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Polyakov loop in a non-covariant operator formalism

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We discuss a Polyakov loop in a non-covariant operator formalism that consists of only physical degrees of freedom at finite temperature. It is pointed out that although the Polyakov loop is expressed by a Euclidean time component of gauge fields in a covariant path integral formalism, there is no direct counterpart of the Polyakov loop operator in the operator formalism because the Euclidean time component of gauge fields is not a physical degree of freedom. We show that by starting with an operator that is constructed in terms of only physical operators in the non-covariant operator formalism, the vacuum expectation value of the operator calculated by the trace formula can be rewritten into the familiar form of an expectation value of the Polyakov loop in a covariant path integral formalism at finite temperature for the cases of the axial and Coulomb gauge.

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1. Introduction

Gauge invariance is undoubtedly one of the fundamental principles in particle physics. Both local and non-local gauge-invariant operators play an important role in gauge theory. An example of such non-local operators is the Wilson loop, which will be given by a line integral along a rectangular contour such that one side is taken to be in a space-like direction and another side to be in a time direction. A non-trivial expectation value of the Wilson loop operator will provide a signal of "quark" confinement in non-Abelian gauge theories [1].

Another example of non-local gauge-invariant operators is a Polyakov loop, which is similar to a Wilson loop but is given by a line integral along a Euclidean time axis. In this paper, we will focus on the Polyakov loop operator whose explicit form is given by $\mathrm{tr}\,\mathcal{P}\exp\left(ig\int_0^\beta d\tau A_\tau\right)$, where $\mathcal{P}$ denotes a path-ordered symbol, β is an inverse temperature, and A_τ is a Euclidean time component of gauge fields. The Polyakov loop is known as an order parameter of the confinement–deconfinement phase transition at finite temperature [2,3]. Furthermore, it could provide an order parameter of gauge symmetry breakings [2,4–9].

In order to confirm that the Polyakov loop is a physical observable at finite temperature, one might verify that a zero mode of the Euclidean time component A_τ in the Polyakov loop cannot be eliminated by gauge transformations due to the periodicity with respect to the Euclidean time

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at finite temperature. However, the statement that the Polyakov loop is physical seems to be less obvious than we thought.[1]

One reason why the above statement is not so obvious is that the Minkowski time component A_0 of gauge fields[2] is not a dynamical degree of freedom in gauge theory. The A_0 component can be removed in an operator formalism, and then the A_0 degree of freedom disappears completely from the Hilbert space of the theory, as explained below in some detail. One might stress that the A_0 (or A_τ) component is included in a covariant path integral representation of gauge theory. However, the A_0 (or A_τ) degree of freedom turns out to be introduced as *an auxiliary field* in a path integral formalism (see Sect. 3).

Quantization of gauge theory cannot be performed in a straightforward fashion because it contains redundant gauge degrees of freedom due to gauge invariance. Several ways to quantize it have been described and are expected to be physically equivalent.[3]

One reliable method to quantize gauge theory is to remove all unphysical degrees of freedom by explicitly imposing a gauge-fixing condition [13]. If the theory includes only physical degrees of freedom, the quantization of the theory is rather straightforward, though the form of the Hamiltonian would be messy.

For instance, in the axial gauge $A_3 = 0$ [14,15], which will be discussed in this paper, the theory can be described only by the physical degrees of freedom A_1 and A_2 for the gauge fields because A_3 is taken to be zero by the axial gauge condition, and A_0 is also removed by use of the equation of motion for A_0, which should be regarded as a constraint equation since it includes no time derivative of A_0.[4] Therefore, the gauge part of the Hilbert space in the axial gauge formalism can be spanned by the states $\{\, |\, A_1, A_2\, \rangle\, \}$, and any physical operators have to be constructed in terms of A_k $(k = 1, 2)$ and their conjugate momenta Π_k (and also fermion/scalar fields).

Now we can raise the issue of how to construct a Polyakov loop operator in the axial gauge operator formalism. Since the time component is completely eliminated from the spectrum in the operator formalism, it seems non-trivial to construct a Polyakov loop operator without A_0 or A_τ.

One of our main purposes is to present an explicit operator form corresponding to a Polyakov loop operator in the operator formalism that contains only physical degrees of freedom: A_k, Π_k $(k = 1, 2)$ and matter fields. Another purpose of this paper is to show that an expectation value of the above operator given in the non-covariant operator formalism can be rewritten into a familiar form of an expectation value of a Polyakov loop in a covariant path integral formalism at finite temperature. In the rewriting, we will see where the Euclidean time component A_τ comes from and also clarify how the operator given by physical operators turns into the standard form of the Polyakov loop given by A_τ.

[1] One might notice that Wilson lines along non-contractible loops in extra dimensions [10,11] resemble a Polyakov loop along the Euclidean time axis. Formally there seems no difference between them in a Euclidean path integral formulation. However, there is an important difference. A Polyakov loop becomes meaningless at zero temperature but Wilson lines do not. This is because the Euclidean time component of gauge fields can be removed completely by gauge transformations at zero temperature but the extra dimensional components cannot be.

[2] In this paper, we will use A_τ and A_0 for the Euclidean and Minkowski time components of gauge fields, respectively.

[3] We will not discuss the Gribov problem [12] in this paper. The problem is not expected to be related directly to our issue.

[4] If we take the temporal gauge $A_0 = 0$ [16–18], A_0 can be eliminated rather directly.

This paper is organized as follows. In the next section, we take the axial gauge to quantize a non-Abelian gauge theory, and remove all the unphysical degrees of freedom from a Lagrangian. Then, we derive a Hamiltonian consisting of only physical ones. In Sect. 3, we first construct a finite temperature path integral representation in the axial gauge by use of the trace formula [19,20], and then rewrite it into a covariant path integral form with all the gauge degrees of freedom. We will see that the Euclidean time component of the gauge fields is recovered as a Gaussian auxiliary field. Our main results will be given in Sect. 4. We propose an operator that is expected to represent a Polyakov loop operator in the axial gauge and is written in terms of physical operators only. We show that the operator turns into the standard form of the Polyakov loop in a covariant Euclidean path integral formula at finite temperature. In Sect. 5, we reformulate the analysis in the Coulomb gauge. Section 6 is devoted to conclusions. Some details of the calculations will be given in the appendices.

2. Hamiltonian in the axial gauge

In this section, we present a canonical formulation of non-Abelian gauge theory coupled with a fermion in the axial gauge. Although the results given in Sects. 2 and 3 will not be new [13], we shall explain them to make our discussions in Sect. 4 comprehensible. We consider an $SU(N)$ gauge theory whose Lagrangian is given by

$$\mathcal{L} = \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu \psi, \tag{2.1}$$

where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \quad D_\mu \psi \equiv \left(\partial_\mu - igA^a_\mu T^a\right)\psi. \tag{2.2}$$

The representation of the fermion ψ under the $SU(N)$ gauge group can be arbitrary, but for simplicity we take it to be the fundamental representation. Throughout this paper, we use the convention that repeated indices are summed over (unless otherwise stated), and $\mu, \nu, \ldots = 0, 1, 2, 3$ (or $\tau, 1, 2, 3$); $i, j = 1, 2, 3$; $k, l, \ldots = 1, 2$; $a, b, \ldots = 1, 2, \ldots, N^2 - 1$. The equation of motion for the gauge field A^a_μ following from the Lagrangian (2.1) is

$$(D_\mu)^{ab} F^{b\mu\nu} + J^{a\nu}_F = 0, \tag{2.3}$$

where $J^{a\nu}_F \equiv g\bar{\psi}\gamma^\nu T^a \psi$ and $(D_\mu)^{ab} = \delta^{ab}\partial_\mu + g f^{acb} A^c_\mu$. We impose the axial gauge

$$A^a_3 = 0. \tag{2.4}$$

Then, we can formally solve the $\nu = 0$ component of the equation of motion for A^a_0 and obtain that

$$A^a_0 = (\mathcal{D}^{-1})^{ab}\Big((D_k \dot{A}_k)^b + J^{b0}_F\Big) \qquad (k = 1, 2). \tag{2.5}$$

Here, the operator $(\mathcal{D}^{-1})^{ab}$ is formally defined as the inverse of $(\mathcal{D})^{ab} \equiv (D^2_k)^{ab} + \delta^{ab}(\partial_3)^2$.

