

On the 4D generalized Proca action for an Abelian vector field

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Received May 27, 2016

Accepted September 2, 2016

Published September 19, 2016

Abstract. We summarize previous results on the most general Proca theory in 4 dimensions containing only first-order derivatives in the vector field (second-order at most in the associated Stückelberg scalar) and having only three propagating degrees of freedom with dynamics controlled by second-order equations of motion. Discussing the Hessian condition used in previous works, we conjecture that, as in the scalar galileon case, the most complete action contains only a finite number of terms with second-order derivatives of the Stückelberg field describing the longitudinal mode, which is in agreement with the results of *JCAP* **05** (2014) 015 and *Phys. Lett. B* **757** (2016) 405 and complements those of *JCAP* **02** (2016) 004. We also correct and complete the parity violating sector, obtaining an extra term on top of the arbitrary function of the field A_μ , the Faraday tensor $F_{\mu\nu}$ and its Hodge dual $\tilde{F}_{\mu\nu}$.

Keywords: gravity, modified gravity, particle physics - cosmology connection

ArXiv ePrint: [1605.08355](https://arxiv.org/abs/1605.08355)



Contents

1	Introduction	1
2	Present status	2
2.1	Generalized Abelian Proca theory	2
2.2	Investigation procedures	2
3	Generic structure	3
4	Parity conserving terms	5
5	Parity violating terms	5
5.1	A useful identity	6
5.2	Simplification of \mathcal{L}_5^c and \mathcal{L}_6^c	7
6	Final model	7

1 Introduction

Along the line of modifying gravity in a scalar-tensor way, many proposals have been made to write down theories whose dynamics stem from second order equations of motion for both the tensor and the scalar degrees of freedom [1–5], thus generalizing an old proposal [6]; such theories have been dubbed Galileons. The obvious next move consists in obtaining a similar general action for a vector field [7] (see also in refs. [8–11]), thereby forming the vector Galileon case [12, 13], which was investigated thoroughly [14–21]. Demanding U(1) invariance led to a no-go theorem [22] which can be by-passed essentially by dropping the U(1) invariance hypothesis. Cosmological implications of such a model can be found e.g. in refs. [23–35].

Recent papers [12, 36, 37] have derived the most general action containing a vector field, with different conclusions as to the number of possible terms given the underlying hypothesis. In refs. [12, 37], the Lagrangian was built from contractions of derivative terms with Levi-Civita tensors, whereas ref. [36] used a more systematic approach based on the Hessian condition. It appears that a consensus has finally been reached, suggesting only a finite number of terms in the theory, all of them being given in an explicit form. To describe this consensus and complete the discussion, we examine in the present paper an alternative explanation for the presence of, allegedly, only a finite number of terms in the generalized Proca theory, using the tools developed in ref. [36] where the infinite series of terms was conjectured. This discussion also allows us to compare the systematic procedure used in ref. [36] with the construction based on Levi-Civita tensors of refs. [12, 37]. We then summarize these previously obtained results and settle the whole point in as definite a manner as possible.

Focusing on the parity violating sector of the model, not thoroughly investigated in refs. [12, 37], certain terms obtained in ref. [36] should not appear according to the above-mentioned discussion. Indeed, we show that, because of an identity not taken into account in ref. [36], those unexpected parity-violating terms are either merely vanishing or can be combined into a simple scalar formed with the field A^μ , the Faraday tensor $F^{\mu\nu}$ and its Hodge

dual $\tilde{F}^{\mu\nu}$. This closes the gap, hence providing an even firmer footing to the conjecture according to which the most general theory is in fact given by eq. (6.2), which is, up to a new term uncovered in this paper, eq. (12) of ref. [37] in Minkowski space, or eq. (28) in an arbitrary curved spacetime.

In section 2, we summarize the results previously obtained, together with the associated investigation procedures. We then present the generic structure in section 3, emphasizing how it permits an automatic implementation of the Hessian condition, and argue that the number of acceptable Lagrangian structures satisfying the usual physical requirements is finite, up to arbitrary functions. Splitting the possible terms into parity conserving and violating contributions, we motivate our conclusion in sections 4 and 5, resolving the apparent disagreement between the present conjecture and the conclusions of a previous work [36]; we conclude in section 6 by explicitly writing down the final 4D vector action.

