



Cosmology in 5D and 4D Einstein–Gauss–Bonnet gravity

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Abstract We consider the five-dimensional Einstein–Gauss–Bonnet gravity, which can be obtained by means of an appropriate choice of coefficients in the five-dimensional Lanczos–Lovelock gravity theory. The Einstein–Gauss–Bonnet field equations for the Friedmann–Lemaître–Robertson–Walker metric are found as well as some of their solutions. The hyperbolicity of the corresponding equations of motion is discussed. A four-dimensional gravity action is obtained from the Gauss–Bonnet gravity using the Randall–Sundrum compactification procedure and then it is studied the implications of the compactification procedure in the cosmological solutions. The same procedure is used to obtain gravity in four dimensions from the five-dimensional AdS–Chern–Simons gravity to then study some cosmological solutions. Some aspects of the construction of the four-dimensional action gravity, as well as a brief review of Lovelock gravity in 5D are considered in an Appendix.

1 Introduction

The 5-dimensional action for Lanczos–Lovelock gravity theory [1–4], is a polynomial of degree 2 in curvature, which can be written in terms of the Riemann curvature R^{ab} and the vielbein e^a as

$$S_{LL}^{(5D)} = \frac{1}{8\kappa_5} \int \varepsilon_{abcde} \left(\alpha R^{ab} R^{cd} e^e + \frac{2}{3} R^{ab} e^c e^d e^e + \beta e^a e^b e^c e^d e^e \right), \quad (1)$$

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where (i) α, β are arbitrary constants, (ii) $e^a = e^a_\mu dx^\mu$, $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ are the fünfbein fields and spin connection, respectively, (iii) $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$ is the 2-form curvature and $\kappa_5 = 12\pi^2 G_5$, where G_5 is the 5-dimensional Newton constant.

Comparing the action (1), when $\alpha = 0$, with the Einstein–Hilbert–Cartan action with cosmological constant in 5D

$$S_{EHC}^{(5D)} = \frac{1}{12\kappa_5} \int \varepsilon_{abcde} \left(R^{ab} e^c e^d e^e - \frac{\Lambda_{5D}}{10} e^a e^b e^c e^d e^e \right), \quad (2)$$

we can see that the action (1) matches action (2) only if $\beta = -\Lambda_{5D}/15$. With this choice of constant β , the action (1) takes the form of Einstein–Gauss–Bonnet (EGB) action with cosmological constant

$$S_{EGB}^{(5D)} = \frac{1}{8\kappa_5} \int \varepsilon_{abcde} \left(\alpha R^{ab} R^{cd} e^e + \frac{2}{3} R^{ab} e^c e^d e^e - \frac{\Lambda_{5D}}{15} e^a e^b e^c e^d e^e \right). \quad (3)$$

In presence of matter, the action is given by

$$S^{(5D)} = S_{EGB}^{(5D)} + S_M^{(5D)}, \quad (4)$$

where $S_M^{(5D)} = S_M^{(5D)}(e^a, \omega^{ab})$ is the matter action whose variation leads to

$$\delta S_M^{(5D)} = \frac{\delta L_M^{(5D)}}{\delta e^a} \delta e^a + \frac{\delta L_M^{(5D)}}{\delta \omega^{ab}} \delta \omega^{ab}, \quad (5)$$

where $\delta L_M^{(5D)}/\delta e^a$ and $\delta L_M^{(5D)}/\delta \omega^{ab}$ are related to the anholonomic forms (in an orthonormal frame) of the energy–momentum tensor T_{ab} and the spin tensor S^c_{ab} respectively. This means that the variation of the action (4) leads to the

following field equations

$$\begin{aligned} \varepsilon_{abcde} \left(\alpha R^{bc} R^{de} + 2R^{bc} e^d e^e - \frac{\Lambda_{5D}}{3} e^b e^c e^d e^e \right) \\ = -8\kappa_5 \frac{\delta L_M^{(5D)}}{\delta e^a}, \end{aligned} \tag{6}$$

$$2\varepsilon_{abcde} T^c \left(\alpha R^{de} + e^d e^e \right) = -4\kappa_5 \frac{\delta L_M^{(5D)}}{\delta \omega^{ab}}, \tag{7}$$

where $T^a = De^a = de^a + \omega^a_b e^b$ is the 2-form torsion. When the spin tensor is zero, one solution is the zero torsion ($T^a = De^a = 0$).

