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Critical collapse for the Starobinsky R^2 model

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Abstract: We study gravitational collapse for the Starobinsky R^2 model, a particular example of an $f(r)$ theory, in a spherically symmetric spacetime. We add a massless scalar field as matter content to the spacetime. We work in the Einstein frame, where an additional scalar field arises due to the conformal transformation. As in general relativity, depending on the initial data, we found that the gravity scalar field and the physical scalar field can collapse, forming a black hole, in which the final solution is the Schwarzschild metric. We found the threshold of black hole formation through a fine-tuning method and studied critical collapse near this regime.

Keywords: Black Holes, Classical Theories of Gravity, Spacetime Singularities

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Contents

1 Introduction

Einstein's theory of general relativity (GR) has been our best classical theory to describe gravity. All direct experimental tests are consistent with the theory [1]. However, convincing evidence tells us that GR is not the final theory to describe gravity. Many arguments indicate that GR is not renormalizable in the standard quantum field theory sense [2–4]. It is geodesically incomplete for most of its solutions, like black holes that contain spacetime singularity, and does not explain many inconsistencies in the early Universe [5].

Many modifications of the theory have been considered in the literature. The most usual modifications consist of adding new degrees of freedom, by adding new fundamental fields in the action, or considering a spacetime with higher dimensions [6]. Another alternative is to modify the scalar curvature term in the Einstein-Hilber action. In this modification, known as $f(R)$ gravity theories [7–9], the standard Ricci scalar in the Einstein-Hilbert action is replaced by a more general function of the scalar curvature R. The action reads

$$\mathcal{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\; f(R) + S_m, \tag{1.1}$$

where G is the Newtonian constant, and S_m is the action describing the matter content in spacetime. $f(R)$ is an alternative theory of gravity that generalize Einstein's theory of GR, allowing for deviations in the strong-field regime; see refs. [10, 11] for reviews of this theory. The above action is written in the so-called Jordan frame. However, we can simplify the study of the dynamics of gravitational equations by making a conformal transformation to the Einstein frame. This frame provides a more detailed interpretation of the theory as a scalar-tensor theory, in which the scalar degree of freedom interacts with ordinary matter and modifies the gravitational interaction. $f(R)$ theories have been extensively studied in the context of cosmology, as they provide an alternative explanation for the accelerated expansion of the universe without the need for dark energy [7, 12–16].

In this paper, we are interested in studying the formation of black holes in $f(R)$ theories. It is well known that astrophysical black holes are formed by the gravitational collapse of matter. Black hole (BH) physics and spherical collapse have been key routes to understanding gravity. A remarkable property of a BH is that only a few parameters entirely describe its gravitational field. This is the so-called no-hair conjecture, or the uniqueness theorem [17]. Nonetheless, modifying Einstein-Hilbert's action could give rise to new BH solutions that can violate the uniqueness theorem (see ref. [18], and references therein, for static and spherically symmetric BH solutions in $f(R)$ gravity).

Understanding gravitational collapse phenomena have been a critical point in understanding the nature of the gravitational field. Based on GR, the dynamics of an isolated system that interacts only through the gravitational field typically end up either with the formation of a single black hole or a complete dispersion of the mass-energy to infinity. The numerical study of spherical gravitational collapse has a long history. However, one of the pioneering works in this area was performed by M. Choptuik [19], who solved the dynamical evolution of a scalar field minimally coupled to gravity from the numerical point of view. His investigations were inspired by previous analytical studies made by D. Christodoulou [20–22].

Choptuik [19] has shown that if p is a parameter describing some aspect of the initial distribution of scalar field energy, there exists a critical value $p^\star$ which denotes the threshold of black hole formation. For $p < p^\star$, the scalar field disperses to infinity, while for $p > p^\star$ a black hole is formed. In the supercritical regime, meaning for $p > p^\star$ but very close to the threshold, a universal behavior appears (i.e., independent of the initial data) relating the mass M of black holes to a universal scaling behavior

$$M \propto (p - p^\star)^\gamma\,, \quad \gamma \simeq 0.37\,. \tag{1.2}$$

This solution has been repeatedly verified by using a fully 3D code [23]. Adding a mass term to the theory [24], which introduces a length scale, also produces a universal behavior but with a different scaling parameter γ.

The pioneering work of Choptuik opened a new line of investigation to study the gravitational collapse of other fields. For example, the previous work has been extended to massive scalar fields [24, 25], the gravitational collapse of radiation [26] and perfect fluid [27], a complex scalar field with angular momentum [28], K-essence models [29], the Einstein-Massless-Dirac system [30], and massive vector field [31], to mention some of the many investigations carried out to date.