2 Present status

Let us first introduce the vector theory, the hypothesis and results obtained thus far. We assume in what follows the Minkowski metric to take the form $g^{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and set $(\partial \cdot A) \equiv \partial_\mu A^\mu$ and $X = A_\mu A^\mu$ for simplification and notational convenience.

2.1 Generalized Abelian Proca theory

One seeks to generalize Proca theory, namely that stemming from the action

$$\mathcal{S}_{\text{Proca}} = \int \mathcal{L}_{\text{Proca}} d^4x = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 X \right) d^4x, \quad (2.1)$$

with A^μ being a massive vector field, not subject to satisfy a U(1) invariance, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ being the associated Faraday tensor. The generalization of this action can be made by considering all “safe” terms containing the vector field and its first derivative. To explicit what safe means in this context, one decomposes the field into a scalar π and pure vector \bar{A} parts according to

$$A_\mu = \partial_\mu \pi + \bar{A}_\mu \quad \text{or} \quad A = d\pi + \bar{A}, \quad (2.2)$$

where π is commonly referred to as the Stückelberg field, and \bar{A}^μ is divergence-free. One then demands the equations of motion for A^μ , and for both π and \bar{A}^μ , are second order, and that the Proca field propagates only three degrees of freedom [38]. These conditions are discussed in full depth in refs. [12, 36, 37]. The first condition ensures that the model can be stable [39–41], while the second stems from the fact that a massive field of spin s propagates $2s + 1$ degrees of freedom.

Note that the scalar field will appear in two different parts, one containing only the Stückelberg field itself, and one containing also the pure vector contribution \bar{A} . Examining the decoupling limit of the theory, one recovers for the pure scalar part of the Lagrangian the exact requirements of the Galileon theory [1–4], and so this part of the Lagrangian must reduce to this well-studied class of model.

2.2 Investigation procedures

Two different but equivalent procedures have been devised to write down the most general theory sought for. The first, originally proposed and explained in refs. [12, 37], consists in

a systematic construction of scalar Lagrangians in terms of contractions of two Levi-Civita tensors with derivatives of the vector field. This permits an easy comparison with the Galileon theory, as the same structure automatically ensues. The condition that only three degrees of freedom propagate is then verified on the relevant terms.

The second procedure, put forward in ref. [36], works somehow the other way around by systematically constructing all possible scalar Lagrangians propagating only three degrees of freedom. To achieve this requirement, a condition on the Hessian of the Lagrangian \mathcal{L} (or each independent such Lagrangian) considered, namely

$$\mathcal{H}^{\mu\nu} = \frac{\partial^2 \mathcal{L}}{\partial(\partial_0 A_\mu) \partial(\partial_0 A_\nu)}, \quad (2.3)$$

is imposed. As discussed in ref. [36], in order that the timelike component of the vector field does not propagate in non trivial theories, the components $\mathcal{H}^{0\mu}$ must vanish. All possible terms satisfying this constraint are considered at each order.

There are two crucial points concerning the latter method that still need to be checked once the terms satisfying all other requirements have been obtained: not only they must reduce to the scalar galileon Lagrangians in the pure scalar sector, but they must imply second class constraints. Moreover, given that it is a systematic expansion in terms of scalars built out of vectors with derivatives, one must make sure they are not either identically vanishing or mere total derivatives. In other words, although the method ensures that all possible terms will be found, they are somehow too numerous and there may remain some redundancy that must be tracked down and eliminated.

3 Generic structure

As all contractions between vector derivatives and δ and ϵ can always be written in terms of ϵ only, a complete basis for expanding the general category of Lagrangians of interest is made up with terms of the form

$$\mathcal{T}_N^i = \underbrace{\epsilon^- \dots \epsilon^-}_{N} \partial.A.\partial.A.\dots, \quad (3.1)$$

where indices appearing in the field derivatives are contracted only with corresponding indices in the Levi-Civita tensors, the remaining indices being contracted possibly in between Levi-Civita tensors in such a way as to yield a scalar. Each index i reflects the fact that there can be more than one way to contract the N Levi-Civita tensor to form a scalar. These terms form a complete basis for the Lagrangians containing an arbitrary number of field derivatives.