Summarizing, we have considered the 5-dimensional Lanczos–Lovelock gravity, which for an appropriate choice of coefficients leads to the EGB gravity action. This work is organized as follows: In Sect. 2 we find the EGB gravitational field equations for the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. A discussion about the hyperbolicity of the metric ends this section. Section 3 is devoted to find a 4-dimensional gravity action from the Gauss–Bonnet gravity using the Randall–Sundrum compactification procedure and then we study the implications in the cosmological solutions of the compactification procedure. In Sect. 4 we use the same procedure to obtain gravity in 4D from the 5-dimensional AdS–Chern–Simons gravity and then we study some of its cosmological implications. Finally Concluding Remarks are presented in Sect. 5. An Appendix is included, where is considered a brief gravity review of Lovelock gravity in 5D, as well as some aspects of the construction of the 4-dimensional action gravity.

2 Cosmology in Einstein–Gauss–Bonnet gravity without cosmological constant

Consider the action (4) without cosmological constant, which means that the lagrangian is given by

$$L^{(5D)} = L_{EGB}^{(5)}|_{\Lambda_5=0} + L_M^{(5D)}, \tag{8}$$

with

$$\begin{aligned} L_{EGB}^{(5)}(e, \omega)|_{\Lambda_5=0} = \frac{2}{3} \varepsilon_{abcde} R^{ab} e^c e^d e^e \\ + \alpha \varepsilon_{abcde} R^{ab} R^{cd} e^e, \end{aligned} \tag{9}$$

being $\alpha = |\alpha| \operatorname{sgn}(\alpha)$ a constant and $L_M^{(5D)}$ represents a matter lagrangian. Later we show two cosmological scenarios associated with $\operatorname{sgn}(\alpha)$.

The variation of the action $S^{(5D)}$ with respect to the vielbein e^a and the spin connection ω^{ab} leads to the equations

$$\varepsilon_{abcde} \left(2R^{bc} e^d e^e + |\alpha| \operatorname{sgn}(\alpha) R^{bc} R^{de} \right) = -8\kappa_5 \frac{\delta L_M^{(5D)}}{\delta e^a}, \tag{10}$$

$$\varepsilon_{abcde} \left(T^c e^d e^e + |\alpha| \operatorname{sgn}(\alpha) R^{cd} T^e \right) = -4\kappa_5 \frac{\delta L_M^{(5D)}}{\delta \omega^{ab}}. \tag{11}$$

If the matter under consideration has no spin, then

$$\frac{\delta L_M}{\delta \omega^{ab}} = 0 \quad \text{and} \quad T^a = 0. \tag{12}$$

Since

$$\frac{\delta L_M}{\delta e^a} = \frac{1}{4!} T_a^\mu \varepsilon_{\mu bcde} e^b e^c e^d e^e, \tag{13}$$

where $T_{\mu\nu}$ is the energy–momentum tensor, we have that Eq. (10) takes the form

$$\begin{aligned} \varepsilon_{abcde} \left(2R^{bc} e^d e^e + |\alpha| \operatorname{sgn}(\alpha) R^{bc} R^{de} \right) \\ = -\frac{\kappa_5}{3} T_a^\mu \varepsilon_{\mu bcde} e^b e^c e^d e^e. \end{aligned} \tag{14}$$

2.1 Field equations and cosmology

We consider a flat FLRW metric

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \tag{15}$$

where $a(t)$ is the cosmic scale factor and $i, j = 1, 2, 3, 4$. After some calculations, the 2-form curvature turns out to be

$$R^{0p} = \frac{\ddot{a}}{a} e^0 e^p = (\dot{H} + H^2) e^0 e^p = -q H^2 e^0 e^p, \tag{16}$$

$$R^{pq} = H^2 e^p e^q, \tag{17}$$

where $p, q = 1, 2, 3, 4$, $H = \dot{a}/a$ is the Hubble parameter, $\dot{H} = dH/dt = \ddot{a}/a - H^2$ and $q = -(1 + \dot{H}/H^2)$ is the deceleration parameter. From here, it is direct to see that when $q < 0$ we have $\ddot{a} > 0$ and if $q > 0$ then $\ddot{a} < 0$.