Research into gravitational collapse in $f(R)$ gravity has been undertaken in various specific cases [32–34]. For instance, refs. [35, 36] investigated the collapse of a spherical scalar field in the Hu-Sawicki model, the Starobinsky R^2 model, and the $R \ln R$ model, using double-null coordinates in the Einstein frame to focus on the behavior in the vicinity of the

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singularity of the formed BH. In addition, gravitational fluid collapse in $f(R)$ gravity was examined in refs. [37–42]. Hwang [43] investigated the gravitational collapse of a charged black hole in $f(R)$ gravity, utilizing the double-null formalism and exploring the mass inflation of the Cauchy horizon. The formation of a uniformly collapsing cloud of self-gravitating dust particles in an $f(R)$ model was studied in ref. [44], which shares analogies with the formation of large-scale structures in the early Universe and with the formation of stars in a molecular cloud experiencing gravitational collapse. Furthermore, ref. [45] explored the formation of dark matter halos, while ref. [46] analyzed the gravitational collapse of massive stars in $f(R)$ gravity.

In this paper, we study the spherically symmetric gravitational collapse of a massless scalar matter field in asymptotic flat spacetime in the Starobinsky R^2 gravity, one specific model in the $f(R)$ gravity. A similar analysis was performed in [47], but we extend the previous analysis by exploring the critical phenomena in the vicinity of the formation of the apparent horizon.

In addition to the power-law scaling relation, Choptuik gravitational collapse is well-known for exhibiting another exciting property: self-similarity. In the Choptuik model, discrete self-similarity means that there are specific control parameter values for which the collapsing mass distribution produces a series of self-similar solutions that repeat themselves at increasingly fine scales. Although this is an exciting property to investigate, in this paper, we focus only on the power-law scaling relation at the threshold of BH formation.

The organization of the paper is as follows: in section 2, we will derive the equations of motion for a general $f(R)$ theory. We start from the Jordan frame and make the conformal transformation to write this theory as a Scalar-Tensor theory. In section 3, we assume a spherically symmetric spacetime and derive the equations for gravitational collapse. Then in section 4, we will explain the numerical method. We will report numerical results in section 5. In the final section, we will present conclusions and discussions.

2 Framework of $f(R)$ gravity

This section briefly summarized the equations derived from the $f(R)$ theory. We start with the action (1.1) written in the Jordan frame [10, 11]. Variation of the action with respect to the metric tensor g gives us

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) f' = 8\pi G\, T^{(M)}_{\mu\nu}, \tag{2.1}$$

where $f' \equiv \partial_R f(R)$ denotes the derivative of the function f with respect to its argument R, ∇ is the covariant derivative, $\Box$ is the usual notation for the covariant D'Alembert operator $\Box \equiv \nabla_\alpha \nabla^\alpha$, and $T^{(M)}_{\mu\nu}$ is the energy-momentum tensor for the matter field. Defining

$$\chi \equiv \frac{\partial f(R)}{\partial R}\,, \qquad U'(\chi) \equiv \frac{dU}{d\chi} = \frac{1}{3}(2f - \chi R), \tag{2.2}$$

one can write the trace of the eq. (2.1) as

$$\Box\chi = U'(\chi) + \frac{8\pi G}{3} T. \tag{2.3}$$

Thus, $\chi = f'(R)$ is a new scalar degree of freedom that is not present in GR. In fact, for Einstein's gravity, $f(R) = R \Longrightarrow \chi \equiv 1$ and $\Box\chi \equiv 0$, the above equation reduces to the trace of the usual Einstein equations.

In the Jordan frame, the gravitational Lagrangian is an arbitrary function of R, and Einstein's equations are usually fourth-order in the metric. In order to make the formalism less involved, we transform $f(R)$ gravity from the current frame into the Einstein frame. In the latter, the second-order derivatives of $f'(R)$ are absent in the equations of motion for the metric components. We will see that the formalism can be treated as Einstein's gravity coupled to a scalar field. Therefore, we can use some results that have been developed in the numerical relativity community.

Taking the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where the conformal factor is given by

$$\Omega^2 = f'(R) = \exp\left(\sqrt{\frac{2}{3}}\kappa\phi\right) = \chi, \tag{2.4}$$

one obtains the corresponding action of $f(R)$ gravity in the Einstein frame

$$\mathcal{S}_E = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - V(\phi)\right] + \int \mathrm{d}^4x\mathcal{L}_M(\tilde{g}_{\mu\nu}/\chi(\phi), \psi), \tag{2.5}$$

where $\kappa^2 = 8\pi G$, and a tilde denotes quantities in the Einstein frame; thus, $\tilde{\nabla}$ and $\tilde{R}$ are the covariant derivative and the Ricci scalar associated to the metric $\tilde{g}_{\mu\nu}$. In this action, ϕ is a new scalar field that takes into account the modifications of Einstein gravity (in this paper, usually referred to as the gravity scalar field), the potential is given by

$$V(\phi) \equiv \frac{f'R - f}{2\kappa^2 f'^2}\,, \tag{2.6}$$

and ψ is the matter field that, in the Einstein frame, is non-minimally coupled to gravity.