The general Lagrangian will then be of the form

$$\mathcal{L} = \sum_{i,N} f_N^i(X) \mathcal{T}_N^i, \quad (3.2)$$

where we consider only prefactors that are functions of X : one could envisage contracting a vector field itself with the derivative terms involved in eq. (3.2), but that would lead to an equivalent basis up to integrations by parts [36].¹ When written in terms of the Stückelberg

¹We have found one special case, discussed below eq. (6.1), for which the total derivative of the integration by parts would actually vanish for symmetry reasons; we included and discussed this special term in our final form of the action.

field only, i.e. setting $A_\mu \rightarrow \partial_\mu \pi$, and restricting attention to $N = 2$, eq. (3.2) automatically yields the subclass of the generalised galileon theory [3] containing only derivatives of the scalar field.²

The terms thus built in eq. (3.1) now fall into two distinct categories, depending on how they behave under a $U(1)$ gauge transformation. Those invariant under such transformations contracts all field derivative indices to one and only one Levi-Civita tensor, i.e. they take the form

$$\epsilon^{\mu\nu-} \epsilon^{\rho\sigma-} \dots \partial_\mu A_\nu \partial_\rho A_\sigma \dots ,$$

which can all be equivalently expressed as functions of scalar invariants made out of the Faraday tensor $F_{\mu\nu}$ and its Hodge dual $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. Indeed, written in this form, one can identically replace all $\partial_\mu A_\nu$ by $\frac{1}{2} F_{\mu\nu}$. Conversely, since the following identities

$$F^{\mu\nu} F_{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta} \epsilon_{\rho\sigma\alpha\beta} \partial_\mu A_\nu \partial^\rho A^\sigma , \quad (3.3)$$

and

$$\tilde{F}^{\mu\nu} F_{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma , \quad (3.4)$$

hold, any function of F and \tilde{F} can be expressed as a term such as discussed above.

This leads to the first Lagrangian compatible with our requirements, namely the so-called \mathcal{L}_2 , containing all possible scalars made by contracting A_μ , $F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}$. Such terms can always be expressed [42] as functions of the scalars X , $F^2 \equiv F_{\mu\nu} F^{\mu\nu}$, $F \cdot \tilde{F} \equiv F_{\mu\nu} \tilde{F}^{\mu\nu}$ and

$$\left(A \cdot \tilde{F} \right)^2 \equiv A_\alpha \tilde{F}^{\alpha\sigma} A_\beta \tilde{F}^{\beta\sigma} = A_\alpha A_\beta \epsilon^{\alpha\sigma\mu\nu} \epsilon^\beta_{\sigma\kappa\delta} \partial_\mu A_\nu \partial^\kappa A^\delta , \quad (3.5)$$

again up to integrations by part. The Lagrangian \mathcal{L}_2 always satisfies the conditions discussed in the previous section, and in particular yields a trivially vanishing Hessian condition $\mathcal{H}^{0\mu}$: varying \mathcal{L}_2 with respect to $\partial_0 A_0$ [see eq. (2.3)] yields a factor containing ϵ^{00-} , which vanishes identically. It also gives second order equation of motion both for π and \bar{A}_μ as it contains neither $\partial\partial\pi$ nor $\partial\partial\bar{A}_\mu$ terms. We should emphasize at this point that \mathcal{L}_2 contains parity conserving as well as parity violating terms; we shall not consider them any more, but they should be assumed always present in the forthcoming discussion.