We define the canonical momentum for the gauge field A^a_k $(k = 1, 2)$ by

$$\Pi^{ak}_A \equiv \frac{\partial \mathcal{L}(A_k, \psi)_{\text{axial}}}{\partial \dot{A}^a_k}, \tag{2.6}$$

where $\mathcal{L}(A_k, \psi)_{\text{axial}}$ is obtained by imposing the axial gauge on the Lagrangian (2.1) and eliminating A^a_0 in the Lagrangian (2.1) by using Eq. (2.5), so that it is written in terms of only physical degrees of freedom. The explicit form of $\mathcal{L}(A_k,\psi)_{\rm axial}$ is given by Eq. (A.1) in Appendix A, from which the canonical momentum for A_k^a is calculated as

$$\Pi_A^{ak} = (M)_{kl}^{ab}\dot{A}_l^b - (D_k\mathcal{D}^{-1}J_F^0)^a \qquad (l=1,2), \tag{2.7}$$

where

$$(M)_{kl}^{ab} = \delta^{ab}\delta_{kl} - (D_k\mathcal{D}^{-1}D_l)^{ab}. \tag{2.8}$$

By solving Eq. (2.7) for $\dot{A}_k^a$, we have

$$\dot{A}_k^a = (M^{-1})_{kl}^{ab}\Big(\Pi_A^{bl} + (D_l\mathcal{D}^{-1}J_F^0)^b\Big), \tag{2.9}$$

where $(M^{-1})_{kl}^{ab}$ is defined as the inverse of $(M)_{kl}^{ab}$, whose explicit form is

$$(M^{-1})_{kl}^{ab} = \delta^{ab}\delta_{kl} + (D_k(\partial_3)^{-2}D_l)^{ab}. \tag{2.10}$$

It is straightforward to check that $(MM^{-1})_{km}^{ab} = \delta^{ab}\delta_{km}$ by using $(\mathcal{D})^{ab} = (D_k^2)^{ab} + \delta^{ab}(\partial_3)^2$. As for the fermion ψ, we take the right derivative with respect to the Grassmann variable, and the canonical momentum for ψ is

$$\Pi_\psi \equiv \frac{\partial\mathcal{L}(A_k,\psi)_{\rm axial}}{\partial\dot{\psi}} = \bar{\psi}i\gamma^0. \tag{2.11}$$

Then, the Hamiltonian is

$$\begin{aligned}\mathcal{H}_{\rm axial} &\equiv \Pi_A^{ak}\dot{A}_k^a + \Pi_\psi\dot{\psi} - \mathcal{L}(A_k,\psi)_{\rm axial}\\ &= \frac{1}{2}\Pi_A^{ak}(M_{kl}^{-1}\Pi_A^l)^a + \Pi_A^{ak}(M_{kl}^{-1}D_l\mathcal{D}^{-1}J_F^0)^a\\ &\quad + \frac{1}{2}(D_k\mathcal{D}^{-1}J_F^0)^a(M_{kl}^{-1}D_l\mathcal{D}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a\\ &\quad + \frac{1}{4}F_{kl}^aF_{kl}^a + \frac{1}{2}\partial_3A_k^a\partial_3A_k^a - \bar{\psi}i\gamma^kD_k\psi - \bar{\psi}i\gamma^3\partial_3\psi.\end{aligned} \tag{2.12}$$

In deriving the Hamiltonian, we have performed the partial integration with respect to $(M^{-1})_{kl}^{ab}$. The Hamiltonian is written in terms of only physical degrees of freedom A_k^a, Π_A^{ak}, ψ, and Π_ψ as it should, though it is messy, including the non-local terms. It may be appropriate to mention here that one is able to rewrite Eq. (2.12) into another form presented by Eq. (A.6) in Appendix A, in which the self-energy of the color charge densities of the fermion manifestly appears. We shall use the Hamiltonian (2.12) in the discussions given below.

3. Trace formula in the axial gauge at finite temperature

Let us first define the vacuum expectation value of an operator $\hat{\mathcal{O}}$ at finite temperature by the trace formula,

$$\langle\hat{\mathcal{O}}\rangle_\beta \equiv {\rm Tr}\left(\hat{\mathcal{O}}\,e^{-\beta\hat{H}}\right) \quad \text{with} \quad \hat{H} \equiv \int d^3\mathbf{x}\,\hat{\mathcal{H}}, \tag{3.1}$$

where β stands for the inverse of the temperature T. If one chooses a non-covariant gauge such as the axial or Coulomb gauge, the operator $\hat{\mathcal{O}}$ and the Hamiltonian $\hat{H}$ are written in terms of only physical degrees of freedom, i.e. $\hat{A}^a_k$, $\hat{\psi}$ and their canonical momenta, $\hat{\Pi}^{ak}_A$, $\hat{\Pi}_\psi$, so that the trace in Eq. (3.1) must be taken over the physical states alone, denoted here by $A_{\rm phys} = \{A^a_k, \psi\}$:

$$\begin{aligned}&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})\; e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}\right)_{\rm gauge}\\&= \int \mathcal{D}A_{\rm phys} \langle A_{\rm phys}|\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})\; e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}|A_{\rm phys}\rangle_{\rm gauge}. \end{aligned} \tag{3.2}$$

It should be understood that bosons (fermions) must obey the (anti-) periodic boundary condition for the Euclidean time direction because of the quantum statistics at finite temperature.

Our aim is to find the operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ which satisfies

$$\begin{aligned}&\mathrm{Tr}\left(\mathcal{O}(\hat{A}^a_k, \hat{\psi}, \hat{\Pi}^{ak}_A, \hat{\Pi}_\psi) e^{-\beta H(\hat{A}^a_k, \hat{\psi}, \hat{\Pi}^{ak}_A, \hat{\Pi}_\psi)}\right)_{\rm gauge} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\; \det(M^{\rm gauge}_{\rm FP}) \prod_{x,a} \delta(\chi^a_{\rm gauge}(x))\\&\times\; \mathrm{tr}\left(\mathcal{P}\, e^{ig\int_0^\beta d\tau A_\tau(\tau, \mathbf{x}_0)}\right) \exp \int_0^\beta d\tau \int d^3\mathbf{x} \left\{\frac{-1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \bar{\psi} i\gamma_\mu D_\mu \psi\right\}, \end{aligned} \tag{3.3}$$

where $M^{\rm gauge}_{\rm FP}$ and $\delta(\chi^a_{\rm gauge}(x))$ are the Faddeev–Popov determinant and the gauge condition for the chosen gauge, respectively, and $\mu, \nu = \tau, 1, 2, 3$. We shall consider the cases of the axial and Coulomb gauges in this paper. The right-hand side of Eq. (3.3) is nothing but the vacuum expectation value of the Polyakov loop operator in the covariant path integral form under the chosen gauge condition. It should be noticed that the left-hand side of Eq. (3.3) is given by the trace formula in the non-covariant operator formalism by use of the physical degrees of freedom alone, not including A_τ, particularly that the operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ has to be constructed by the physical operators only. Therefore, it seems to be a non-trivial problem whether or not there exists such an operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ satisfying Eq. (3.3). Furthermore, we need to clarify where the unphysical degree of freedom A_τ comes from in deriving the right-hand side of Eq. (3.3) from its left-hand side, and manage the path-ordered product of the Polyakov loop in the trace formula.