From the action (2.5), we can proceed as usual. Variation with respect to the metric field, $\tilde{g}_{\mu\nu}$, give us the Einstein field equations

$$\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \kappa^2\left[\tilde{T}^{(\phi)}_{\mu\nu} + \tilde{T}^{(M)}_{\mu\nu}\right], \tag{2.7}$$

where we have defined the energy-momentum tensors as

$$\tilde{T}^{(\phi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \tilde{g}_{\mu\nu}\left[\frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right], \qquad \tilde{T}^{(M)}_{\mu\nu} = \frac{T^{(M)}_{\mu\nu}}{\chi}. \tag{2.8}$$

We are interested in the gravitational collapse of the most simple type of matter, a massless scalar field ψ. The Lagrangian density is given by

$$\mathcal{L}_M = -\frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi; \tag{2.9}$$

thus, the energy-momentum tensor for the matter field, in the Einstein frame, is

$$\begin{aligned}\tilde{T}^{(M)}_{\mu\nu} &= \frac{1}{\chi}\left(\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi\right) = \frac{1}{\chi}\left(\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi\right)\\ &= \exp\left(-\sqrt{\frac{2}{3}}\kappa\phi\right)\left(\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi\right)\,.\end{aligned} \tag{2.10}$$

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Using eq. (2.9), variation of the action (2.5) with respect to the scalar fields, gives us

$$\tilde{\Box}\phi - \frac{\partial V(\phi)}{\partial\phi} - \frac{\kappa}{\sqrt{6}}\tilde{T}^{(M)} = 0, \tag{2.11}$$

$$\tilde{\Box}\psi - \sqrt{\frac{2}{3}}\,\kappa\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\psi = 0, \tag{2.12}$$

where $\tilde{T}^{(M)}$ is the trace of the energy-momentum tensor of the matter field. For the massless scalar field, Lagrangian (2.9), we obtain $\tilde{T}^{(M)} = -\exp\left(-\sqrt{\frac{2}{3}}\kappa\phi\right)\tilde{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi$. Henceforth, we work only with equations of motion in the Einstein frame; thus, we drop the tilde over the quantities.

As stated, in $f(R)$ gravity, the scalar field account for modifications of the gravitational interaction. In the absence of a scalar matter field, the action reduces to a standard scalar-tensor theory of gravity, vacuum $f(R)$. Thus, the action suggests that critical phenomena can occur in the absence of a matter field, which raises the question of whether the matter field would introduce new physics at the critical regime. In order to investigate this question further, one could analyze the behavior of the critical exponents in this scenario. As we know, critical exponents describe the power-law behavior of various quantities and can provide insight into the underlying physics. By studying the critical exponents in the absence of a scalar matter field and comparing them to those in the presence of a matter field, one can determine whether the matter field introduces new physics and alters the critical regime. This analysis can help shed light on the behavior of $f(R)$ gravity and its relationship to other theories of gravity.

3 Metric ansatz and equations

As mentioned before, we are interested in studying the formation of event horizon in $f(R)$ theories. We use the metric in polar-areal coordinates [19], which takes the form

$$\mathrm{d}s^2 = -\alpha^2(t,r)\mathrm{d}t^2 + a^2(t,r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,, \tag{3.1}$$

where, α and a are the metric functions that depend on *time* and the *radial* coordinates in order to study the evolution. Using this metric ansatz, the equations of motion for the scalar fields (2.11), (2.12) are second-order PDEs. In order to reduce the field equations to a system of first-order PDEs, we define the following auxiliary fields [19, 47]

$$\begin{aligned} Q(t,r) &= \partial_r\phi(t,r)\,, \qquad & P(t,r) &= \frac{a}{\alpha}\partial_t\phi(t,r)\,, \\ \Phi(t,r) &= \partial_r\psi(t,r)\,, \qquad & \Pi(t,r) &= \frac{a}{\alpha}\partial_t\psi(t,r)\,. \end{aligned} \tag{3.2}$$

Thus, eqs. (2.11) and (2.12) are written as[1]

$$\partial_t P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a} Q\right) - a\alpha \frac{\partial V(\phi)}{\partial \phi} - e^{-\sqrt{\frac{2}{3}}\phi}\frac{\kappa\alpha}{\sqrt{6}a}\left(\Pi^2 - \Phi^2\right), \tag{3.3}$$

$$\partial_t Q = \frac{\partial}{\partial r}\left(\frac{\alpha}{a} P\right), \tag{3.4}$$

$$\partial_t \Pi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a}\Phi\right) + \sqrt{\frac{2}{3}}\frac{\kappa\alpha}{a}(P\Pi - Q\Phi), \tag{3.5}$$

$$\partial_t \Phi = \frac{\partial}{\partial r}\left(\frac{\alpha}{a}\Pi\right). \tag{3.6}$$