All the terms contained in eq. (3.2) but not of the form discussed in the previous paragraph read

$$\mathcal{L}_N^i = f_N^i(X) \underbrace{\epsilon^{\mu-} \dots \epsilon^{\nu-}}_N \partial_\mu A_\nu \dots , \quad (3.6)$$

where at least one field derivative has indices contracted with two distinct Levi-Civita tensors and the $f_N^i(X)$ are arbitrary functions of the gauge vector magnitude $X = A^2$. For $N \leq 2$, the Hessian condition is automatically satisfied: $\mathcal{H}^{0\mu}$ stems for a variation of the Lagrangian with respect to $\partial_0 A_0$ and $\partial_0 A_\mu$. This demands three equal “0” indices distributed on at most two Levi-Civita tensors, resulting in a vanishing contribution for symmetry reasons. The other requirements, such as the order of the equations of motion these terms lead to, are discussed in length in sections 4 and 5.

For $N > 2$, the situation is less clear, as the Hessian does not then identically vanish. Instead, the condition $\mathcal{H}^{0\mu} = 0$ then implies that the coefficients of all the linearly independent terms stemming from this condition vanish. The number of such linearly independent terms increases with the number of field derivatives allowed for in the Lagrangian, and it

²The full generalized galileon theory is recovered if one also makes the replacement $f_N^i(X) \rightarrow f_N^i(\pi, \partial\pi)$.

is therefore to be expected that, above a given threshold value $N > N_{\text{thr}}$, up to unforeseeable fortuitous cancellations, no new term will be obtainable that could possibly satisfy the requirements of a safe theory. We conjecture that, as in the scalar galileon case, $N_{\text{thr}} = 2$; the following sections detail the reasons hinting to such a conjecture, splitting into parity conserving (N even) and violating (N odd) contributions. Note that there exists a general argument, based on the fact that the Lagrangian contains second-order derivatives of the field, for which the scalar galileon theory automatically stops at $N = 2$ [3], whereas in the vector case, no such argument can be found, the Lagrangian containing only first-order derivatives and there could exist terms which vanish when $A_\mu \rightarrow \partial_\mu \pi$ while still satisfying all other hypothesis. As a result, the arguments below are different from those needed to show $N_{\text{thr}} = 2$ in the scalar galileon case.

4 Parity conserving terms

Previous works discussed parity conserving actions with $N = 2$ including up to 4 field derivatives ∂A , the so-called \mathcal{L}_n , with $n = 3, \dots, 5$ [12], and $n = 6$ [36], n counting the number of field derivative plus two (this convention, bizarre in the vector case, is meaningful in the original galileon construction). Up to $n = 5$, the Lagrangians satisfy the condition that the scalar part of the vector field corresponds only to non trivial total derivative interactions, a condition which, once relaxed, yields the extra $n = 6$ term: in the latter situation, one can always factorize the action by some factors involving the Faraday tensor and its dual, ensuring it vanishes in the pure scalar sector. All these terms were shown to be of the form presented in eq. (3.6) above with $N = 2$, thus agreeing with our conjecture. They also comply with all the necessary requirements we asked for the theory to be physically meaningful, with second-order equations of motion and only three propagating degrees of freedom [36, 37].

In ref. [36], new terms were also suggested which, similarly to \mathcal{L}_6 , were of the form $(\partial A)^p F^q \tilde{F}^r$ (with r even to ensure parity conservation), and therefore vanishing in the pure scalar sector. It was even argued that an infinite tower of such terms could be generated. A further examination of these terms however revealed a different, and somehow more satisfactory, picture: some new terms, by virtue of the Cayley-Hamilton theorem, vanish identically in 4 dimensions, a conclusion that can also be reached by rewriting the relevant terms in the form presented in eq. (3.6), but with Levi-Civita tensors having more than 4 indices [37], explaining why the new terms identically vanish in 4 dimensions to which the present analysis is restricted: in a way similar to Lovelock theory for a spin 2 field [43], one can imagine that for each number of dimensions, a finite number of new terms can be generated.

In conclusion of this short section, suffices it to say that extra parity preserving terms involving more fields and not already present in \mathcal{L}_2 have been actively searched for, and never found; although this does not prove that such terms cannot be found, this provides a sufficiently solid basis to assume this statement as a conjecture, which will only make sense provided a similar conclusion can be reached for the parity-violating terms to which we now turn.