Before we construct the operator $\hat{\mathcal{O}}$ explained above, let us first study the partition function at finite temperature, say $\hat{\mathcal{O}} = \hat{\mathbf{1}}$, in the case of the axial gauge. In this case, what we would like to show is that for the Hamiltonian (2.12) the equation

$$\begin{aligned}&\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}\right)_{\rm axial}\\&= \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\; \det(\partial_3) \prod_{x,a} \delta(A^a_3) \exp \int_0^\beta d\tau \int d^3\mathbf{x} \left\{\frac{-1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \bar{\psi} i\gamma_\mu D_\mu \psi\right\} \end{aligned} \tag{3.4}$$

holds, where the right-hand side of Eq. (3.4) is the standard path integral representation for the partition function at finite temperature in the axial gauge. This may be too pedagogical, but helpful for the later discussions. Using the completeness relation for the canonical momenta Π^{ak}_A and Π_ψ, the left-hand side of Eq. (3.4) can be written in the path integral form as

$$\begin{aligned}&\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}\right)_{\rm axial}\\&= \int \mathcal{D}\Pi^{ak}_A \mathcal{D}A^a_k \mathcal{D}\Pi_\psi \mathcal{D}\psi\; \exp \int_0^\beta d\tau \int d^3\mathbf{x} \left\{i\Pi^{ak}_A \dot{A}^a_k + i\Pi_\psi \dot{\psi} - \mathcal{H}_{\rm axial}\right\}, \end{aligned} \tag{3.5}$$

where $\mathcal{H}_{\rm axial}$ is given by Eq. (2.12) and $\dot{A}^a_k$, $\dot{\psi}$ stands for the derivative with respect to the Euclidean time τ, $\dot{A}^a_k = \partial_\tau A^a_k$, $\dot{\psi} = \partial_\tau \psi$. This notation will be used hereafter. Let us note that the imaginary unit i in front of $\Pi^{ak}_A \dot{A}^a_k$ and $\Pi_\psi \dot{\psi}$ does not come from the Euclideanization, but from taking the inner product of the "coordinate" $\phi(\mathbf{x})$ and the "momentum" $\Pi_\phi(\mathbf{x})$,

$$\langle \phi | \Pi_\phi \rangle \propto \exp\Big\{ i \int d^3\mathbf{x}\; \Pi_\phi(\mathbf{x})\phi(\mathbf{x}) \Big\}. \tag{3.6}$$

The exponent of the right-hand side of Eq. (3.5) is quadratic with respect to Π^{ak}_A, whose explicit form is given by Eq. (A.7) in Appendix A, so that we can perform the Gaussian integration $\mathcal{D}\Pi^{ak}_A$. Then, we arrive at

$$\begin{aligned}
\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})} \right)_{\rm axial} &= \int \mathcal{D}A^a_k \mathcal{D}\Pi_\psi \mathcal{D}\psi \; \det{}^{-\frac{1}{2}}(M^{-1}) \exp \int_0^\beta d\tau \int d^3\mathbf{x} \\
&\times \Big\{ -\frac{1}{2}\dot{A}^a_k \dot{A}^a_k - \frac{1}{2}(D_k \dot{A}_k)^a (\mathcal{D}^{-1} D_l \dot{A}_l)^a - i\dot{A}^a_k (D_k \mathcal{D}^{-1} J^0_F)^a + \frac{1}{2} J^{a0}_F (\mathcal{D}^{-1} J^0_F)^a \\
&- \frac{1}{4} F^a_{kl} F^a_{kl} - \frac{1}{2}\partial_3 A^a_k \partial_3 A^a_k + i\Pi_\psi \dot{\psi} + \bar{\psi} i\gamma^k D_k \psi + \bar{\psi} i \gamma^3 \partial_3 \psi \Big\}.
\end{aligned} \tag{3.7}$$

In order to restore the A_τ degree of freedom, let us consider the Gaussian integral given by

$$\begin{aligned}
1 = \int \mathcal{D}A^a_\tau \det{}^{\frac{1}{2}}(\mathcal{D}) \exp \int_0^\beta d\tau \int d^3\mathbf{x} \Big\{ -\frac{1}{2} \Big(iA^a_\tau - (\mathcal{D}^{-1})^{ac} \big(i(D_k \dot{A}_k)^c + J^{c0}_F \big) \Big) \\
\times (\mathcal{D})^{ad} \Big(iA^d_\tau - (\mathcal{D}^{-1})^{de} \big(i(D_l \dot{A}_l)^e + J^{e0}_F \big) \Big) \Big\}.
\end{aligned} \tag{3.8}$$

By inserting Eq. (3.8) into Eq. (3.7), some of the terms in the exponent in Eqs. (3.7) and (3.8) cancel each other, as shown in Appendix A.2, where we explicitly present the expansion of the exponent in the Gaussian integral (3.8) by Eq. (A.9). Then, we obtain that

$$\begin{aligned}
\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}^k_A, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})} \right)_{\rm axial} &= \int \mathcal{D}A^a_\tau \mathcal{D}A^a_k \mathcal{D}\Pi_\psi \mathcal{D}\psi \; \det{}^{-\frac{1}{2}}(M^{-1}) \det{}^{\frac{1}{2}}(\mathcal{D}) \\
&\times \exp \int_0^\beta d\tau \int d^3\mathbf{x} \Big\{ -\frac{1}{2}\dot{A}^a_k \dot{A}^a_k - \frac{1}{4} F^a_{kl} F^a_{kl} - \frac{1}{2}\partial_3 A^a_k \partial_3 A^a_k \\
&- \frac{1}{2}(D_k A_\tau)^a (D_k A_\tau)^a - \frac{1}{2}\partial_3 A^a_\tau \partial_3 A^a_\tau + \dot{A}^a_k (D_k A_\tau)^a \\
&+ i\Pi_\psi \dot{\psi} + \bar{\psi} i\gamma^k D_k \psi + \bar{\psi} i\gamma^3 \partial_3 \psi + iA^a_\tau J^{a0}_F \Big\}.
\end{aligned} \tag{3.9}$$

Note that A^a_τ is recovered as the auxiliary field through the Gaussian integral (3.8). We finally introduce the factor

$$1 = \int \mathcal{D}A^a_3 \prod_{x,a} \delta(A^a_3) \tag{3.10}$$

in order to restore the A^a_3 degree of freedom. By inserting it into Eq. (3.9), the exponent in Eq. (3.9) is summarized into the covariant form of the Lagrangian thanks to the delta function in Eq. (3.10). The determinants in Eq. (3.9) are evaluated as

$$\det{}^{-\frac{1}{2}}(M^{-1}) \det{}^{\frac{1}{2}}(\mathcal{D}) = \det(\partial_3), \tag{3.11}$$

which is shown in Appendix C.1. Hence, we finally obtain that

$$\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{axial}}$$
$$= \int \mathcal{D}A^a_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \det(\partial_3)\prod_{x,a}\delta(A^a_3)\ \exp\int_0^\beta d\tau \int d^3\mathbf{x}\left\{\frac{-1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \bar{\psi}i\gamma_\mu D_\mu\psi\right\}, \quad (3.12)$$

where we have used $\mathcal{D}\Pi_\psi = \mathcal{D}\bar{\psi}$ and defined $\gamma_\tau \equiv -i\gamma^0$. This is Eq. (3.4) that we would like to prove.

4. Polyakov loop in the axial gauge

Let us discuss the Polyakov loop operator in the axial gauge, which is one of our main purposes in the paper. As explained in Sect. 1, the gauge part of the Hilbert space in the axial gauge does not contain the gauge field A_τ, from which the Polyakov loop operator is usually defined. Thus, the problem is how to construct the operator written in terms of only physical degrees of freedom in the axial gauge, corresponding to the Polyakov loop operator. Our aim is to present an explicit form of the operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ that satisfies

$$\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})\ e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{axial}}$$
$$= \int \mathcal{D}A^a_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \det(\partial_3)\prod_{x,a}\delta(A^a_3)\ \mathrm{tr}\Big(\mathcal{P}\exp\Big\{ig\int_0^\beta d\tau A_\tau(\tau,\mathbf{x}_0)\Big\}\Big)$$
$$\times\ \exp\int_0^\beta d\tau\int d^3\mathbf{x}\left\{\frac{-1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \bar{\psi}i\gamma_\mu D_\mu\psi\right\}, \quad (4.1)$$

where the operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ must be written in terms of only physical degrees of freedom in the axial gauge. The trace of the left-hand side is taken over the physical state alone in the axial gauge where there is no A^a_τ degree of freedom. The right-hand side is, of course, the vacuum expectation value of the Polyakov loop operator in the covariant path integral form with the axial gauge.