To fix the evolution of the system, we need the equations for the metric variables. These are obtained from eq. (2.7). The $tt-$ and $rr-$components of the Einstein equations yield the following expressions

$$\frac{a'}{a} = \frac{1-a^2}{2r} + \frac{\kappa^2}{4} r \left(e^{-\sqrt{\frac{2}{3}}\phi}\left(\Pi^2 + \Phi^2\right) + P^2 + Q^2\right) + \frac{\kappa^2}{2} r a^2 V(\phi), \tag{3.7}$$

$$\frac{\alpha'}{\alpha} = \frac{a^2-1}{2r} + \frac{\kappa^2}{4} r \left(e^{-\sqrt{\frac{2}{3}}\phi}\left(\Pi^2 + \Phi^2\right) + P^2 + Q^2\right) - \frac{\kappa^2}{2} r a^2 V(\phi), \tag{3.8}$$

here, a prime denotes a derivative with respect to the radial coordinate. These equations do not depict time evolution. However, they serve as constraint equations that must hold for every instance in time. Consequently, the values of the auxiliary fields within a given hypersurface determine the evolution of the metric functions for that particular hypersurface or time.

As in GR, it can be checked that the system of PDEs (2.7) is over-determined and can provide another equation for one of the metric functions. In fact, the $\theta\theta-$component of eq. (2.7) gives an evolution equation for metric variable a. We do not solve this equation, but we use it to monitor its convergence to zero as a check for the correctness of our solution.

4 Numerical setup

In this section, we provide a detailed account of the numerical formalisms employed for solving the system of PDEs under consideration. We give a comprehensive discussion on the boundary conditions, initial conditions, and methods utilized to detect the formation of a BH.

4.1 Initial conditions

The foliation of spacetime allows us to rewrite Einstein's equations as a set of PDEs that can be solved numerically. To completely specify the Cauchy problem, we need to provide initial data for the gravitational field and matter distribution on an initial hypersurface. Once the initial data is specified, we can use numerical methods to evolve the spacetime forward

[1] Eqs. (3.4), (3.6) come from the definitions of the auxiliary fields, and eqs. (3.3), (3.5) from the Klein-Gordon equations of the fields.

in time, solving for the metric and matter fields at each point in spacetime. We follow the same scheme that Choptuik used in [19], in which a smooth function parameterizes the initial distribution of the scalar field.

At initial time ($t = t_0 = 0$), we fix the form of each scalar field to be a Gaussian distribution

$$\phi(0,r) = p_o^{(\phi)} \exp\left[-\left(\frac{r-r_o}{d}\right)^2\right], \qquad \psi(0,r) = p_o^{(\psi)} \exp\left[-\left(\frac{r-r_o}{d}\right)^2\right] \tag{4.1}$$

where $\left(p_o^{(\phi)},\, p_o^{(\psi)},\, r_o,\, d\right)$ are constant parameters. We keep only $\left(p_o^{(\phi)},\, p_o^{(\psi)}\right)$ as free parameters, the others are fixed to ($r_0 = 20$, $d = 3$). Once the initial data for both scalar fields are established, we are free to choose the initial data for the auxiliary fields (3.2). For example, we set

$$\begin{aligned} \Phi(0,r) &= \Pi(0,r) = \partial_r\psi(0,r)\,, \\ Q(0,r) &= P(0,r) = \partial_r\phi(0,r)\,, \end{aligned} \tag{4.2}$$

which means that each scalar field is initially ingoing, ϕ and ψ approach to the origin.

4.2 Boundary conditions

In this paper, the range for the spatial coordinate is $r \in (0,\, r_{\max})$; where $r_{\max}$ simulates the infinity and numerically has to be chosen in a way that our solutions do not depend on this parameter. This can be done by imposing boundary conditions at the edges of this range.

The first boundary is located at $r = 0$, which is the origin of the spherically symmetric coordinate system. To ensure that the fields are well-defined at this point, we typically impose regularity conditions that ensure the fields are finite and smooth. Therefore, analyzing the system of equations near $r \to 0$ and imposing spacetime to be locally flat there, we enforce ϕ and ψ to satisfy the following conditions:

$$\partial_r\phi(0,r) = 0\,, \qquad \partial_r\psi(0,r) = 0\,. \tag{4.3}$$

Also, the regularity of the system imposes $a(t, r = 0) = 1$.