5 Parity violating terms

Parity-violating terms can be written as in eq. (3.6) with an odd number N of Levi-Civita tensors. For $N = 1$, it leads to an action built from eq. (3.4), and hence is already included in \mathcal{L}_2 discussed above. One thus expects no terms not included in \mathcal{L}_2 since those terms would

contain at least three Levi-Civita symbols. In ref. [36] however, two extra such terms were found to satisfy all the physically motivated requirements, obtained through the systematic Hessian method. They read

$$\mathcal{L}_5^\epsilon = F_{\mu\nu} \tilde{F}^{\mu\nu} (\partial \cdot A) - 4 \left(\tilde{F}_{\rho\sigma} \partial^\rho A_\alpha \partial^\alpha A^\sigma \right), \quad (5.1)$$

and

$$\mathcal{L}_6^\epsilon = \tilde{F}_{\rho\sigma} F^\rho F^\sigma \partial^\alpha A^\beta. \quad (5.2)$$

According to our conjecture, they should either vanish or be contained in the previous terms up to a total derivative. We show below that it is indeed the case, and for that purpose we first recall an identity derived and first reported, to our knowledge, in ref. [42]; this completes the proof that the systematic procedure could not find terms having up to 4 field derivatives that are not contained in \mathcal{L}_2 .

5.1 A useful identity

Let $A_{\mu\nu}$ and $B_{\mu\nu}$ be two antisymmetric tensors in a four-dimensional spacetime with mostly positive signature. One has

$$A^{\mu\alpha} \tilde{B}_{\nu\alpha} + B^{\mu\alpha} \tilde{A}_{\nu\alpha} = \frac{1}{2} \left(B^{\alpha\beta} \tilde{A}_{\alpha\beta} \right) \delta_\nu^\mu, \quad (5.3)$$

where $\tilde{X}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} X_{\alpha\beta}$ is the Hodge dual of X [42].

In order to prove this identity, one uses the relation (see, e.g., ref. [44])

$$\epsilon^{\alpha_1 \dots \alpha_k \delta_{(k+1)} \dots \delta_n} \epsilon_{\beta_1 \dots \beta_k \delta_{(k+1)} \dots \delta_n} = (-1)^s (n-k)! k! \delta_{\beta_1 \dots \beta_k}^{[\alpha_1 \dots \alpha_k]}, \quad (5.4)$$

where s counts the number of minus signs in the signature of the metric and n the dimension of spacetime. One gets

$$\epsilon^{\alpha_1 \alpha_2 \alpha_3 \delta} \epsilon_{\beta_1 \beta_2 \beta_3 \delta} = -3! \delta_{\beta_1}^{[\alpha_1} \delta_{\beta_2}^{\alpha_2} \delta_{\beta_3}^{\alpha_3]}, \quad (5.5)$$

and

$$\epsilon^{\alpha_1 \alpha_2 \delta_1 \delta_2} \epsilon_{\beta_1 \beta_2 \delta_1 \delta_2} = -2! 2! \delta_{\beta_1}^{[\alpha_1} \delta_{\beta_2}^{\alpha_2]}, \quad (5.6)$$

in the $n = 4$ -dimensional case, leading to

$$X^{\alpha\beta} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \tilde{X}_{\mu\nu}, \quad (5.7)$$

to express a tensor from its Hodge dual.

Beginning from the left-hand side of the identity we wish to prove, we get

$$A^{\mu\alpha} \tilde{B}_{\nu\alpha} = -\frac{1}{4} \epsilon^{\gamma\epsilon\mu\alpha} \epsilon_{\nu\sigma\rho\alpha} \tilde{A}_{\gamma\epsilon} B^{\sigma\rho}, \quad (5.8)$$

which, upon using eq. (5.5), yields $A^{\mu\alpha} \tilde{B}_{\nu\alpha} = \frac{3}{2} \delta_\nu^{[\gamma} \delta_\sigma^{\epsilon} \delta_\rho^{\mu]} \tilde{A}_{\gamma\epsilon} B^{\sigma\rho}$. Expanding and simplifying the relevant terms, one finally obtains eq. (5.3), as desired.