We further consider an energy–momentum tensor corresponding to a perfect fluid

$$T_\mu^\nu = \operatorname{diag}(-\rho, p, p, p, p). \tag{18}$$

After replacing (16) and (18) in (14) we obtain the Friedmann constraint and the conservation equation, respectively,

$$6H^2 + 3|\alpha| \operatorname{sgn}(\alpha) H^4 = \kappa_5 \rho, \tag{19}$$

$$(1+z) \frac{d\rho}{dz} = 4(\rho + p), \tag{20}$$

where we have introduced the redshift parameter defined as $1+z = a_0/a$ and $a_0 = a(t_0)$. Choosing $\kappa_5 = 1$ unities and using the barotropic equation of state $p = \omega\rho$, we write the Eqs. (19) and (20) in the form

$$6H^2 \left(1 + \frac{1}{2} |\alpha| \operatorname{sgn}(\alpha) H^2 \right) = \rho, \tag{21}$$

$$\rho(z) = \rho(0) (1+z)^{4(1+\omega)}, \tag{22}$$

such that

$$\operatorname{sgn}(\alpha) = 1 \implies H^2(z)$$

$$= \frac{1}{|\alpha|} \left(\sqrt{1 + \frac{|\alpha|}{3} \rho(z)} - 1 \right), \tag{23}$$

$$\begin{aligned} \text{sgn}(\alpha) = -1 &\implies H_{\pm}^2(z) \\ &= \frac{1}{|\alpha|} \left(1 \pm \sqrt{1 - \frac{|\alpha|}{3} \rho(z)} \right). \end{aligned} \tag{24}$$

The last case shows an upper bound for ρ , that is, $\rho(z_s) = 3/|\alpha|$ and so

$$H_{\pm}(z_s) = \sqrt{\frac{1}{|\alpha|}}, \tag{25}$$

being

$$z_s = -1 + \left(\frac{3}{|\alpha| \rho(0)} \right)^{1/4(1+\omega)}. \tag{26}$$

Replacing (26) into (24), we write

$$H_{\pm}^2(z) = \frac{1}{|\alpha|} \left(1 \pm \sqrt{1 - \left(\frac{1+z}{1+z_s} \right)^{4(1+\omega)}} \right), \tag{27}$$

so that we have a solution for $H_{\pm}(z)$ if $z \leq z_s$. According to (26) and (27) and considering $\omega = 0$ (cold dark matter), we have

$$z_s = -1 + \left(\frac{3}{|\alpha| \rho(0)} \right)^{1/4} \geq 0. \tag{28}$$

If $-1 < z_s < 0$, we have $\rho(z \rightarrow z_s) \rightarrow 3/|\alpha| \implies H_{\pm}(z \rightarrow z_s) \rightarrow \sqrt{1/|\alpha|}$, i.e., a future de Sitter evolution, unlike in 4D- Λ CDM where a de Sitter evolution is reached when $z \rightarrow -1$.

If $z_s > 0$, we have an unrealistic situation given that $\rho(z \rightarrow z_s) \rightarrow 3/|\alpha| \implies H_{\pm}(z \rightarrow z_s) \rightarrow \sqrt{1/|\alpha|}$, that is, a past de Sitter evolution.

If $z_s = 0$, we write

$$H_{\pm}^2(z) = \frac{\rho(0)}{3} \left(1 \pm \sqrt{1 - (1+z)^4} \right), \tag{29}$$

so that, $H_{\pm}^2(0) = \rho(0)/3$ and $H_{\pm}^2(z \rightarrow -1) \rightarrow 2\rho(0)/3$. Recalling that in the present discussion, see (9), there is not cosmological constant (thinking in a de Sitter evolution).

In the absence of ρ , from (24) we obtain a self-accelerating solution given by

$$H_{+} = \sqrt{\frac{2}{|\alpha|}}. \tag{30}$$

Substituting (22) into (23), it is straightforward to show that $\omega > -1 \implies \rho(z \rightarrow -1) \rightarrow 0$ and $H(z \rightarrow -1) \rightarrow 0$,

$$\tag{31}$$

typical behavior of cosmic components not associated with dark energy.