Our second boundary is located at $r = r_{\max}$. In this boundary, we impose the so-called "zero amplitude" or "radiative" boundary condition. This condition requires that any signals entering the integration domain from outside the domain have zero amplitude at the boundary. Therefore, we impose radiation boundary conditions [48] at $r = r_{\max}$. This condition is important for ensuring that the simulated waveforms are physically meaningful and do not include spurious signals that arise from reflections or echoes at the boundaries of the simulation domain. Mathematically, this condition is given by imposing

$$\frac{\partial\Psi(t,r_{\max})}{\partial t} + \frac{\partial\Psi(t,r_{\max})}{\partial r} + \frac{\Psi(t,r_{\max})}{r} = 0\,, \tag{4.4}$$

where Ψ denotes any of the scalar fields (ϕ,ψ). Notice that this boundary condition is exact only for spherically symmetric waves, characterized by the property that they are incident normal to the boundary $r = r_{\max}$.

It can be mentioned that, in our numerical evolution, this boundary is relevant for those simulations in which there is no BH formation and the matter field is dispersed to infinity. We fix the value of $r_{\max}$ in such a way that when a BH is formed, the reflected matter does not have enough time to reach the outer boundary.

4.3 Black hole detection

In the weak-field regime (when the amplitudes of the scalar fields are close to zero), it is expected that the matter content travels toward the origin and then, it is dispersed to infinity. As we increase the initial amplitude of the matter field $p_o^{(\psi)}$ or the initial amplitude of the gravity scalar field $p_o^{(\phi)}$, a BH can be formed since we increase the amount of matter in spacetime and/or the gravity potential becomes stronger.

Varying $\left\{p_o^{(\phi)}, p_o^{(\psi)}\right\}$ parameters, we can reach the limit where a BH forms in the evolution. However, as we stated before, our metric —written in the polar/areal coordinates (3.1)— cannot penetrate apparent horizons.

Nevertheless, we can detect signals that indicate the formation of a BH: the lapse function goes to zero, signaling that the coordinate system starts to become singular near what would be the radius of the BH. In a spherically symmetric spacetime, we can monitor the *Misner-Sharp mass* [49]

$$m_{MS}(t,r) = \frac{r}{2}\left(1 - (\nabla r)^2\right) = \frac{r}{2}\left(1 - \frac{1}{a^2(t,r)}\right). \tag{4.5}$$

Physically, the Misner-Sharp mass can be interpreted as the mass contained within a sphere of radius r centered at the origin. In a collapse simulation where a BH is formed, we will see that the function $2m_{MS}/r$ rapidly tends to 1 at some specific radius, $r = R_{BH}$, where R_{BH} is precisely the radius of the BH. Numerically, we say that we have detected a black hole if $2m_{MS}/r \geq 0.995$.

5 Model and results

For numerical purposes, we need to specify the exact form of $f(R)$ gravity. In this paper, we focused on the Starobinsky R^2 model

$$f(R) = R + \beta R^2 \tag{5.1}$$

where β is a constant parameter (specifically, we chose $\beta = 1$). This model was proposed as an inflationary model that could explain the observed large-scale homogeneity and isotropy of the universe. For details of the model, see [50, 51] and the review [11, 52].

In this model, the Ricci scalar in terms of the gravity scalar field is given by $R = (\chi - 1)/2$, and the potential (2.6), in the Einstein frame, can be written as

$$V(\phi) = \frac{1}{8}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2 \tag{5.2}$$

and is plotted in figure 1. The exponential function in the potential ensures that $V(\phi)$ is flat for large values of ϕ, leading to a long period of slow-roll inflation. Thus, inflation is a natural phenomenon in the R^2 model [53].

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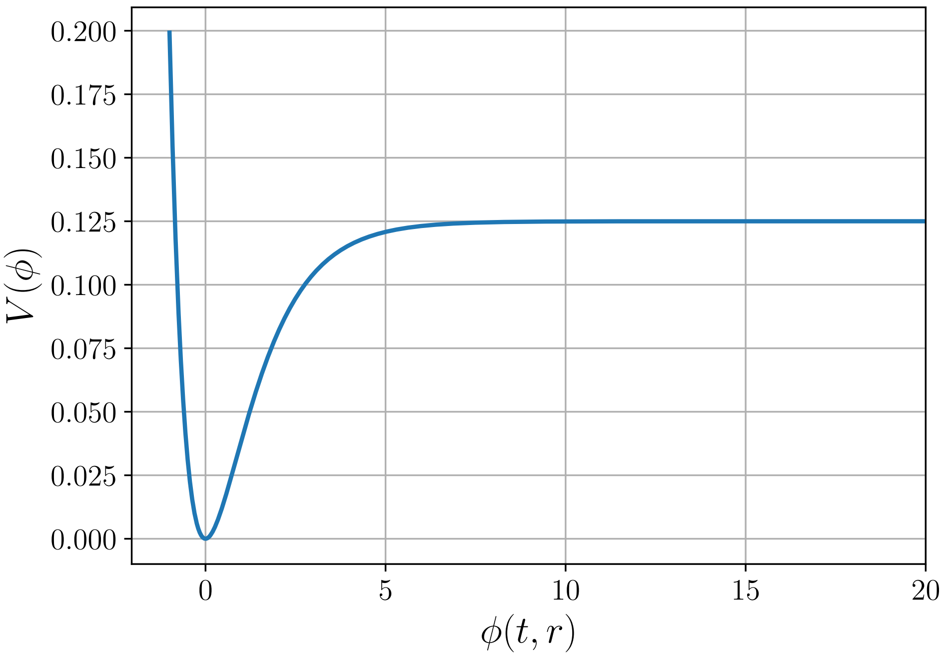