As a direct consequence, we can easily deduce the identities

$$F^{\mu\alpha} F_{\nu\alpha} - \tilde{F}^{\mu\alpha} \tilde{F}_{\nu\alpha} = \frac{1}{2} \left(F^{\alpha\beta} F_{\alpha\beta} \right) \delta_\nu^\mu, \quad (5.9)$$

and

$$F^{\mu\alpha} \tilde{F}_{\nu\alpha} = \frac{1}{4} \left(F^{\alpha\beta} \tilde{F}_{\alpha\beta} \right) \delta_\nu^\mu, \quad (5.10)$$

which follows from substituting $A_{\mu\nu} = F_{\mu\nu}$, $B_{\mu\nu} = \tilde{F}_{\mu\nu}$ and $A_{\mu\nu} = B_{\mu\nu} = F_{\mu\nu}$ respectively in eq. (5.3).

5.2 Simplification of \mathcal{L}_5^ϵ and \mathcal{L}_6^ϵ

We now use the above identities to first expand \mathcal{L}_5^ϵ . One has

$$\mathcal{L}_5^\epsilon = F_{\mu\nu}\tilde{F}^{\mu\nu}\partial \cdot A - 4\tilde{F}_{\rho\sigma}\partial^\rho A^\alpha\partial_\alpha A^\sigma = F_{\mu\nu}\tilde{F}^{\mu\nu}\partial \cdot A - 4\tilde{F}_{\rho\sigma}(F^{\rho\alpha} + \partial^\alpha A^\rho)\partial_\alpha A^\sigma, \quad (5.11)$$

whose last term can be transformed into

$$\tilde{F}_{\rho\sigma}(F^{\rho\alpha} + \partial^\alpha A^\rho)\partial_\alpha A^\sigma = \frac{1}{4}F\tilde{F}\delta_\sigma^\alpha\partial_\alpha A^\sigma + \tilde{F}_{\rho\sigma}\partial^\alpha A^\rho\partial_\alpha A^\sigma, \quad (5.12)$$

so that, finally, one ends up with

$$\mathcal{L}_5^\epsilon = -4\tilde{F}_{\rho\sigma}\partial^\alpha A^\rho\partial_\alpha A^\sigma = 0, \quad (5.13)$$

being a contraction between a fully symmetric and a fully antisymmetric tensor.

As for \mathcal{L}_6^ϵ , one has

$$\mathcal{L}_6^\epsilon = \left(\tilde{F}_{\rho\sigma}F^{\sigma\alpha}\right)F^{\rho\beta}\partial_\alpha A_\beta = \left(-\frac{1}{4}F\tilde{F}\delta_\rho^\alpha\right)F^{\rho\beta}\partial_\alpha A_\beta. \quad (5.14)$$

A few straightforward manipulations then yield

$$\mathcal{L}_6^\epsilon = -\frac{1}{8}\left(F\tilde{F}\right)F^2, \quad (5.15)$$

showing \mathcal{L}_6^ϵ is not in fact a new term but is already included in the Lagrangian \mathcal{L}_2 .

6 Final model

The two extra parity-violating terms obtained in ref. [36] have been shown here to be either vanishing or already included in a previous Lagrangian. As mentioned below eq. (3.2), we also found that the term

$$\mathcal{L}_4^{\text{bis}} = g_4(X)A^\mu A_\lambda\tilde{F}_{\mu\nu}\partial^\lambda A^\nu = g_4(X)A^\mu A_\lambda\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma\partial^\lambda A^\nu, \quad (6.1)$$

is compatible with all the conditions we demand and could therefore be included in the general analysis. The corresponding term in eq. (3.2) would be proportional to $\tilde{F}_{\mu\nu}S^{\mu\nu}$, with $S^{\mu\nu} = \partial^\mu A^\nu + \partial^\nu A^\mu$ being the symmetric counterpart of the Faraday tensor which clearly vanishes identically. The line of reasoning leading to a small number of possible terms of the form (3.2) should apply to higher order terms of the kind (6.1); we did not find any such terms.