In our trials to find operators satisfying Eq. (4.1), we have observed that the non-Abelian nature of the path-ordered product of the Polyakov loop in Eq. (4.1) is an obstacle to rewriting the expectation value of the operator $\mathcal{O}(\hat{\Pi}^{ak}_A, \hat{A}^a_k, \hat{\Pi}_\psi, \hat{\psi})$ in the trace formula of the left-hand side of Eq. (4.1) into the covariant path integral form in the right-hand side of Eq. (4.1). Thus, it seems to be necessary to transform the right-hand side of Eq. (4.1) into a path integral form without the path-ordered product.

To this end, let us consider gauge transformations such as

$$A^a_\tau(\tau,\mathbf{x})T^a = U(\tau)\Big(A'^a_\tau(\tau,\mathbf{x})T^a + \frac{i}{g}\partial_\tau\Big)U^\dagger(\tau), \quad (4.2)$$

$$A^a_i(\tau,\mathbf{x})T^a = U(\tau)A'^a_i(\tau,\mathbf{x})T^aU^\dagger(\tau), \quad (4.3)$$

$$\psi(\tau,\mathbf{x}) = U(\tau)\psi'(\tau,\mathbf{x}), \quad (4.4)$$

where the unitary matrix $U(\tau)$ is assumed to depend only on the Euclidean time τ with the periodic boundary condition $U(\tau+\beta) = U(\tau)$ in order not to contradict with the quantum statistics at finite temperature. We notice that there is no inhomogeneous term in the right-hand side of Eq. (4.3) because $U(\tau)$ does not depend on the spatial coordinate $\mathbf{x}$. If we formally write the delta functions

$\prod_{x,a}\delta(A_3^a(\tau,\mathbf{x}))$ as $\prod_{x}\delta(A_3(\tau,\mathbf{x}))$ with $A_3 = A_3^a T^a$, it will be obvious that the delta functions are invariant under the gauge transformation (4.3), i.e.

$$\prod_x \delta(A_3(\tau,\mathbf{x})) = \prod_x \delta(A_3'(\tau,\mathbf{x})). \tag{4.5}$$

We choose the unitary matrix $U(\tau)$ to diagonalize the Polyakov loop by the gauge transformation (4.2) as follows: The path-ordered product $\mathcal{P}\ e^{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)T^a}$ transforms under the gauge transformation (4.2) as

$$\begin{aligned}\mathcal{P}\ e^{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)T^a} &= U(\beta)\left[\mathcal{P}\ e^{ig\int_0^\beta d\tau A'^a_\tau(\tau,\mathbf{x}_0)T^a}\right]U^\dagger(0)\\ &= U(0)\left[\mathcal{P}\ e^{ig\int_0^\beta d\tau A'^a_\tau(\tau,\mathbf{x}_0)T^a}\right]U^\dagger(0).\end{aligned} \tag{4.6}$$

Since $\mathcal{P}\ e^{ig\int_0^\beta d\tau A'^a_\tau(\tau,\mathbf{x}_0)T^a}$ is a unitary matrix fixed at $\mathbf{x}_0$, it can be diagonalized by a unitary matrix $U(0)$. Then, we can rewrite the Polyakov loop into the form

$$\begin{aligned}\mathrm{tr}\Big(\mathcal{P}\ e^{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)T^a}\Big) &= \mathrm{tr}\Big(\exp\Big\{ig\int_0^\beta d\tau A'^{\tilde{a}}_\tau(\tau,\mathbf{x}_0)T^{\tilde{a}}\Big\}\Big)\\ &= \sum_{\alpha=1}^{D}\exp\Big\{ig\int_0^\beta d\tau A'^{\tilde{a}}_\tau(\tau,\mathbf{x}_0)(T^{\tilde{a}})_{\alpha\alpha}\Big\},\end{aligned} \tag{4.7}$$

where $\{T^{\tilde{a}},\ \tilde{a}=1,2,\ldots,N-1\}$ are generators of the Cartan subalgebra of $SU(N)$ with the diagonal form

$$(T^{\tilde{a}})_{\alpha\beta} = (T^{\tilde{a}})_{\alpha\alpha}\delta_{\alpha\beta} \qquad (\alpha,\beta = 1,2,\ldots,D). \tag{4.8}$$

Here, D stands for the dimension of the representation of $\{T^a\}$ ($D=N$ for the fundamental representation). Thus, the Polyakov loop turns out to reduce to the one defined for the $U(1)$ subgroups in the $SU(N)$ gauge group. Thanks to the reduction, the trace in the original expression (4.1) is replaced by the summation with respect to the Abelian part of the Polyakov loop in Eq. (4.7) (note that $(T^{\tilde{a}})_{\alpha\alpha}$ is just a number) and one does not need to take care of the path-ordered integral because of the Abelian nature of the $U(1)$ gauge group.

It is convenient for transparent calculations to use the notation defined by

$$(\tilde{T}^a)_{\alpha\alpha} \equiv \begin{cases} (T^{\tilde{a}})_{\alpha\alpha} & \text{for } a=\tilde{a} \quad (\text{not summed over } \alpha),\\ 0 & \text{for } a\neq\tilde{a}.\end{cases} \tag{4.9}$$

By taking account of the above discussions, Eq. (4.1) becomes

$$\begin{aligned}&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}_A^{ak},\hat{A}_k^a,\hat{\Pi}_\psi,\hat{\psi})\ e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\text{axial}}\\ &= \sum_{\alpha=1}^{D}\int \mathcal{D}A_\mu^a\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \det(\partial_3)\prod_{x,a}\delta(A_3^a)\ \exp\left\{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)(\tilde{T}^a)_{\alpha\alpha}\right\}\\ &\quad\times\ \exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{\frac{-1}{4}F_{\mu\nu}^aF_{\mu\nu}^a - \bar{\psi}i\gamma_\mu D_\mu\psi\Big\},\end{aligned} \tag{4.10}$$

where we have removed the prime $'$ from all the fields. Thus, our aim is now to find the operator $\mathcal{O}(\hat{\Pi}_A^{ak}, \hat{A}_k^a, \hat{\Pi}_\psi, \hat{\psi})$ that satisfies the above relation.

We propose the operator $\mathcal{O}(\hat{\Pi}_A^{ak}, \hat{A}_k^a, \hat{\Pi}_\psi, \hat{\psi})$ satisfying Eq. (4.10) as

$$\mathcal{O}(\hat{\Pi}_A^{ak}, \hat{A}_k^a, \hat{\Pi}_\psi, \hat{\psi}) = \sum_{\alpha=1}^{D} \exp\left\{g\int_0^\beta d\tau \int d^3\mathbf{x} (\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\partial_3)^{-2}\big((\hat{D}_k\hat{\Pi}_A^k)^a + \hat{J}_F^{a0}\big)\Big)\right\}, \tag{4.11}$$

where $(\tilde{T}^a)_{\alpha\alpha}$ is defined in Eq. (4.9). It should be emphasized that the above operator is described by only the physical operators but not A_τ. This is a main result of our paper. The above form of the operator might be guessed from Eqs. (2.5) and (2.9), but the proof that Eq. (4.11) leads to the relation (4.10) seems to be far from trivial, as we will see below.