Figure 1. Potential for the Starobinsky R^2 model, $f(R) = R + R^2$.

In this paper, we work in units in which $\kappa = 1$. The system is evolved between $r = 10^{-50}$ and $r = r_{\max} = 50$ from $t = 0$ until it forms an apparent horizon (r_H). We solve numerically the system of PDEs using the fourth-order Runge-Kutta method in both time and spatial directions. We work with a fixed grid in which the r-spacing is given by $\Delta r = 10^{-4}$. The time resolution is also fixed but satisfies the Courant-Friedrichs-Lewy condition $\Delta t = \Delta r/5$.

5.1 Weak field regime

When the amplitude of the matter field and the amplitude of the gravity scalar field are small ($p_o^{(\phi)} \ll 1$, $p_o^{(\psi)} \ll 1$), the gravitational interaction is weak and the matter field bounces at the origin ($r = 0$) and then disperses to infinity. In figure 2, we plotted both scalar fields at different time[2] for an evolution in which eq. (4.1) gives the initial data. Specifically, we fixed $p_o^{(\psi)} = 0.01$ and $p_o^{(\phi)} = 0.03$. In this figure, we also plotted the evolution for the Choptuik case, $f(R)$ model (5.1) with $\beta = 0$.

Based on the evolution, we see that gravitational potential does not capture enough matter to form a black hole, and after the interaction near the origin, the matter field is dispersed to infinity. In the same figure, we can notice that the term R^2 does not significantly modify the gravitational collapse compared to GR, the Choptuik gravitational collapse. When $\beta \neq 0$ and near the critical surface, the matter field takes more time to bounce, reaching infinity a bit later than the Choptuik case. This effect is no longer

[2] Time is the iteration step and not proper time at $r = 0$.

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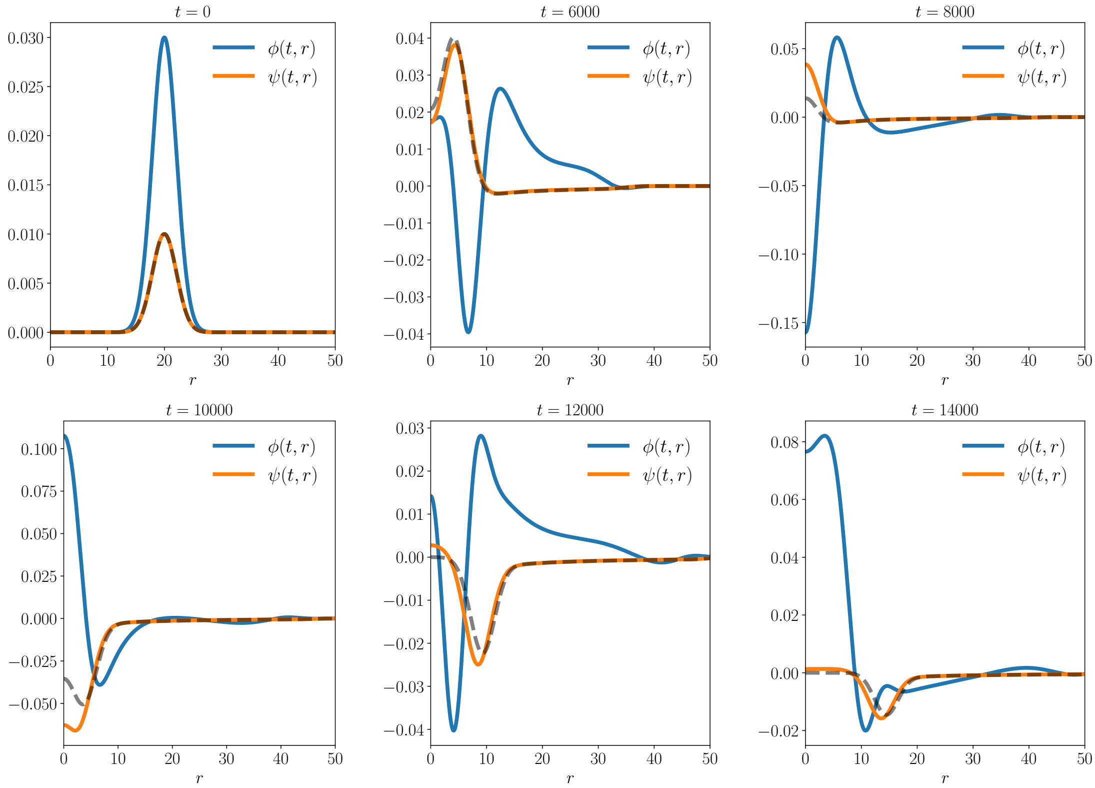