According to all the above discussions, it seems safe to conjecture that the final complete action for a Proca vector field involving only first-order derivatives in 4 dimensions is that given by eq. (12) of ref. [37], together with eq. (6.1). The complete formulation of the parity-conserving terms were also derived and written in a simple form in refs. [12, 36, 37]. We merely repeat the full action below:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=2}^6\mathcal{L}_i + \mathcal{L}_4^{\text{bis}}\right), \quad (6.2)$$

with

$$\begin{aligned}
\mathcal{L}_2 &= f_2 \left(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu} \right) = f_2 \left[X, F^2, F \cdot \tilde{F}, \left(A \cdot \tilde{F} \right)^2 \right], \\
\mathcal{L}_3 &= f_3(X) \partial \cdot A = \frac{1}{2} f_3(X) S_\mu{}^\mu, \\
\mathcal{L}_4 &= f_4(X) \left[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho \right] = \frac{1}{4} f_4(X) \left\{ \left[(S_\mu{}^\mu)^2 - S_\rho{}^\sigma S_\sigma{}^\rho \right] + F_{\mu\nu} F^{\mu\nu} \right\}, \\
\mathcal{L}_5 &= f_5(X) \left[(\partial \cdot A)^3 - 3(\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho + 2\partial_\rho A^\sigma \partial_\gamma A^\rho \partial_\sigma A^\gamma \right] + g_5(X) \tilde{F}^{\alpha\mu} \tilde{F}^\beta{}_\mu \partial_\alpha A_\beta \\
&= \frac{1}{8} f_5(X) \left[(S_\mu{}^\mu)^3 - 3(S_\mu{}^\mu) S_\rho{}^\sigma S_\sigma{}^\rho + 2S_\rho{}^\sigma S_\sigma{}^\gamma S_\gamma{}^\rho \right] + \frac{1}{4} [2g_5(X) - 3f_5(X)] \tilde{F}^{\alpha\mu} \tilde{F}^\beta{}_\mu S_{\alpha\beta}, \\
\mathcal{L}_6 &= g_6(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} \partial_\alpha A_\mu \partial_\beta A_\nu = \frac{1}{4} g_6(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} (S_{\alpha\mu} S_{\beta\nu} + F_{\alpha\mu} F_{\beta\nu}).
\end{aligned} \tag{6.3}$$

In eq. (6.3), f_2 is an arbitrary function of all possible scalars made out of A_μ , $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, containing both parity violating and preserving terms, while f_3 , f_4 , f_5 , g_5 and g_6 are arbitrary functions of X only. Note that this dependence is compatible with our basis choice in eq. (3.2), so that any other choice, for instance $g_k(X, F^2)$, would spoil the Hessian condition. We assume also that the standard kinetic term, $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, does not appear in f_2 , in order that the normalization of the vector field follows that of standard electromagnetism and thus we have pushed it out in eq. (6.2). The Lagrangians of eq. (6.3) are expressed in terms of either the ordinary derivatives $\partial_\mu A_\nu$, or in terms of its symmetric $S_{\mu\nu}$ and antisymmetric $F_{\mu\nu}$ parts:³ the second formulation, obtained by setting $\partial_\mu A_\nu = \frac{1}{2} (S_{\mu\nu} + F_{\mu\nu})$ and making use, in the case of \mathcal{L}_5 , of eq. (5.9), induces extra terms in \mathcal{L}_4 and \mathcal{L}_6 which can be absorbed in the parity-preserving part of \mathcal{L}_2 , being functions of A_μ and $F_{\mu\nu}$.