For the operator (4.11), the path integral representation for the left-hand side of Eq. (4.10) is

$$\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}_A^{ak}, \hat{A}_k^a, \hat{\Pi}_\psi, \hat{\psi})\, e^{-\beta H(\hat{\Pi}_A^k, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}\right)_{\mathrm{axial}} = \sum_{\alpha=1}^{D}\int \mathcal{D}\Pi_A^{ak}\mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi \\
&\times\ \exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\partial_3)^{-2}\big((D_k\Pi_A^k)^a + J_F^{a0}\big)\Big) \\
&+ i\Pi_A^{ak}\dot{A}_k^a + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\mathrm{axial}}\Big\},
\end{aligned} \tag{4.12}$$

where the Hamiltonian $\mathcal{H}_{\mathrm{axial}}$ is given by Eq. (2.12). Since the exponent of the right-hand side in Eq. (4.12) is quadratic with respect to Π_A^{ak}, whose explicit calculations and forms are given in Appendix A.3, we can perform the Gaussian integration $\mathcal{D}\Pi_A^{ak}$, and we obtain that

$$\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}_A^{ak}, \hat{A}_k^a, \hat{\Pi}_\psi, \hat{\psi})\, e^{-\beta H(\hat{\Pi}_A^k, \hat{A}_k, \hat{\Pi}_\psi, \hat{\psi})}\right)_{\mathrm{axial}} \\
&= \sum_{\alpha=1}^{D}\int \mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi\ \det{}^{-\frac{1}{2}}(M^{-1})\ \exp\int_0^\beta d\tau\int d^3\mathbf{x} \\
&\times\ \Big\{-\frac{1}{2}\dot{A}_k^a\dot{A}_l^a - \frac{1}{2}(D_k\dot{A}_k)^a(\mathcal{D}^{-1}D_l\dot{A}_l)^a - i\dot{A}_k^a(D_k\mathcal{D}^{-1}J_F^0)^a + \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a \\
&- \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}\partial_3A_k^a\partial_3A_k^a + i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3\partial_3\psi \\
&+ g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J_F^0)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}D_k\dot{A}_k)^a \\
&- \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
&+ \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big\}.
\end{aligned} \tag{4.13}$$

We next insert the Gaussian integral,

$$\begin{aligned}
1 &= \int \mathcal{D}A_\tau^a\ \det{}^{\frac{1}{2}}(\mathcal{D})\ \exp\int_0^\beta d\tau\int d^3\mathbf{x} \\
&\Big\{-\frac{1}{2}\Big(iA_\tau^a - (\mathcal{D}^{-1})^{ac}\big(i(D_k\dot{A}_k)^c + J_F^{c0} + g(\tilde{T}^c)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big) \\
&\times (\mathcal{D})^{ad}\Big(iA_\tau^d - (\mathcal{D}^{-1})^{de}\big(i(D_l\dot{A}_l)^e + J_F^{e0} + g(\tilde{T}^e)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\Big\},
\end{aligned} \tag{4.14}$$

into Eq. (4.13) in order to recover the A^a_τ degree of freedom. Note that we add new terms of the point color charge densities $g(\tilde{T}^c)_{\alpha\alpha}\delta(\mathbf{x}-\mathbf{x}_0)$, compared with Eq. (3.8). If we expand the exponent of Eq. (4.14), which is given by Eq. (A.16) in Appendix A, we see that some of the terms in the exponent of Eq. (4.13) cancel with those in Eq. (4.14). Then, we have

$$
\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A,\hat{A}^a_k,\hat{\Pi}_\psi,\hat{\psi})\, e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\text{axial}}\\
&=\sum_{\alpha=1}^{D}\int \mathcal{D}A^a_\tau\mathcal{D}A^a_k\mathcal{D}\Pi_\psi\mathcal{D}\psi\ \det{}^{-\frac{1}{2}}(M^{-1})\det{}^{\frac{1}{2}}(\mathcal{D})\ \exp\int_0^\beta d\tau\int d^3\mathbf{x}\\
&\Big\{-\frac{1}{2}\dot{A}^a_k\dot{A}^a_k-\frac{1}{4}F^a_{kl}F^a_{kl}-\frac{1}{2}\partial_3A^a_k\partial_3A^a_k+i\Pi_\psi\dot{\psi}+\bar{\psi}i\gamma^kD_k\psi+\bar{\psi}i\gamma^3\partial_3\psi+iA^a_\tau J^{a0}_F\\
&-\frac{1}{2}(D_kA_\tau)^a(D_kA_\tau)^a-\frac{1}{2}\partial_3A^a_\tau\partial_3A^a_\tau+\dot{A}^a_k(D_kA_\tau)^a\\
&+igA^a_\tau(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\\
&-\frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big\}. \qquad (4.15)
\end{aligned}
$$

We note that A^a_τ is recovered as the auxiliary field through the Gaussian integral (4.14). The fourth line of Eq. (4.15) is just what we wanted, and corresponds to the Polyakov loop in Eq. (4.10). Since the term comes from the cross terms in the Gaussian integral (4.14), it turns out to be difficult to get the Polyakov loop in the original path-ordered form of Eq. (4.1) from the above procedure. This is the reason why we rewrite the Polyakov loop of Eq. (4.1) into the Abelian form given in Eq. (4.10).

We finally restore the A^a_3 degree of freedom by Eq. (3.10), and the terms in the exponent (4.15) are summarized into the covariant form of the Lagrangian. The determinants are evaluated, as before, like Eq. (3.11). Then, we arrive at

$$
\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A,\hat{A}^a_k,\hat{\Pi}_\psi,\hat{\psi})\, e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\text{axial}}\\
&=\sum_{\alpha=1}^{D}\int \mathcal{D}A^a_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \det(\partial_3)\prod_{x,a}\delta(A^a_3)\ \exp\left\{ig\int_0^\beta d\tau A^a_\tau(\tau,\mathbf{x}_0)(\tilde{T}^a)_{\alpha\alpha}\right\}\\
&\quad\times\ \exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{\frac{-1}{4}F^a_{\mu\nu}F^a_{\mu\nu}-\bar{\psi}i\gamma_\mu D_\mu\psi\Big\}. \qquad (4.16)
\end{aligned}
$$

Hence, we have proved Eq. (4.10) for the operator $\hat{\mathcal{O}}$ defined by Eq. (4.11). One can come back to Eq. (4.1) by the inverse gauge transformations for Eqs. (4.2)–(4.4). One can say that we prove Eq. (3.3) in the axial gauge for the operator $\mathcal{O}$ defined by Eq. (4.11).

Before closing this section, let us discuss the last term in Eq. (4.15), which is the divergent self-energy of the point color charge densities due to introducing the Polyakov loop operator (4.11). If we define a new operator $\hat{\mathcal{O}}_{\text{new}}$ in such a way that we subtract the self-energy part from the beginning by the counter term, δ^a_{self},

$$
\hat{\mathcal{O}}_{\text{new}}\equiv\sum_{\alpha=1}^{D}\exp\left\{g\int_0^\beta d\tau\int d^3\mathbf{x}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\partial_3)^{-2}\big((\hat{D}_k\hat{\Pi}^k_A)^a+\hat{J}^{a0}_F+\delta^a_{\text{self}}\big)\Big)\right\}, \quad (4.17)
$$

where

$$\delta^a_{\rm self} \equiv \frac{g}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0), \tag{4.18}$$

then the divergent term does not appear in the expression (4.15). We note that the term subtracted by the counter term, $(\partial_3)^{-2}\delta^a_{\rm self}$, obviously does not depend on the field.

5. Hamiltonian in the Coulomb gauge

In this section, the following two subsections reformulate the analyses done in Sects. 2, 3, and 4 for the case of the Coulomb gauge.

The Coulomb gauge is given by

$$\partial_i A^{ai} = 0 \qquad (i = 1, 2, 3). \tag{5.1}$$

One of the components, say A^a_3, is eliminated by Eq. (5.1) like

$$A^a_3 = -(\partial_3)^{-1}\partial_k A^a_k. \tag{5.2}$$

We impose the Coulomb gauge on the $\nu = 0$ component of the equation of motion (2.3) and formally solve it for A^a_0. Then, we have

$$A^a_0 = (\mathcal{D}^{-1})^{ab}\Big((C_k\dot{A}_k)^b + J^{b0}_F\Big) \qquad (k = 1, 2). \tag{5.3}$$

The operator $(\mathcal{D}^{-1})^{ab}$ is defined as the inverse of $(\mathcal{D})^{ab} \equiv (D^2_i)^{ab}$. We find it useful to leave A^a_3 in the covariant derivative $(D_3)^{ab}$ as it is, in order to perform calculations as clearly as possible, keeping Eq. (5.2) in mind. $(C_k)^{ab}$ in Eq. (5.3) is defined by

$$(C_k)^{ab} \equiv (D_k)^{ab} - (D_3)^{ab}(\partial_3)^{-1}\partial_k. \tag{5.4}$$

The partial integration with respect to $(C_k)^{ab}$ yields a new operator, accompanying a minus sign, given by

$$(\tilde{C}_k)^{ab} \equiv (D_k)^{ab} - \partial_k(\partial_3)^{-1}(D_3)^{ab} \tag{5.5}$$

and vice versa.

