any model that satisfies our assumptions, we need to calculate $|A_0^{SD}|^2$ and $\sin^2\phi_0^{SD}$ in order to make a prediction for $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$.

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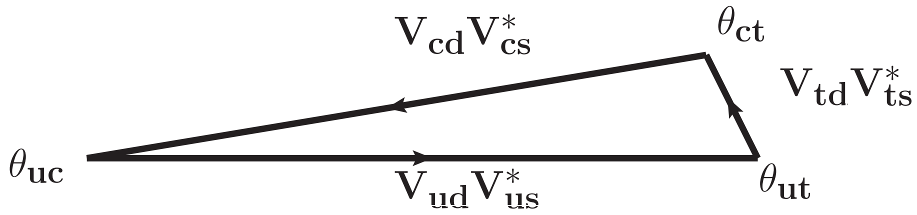

Figure 1. The "ds" unitarity triangle, see refs. [13, 71]. The plot is not to scale.

5.2 SM calculation

Next we discuss the situation in the SM and remark on more generic models. The SM short-distance prediction has been discussed in ref. [59]. Here we do not present any new arguments, but instead we review the results in the literature, explicitly stating the assumptions made, and present the results in a basis independent way.

In order to discuss the situation in the SM we look at the "ds" unitarity triangle, that we plot in figure 1. The angles are given as [13]:

$$\theta_{ct} \equiv \arg\left(-\frac{V_{td}V^*_{ts}}{V_{cd}V^*_{cs}}\right) = \pi - \beta - \beta_s \sim \lambda^0 \,, \tag{5.6}$$

$$\theta_{ut} = \arg\left(-\frac{V_{ud}V^*_{us}}{V_{td}V^*_{ts}}\right) = \beta + \beta_s - \theta_{uc} \sim \lambda^0 \,, \tag{5.7}$$

$$\theta_{uc} = \arg\left(-\frac{V_{cd}V^*_{cs}}{V_{ud}V^*_{us}}\right) \sim \lambda^4 \,. \tag{5.8}$$

In what follows, when we discuss the SM prediction, we make one more approximation:

(*iv*) We neglect effects of $\mathcal{O}(\lambda^4)$. In particular, we set $\theta_{uc} = 0$.

With this approximation we then write

$$\frac{q}{p} = -\left(\frac{V_{cd}V^*_{cs}}{V^*_{cd}V_{cs}}\right)\left[1 + \mathcal{O}(\lambda^4)\right] \approx -\left(\frac{V_{cd}V^*_{cs}}{V^*_{cd}V_{cs}}\right) , \tag{5.9}$$

where in the last step we used $\theta_{uc} = 0$.

We are now ready to show that in the SM the long-distance amplitude is CP conserving, complying with assumption (iii) above. The claim is that the CKM factors in the long-distance amplitudes are to a good approximation $V_{us}V^*_{ud}$. The reason is that rescattering effects, which are what results in the long-distance contributions, are dominated by tree level decays followed by QCD rescattering. The most important one is $K \to \gamma\gamma$, which is dominated by the π^0 poles [59, 62]. We thus have

$$\lambda_0^{LD} = \frac{q}{p}\frac{\overline{A}_0^{LD}}{A_0^{LD}} = -\left(\frac{V_{cd}V^*_{cs}}{V^*_{cd}V_{cs}}\right)\left(\frac{V_{ud}V^*_{us}}{V^*_{ud}V_{us}}\right) \quad \Rightarrow \quad \mathcal{I}m(\lambda_0^{LD}) = 0 \,, \tag{5.10}$$

where in the last step we use $\theta_{uc} = 0$. The fact that $\mathcal{I}m(\lambda_0^{LD}) = 0$ implies that the long-distance amplitude is CP conserving.

We next discuss working assumption (ii) above, that is, that CP violation enters only for $\ell = 0$. Within the SM the short-distance effects are due to the following Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2\theta_W} \left[V_{cs}^* V_{cd} Y_{NL} + V_{ts}^* V_{td} Y(x_t)\right] \left[(\bar{s}d)_{V-A}(\bar{\mu}\mu)_{V-A}\right] + h.c., \tag{5.11}$$

with $x_t = m_t^2/m_W^2$, and the loop function $Y(x_t) \approx 0.950 \pm 0.049$ and $Y_{NL} = \mathcal{O}(10^{-4})$ [60, 72]. Thus the leading SM short-distance physics operator is

$$(\bar{s}d)_{V-A}(\bar{\mu}\mu)_{V-A} + h.c.\,. \tag{5.12}$$

This operator contributes only to the $\ell = 0$ final state [59]. For completeness, we provide a short derivation of this known result in appendix C.

A few comments are in order:

1. Scalar operators could also lead to CP violation in the $\ell = 1$ amplitude through short-distance effects. However, in the SM, the contribution of these operators to the rate are suppressed with respect to the operator in eq. (5.11) by a factor of $(m_K/m_W)^2 \sim 10^{-5}$ [73], and can be safely neglected for the extraction of SM parameters.