Figure 2. Plots of the scalar field profiles (in the weak field regime) at different times of the evolution. $\phi(t,r)$ (blue line) is the gravity scalar field. $\psi(t,r)$ (orange line) is the physical scalar field. The dashed line corresponds to the evolution of GR, i.e., the Choptuik gravitational collapse.

appreciated for smaller values of $p_o^{(\phi)}$ and $p_o^{(\psi)}$. When these initial values move away from the critical surface, we can conclude that GR governs the collapse.

5.2 Strong field regime

It is expected that as the initial mass of the matter field (ψ) is increased in a gravitational collapse scenario, the resulting outcome will eventually be a black hole. This is because as the mass of the collapsing matter increases, it produces a stronger gravitational field that causes more matter to collapse toward the center. Also, if we fix the matter field amplitude $p_o^{(\psi)}$ but increase the amplitude of the gravity scalar field $p_o^{(\phi)}$, it is expected that potential gravity becomes stronger, also forming a BH.

In this paper, we run simulations in an extended parameter space $\left(p_o^{(\psi)},\ p_o^{(\phi)}\right)$ from values in which all fields are dispersed to infinity, to values where the final result of the evolution is the creation of an event horizon. In figure 3, we plot the radius of the formed BH at the end of each evolution for the parameter space explored in this paper. It can be noticed that for small values of both scalar fields, there is no BH formation. When we increase one of the two amplitude parameters, inevitably, the evolution stops, signaling the formation of a BH. This is consistent with our understanding that as the mass of the collapsing matter increases, the gravitational field becomes stronger and a black hole is

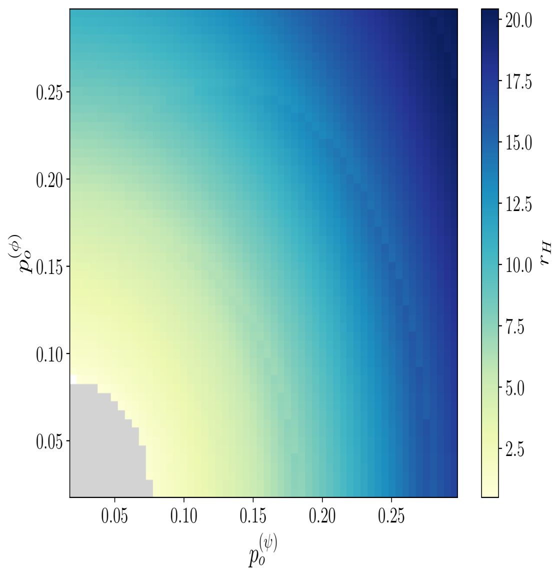

Figure 3. Radius of the formed BH (r_h) depending on the value for the initial data $\left(p_o^{(\psi)},\ p_o^{(\phi)}\right)$.

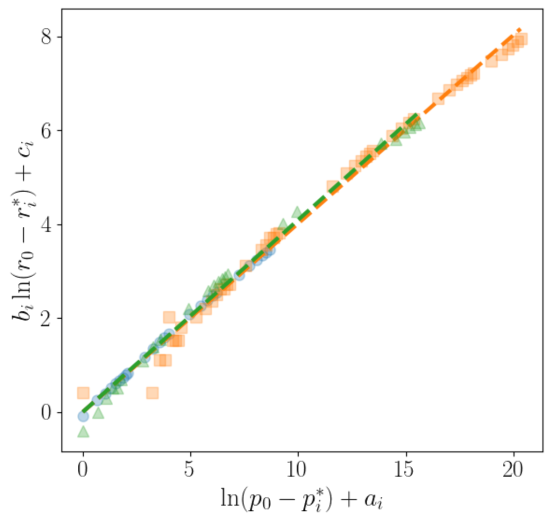

Figure 4. Radius scaling relation (5.3). Blue-circle and orange-squared data are obtained by fixing $p_o^{(\phi)}$ and varying $p_o^{(\psi)}$. Green-triangle data is the vacuum collapse in the Starobinsky model.

more likely to form. The results also show that the radius of the BH increases as the initial mass of the matter fields becomes stronger. This is also expected since the strength of the gravitational field produced by the collapsing matter is directly related to its mass, and a stronger field leads to a larger BH.