We end this section by highlighting what is shown in (24), that is, an upper bound for the present energy density $\rho(z_s) = 3/|\alpha| \iff H_{\pm}(z_s) = \sqrt{1/|\alpha|}$. For comparison, in 4D, the Λ CDM model tell us that $\rho(z \rightarrow -1) \rightarrow 0 \implies H(z \rightarrow -1) \rightarrow \sqrt{\Lambda/3}$. We also highlight what is shown in (30), a self-accelerating solution.

The deceleration parameter can be written as

$$q = -1 + \frac{1}{2} (1+z) \frac{dH^2}{H^2 dz}, \tag{32}$$

that after using (21)–(22), it is straightforward to obtain

$$q(z) = -1 + 2(1+\omega) f(z), \tag{33}$$

where we have defined

$$f(z) = \frac{1}{1 + \alpha H^2(z)} \left(1 + \frac{1}{2} \alpha H^2(z) \right), \tag{34}$$

and $f(z) < 1$. The Friedmann constraint tells us that, regardless of $\text{sgn}(\alpha)$, the following inequality must be satisfied

$$1 + \frac{1}{2} \alpha H^2(z) > 0, \tag{35}$$

so that

$$q(z) < 0 \implies 2(1+\omega) < \frac{1}{f(z)}. \tag{36}$$

Considering $\omega \geq 0$ (fluids that by themselves generate deceleration, $q = 1 + 2\omega > 0$), it is evident that $2(1+\omega) > 1$, so that the inequality given in (36) is satisfied.

In good accounts, it is perfectly possible to satisfy $q(z) < 0$ (36) so that we have consistency with the so-called Lorentzian metric condition. In fact, according to Ref. [5], (see also [6–11]), “a necessary condition for hyperbolic EOM is that effective metric be Lorentzian” and that the conditions for said metric to be Lorentzian is that

$$\text{sgn} \left(1 + 4\lambda_2 H^2 \right) = \text{sgn} \left(1 - 4\lambda_2 q H^2 \right), \tag{37}$$

where now $\lambda_2 = \alpha/4l_p^3$ (see Appendix), as can be seen from equations (3.11) and (3.12) of the aforementioned Ref. [5]. In this same reference it is established that if this equality is not fulfilled, then the aforementioned metric will not be Lorentzian. This means that if $\lambda_2 > 0$ and $4\lambda_2 q H^2 > 1$, we have a non-Lorentzian metric. However, if $q < 0$ the metric is Lorentzian.

3 Gravity in 4D from Einstein–Gauss–Bonnet gravity

The existence of new dimensions may have non trivial effects in our understanding of the cosmology of the early Universe, among many other issues. By convention, it has always been assumed that such extra dimensions should be compactified to manifolds of small radii with sizes of the order of the Planck length.

It was only in the last years of the twentieth century when people started to ask the question of how large could these extra dimensions be without getting into conflict with observations. In this context, of particular interest are the Randall and Sundrum models [12, 13] for warped backgrounds, with compact or even infinite extra dimensions. Randall and Sundrum proposed that the metric of the spacetime is given by

$$ds^2 = e^{-2kr_c} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2, \tag{38}$$

i.e. a 4-dimensional metric multiplied by a ‘‘warp factor’’ which is a rapidly changing function of an additional dimension, k is a scale of the order of Planck scale, x^μ are coordinates for the familiar 4-dimensions, while $0 \leq \phi \leq \pi$ is the coordinate for an extra dimension, which is a finite interval whose size is set by r_c , known as ‘‘compactification radius’’. Randall and Sundrum showed that this metric is a solution to Einstein’s equations.

3.1 4-dimensional gravity from the Einstein–Gauss–Bonnet gravity

From Eq. (3) we can see that the Lagrangian contains the Gauss–Bonnet term, the Einstein–Hilbert term and a cosmological term. Following the procedure given in the Appendix, we find that the 5-dimensional action gravity compactified to 4-dimensions is given by

$$\tilde{S}[\tilde{e}] = \frac{1}{8\kappa_5} \int_{\Sigma_4} \tilde{\epsilon}_{mnpq} \left(A \tilde{R}^{mn} \tilde{R}^{pq} + B \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + C \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right), \tag{39}$$

where

$$A = r_c \int_0^{2\pi} d\phi \tag{40}$$

$$B = 2r_c \int_0^{2\pi} d\phi e^{2f(\phi)} \left(1 - \frac{\alpha}{r_c^2} (3f'^2 + 2f'') \right), \tag{41}$$

and

$$C = r_c \int_0^{2\pi} d\phi e^{4f(\phi)} \left(\frac{\alpha}{r_c^4} f'^2 (5f'^2 + 4f'') - \frac{2}{3r_c^2} (5f'^2 + 2f'') - \frac{\Lambda_{5D}}{3} \right). \tag{42}$$