2. Only the axial-times-axial part of the hadronic times leptonic currents of eq. (5.12) is relevant for $K \to \mu^+\mu^-$ (see appendix C).

We conclude that the approximations and assumptions we work under are valid in the SM up to very small deviations, of order $\lambda^4 \sim \epsilon_K \sim 10^{-3}$. Thus, within the SM, the only source of a CP violating phase is the weak effective Hamiltonian given in eq. (5.11). Moreover, any extension of the SM in which the leptonic operator remains vectorial rather than a scalar would satisfy our set of assumptions. For example also models with right-handed currents fall under this category. Thus, within the SM and any such extension it is straightforward to extract a prediction for $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ purely from short-distance physics.

We are now ready to discuss the SM prediction for $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$. We recover the result, given in ref. [59], using phase convention independent expressions (see appendix B). We first redefine A_0^{SD} by subtracting the charm contribution, which is relatively real under the approximation $\theta_{uc} = 0$. Then we can write

$$\lambda_0^{SD} = \frac{q}{p}\frac{\overline{A}_0^{SD}}{A_0^{SD}} = -\left(\frac{V_{cd}V_{cs}^*}{V_{cd}^*V_{cs}}\right)\left(\frac{V_{td}^*V_{ts}}{V_{td}V_{ts}^*}\right) = -e^{-2i\theta_{ct}} \quad \Rightarrow \quad \sin^2\phi_0^{SD} = \sin^2\theta_{ct}. \tag{5.13}$$

The calculation of $|A_0^{SD}|^2$ and the phase space integral is reviewed in appendix B. The result is given in eq. (B.8):

$$\begin{aligned}\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} &= \frac{\beta_\mu \tau_S}{16\pi m_K}\left|\frac{G_F}{\sqrt{2}}\frac{2\alpha_{em}}{\pi \sin^2\theta_W} m_K m_\mu \times Y(x_t) \times f_K \times V_{ts}V_{td}\sin\theta_{ct}\right|^2 \\ &\approx 1.64 \cdot 10^{-13} \times \left|\frac{V_{ts}V_{td}\sin\theta_{ct}}{(A^2\lambda^5\bar{\eta})_{\text{best fit}}}\right|^2,\end{aligned} \tag{5.14}$$

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where we use

$$(A^2\lambda^5\bar{\eta})_{\text{best fit}} = 1.33 \cdot 10^{-4}. \tag{5.15}$$

Eq. (5.14) is very precise. There are a few sources of uncertainties that enter here. They are all under control:

1. The only hadronic parameter is the kaon decay constant, which is well known from charged kaon decays. Isospin breaking effects can also be incorporated in lattice QCD if needed [74], reducing the ultimate hadronic uncertainties below the 1% level.

2. We have neglected subleading terms, that is, we neglected the term proportional to $Y_{NL} \sim 10^{-4}$ from eq. (5.11), as well as CPV effects of order ϵ_K.

3. Parametric errors, including the dependence of the loop function $Y(x_t)$ on m_t/m_W, are small, as the errors on the top and W masses are below the 1% level.

4. Only leading order results in the loop expansion are used. Higher order terms are expected to be suppressed by a loop factor, which is of order 1%. If needed, higher orders in the loop function can be incorporated in order to reduce this uncertainty.

5. We only consider the leading SM operator, which is vectorial. At higher order scalar operators are also present, but these effects are suppressed by $\mathcal{O}(m_K^2/m_W^2)$ [73].

We conclude that a measurement of $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ would be a very clean independent measurement of the following combination of CKM elements

$$|V_{ts}V_{td}\,\sin\theta_{ct}| = |V_{ts}V_{td}\,\sin(\beta+\beta_s)| \approx A^2\lambda^5\bar{\eta}\,, \tag{5.16}$$

which coincides with eq. (1.1).

A similar analysis can be done in any model that satisfies the assumptions we have made. In particular, these results hold in any model that generates the same operator as in the SM. In such a model the prediction would be amended by replacing the SM values for the CKM parameters and the loop function with the respective values in the model under consideration.

We end this section with two remarks

1. There are models where we can have a significant contribution to the CP-odd amplitude from scalar operators [37], in which case our assumption (*ii*) is not satisfied.

2. In addition to our quantity of interest, $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$, under the same set of assumptions it is also possible to calculate the short-distance contribution to $A_0^{\text{CP-even}}$, that is, $A(K_L \to \mu^+\mu^-)_{\ell=0}^{SD}$. Then, assuming given values for the CKM parameters, the measurement of the interference terms is also a measurement of the long-distance amplitude $A(K_L \to \mu^+\mu^-)_{\ell=0}^{LD}$, and in particular of its unknown sign [14].