As in the case for the axial gauge, we eliminate A^a_0 by Eq. (5.3) and A^a_3 by Eq. (5.2) in the Lagrangian (2.1). Then, we obtain the Lagrangian $\mathcal{L}(A_k, \psi)_{\rm Coul}$ written in terms of only physical degrees of freedom, whose explicit form is presented by Eq. (B.5) in Appendix B. Then, the canonical momentum for A^a_k is given by

$$\Pi^{ak}_A \equiv \frac{\partial\mathcal{L}(A_k, \psi)_{\rm Coul}}{\partial\dot{A}^a_k} = (N)^{ab}_{kl}\dot{A}^b_l - (\tilde{C}_k\mathcal{D}^{-1}J^0_F)^a, \tag{5.6}$$

where

$$(N)^{ab}_{kl} \equiv (\delta_{kl} + \partial_k(\partial_3)^{-2}\partial_l)\delta^{ab} - (\tilde{C}_k\mathcal{D}^{-1}C_l)^{ab}. \tag{5.7}$$

We obtain from Eq. (5.6) that

$$\dot{A}_k^a = (N^{-1})_{kl}^{ab}\Big(\Pi_A^{bl} + (\tilde{C}_l \mathcal{D}^{-1} J_F^0)^b\Big), \tag{5.8}$$

where $(N^{-1})_{kl}^{ab}$ is defined as the inverse of $(N)_{kl}^{ab}$ and its explicit form is

$$(N^{-1})_{kl}^{ab} = (\delta_{kl} - \partial_k \Delta^{-1} \partial_l)\delta^{ab} + \big(\tilde{C}'_k \tilde{\Delta}^{-1} C'_l\big)^{ab}, \tag{5.9}$$

where

$$(\tilde{C}'_k)^{ab} = (D_k)^{ab} - \partial_k \Delta^{-1} \partial \cdot (D)^{ab}, \tag{5.10}$$

$$(C'_k)^{ab} = (D_k)^{ab} - (D)^{ab} \cdot \partial \Delta^{-1} \partial_k, \tag{5.11}$$

$$\tilde{\Delta}^{-1} \equiv \big(D \cdot \partial \Delta^{-1} \partial \cdot D\big)^{-1}, \tag{5.12}$$

$$\Delta^{-1} \equiv (\partial_i^2)^{-1}. \tag{5.13}$$

We have also used the notation $\partial \cdot D \equiv \partial_i D_i$. Let us note that $\partial \cdot D = D \cdot \partial$ in the Coulomb gauge. We present the relations among the operators, (5.4), (5.5), (5.10), (5.11), and the proof of Eq. (5.9) in Appendix B. As for the fermion, we take the right derivative with respect to the Grassmann variable, so that we have

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}(A_k, \psi)_{\text{Coul}}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0. \tag{5.14}$$

Then, the Hamiltonian is obtained as

$$\begin{aligned}
\mathcal{H}_{\text{Coul}} &= \Pi_A^{ak} \dot{A}_k^a + \Pi_\psi \dot{\psi} - \mathcal{L}(A_k, \psi)_{\text{Coul}} \\
&= \frac{1}{2} \Pi_A^{ak} (N_{kl}^{-1} \Pi_A^l)^a + \Pi_A^{ak} (N_{kl}^{-1} \tilde{C}_l \mathcal{D}^{-1} J_F^0)^a \\
&\quad + \frac{1}{2} (\tilde{C}_k \mathcal{D}^{-1} J_F^0)(N_{kl}^{-1} \tilde{C}_l \mathcal{D}^{-1} J_F^0)^a - \frac{1}{2} J_F^{a0} (\mathcal{D}^{-1} J_F^0)^a \\
&\quad + \frac{1}{4} F_{kl}^a F_{kl}^a + \frac{1}{2} F_{3k}^a F_{3k}^a - \bar{\psi} i \gamma^k D_k \psi - \bar{\psi} i \gamma^3 D_3 \psi.
\end{aligned} \tag{5.15}$$

This Hamiltonian is written in terms of only physical degrees of freedom, as it should. If we compare the Hamiltonian (5.15) with that in the axial gauge (2.12), we observe the correspondences among the operators in the Hamiltonians such as $M^{-1} \leftrightarrow N^{-1}$, $D_k \leftrightarrow \tilde{C}_k$, even though their explicit forms are quite different. The similarity between them may be the consequence of the fact that in both cases the physical degrees of freedom are A_k, Π_A^{ak}, ψ, and Π_ψ alone. The Hamiltonian (5.15) can be recast into another form, which is presented by Eq. (B.17) in Appendix B, and the self-energy of the point color charge densities of the fermion in the Coulomb gauge becomes manifest. We shall use the Hamiltonian (5.15) in the discussions below.

5.1. Trace formula in the Coulomb gauge at finite temperature

In this section we would like to repeat the same analyses as in the case of the axial gauge in Sect. 3. We first show that the Hamiltonian (5.15) can reproduce the well-known path integral representation for the partition function in the Coulomb gauge at finite temperature,

$$\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\psi}\mathcal{D}\psi \;\det(\partial_i D_i)\prod_{x,a}\delta(\partial_i A_i^a) \times \;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{\frac{-1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \bar{\psi}i\gamma_\mu D_\mu\psi\Big\}. \quad (5.16)$$

The discussions here may be too pedagogical again, but are helpful for the discussions in the next subsection. The path integral representation for the left-hand side of Eq. (5.16) is

$$\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} = \int \mathcal{D}\Pi_A^{ak}\mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{i\Pi_A^{ak}\dot{A}_k^a + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\mathrm{Coul}}\Big\}. \quad (5.17)$$

Since the exponent in Eq. (5.17) is quadratic with respect to Π_A^{ak}, one can perform the Gaussian integration. The detailed expression for the quadratic form is given by Eq. (B.18) in Appendix B. Then, we have

$$\begin{aligned}\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} &= \int \mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi\;\det{}^{-\frac{1}{2}}(N^{-1})\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\\ &\times\;\Big\{-\frac{1}{2}\dot{A}_k^a\dot{A}_k^a - \frac{1}{2}\big((\partial_3)^{-1}\partial_k\dot{A}_k^a\big)\big((\partial_3)^{-1}\partial_l\dot{A}_l^a\big) + \frac{1}{2}\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}C_l\dot{A}_l)^a\\ &- i\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}J_F^0)^a + \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a\\ &- \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}F_{3k}^aF_{3k}^a + i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3D_3\psi\Big\}.\end{aligned} \quad (5.18)$$