Since $f(\phi)$ is arbitrary and continuously differentiable function, and since we are working with a cylindrical variety, we find that (40), (41) lead to

$$A = 2\pi r_c \tag{43}$$

$$B = 2\pi r_c \left(1 + \frac{\alpha}{r_c^2} \right), \tag{44}$$

and

$$C = -\frac{\pi}{4r_c} \left(\frac{\alpha}{r_c^2} - 2 + \Lambda_{5D} r_c^2 \right), \tag{45}$$

where we have choose $f(\phi) = \ln(\sin \phi)$.

Note that in the action (39) there is a quadratic term in the curvature given by $A \tilde{\epsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq}$, which represents the 4-dimensional Gauss–Bonnet term. This term is a topological one, so that it does not contribute to the dynamics and it can be eliminated. This means that compactification avoids the problems cited in Ref. [14] (see also [15–18]). Equation (1) of this reference agrees with the Lagrangian (39), except for the cosmological term, when $\lambda_1 = \alpha$, $\lambda_2/\lambda_1 = -4$ and $\lambda_3/\lambda_1 = 1$.

Taking into account that the action (39) should lead to the four-dimensional Einstein–Hilbert–Cartan action, namely

$$\tilde{S}_{EHC}^{(4D)} = \frac{1}{4\kappa_4} \int_{\Sigma_4} \tilde{\epsilon}_{mnpq} \left(\tilde{R}^{mn} \tilde{e}^p \tilde{e}^q - \frac{\Lambda_{4D}}{6} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right), \tag{46}$$

where $\kappa_4 = 8\pi G$, it is direct to see that this occurs when

$$B = 3\pi \frac{G_5}{G}, \quad C = -\frac{\pi}{2} \Lambda_{4D} \frac{G_5}{G}. \tag{47}$$

On the other hand we know that if G_D is Newton’s constant in D -dimensions and if G is the usual Newton’s constant, then

$$G_D = (l_C)^{D-4} G,$$

where l_C is the length of the extra compact dimension [19]. In our particular case, $D = 5$ and then $l_C = 2\pi r_c$. This means that $G_5 = 2\pi r_c G$. So that (47) takes the form

$$B = 6\pi^2 r_c, \quad C = -\pi^2 \Lambda_{4D} r_c. \tag{48}$$

Now, from (43), (45) and (48), it is direct to see that

$$\frac{\alpha}{r_c^2} = 3\pi - 1, \tag{49}$$

and then

$$\Lambda_{4D} = \Lambda_{4D}(r_c, \Lambda_{5D}) = \frac{1}{4\pi} \left(\Lambda_{5D} + \frac{3(\pi - 1)}{r_c^2} \right). \tag{50}$$

Introducing (48) into (39) we obtain the action (46) where now Λ_{4D} is given by (50).

In tensor language the two terms in (39) can be written as

$$\begin{aligned} \tilde{\epsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q &= -2 \int d^4 \tilde{x} \sqrt{-\tilde{g}} \tilde{R}, \\ \tilde{\epsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q &= -24 \int d^4 \tilde{x} \sqrt{-\tilde{g}}, \end{aligned} \tag{51}$$

where \tilde{g} is the determinant of the 4-dimensional metric tensor $\tilde{g}_{\mu\nu}$ and \tilde{R} is the Ricci scalar. Thus, the action (39) is now written as

$$\tilde{S}[\tilde{g}] = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left(\tilde{R} + 2\Lambda_{4D} \right), \tag{52}$$

whose field equations are

$$G_{\mu\nu} = \Lambda_{4D} g_{\mu\nu}, \tag{53}$$

and $\Lambda_{4D} = \Lambda_{4D}(r_c, \Lambda_{5D})$.

According to (40), we can say little or nothing about the presence of r_c . The only thing we can “speculate” is to say that Λ_{4D} originates from the compactification radius and the 5-dimensional cosmological constant, and nothing else.