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6 Experimental considerations

We now turn to discuss the feasibility of the extraction of $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$. As is apparent from eq. (4.5) we need to experimentally extract $C_{\text{Int.}}$ and C_L. Of these, C_L has already been measured, and we can expect that in the future it will be measured with even higher precision. The question is how well can $C_{\text{Int.}}$ be extracted.

Below we estimate the number of kaons that is needed to perform the measurement assuming the SM. For that we need the values in the SM of the relevant amplitudes. While the method we discuss does not require any estimation of the amplitudes, we use these estimates to illustrate the expected magnitude of the interference terms, and to estimate the needed statistics to perform the measurements. Of the three amplitudes, $|A_0^{\text{CP-odd}}|$ can be calculated perturbatively, $|A_0^{\text{CP-even}}|$ can be extracted directly from the measured value of $\mathcal{B}(K_L \to \mu^+\mu^-)$, and $|A_1^{\text{CP-even}}|$ can only be estimated a priori by relying on non-perturbative calculations from the literature, that suffer from large hadronic uncertainties. We provide the details of these estimations in appendix B. They result in the following values for the experimental parameters:

$$\begin{aligned}
(C_L^{K^0})_{\text{SM}} &= |A_0^{\text{CP-even}}|^2 \equiv 1, \\
(C_S^{K^0})_{\text{SM}} &= |A_0^{\text{CP-odd}}|^2 + \beta_\mu^2 |A_1^{\text{CP-even}}|^2 \approx 0.43, \\
(C_{\text{Int.}}^{K^0})_{\text{SM}} &= |A_0^{\text{CP-even}}||A_0^{\text{CP-odd}}| \approx 0.12,
\end{aligned} \tag{6.1}$$

where we have used a normalization such that the coefficient $(C_L^{K^0})_{\text{SM}}$ is set to be unity. Using these estimates, we plot the time dependence of the rate in figure 2, for two values of the unknown phase, $\varphi_0 = \arg(A_0^{\text{CP-odd}*} A_0^{\text{CP-even}})$. For illustration, we also plot the time dependence excluding the interference terms (see caption). We use the range $t \lesssim 6\tau_S$ as for larger times the beam is almost a pure K_L beam. The relative magnitude of the interference terms is apparent in the difference between the two plotted curves. We find the relative integrated effect to be of order 3% to 6%, depending on the value of φ_0.

Based on the above, we can roughly estimate the number of required kaons. We have $\mathcal{B}(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \cdot 10^{-9}$ [67], and only about 1% of the K_L particles decay inside our region of interest, $t \lesssim 6\tau_S$. Since the coefficients in eq. (6.1) are not very small, we can use this to estimate that the number of useful events is roughly a fraction of 10^{-10} out of the kaons. Thus, for example, in order to get $\mathcal{O}(1000)$ events in the interesting region we need $\mathcal{O}(10^{13})$ K^0 particles to start with. We do not expect this preliminary estimate to be strongly affected by backgrounds or reconstruction efficiencies.

Experimentally, it is not easy to produce a pure neutral kaon beam. Experiments currently running enjoy a very high luminosity of kaons of order 10^{14} kaons a year (see ref. [75] for NA62, ref. [9] for KOTO, and ref. [76] for LHCb). However, these kaons are either charged (NA62), or to a good approximation a pure K_L (KOTO), or come with an almost equal mix of K^0 and $\overline{K}^0$ (LHCb).

Thus, for the purpose of the analysis we are considering, we need to turn to a mixed beam or a regenerated beam. As discussed in appendix A, in the case of a mixed beam with non-zero production asymmetry, the sensitivity to the interference terms is diluted by a