We next consider the Gaussian integral given by

$$\begin{aligned}1 = \int \mathcal{D}A_\tau^a\;\det{}^{\frac{1}{2}}(\mathcal{D})\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{&-\frac{1}{2}\Big(iA_\tau^a - (\mathcal{D}^{-1})^{ac}\big(i(C_k\dot{A}_k)^c + J_F^{c0}\big)\Big)\\ &\times(\mathcal{D})^{ad}\Big(iA_\tau^d - (\mathcal{D}^{-1})^{de}\big(i(C_l\dot{A}_l)^e + J_F^{e0}\big)\Big)\Big\}\end{aligned} \quad (5.19)$$

in order to restore the A_τ degree of freedom. Inserting the Gaussian integration (5.19) into Eq. (5.18), some of the terms in the exponents of Eqs. (5.18) and (5.19) cancel each other. We present the expansion of the exponent of Eq. (5.19) in Eq. (B.20) in Appendix B. Then, we have

$$\begin{aligned}\mathrm{Tr}\left(e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} &= \int \mathcal{D}A_\tau^a\mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi\;\det{}^{-\frac{1}{2}}(N^{-1})\;\det{}^{\frac{1}{2}}(\mathcal{D})\\ &\times\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{-\frac{1}{2}\dot{A}_k^a\dot{A}_k^a - \frac{1}{2}\big((\partial_3)^{-1}\partial_k\dot{A}_k^a\big)\big((\partial_3)^{-1}\partial_l\dot{A}_l^a\big) - \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}F_{3k}^aF_{3k}^a\\ &- \frac{1}{2}(D_iA_\tau)^a(D_iA_\tau)^a + \dot{A}_k^a(D_kA_\tau)^a - (D_3A_\tau)^a((\partial_3)^{-1}\partial_k\dot{A}_k^a)\\ &+ i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3D_3\psi + iA_\tau^aJ_F^{a0}\Big\}.\end{aligned} \quad (5.20)$$

As in the case of the axial gauge, A_τ is recovered as the auxiliary field through the Gaussian integral (5.19). In order to restore the A_3^a degree of freedom, let us consider the identity

$$1 = \int \mathcal{D}A_3^a \prod_{x,a} \delta(A_3^a + (\partial_3)^{-1}\partial_k A_k^a) \Big(= \int \mathcal{D}A_3^a \det(\partial_3) \prod_{x,a} \delta(\partial_i A_i^a)\Big). \tag{5.21}$$

Then, the exponent (5.20) is summarized into the covariant form of the Lagrangian thanks to the delta function in Eq. (5.21). And as shown in Appendix C.2, the determinants are evaluated as

$$\det^{-\frac{1}{2}}(N^{-1})\det^{\frac{1}{2}}(\mathcal{D})\det(\partial_3) = \det(\partial_i D_i). \tag{5.22}$$

Hence, we obtain that

$$\begin{aligned}\mathrm{Tr}\left(e^{-\beta H(\hat\Pi_A^k,\hat A_k,\hat\Pi_\psi,\hat\psi)}\right)_{\rm Coul} &= \int \mathcal{D}A_\mu^a \mathcal{D}\bar\psi\mathcal{D}\psi \, \det(\partial_i D_i) \prod_{x,a} \delta(\partial_i A_i^a) \\ &\times \exp\int_0^\beta d\tau \int d^3\mathbf{x}\Big\{\frac{-1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \bar\psi i\gamma_\mu D_\mu \psi\Big\},\end{aligned} \tag{5.23}$$

where we have used $\mathcal{D}\Pi_\psi = \mathcal{D}\bar\psi$ and defined $\gamma_\tau \equiv -i\gamma^0$. We have finished proving Eq. (5.16).

5.2. *Polyakov loop in the Coulomb gauge*

Let us proceed to discuss the Polyakov loop in the Coulomb gauge.[5] We no longer have the A_τ degree of freedom, from which the Polyakov loop is usually defined, in the operator formalism of the Coulomb gauge. Nevertheless, it is natural to expect that the operator written in terms of only physical degrees of freedom corresponding to the Polyakov loop should exist and is to be defined.

Our aim is to present the operator $\hat{\mathcal{O}}$ satisfying Eq. (3.3) for the case of the Coulomb gauge, that is, it satisfies

$$\begin{aligned}&\mathrm{Tr}\left(\mathcal{O}(\hat\Pi_A^{ak},\hat A_k^a,\hat\Pi_\psi,\hat\psi)\, e^{-\beta H(\hat\Pi_A^k,\hat A_k,\hat\Pi_\psi,\hat\psi)}\right)_{\rm Coul} \\ &= \sum_{\alpha=1}^{D}\int \mathcal{D}A_\mu^a \mathcal{D}\bar\psi\mathcal{D}\psi \, \det(\partial_i D_i) \prod_{x,a} \delta(\partial_i A_i^a) \exp\left\{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)(\tilde T^a)_{\alpha\alpha}\right\} \\ &\times \exp\int_0^\beta d\tau \int d^3\mathbf{x}\Big\{\frac{-1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \bar\psi i\gamma_\mu D_\mu \psi\Big\},\end{aligned} \tag{5.24}$$

where, as in the case of the axial gauge, the Polyakov loop is reduced to the $U(1)$ gauge sector of the $SU(N)$ gauge group by the gauge transformations (4.2)–(4.4). Note again that $\hat{\mathcal{O}}$ should be written in terms of only physical degrees of freedom.

We shall show that

$$\hat{\mathcal{O}} \equiv \sum_{\alpha=1}^{D} \exp\left\{g\int_0^\beta d\tau \int d^3\mathbf{x} (\tilde T^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\hat{\hat\Delta}^{-1})^{ab}\big((\hat C_k'\hat\Pi_A^k)^b + \hat J_F^{b0} + \delta_{\rm self}^b\big)\Big)\right\} \tag{5.25}$$

satisfies Eq. (5.24). Here we have inserted the counter term defined by

$$\delta_{\rm self}^b \equiv \frac{g}{2}(\tilde T^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) \tag{5.26}$$

[5] As a different approach from ours, a Polyakov loop in the temporal gauge has been discussed in Ref. [21] and treated as a Wilson line of the Hosotani mechanism [10,11] by exchanging the role of a spatial coordinate and the Euclidean time.

in the definition (5.25). Note that the counter term (5.26) is the same form as the one for the axial gauge (4.18), and it does not depend on fields. $\delta^b_{\rm self}$ cancels the divergent self-energy of the point color charge densities arising from introducing the operator (5.25), that is, the Polyakov loop operator as we will see below. It may be worthwhile pointing out that the term subtracted by the counter term is $(\tilde{\Delta}^{-1})^{ab}\delta^b_{\rm self}$, so that, unlike the case for the axial gauge, it depends on the gauge field through the operator $\tilde{\Delta}^{-1}$. The above form of the operator (5.25) might be guessed from Eqs. (5.3) and (5.8), but it seems to be far from trivial that the operator actually satisfies Eq. (5.24), as we will see below.

The path integral representation for the left-hand side of Eq. (5.24) is

$$
\begin{aligned}
&{\rm Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A,\hat{A}^a_k,\hat{\Pi}_\psi,\hat{\psi})\; e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\rm Coul}\\
&=\sum_{\alpha=1}^{D}\int \mathcal{D}\Pi^{ak}_A\mathcal{D}A^a_k\mathcal{D}\Pi_\psi\mathcal{D}\psi\\
&\quad\times\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\tilde{\Delta}^{-1})^{ab}\big((C'_k\Pi^k_A)^b+J^{b0}_F+\delta^b_{\rm self}\big)\Big)\\
&\quad+i\Pi^{ak}_A\dot{A}^a_k+i\Pi_\psi\dot{\psi}-\mathcal{H}_{\rm Coul}\Big\},
\end{aligned}
\tag{5.27}
$$

where the Hamiltonian $\mathcal{H}_{\rm Coul}$ is given by Eq. (5.15). The exponent in Eq. (5.27) is quadratic with respect to Π^{ak}_A, so that we can perform the integration $\mathcal{D}\Pi^{ak}_A$. The detailed calculations and the explicit quadratic form for the exponent in Eq. (5.27) are given in Appendix B.3. Then, we have

$$
\begin{aligned}
&{\rm Tr}\left(\mathcal{O}(\hat{\Pi}^{ak}_A,\hat{A}^a_k,\hat{\Pi}_\psi,\hat{\psi})\; e^{-\beta H(\hat{\Pi}^k_A,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\rm Coul}=\sum_{\alpha=1}^{D}\int \mathcal{D}A^a_k\mathcal{D}\Pi_\psi\mathcal{D}\psi\;{\rm det}^{-\frac{1}{2}}(N^{-1})\\
&\times\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{-\frac{1}{2}\dot{A}^a_k\dot{A}^a_k-\frac{1}{2}\big((\partial_3)^{-1}\partial_k\dot{A}^a_k\big)\big((\partial_3)^{-1}\partial_l\dot{A}^a_l\big)+\frac{1}{2}\dot{A}^a_k(\tilde{C}_k\mathcal{D}^{-1}C_l\dot{A}_l)^a\\
&-i\dot{A}^a_k(\tilde{C}_k\mathcal{D}^{-1}J^0_F)^a+\frac{1}{2}J^{a0}_F(\mathcal{D}^{-1}J^0_F)^a-\frac{1}{4}F^a_{kl}F^a_{kl}-\frac{1}{2}F^a_{3k}F^a_{3k}\\
&+i\Pi_\psi\dot{\psi}+\bar{\psi}i\gamma^kD_k\psi+\bar{\psi}i\gamma^3D_3\psi\\
&+g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\tilde{\Delta}^{-1})^{ab}\delta^b_{\rm self}\\
&+g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J^0_F)^a+ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}C_k\dot{A}_k)^a\\
&-\frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\\
&+\frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big\}.
\end{aligned}
\tag{5.28}
$$