As we saw in figure 3, once we cross the critical surface in a gravitational collapse scenario, a BH will always form. However, ref. [47] reports an interesting finding that there exist initial data on the black hole formation side of the critical surface that does not produce a BH. This finding is based on numerical simulations of gravitational collapse in a specific parameter space, and is supported by figure 3 of that paper, which shows regions in the parameter space where BH formation does not occur despite being expected. One possible explanation for this discrepancy is that these non-black-hole-forming regions represent small patches in the parameter space that are not well-resolved by our numerical simulations.[3]

Although we do not show it here, every time a BH forms, the curvature scalar R and the Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$ are zero at the last time of the evolution and outside the event horizon, $r > r_H$; thus the exterior solution is Schwarzschild, which is consistent with the no-hair theorem and previous results showed in [54].

The Choptuik model is a simplified mathematical model used to study the formation of black holes in gravitational collapse. In this model, the critical collapse is of type II, which means that as the critical limit ($p^\star$) is approached, the radius of the black hole formed in the collapse becomes smaller and smaller, eventually approaching zero. This phenomenon is known as the "critical scaling" of the Choptuik model. One exciting feature of this critical

[3] It can also be mentioned that the last term in eq. (13) of ref. [47] seems to have the wrong sign. It remains to be checked if it is a typo.

scaling is that it allows the formation of infinitesimal black holes. For the Starobinsky model, we observe a similar behavior. The radius of the formed BH near the critical limit seems to start at a small value, near the critical surface $r_H \approx 10^{-3}$.

Performing a fine-tuning method between the weak-field regime and the strong-field regime, we found the critical surface. Near this limit, the radius of the BH satisfies a scaling behavior

$$r_H \propto (p - p^{\star})^{\gamma}\,, \tag{5.3}$$

similar to the Choptuik case [19]. Our results show that the scaling exponent, denoted as γ in eq. (5.3), is approximately 0.40 for the Starobinsky model. This scaling relation is plotted in figure 4. This result was obtained by varying the initial amplitude of the matter-scalar field while keeping the initial amplitude of the gravity-scalar field fixed. The study presented results for two different initial amplitudes of the matter-scalar field: 0.02 (orange-square data, figure 4) and 0.06 (green-square data, figure 4).

Furthermore, it has to be mentioned that the same scaling relation was obtained in the gravitational vacuum collapse of the Starobinsky model, where the matter-scalar field was set to zero, and only the gravity-scalar field was varied. The blue-circle data in figure 4 represent the results for this case. Overall, the study suggests that the scaling exponent of the Starobinsky model in gravitational collapse is independent of the initial amplitude of both scalar fields.

6 Conclusions and discussions

In this paper, we have studied gravitational collapse for the Starobinsky R^2 model (a particular example of $f(R)$ theory) coupled to a massless scalar field in a spherically symmetric spacetime. The Starobinsky model has been used as a possible explanation for inflation in the early universe. Also, it is considered the first quantum correction to the vacuum Einstein equations and was expected to avoid the singularity problem at the center of black holes.

We have presented the framework of $f(R)$ theory in the Jordan frame and performed a conformal transformation to write this theory as a Scalar-Tensor theory. In the new frame, called the Einstein frame, besides the standard matter field added to the spacetime, gravity is coupled to a new scalar field which drives the modifications from the Einstein gravity.

The gravitational collapse for the Starobinsky R^2 model is very similar to the Choptuik cases [$f(R) = R$], in the sense that depending on the initial conditions, the final state of the system is the formation of a BH or dispersion of the matter field to infinity. Our results show that in a spherically symmetric spacetime, when the amplitude of the initial matter scalar field and the gravity scalar field are weak, the final result of the evolution is a Minkowski spacetime; the matter field is dispersed to infinity. A BH is formed when we increase the amplitude of one of the scalar fields. Increasing the matter field amplitude adds more mass to the spacetime at the initial time; while increasing the amplitude of the gravity scalar field, the gravitational potential capture more matter, forming a BH in both cases. When a BH is formed, the final solution is the Schwarzschild metric. These results are consistent with those obtained in ref. [54].

Furthermore, the study analyzed the critical phenomena near the formation of the event horizon and found that in the critical limit, there is a power-law scaling for the radius of the event horizon (r_H) with a critical exponent of approximately 0.40 (denoted as γ). This critical exponent is a characteristic of the gravitational collapse and is independent of the specific scalar field being studied. Overall, the study provides insights into the formation of black holes in spherically symmetric spacetimes with weak initial scalar fields and sheds light on the critical behavior near the formation of event horizons for the Starobinsky R^2 model.

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