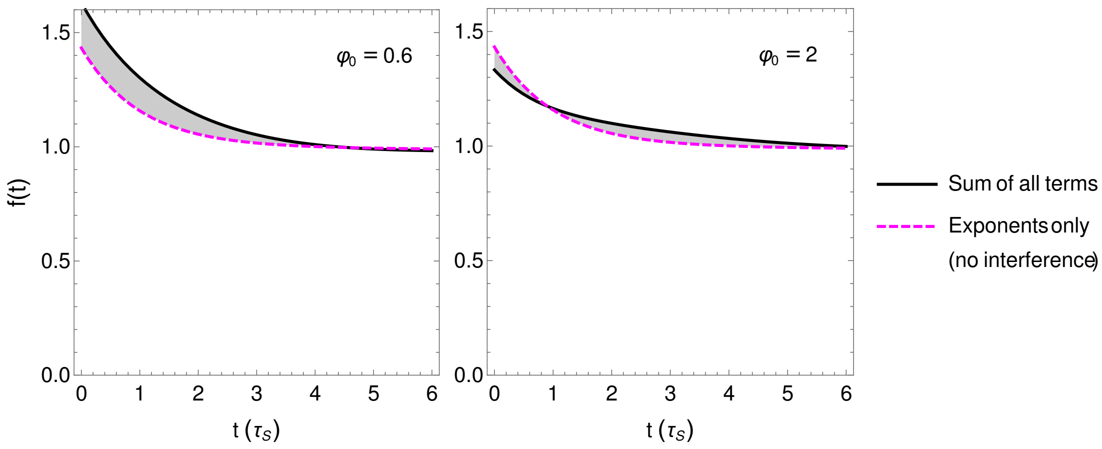

Figure 2. The expected approximate time dependence within the SM, using the coefficients of eq. (6.1), for two values of $\varphi_0 = \arg(A_0^{\text{CP-odd}\,*} A_0^{\text{CP-even}})$. The difference between the dashed magenta curve and the solid black one is a measure of interference effects.

factor of D. The use of matter effects, for example in the case of a K_L beam going through a regenerator, introduces suppression that is proportional to the regeneration parameter, r. Thus, the number of kaons that are needed in these cases compared to the pure kaon beam, are larger by roughly $1/D$ or $1/r$ as we need to overcome these suppression effects.

Several approaches that could be useful in acquiring the needed sensitivity to the interference terms appear in the literature:

1. There are cases with QCD production where both K^0 and $\overline{K}^0$ are produced, but there is an asymmetry, that is $D \neq 0$. One example is the "high intensity K_S-run" at the NA48 experiment, which reported 10^{10} K_S decays with $D \sim 0.3$ [77].

2. Regeneration in K_L beams [78–81]. Numerically, typical values for r range from $\mathcal{O}(10^{-2})$ to a few times 10^{-1}, depending on the material and on the relevant kaon momentum.

3. The use of a charge exchange target in order to generate pure K^0 beams from K^+ beams [82, 83].

4. Post-selection using tagging in high energy production, for example, by looking at the charge of the pion in K^* decays, or by tagging Λ^0 and K^- in $pp \to K^0 K^- X$ and $pp \to K^0 \Lambda^0 X$ decays [14].

We do not discuss these options in any detail. The high yields of currently running experiments is encouraging in terms of the ability of future endeavors to reach the desired sensitivity, should some of these methods be implemented. Clearly, a detailed study of the experimental requirements is needed in order to arrive at a reliable estimate for the expected sensitivity.

We close this section with a remark about the time dependence. A measurement of the full time dependence would result in the best sensitivity. However, in principle, a measurement of the integral over four different time intervals suffices to get the needed information. In practice, C_L is already known, C_S can be extracted from a beam with $D = 0$, and then we would need two time intervals using a beam with $D \neq 0$ or $r \neq 0$.

7 Conclusion and outlook

We have demonstrated how, under well-motivated approximations and assumptions, it is possible to cleanly test the SM using a measurement of the time-dependent decay rate of $K \to \mu^+\mu^-$. A necessary ingredient is sensitivity to the interference between the K_L and K_S amplitudes, as can be seen from eq. (4.5), which is our main result. The relevant SM parameter of interest is

$$|V_{ts}V_{td}\sin(\beta + \beta_s)|\,, \tag{7.1}$$

which is exactly the CKM parameter combination that appears in $K_L \to \pi^0\nu\bar{\nu}$. Thus, our proposal is to use $K \to \mu^+\mu^-$ as an additional independent measurement of the same SM quantity.

As we discuss in detail, the point to emphasize is that the extraction is theoretically very clean. There are several assumptions that were made in setting up the method, as well as in the calculation within the SM. All of these are valid within the SM to a few per-mill, giving a total uncertainty below the 1% level. This is comparable to the best probing methods for the angle β and related quantities, that is, the CP asymmetries in $B \to \psi K_S$ and the decay rate of $K_L \to \pi^0\nu\bar{\nu}$. The assumptions we rely on are additionally respected by any extension of the SM in which the relevant leptonic current is vectorial.

The approach we discuss can in principle be extended to other decay modes. Most promising are the decays $K \to \pi e^+e^-$ and $K \to \pi\mu^+\mu^-$. The generalization is not trivial as these decays involve more partial waves beyond $\ell = 0, 1$. We plan to discuss these modes in a future publication.

Our very preliminary estimates indicate that these measurements can be carried out in next generation kaon experiments. This is encouraging, and more detailed feasibility studies are called for.

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A Extracting $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ without a pure kaon beam

In the main text, we demonstrated how we can determine $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ for the case of a pure K^0 beam in empty space. Here we present a discussion on two other cases which are more related to realistic experimental situations. The first case is when we have a beam with unequal initial number of K^0 and $\overline{K}^0$. The second case is when we have a pure K_L beam going via a regenerator before the kaons decay. In both cases, it is possible to extract the branching ratio $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ cleanly as in eq. (4.5), with the addition of a dilution factor as in eq. (4.6).