The presence of the new term $\mathcal{L}_4^{\text{bis}}$ is not as surprising as it would appear at first sight when one considers the generic structure of the terms contained in eq. (6.3). For the dynamics of the Lagrangians to be non trivial, up to terms already contained in \mathcal{L}_2 , the functions f_3 , f_4 , f_5 and g_6 must contain at least one factor of $X = g_{\mu\nu} A^\mu A^\nu$ (see also ref. [37]). Assuming $2g_5 - 3f_5$ to also contain such a factor (generic situation, no fine-tuning of the arbitrary functions), we conclude that each term can be written in the form⁴

$$\mathcal{L}_i = A^2 h(X) \langle \mathcal{O}_i \rangle = \tilde{h}(X) \langle A \tilde{\mathcal{O}}_i A \rangle + \partial_\mu J_i^\mu, \tag{6.4}$$

$A^2 h$ standing for the relevant f or g function (this transformation is indeed not possible for $\mathcal{L}_4^{\text{bis}}$). In eq. (6.4), the brackets indicate a trace over spacetime indices, \mathcal{O}_i and $\tilde{\mathcal{O}}_i$ are operators constructed from \tilde{F} 's (possibly none) and at least one S , and J_i^μ is the relevant current to make the identity true.⁵ So, the terms vanishing in the purely scalar case, i.e. those for which \mathcal{O}_i contains at least one factor of \tilde{F} , take the form $\langle AS\tilde{F}A \rangle$, $\langle AS\tilde{F}\tilde{F}A \rangle$ and $\langle AS\tilde{F}S\tilde{F}A \rangle$. The first such term, which is nothing but our $\mathcal{L}_4^{\text{bis}}$, is then seen to appear in a totally natural way.

Our final action is, up to the new term $\mathcal{L}_4^{\text{bis}}$, exactly the same as that of ref. [37]. There is however a subtle difference in the fact that all possible parity-violating terms are also written, being included in f_2 and $\mathcal{L}_4^{\text{bis}}$. Note that the curved space-time generalization of this action is also given in ref. [37], the covariantization of $\mathcal{L}_4^{\text{bis}}$ being obtained by a trivial replacement $\partial \rightarrow \nabla$.

³The relation between our formulations and those in terms of the Levi-Civita tensors is given in ref. [37].

⁴Sometimes also up to terms included in \mathcal{L}_2 .

⁵See ref. [4] for an extensive discussion of these equivalent formulations in the scalar galileon case.

A legitimate question to ask is whether eq. (6.2) is indeed the most general theory that can be written involving a vector field with three propagating degrees of freedom and second-order equations of motion. This has already been conjectured in refs. [12, 37]. Now, the discussion and calculations of the present article correct the conjecture made in ref. [36] about an infinite tower of terms, and also suggests a finite number of terms, even in the parity violating sector. So, there is finally a complete agreement on this point.

An additional indication of the correctness of this conjecture is that the systematic investigation procedure of ref. [36] completed by the calculation of ref. [37] for the parity-conserving sector, and by the present paper in the parity-violating sector, did not find any term other than those shown above up to the orders of \mathcal{L}_6 (parity violating) and \mathcal{L}_7 (parity conserving). However, if there were an infinite tower of possible Lagrangians, one would expect such Lagrangians to appear in our systematic procedure, which is not the case. Note especially that these works show that the parity violating sector contains no other terms than $\mathcal{L}_4^{\text{bis}}$ and those contained in f_2 , which is a very strong constraint, and greatly strengthens the conjecture we have made.

Finally, this work heavily relies on the postulate that spacetime is 4 dimensional. Relaxing this assumption permits to include the extra terms proposed in ref. [36] which, as shown in ref. [37], can be expressed with higher dimensional Levi-Civita tensors. For a given spacetime dimensionality, one thus expects, just like in the Lovelock case for a spin 2 field [43], a finite number of new terms to appear: in practice, in D dimensions, one expects terms containing up to D first order derivatives of the vector field.

Acknowledgments

We acknowledge P. Fleury and C. Pitrou for pointing out the importance of the identity in eq. (5.3). We also wish to thank C. Deffayet, G. Esposito-Farese, L. Heisenberg and J. Beltrán Jiménez for useful clarifications and enlightening discussions. This work was supported by COLCIENCIAS grant numbers 110656933958 RC 0384-2013 and 123365843539 RC FP44842-081-2014. P.P. would like to thank the Labex Institut Lagrange de Paris (reference ANR-10-LABX-63) part of the Idex SUPER, within which this work has been partly done.

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