4 Cosmology in AdS Chern–Simons gravity compactified to 4-dimensions

Consider again the EGB action (3). Choosing $\alpha = l^2$ and $\Lambda = -3/l^2$ in (3), we see that the EGB action takes the form

$$S_{EGB} = \frac{l^2}{8\kappa_5} \int \varepsilon_{abcde} \left(R^{ab} R^{cd} e^e + \frac{2}{3l^2} R^{ab} e^c e^d e^e + \frac{1}{5l^4} e^a e^b e^c e^d e^e \right), \tag{54}$$

where it is straightforward to see that this particular choice for α and Λ in the EGB action leads to the 5-dimensional Chern–Simons gravity action for the AdS algebra, with l interpreted as the radius of the universe.

4.1 4-dimensional gravity from the AdS Chern–Simons gravity

From Eq. (54) we can see that the Lagrangian contains the Gauss–Bonnet term L_{GB} , the Einstein–Hilbert term L_{EH} and a cosmological term L_Λ . Replacing (72), (73) and (74) in (54) we find

$$\tilde{S}[\tilde{e}] = \frac{1}{8\kappa_5} \int_{\Sigma_4} \tilde{\varepsilon}_{mnpq} \left(\tilde{A} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + \tilde{B} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right), \tag{55}$$

where

$$\tilde{A} = \frac{2\pi l^2}{r_c} \left(1 + \frac{r_c^2}{l^2} \right), \tag{56}$$

and

$$\tilde{B} = -\frac{\pi}{4r_c} \left(\frac{l^2}{r_c^2} - 2 - 3\frac{r_c^2}{l^2} \right). \tag{57}$$

It is direct to see that the action (55) lead to the Einstein–Hilbert–Cartan action when

$$\tilde{A} = 6\pi^2 r_c \quad \text{and} \quad \tilde{B} = -\pi^2 \Lambda_{4D} r_c. \tag{58}$$

From (56), (57) and (58) we have

$$\frac{r_c^2}{l^2} = \frac{1}{3\pi - 1}, \tag{59}$$

and then

$$\Lambda_{4D} = \Lambda_{4D}(r_c) = \left(\frac{3\pi - 4}{3\pi - 1} \right) \frac{3}{4r_c^2}. \tag{60}$$

The introduction of (58) into the action (55) leads to the action (46) where now Λ_{4D} is given by (60).

Introducing (51) in (55) we obtain

$$\tilde{S}[\tilde{g}] = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left(\tilde{R} - 2\Lambda_{4D} \right), \tag{61}$$

whose field equations are

$$G_{\mu\nu} = -\Lambda_{4D} g_{\mu\nu}. \tag{62}$$

In order to have a feeling on r_c , from (60) we obtain $r_c \approx \Lambda_{4D}^{-1/2}$. Using $\Lambda_{4D} \sim 10^{-52} [m^{-2}] \sim 3 * 10^{-122} [l_{Planck}^{-2}]$ we have $r_c \sim 10^{61} [l_{Planck}] \approx 10^{26} [m]$, we recalling that $a_0 \approx 10^{26} [m]$ (current causal size of the universe).

According to (58), we obtain $r_c/l \approx 0.34$, i.e., $l \approx 3r_c$. So, interpreting l as the size of the universe appears to be reasonable.

5 Concluding remarks

We have considered the 5-dimensional Lanczos–Lovelock gravity, which for an appropriate choice of coefficients gives the EGB gravity action. It is found the EGB gravitational field equations for the FLRW metric together with some cosmological solutions. And if the deceleration parameter is negative, the so-called Lorentzian metric condition is satisfied (see [5]).

The main purpose of this article was to make the 5-dimensional EGB gravity theory, as well as the 5-dimensional AdS–Chern–Simons, consistent with the idea of a 4-dimensional spacetime, through the replacement of a Randall–Sundrum type metric in the Lagrangian (3), and then to get an interpretation of the 4-dimensional effective cosmological constant.

We have evaluated a 5-dimensional Randall–Sundrum type metric in the Lagrangians (3) and (54), and then we derive an action for a 4-dimensional spacetime embedded in the 5-dimensional spacetime. We have obtained the actions in tensorial language and then we find the corresponding Friedmann equations for homogeneous and isotropic cosmology.