A.1 A mixed beam of K^0 and $\overline{K}^0$

Consider a beam which initially consists of an incoherent mixture of kaons and anti-kaons. We define the production asymmetry,

$$D = \frac{N_{K^0} - N_{\overline{K}^0}}{N_{K^0} + N_{\overline{K}^0}}. \tag{A.1}$$

such that the fractions of K^0 and $\overline{K}^0$ particles are given respectively by

$$\frac{N_{K^0}}{N_{K^0} + N_{\overline{K}^0}} = \frac{1+D}{2}, \qquad \frac{N_{\overline{K}^0}}{N_{K^0} + N_{\overline{K}^0}} = \frac{1-D}{2}. \tag{A.2}$$

Note that $D = 1$ corresponds to a pure K^0 beam, while $D = -1$ corresponds to a pure $\overline{K}^0$ beam.

The decay rate to a final state f is given by the incoherent sum

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{1+D}{2}\left(\frac{\mathrm{d}\Gamma_{K^0}}{\mathrm{d}t}\right) + \frac{1-D}{2}\left(\frac{\mathrm{d}\Gamma_{\overline{K}^0}}{\mathrm{d}t}\right), \tag{A.3}$$

such that its form is given by eq. (2.6) with the following coefficients:

$$\begin{aligned}
C_L &= |A_0^{\text{CP-even}}|^2, \\
C_S &= |A_0^{\text{CP-odd}}|^2 + \beta_\mu^2 |A_1^{\text{CP-even}}|^2, \\
C_{\cos} &= D\,|A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}| \cos\varphi_0, \\
C_{\sin} &= D\,|A_0^{\text{CP-odd}*} A_0^{\text{CP-even}}| \sin\varphi_0.
\end{aligned} \tag{A.4}$$

It is then straightforward to extract our parameter of interest. For $D \neq 0$ we obtain

$$|A_0^{\text{CP-odd}}|^2 = \mathcal{D}_F \frac{{C_{\cos}}^2 + {C_{\sin}}^2}{C_L}, \qquad \mathcal{D}_F = \frac{1}{D^2}\,. \tag{A.5}$$

In terms of the branching ratios we have

$$\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} = \mathcal{D}_F \times \mathcal{B}(K_L \to \mu^+\mu^-) \times \frac{\tau_S}{\tau_L} \times \left(\frac{C_{\text{int}}}{C_L}\right)^2. \tag{A.6}$$

We learn that the beam asymmetry serves as a dilution factor compared to the case of a pure K^0 or $\overline{K}^0$ beam. Note that if $D = 0$ (which means that the beam is an equal admixture of K^0 and $\overline{K}^0$), one cannot use the beam to measure $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$.

We close with a remark regarding the LHCb search for the K_S rate [51]. To a very good approximation at LHCb we have $D = 0$. In that case the interference terms cancel and we are left with just C_L and C_S. Thus, without any further analysis to tag the flavor of the kaon, LHCb is working on extracting the C_S term that includes the decay to both the $\ell = 0$ and $\ell = 1$ states.

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A.2 K_L propagating through a slab of matter

When kaons travel through matter, the time dependence of the kaon wave function is modified via the inclusion of the momentum dependent regeneration parameter [78–81, 84, 85].

We define

$$re^{i\alpha} = -\frac{\pi N}{m}\left(\frac{\Delta f}{\Delta\lambda}\right), \tag{A.7}$$

where r and α are real, and

$$\Delta f \equiv f - \bar{f}, \qquad \Delta\lambda \equiv \Delta m - \frac{i}{2}\Delta\Gamma. \tag{A.8}$$

Here, f $(\bar{f})$ is the difference of forward scattering amplitudes for kaons (anti-kaons), and N is the density of scattering centers in the regenerator. Note that r and α are physical and can be determined from experiment.

Let us consider a pure K_L beam, which is produced by letting the K_S (and interferences) terms decay away. Then we put a regenerator of length L in the path of the beam. Let t_L be the time taken by the kaon to travel through the regenerator. We define $t = 0$ to be the time the kaon emerges from the regenerator. We then study the time dependence of the kaon wave function at later times. For simplicity, in the following we present the result to leading order in r.

The normalized decay rate is given by eq. (2.6), with the coefficients:

$$\begin{aligned}
C_L &= |A_0^{\text{CP-even}}|^2, \\
C_S &= 0, \\
C_{\sin} &= -r\,|A_0^{\text{CP-even}}A_0^{\text{CP-odd}}|\left(\sin(\alpha-\varphi_0) - e^{t_L\Delta\Gamma/2}\sin(\alpha-\varphi_0+\Delta m t_L)\right), \\
C_{\cos} &= r\,|A_0^{\text{CP-even}}A_0^{\text{CP-odd}}|\left(\cos(\alpha-\varphi_0) - e^{t_L\Delta\Gamma/2}\cos(\alpha-\varphi_0+\Delta m t_L)\right).
\end{aligned} \tag{A.9}$$

We can check that for $t_L = 0$ (which means that the regenerator thickness is negligible) or $r = 0$ (the regenerator material is just vacuum), the interference terms vanish as it should.