The quadratic term in the curvature of the action (39) given by $A \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq}$ represents the 4-dimensional Gauss–Bonnet term. This term is a topological one, so that it does not contribute to the dynamics. This means that compactification avoids the problems cited in Ref. [14]. Equation (1) of this last reference agrees with the Lagrangian (39), except for the cosmological term, when $\lambda_1 = \alpha$, $\lambda_2/\lambda_1 = -4$ and $\lambda_3/\lambda_1 = 1$.

Finally, it is important to note that the equations of motion corresponding both the action (9) and the action (39) are second order, so they do not experience instabilities (see details in Ref. [14]).

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6 Appendix: A briefly review the of derivation of the action (39) and of Lovelock gravity in 5D

6.1 Gravity in 4D from EGB gravity

In order to find (39), we will first consider the following 5-dimensional Randall–Sundrum type metric [20]

$$\begin{aligned} ds^2 &= e^{2f(\phi)} \tilde{g}_{\mu\nu}(\tilde{x}) d\tilde{x}^\mu d\tilde{x}^\nu + r_c^2 d\phi^2, \\ &= \eta_{ab} e^a e^b, \\ &= e^{2f(\phi)} \tilde{\eta}_{mn} \tilde{e}^m \tilde{e}^n + r_c^2 d\phi^2, \end{aligned} \tag{63}$$

where $e^{2f(\phi)}$ is the so-called “warp factor”, and r_c is the so-called “compactification radius” of the extra dimension, which is associated with the coordinate $0 \leq \phi < 2\pi$. The symbol \sim denotes 4-dimensional quantities. We will use the usual notation [20,21]

$$\begin{aligned} x^\alpha &= (\tilde{x}^\mu, \phi); \quad \alpha, \beta = 0, \dots, 4; \quad a, b = 0, \dots, 4; \\ \mu, \nu &= 0, \dots, 3; \quad m, n = 0, \dots, 3; \\ \eta_{ab} &= \text{diag}(-1, 1, 1, 1, 1); \quad \tilde{\eta}_{mn} = \text{diag}(-1, 1, 1, 1), \end{aligned} \tag{64}$$

which allows us to write the vielbein

$$\begin{aligned} e^m(\phi, \tilde{x}) &= e^{f(\phi)} \tilde{e}^m(\tilde{x}) \\ &= e^{f(\phi)} \tilde{e}^m_\mu(\tilde{x}) d\tilde{x}^\mu; \quad e^4(\phi) = r_c d\phi, \end{aligned} \tag{65}$$

where \tilde{e}^m is the vierbein.

From the vanishing torsion condition

$$T^a = de^a + \omega^a_b e^b = 0, \tag{66}$$

we obtain the connections

$$\omega^a_{b\alpha} = -e^\beta_b \left(\partial_\alpha e^a_\beta - \Gamma^\gamma_{\alpha\beta} e^a_\gamma \right), \tag{67}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol.

From Eqs. (65) and (66) we find

$$\omega^m_4 = \frac{e^f f'}{r_c} \tilde{e}^m, \tag{68}$$

and the 4-dimensional vanishing torsion condition

$$\tilde{T}^m = \tilde{d}\tilde{e}^m + \tilde{\omega}^m_n \tilde{e}^n = 0, \tag{69}$$

where $f' = \partial f / \partial \phi$, $\tilde{\omega}^m_n = \omega^m_n$ and $\tilde{d} = d\tilde{x}^\mu \partial / \partial \tilde{x}^\mu$.

From (68), (69) and the Cartan’s second structural equation, $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$, we obtain the components of the 2-form curvature [20,21]

$$\begin{aligned} R^{m4} &= \frac{e^f}{r_c} (f'^2 + f'') d\phi \tilde{e}^m, \quad R^{mn} \\ &= \tilde{R}^{mn} - \left(\frac{e^f f'}{r_c} \right)^2 \tilde{e}^m \tilde{e}^n, \end{aligned} \tag{70}$$

where the 4-dimensional 2-form curvature is given by

$$\tilde{R}^{mn} = \tilde{d}\tilde{\omega}^{mn} + \tilde{\omega}^m_p \tilde{\omega}^{pn}. \tag{71}$$

These results allow us to obtain an action for a 4-dimensional gravity from the 5-dimensional EGB action with cosmological constant, whose action is given by (3).