Using the above, we find the dilution factor $\mathcal{D}_F$, for $t_L \neq 0$ and $r \neq 0$ to be

$$\mathcal{D}_F = \frac{1}{2r^2}\left(\frac{\cosh(\Delta\Gamma t_L/2) - \sinh(\Delta\Gamma t_L/2)}{\cosh(\Delta\Gamma t_L/2) - \cos(\Delta m t_L)}\right). \tag{A.10}$$

We learn that the dilution parameter depends on both r and t_L. The extraction of the rate is given by eq. (A.6).

We close with two remarks

1. As we already emphasized, the interference terms are the key to the extraction of $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$. Having $D \neq 0$ or $r \neq 0$ are some of the ways of obtaining interference terms in the time dependence of the kaon beam.

2. More generally one may also have combinations with both non-zero D and r, as well as a general initial state. The calculation is straightforward, though tedious and does not provide much further insight, and so we do not show it here.

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B SM calculations

In the following we first derive the SM prediction for $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$, and then derive approximate numerical estimates for the experimental parameters within the SM.

B.1 SM calculation

Using the standard formula for two body decays [67], as well as the results of eqs. (5.13) and (5.4), we write

$$\mathcal{B}(K_S \to (\mu^+\mu^-)_{\ell=0}) = \frac{\beta_\mu \tau_S}{16\pi m_K} \times 2\sum |\mathcal{M}^{SD}(K_S \to (\mu^+\mu^-)_{\ell=0})|^2 \times \sin^2\theta_{ct}, \tag{B.1}$$

where the sum is over the outgoing spin, as usual. Note that $\mathcal{M}$ is proportional to A, defined in eq. (2.3) but it uses the standard normalization that is used when making calculations.

We write the matrix element for the short-distance contribution as

$$\mathcal{M}^{SD} = g_{\rm SM}\langle\mu\bar{\mu}|\mathcal{O}_\ell|0\rangle \times \langle 0|\mathcal{O}_H|K\rangle, \tag{B.2}$$

where

$$\mathcal{O}_\ell = (\bar{\mu}_L\gamma^\rho\mu_L), \qquad \langle 0|\mathcal{O}_H|K\rangle \equiv -i\, p_K^\rho\, f_K. \tag{B.3}$$

For the kaon decay constant we employ here the convention

$$\langle 0|\bar{s}\gamma_\mu\gamma_5 d|K^0(p)\rangle = ip_\mu f_{K^0}\,. \tag{B.4}$$

The coupling, $g_{\rm SM}$, can be read from eq. (5.11) (note that $(\bar{\mu}\mu)_{V-A} = 2(\bar{\mu}_L\gamma^\rho\mu_L)$)

$$g_{\rm SM} = -\frac{G_F}{\sqrt{2}}\frac{\alpha_{em}}{\pi\sin^2\theta_W}\left[V_{cs}^*V_{cd}Y_{NL} + V_{ts}^*V_{td}Y(x_t)\right]. \tag{B.5}$$

Since under our assumption of $\theta_{uc} = 0$ the part proportional to $V_{cs}^*V_{cd}$ is relatively real, we can further define

$$\tilde{g}_{\rm SM} = -\frac{G_F}{\sqrt{2}}\frac{\alpha_{em}}{\pi\sin^2\theta_W}V_{ts}^*V_{td}Y(x_t). \tag{B.6}$$

As long as what we are after is the CP violating decay rate, we can use $\tilde{g}_{\rm SM}$.

Squaring the amplitude and summing over spins, we find

$$\begin{aligned}
\sum |\mathcal{M}^{SD}_{K\to\mu\mu}|^2 &= \left[-p_K^\rho p_K^\sigma f_K^2\right]|g_{\rm SM}|^2\,{\rm Tr}\left[\bar{u}(k_1)\gamma_\rho P_L v(k_2)\bar{v}(k_2)P_L\gamma_\sigma u(k_1)\right] \\
&= -|g_{\rm SM}|^2\, f_K^2 p_K^\rho p_K^\sigma {\rm Tr}\left[\gamma_\rho P_L(\not{k}_2 - m_\mu)P_L\gamma_\sigma(\not{k}_1 + m_\mu)\right] \\
&= |g_{\rm SM}|^2\, f_K^2 m_\mu^2 p_K^\rho p_K^\sigma {\rm Tr}\left[\gamma_\rho\frac{1}{2}(1-\gamma_5)\gamma_\sigma\right] \\
&= 2|g_{\rm SM}|^2 f_K^2 m_\mu^2 m_K^2.
\end{aligned} \tag{B.7}$$

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Using eq. (B.1) we find

$$\begin{aligned}\mathcal{B}(K_S \to (\mu^+\mu^-)_{\ell=0}) &= \frac{\beta_\mu \tau_S}{16\pi m_K} 4|\tilde{g}_{\rm SM}|^2 f_K^2 m_\mu^2 m_K^2 \sin^2\theta_{ct} \\ &= \frac{\beta_\mu\, \tau_S}{16\pi m_K} \left| \frac{G_F}{\sqrt{2}} \frac{2\alpha_{em}}{\pi \sin^2\theta_W} m_K m_\mu \times Y(x_t) \times f_K \times V_{ts} V_{td} \sin\theta_{ct} \right|^2 ,\end{aligned} \tag{B.8}$$

in agreement with eqs. (37) and (39) of ref. [59].

We next get numerical estimates. We use the lattice QCD result [74] for the hadronic parameter, assuming isospin symmetry:

$$f_K = 155.7 \pm 0.3\,\text{MeV}\,. \tag{B.9}$$

We use the following values for the measured parameters [67],

$$\begin{aligned} m_K &= 497.61\,\text{MeV}, & m_\mu &= 105.658\,\text{MeV}, \\ G_F &= 1.166378 \times 10^{-5}\,\text{GeV}^{-2}, & \alpha_{em} &= 1/129, \\ \sin^2\theta_W &= 0.23, & Y(x_t) &= 0.95, \\ \tau_L &= 5.116 \times 10^{-8}\,\text{s}, & \tau_S &= 8.95 \times 10^{-11}\,\text{s}, \end{aligned} \tag{B.10}$$

and for the CKM values we use

$$|V_{ts} V_{td}\, \sin\theta_{ct}| = A^2 \lambda^5 \bar{\eta}, \qquad \text{with} \quad A = 0.79,\ \lambda = 0.2265,\ \bar{\eta} = 0.357, \tag{B.11}$$

to arrive at the prediction

$$\mathcal{B}(K_S \to (\mu^+\mu^-)_{\ell=0}) \approx 1.64 \cdot 10^{-13} \times \left| \frac{V_{ts} V_{td} \sin\theta_{ct}}{(A^2\lambda^5\bar{\eta})_{\text{best fit}}} \right|^2 , \tag{B.12}$$

with

$$(A^2\lambda^5\bar{\eta})_{\text{best fit}} = 1.33 \cdot 10^{-4}. \tag{B.13}$$

B.2 SM approximate values for the experimental parameters

In order to estimate the magnitude of the effect we are after and to illustrate the expected time dependence, we require approximate values for the remaining two branching ratios within the SM. First, we use the measured branching ratio,

$$\mathcal{B}(K_L \to \mu^+\mu^-)_{\text{exp.}} = \mathcal{B}(K_L \to (\mu^+\mu^-)_{\ell=0})_{\text{exp.}} = (6.84 \pm 0.11) \cdot 10^{-9}, \tag{B.14}$$

which sets the value of the parameter C_L.

The remaining branching ratio, $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=1}$, can only be estimated a priori by relying on non-perturbative calculations from the literature that suffer from large hadronic uncertainties. Nonetheless, we use these results to get an estimate for its magnitude. Below we use the prediction for the long-distance contribution, [14]

$$\mathcal{B}(K_S \to \mu^+\mu^-)^{LD}_{\rm SM} = \mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=1} \approx 4.99 \cdot 10^{-12}. \tag{B.15}$$

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Note that while we quote results to three significant digits, the theoretical uncertainties are much larger. Altogether we have

$$
\begin{aligned}
\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} &\approx 1.64 \cdot 10^{-13}, \\
\mathcal{B}(K_L \to \mu^+\mu^-)_{\ell=0} &\approx 6.84 \cdot 10^{-9}, \\
\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=1} &\approx 4.99 \cdot 10^{-12}.
\end{aligned} \tag{B.16}
$$

The first is the result of the calculation from the SM effective Hamiltonian, the second is the experimental measured value, and the third uses the non-perturbative estimation together with the calculated SM short-distance contribution.

For illustration of the time dependence, we choose to normalize the C coefficients such that $C_L = 1$. The numerical values for the coefficients, as defined in eqs. (2.6) and (2.7), for the case of a pure K^0 or $\overline{K}^0$ beam, are then given by:

$$
\begin{aligned}
(C_L)_{\mathrm{SM}} &\equiv 1, \\
(C_S)_{\mathrm{SM}} &= \frac{\tau_L}{\tau_S} \frac{\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0} + \mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=1}}{\mathcal{B}(K_L \to \mu^+\mu^-)_{\ell=0}} \approx 0.43, \\
(C_{\mathrm{Int.}})_{\mathrm{SM}} &= \sqrt{\frac{\tau_L\, \mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}}{\tau_S\, \mathcal{B}(K_L \to \mu^+\mu^-)_{\ell=0}}} \approx 0.12\,.
\end{aligned} \tag{B.17}
$$

There is one more experimental parameter, the phase φ_0. It is related to the strong phase and we do not provide any estimate for it.