From (3) we can see that the Lagrangian contains the Gauss–Bonnet term L_{GB} , the Einstein–Hilbert term L_{EH} and a cosmological term L_Λ . In fact, replacing (65) and (70) in L_{GB} , L_{EH} , L_Λ and using $\tilde{\varepsilon}_{mnpq} = \varepsilon_{mnpq4}$, we obtain

$$\begin{aligned} L_{GB} &= \varepsilon_{abcde} R^{ab} R^{cd} e^e, \\ &= r_c d\phi \left\{ \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} - \left(\frac{2e^{2f}}{r_c^2} \right) \right. \\ &\quad \times \left(3f'^2 + 2f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q \\ &\quad \left. + \left(\frac{e^{4f}}{r_c^4} f'^2 \right) \left(5f'^2 + 4f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \end{aligned} \tag{72}$$

$$\begin{aligned} L_{EH} &= \varepsilon_{abcde} R^{ab} e^c e^d e^e, \\ &= r_c d\phi \left\{ \left(3e^{2f} \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q \right. \\ &\quad \left. - \left(\frac{e^{4f}}{r_c^2} \right) \left(5f'^2 + 2f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \end{aligned} \tag{73}$$

and

$$\begin{aligned} L_\Lambda &= \varepsilon_{abcde} e^a e^b e^c e^d e^e, \\ &= 5r_c d\phi e^{4f} \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q. \end{aligned} \tag{74}$$

6.2 Lovelock gravity in 5D

En las ecuaciones (2.1) y (2.2) de la Ref. [5] the coefficients λ_k in the Lagrangian (2.2) have dimensions of $[\text{length}]^{(2p-D)}$ and $\delta_{j_1 \dots j_{2p}}^{i_1 \dots i_{2p}}$ are the so-called generalized Kronecker delta. Usually such Lagrangian density is normalized in units of Planck length $\lambda_1 = (16\pi G)^{-1} = l_p^{2-D}$. In 5-dimensions, the Lagrangian is given by the first three terms of the sum

$$\mathcal{L}^{(5D)} = \sqrt{-g}[\lambda_0 + \lambda_1 R + \lambda_2(R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl})], \tag{75}$$

where $\lambda_1 = (16\pi G)^{-1} = l_p^{-3}$.

In the language of differential forms, the five-dimensional Lovelock Lagrangian can be written as [4]

$$\mathcal{L}^{(5)} = \varepsilon_{abcde} \left(\alpha_0 e^a e^b e^c e^d e^e + \alpha_1 R^{ab} e^c e^d e^e + \alpha_2 R^{ab} R^{cd} e^e \right), \tag{76}$$

where α_1, α_2 and α_3 are arbitrary constants.

Taking into account that $\varepsilon_{abcde} e^a e^b e^c e^d e^e = -120 \sqrt{-g} d^5 x$, $\varepsilon_{abcde} R^{ab} e^c e^d e^e = -6 \sqrt{-g} R d^5 x$, $\varepsilon_{abcde} R^{ab} R^{cd} e^e = -\sqrt{-g} (R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}) d^5 x$, we have that (76) can be written in the form

$$\mathcal{L}^{(5)} = -\sqrt{-g} (120\alpha_0 + 6\alpha_1 R + \alpha_2 [R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}]) d^5 x. \tag{77}$$

The comparison of (75) with (77) we see that $\lambda_0 = 120\alpha_0$, $\lambda_1 = 6\alpha_1$, $\lambda_2 = \alpha_2$.

On the another hand, from (3) it is direct to see

$$\mathcal{L}_{EGB}^{(5D)} = \varepsilon_{abcde} \left(\alpha R^{ab} R^{cd} e^e + \frac{2}{3} R^{ab} e^c e^d e^e - \frac{\Lambda_{5D}}{15} e^a e^b e^c e^d e^e \right), \tag{78}$$

where $\alpha = 2\alpha_2/3\alpha_1$, $\beta = 2\alpha_0/3\alpha_1$, which indicates that the coefficients α and λ_2 are proportional. Indeed

$$\alpha = \frac{2\alpha_2}{3\alpha_1} = \frac{4\lambda_2}{\lambda_1} = 64\pi G\lambda_2 = 4l_p^3\lambda_2. \tag{79}$$

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