C The short-distance operator

For completeness, we explain below the well-known results that the short-distance SM amplitude cannot generate an $\ell = 1$ state, and that only the axial parts of both the hadronic and leptonic currents contribute in two-body pseudo-scalar decays.

Our starting point is the factorization of the matrix element

$$
\mathcal{M} = \langle \mu^+\mu^- | O_L^\mu O_{H\mu} | K \rangle = \langle \mu^+\mu^- | O_L^\mu | 0 \rangle \times \langle 0 | O_{H\mu} | K \rangle, \tag{C.1}
$$

where

$$
O_H^\mu = (\bar{s}d)_{V-A}, \qquad O_L^\mu = (\bar{\mu}\mu)_{V-A}. \tag{C.2}
$$

The leading breaking of this factorization is from the photon loop, and thus it is suppressed by roughly $\mathcal{O}(\alpha_{EM}/4\pi) \sim 10^{-3}$.

Considering the leptonic part is sufficient to explain why short-distance physics does not contribute to the $K \to (\mu^+\mu^-)_{\ell=1}$ amplitude. For two spinors ψ and χ, we recall the transformation of the $V - A$ operator under CPT [86]:

$$
\Theta \bar{\psi} \gamma^\mu (1 - \gamma^5) \chi \Theta^\dagger = -\bar{\chi} \gamma^\mu (1 - \gamma^5) \psi\,, \tag{C.3}
$$

where $\Theta = \mathcal{CPT}$ is the CPT operator. This implies

$$
\Theta O_L^\mu \Theta^\dagger = -O_L^\mu. \tag{C.4}
$$

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Using

$$\Theta|\mu^+\mu^-\rangle_\ell = (-1)^{\ell+1}|\mu^+\mu^-\rangle_\ell, \qquad \Theta|0\rangle = |0\rangle, \tag{C.5}$$

we conclude

$$\langle(\mu^+\mu^-)_\ell|O_L^\mu|0\rangle = \langle(\mu^+\mu^-)_\ell|\Theta\Theta^\dagger O_L^\mu\Theta\Theta^\dagger|0\rangle = (-1)^\ell\langle(\mu^+\mu^-)_\ell|O_L^\mu|0\rangle. \tag{C.6}$$

From the above we see that $\mathcal{M}$, defined in eq. (C.1), vanishes when ℓ is odd.

As for the axial part of the hadronic current, the argument is the same as the well-known one for charged pion decay, that we recall below. Consider

$$\langle 0|V^\mu - A^\mu|K(p_K)\rangle. \tag{C.7}$$

The kaon is a pseudo-scalar and the vacuum is parity-even. Thus, the matrix element of V^μ must transform as a pseudovector, and the matrix element of A^μ must transform as a vector. The only available physical observable is the kaon momentum, p_K, which is a vector. It is impossible to construct a product of any number of p_K^μ that transforms like as a pseudovector. We conclude that

$$\langle 0|V^\mu|K\rangle = 0. \tag{C.8}$$

In order to see that also for the leptonic current only the axial part is relevant, we write the matrix element in the following form, leaving the vector and axial-vector components of the leptonic operator general:

$$\mathcal{M} \sim p_K^\alpha\ \bar{u}(k_2)\gamma_\alpha(B + A\gamma^5)v(k_1) \tag{C.9}$$

Then,

$$\begin{aligned}\sum|\mathcal{M}|^2 &\propto \mathrm{Tr}\Big[(\not{k}_2 + m_\mu)\not{p}_K(B + A\gamma^5)(\not{k}_1 - m_\mu)(B^* + A^*\gamma^5)\not{p}_K\Big] \\ &= 4\big(|B|^2 - |A|^2\big)\Big[2(k_1\cdot p_K)(k_2\cdot p_K) - m_K^2(k_1\cdot k_2)\Big] - 4\big(|B|^2 + |A|^2\big)m_\mu^2 m_K^2\end{aligned} \tag{C.10}$$

Using two-body kinematics, we have

$$\begin{aligned}(k_1\cdot p_K) &= (k_2\cdot p_K) = \frac{1}{2}m_K^2, \\ (k_1\cdot k_2) &= \frac{1}{2}m_K^2 - m_\mu^2.\end{aligned} \tag{C.11}$$

Plugging this in to eq. (C.10), the $|B|^2$ terms drop out and we are left only with the terms proportional to $|A|^2$,

$$\sum|\mathcal{M}|^2 \propto |A|^2 m_\mu^2 m_K^2, \tag{C.12}$$

i.e., only the axial-vector part of the operator is relevant.

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