In order to recover the A^a_τ degree of freedom, we insert the Gaussian integral defined by

$$
\begin{aligned}
1&=\int\mathcal{D}A^a_\tau\;{\rm det}^{\frac{1}{2}}(\mathcal{D})\;\exp\int_0^\beta d\tau\int d^3\mathbf{x}\\
&\times\;\Big\{-\frac{1}{2}\Big(iA^a_\tau-(\mathcal{D}^{-1})^{ac}\big(i(C_k\dot{A}_k)^c+J^{c0}_F+g(\tilde{T}^c)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\\
&\quad\times(\mathcal{D})^{ad}\Big(iA^d_\tau-(\mathcal{D}^{-1})^{de}\big(i(C_l\dot{A}_l)^e+J^{e0}_F+g(\tilde{T}^e)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\Big\}
\end{aligned}
\tag{5.29}
$$

into Eq. (5.28). We have added new terms of the point color charge densities $g(\tilde{T}^c)_{\alpha\alpha}\delta(\mathbf{x}-\mathbf{x}_0)$, compared with Eq. (5.19). In Eq. (B.28) of Appendix B, we explicitly present the expansion of the exponent of Eq. (5.29) and we find that some of the terms in the exponent of Eq. (5.29) cancel with those in Eq. (5.28). Then, we obtain that

$$
\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}_A^{ak},\hat{A}_k^a,\hat{\Pi}_\psi,\hat{\psi})\, e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} \\
&= \sum_{\alpha=1}^{D}\int \mathcal{D}A_\tau^a\mathcal{D}A_k^a\mathcal{D}\Pi_\psi\mathcal{D}\psi\ \det{}^{-\frac{1}{2}}(N^{-1})\det{}^{\frac{1}{2}}(\mathcal{D})\ \exp\int_0^\beta d\tau\int d^3\mathbf{x} \\
&\times\ \Big\{-\frac{1}{2}\dot{A}_k^a\dot{A}_k^a-\frac{1}{2}\big((\partial_3)^{-1}\partial_k\dot{A}_k^a\big)\big((\partial_3)^{-1}\partial_l\dot{A}_l^a\big)-\frac{1}{4}F_{kl}^aF_{kl}^a-\frac{1}{2}F_{3k}^aF_{3k}^a \\
&-\frac{1}{2}(D_iA_\tau)^a(D_iA_\tau)^a+\dot{A}_k^a(D_kA_\tau)^a-(D_3A_\tau)^a((\partial_3)^{-1}\partial_k\dot{A}_k^a) \\
&+i\Pi_\psi\dot{\psi}+\bar{\psi}i\gamma^kD_k\psi+\bar{\psi}i\gamma^3D_3\psi+iA_\tau^aJ_F^{a0}+igA_\tau^a(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) \\
&+g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\tilde{\Delta}^{-1})^{ab}\delta_{\mathrm{self}}^b \\
&-\frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big\}. \qquad (5.30)
\end{aligned}
$$

The last two terms are found to cancel with the choice of Eq. (5.26). The A_τ degree of freedom is recovered as the auxiliary field through the Gaussian integral (5.29). The last term in the fourth line of Eq. (5.30) just corresponds to the Polyakov loop in Eq. (5.24).

We finally restore the A_3^a degree of freedom by Eq. (5.21). As before, the exponent is summarized into the covariant form of the Lagrangian and the determinant is evaluated as Eq. (5.22). The result is

$$
\begin{aligned}
&\mathrm{Tr}\left(\mathcal{O}(\hat{\Pi}_A^{ak},\hat{A}_k^a,\hat{\Pi}_\psi,\hat{\psi})\, e^{-\beta H(\hat{\Pi}_A^k,\hat{A}_k,\hat{\Pi}_\psi,\hat{\psi})}\right)_{\mathrm{Coul}} = \sum_{\alpha=1}^{D}\int \mathcal{D}A_\mu^a\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \det(\partial_iD_i)\prod_{x,a}\delta(\partial_iA_i^a) \\
&\times\ \exp\Big\{ig\int_0^\beta d\tau A_\tau^a(\tau,\mathbf{x}_0)(\tilde{T}^a)_{\alpha\alpha}\Big\}\exp\int_0^\beta d\tau\int d^3\mathbf{x}\Big\{\frac{-1}{4}F_{\mu\nu}^aF_{\mu\nu}^a-\bar{\psi}i\gamma_\mu D_\mu\psi\Big\}. \qquad (5.31)
\end{aligned}
$$

Thus, we have proved Eq. (5.24), and this implies that we have verified Eq. (3.3) in the Coulomb gauge for the operator $\hat{\mathcal{O}}$ defined by Eq. (5.25).

It may be worth mentioning here the equivalence between the axial gauge and the Coulomb gauge. The Faddeev–Popov method guarantees the equivalence among different gauge choices in the path integral formalism even at finite temperature. Although the equivalence is believed to be true even in the operator formalism at finite temperature, the operator correspondence between operator formalisms with different gauge choices seems to be less obvious. Actually, our final results (4.16) and (5.31) show that their right-hand sides are mutually transformed by the Faddeev–Popov manner. This fact implies that the results may prove indirectly the equivalence between the axial gauge and the Coulomb gauge in the operator formalism at finite temperature with the corresponding Polyakov loop operators.

6. Conclusions

In quantizing the non-Abelian gauge theory with a non-covariant gauge fixing such as the axial and Coulomb gauges, any physical operator such as the Hamiltonian is written in terms of only physical degrees of freedom, by which the Hilbert space is spanned. A_0 is not a dynamical variable, so it disappears in the Hilbert space. Then, one may wonder how one should define the operator, written in terms of only physical degrees of freedom, corresponding to the Polyakov loop operator when one considers the theory in these gauge fixings.

In order to answer the question, we have first studied the canonical formulation of the non-Abelian gauge theory in the axial and Coulomb gauges, in which the physical degrees of freedom are clarified. We started with the Lagrangian obtained by imposing the gauge condition and by eliminating A_0^a through the equation of motion, from which the canonical momentum for the gauge field is defined. The Hamiltonian obtained in this way is written in terms of only physical degrees of freedom as it should, though it has a complicated form including the non-local operator.

We have constructed the operator corresponding to the Polyakov loop operator in terms of only physical degrees of freedom in the axial and Coulomb gauges. In order to confirm that the defined operator is actually the standard Polyakov loop operator, we have evaluated the trace formula for the operator at finite temperature and showed that it can be rewritten into the covariant path integral form for the usual Polyakov loop operator with all the gauge degrees of freedom under the chosen gauge fixing.

We have encountered a divergent quantity in the process to obtain the covariant path integral form. It is the self-energy of the point color charge densities owing to introducing the operator $\hat{\mathcal{O}}$, which corresponds to the Polyakov loop. Introducing the Polyakov loop is essentially the same as considering the point color charge densities of a fermion. This manifestly appears in the Hamiltonians, (A.6) and (B.17) in the appendices. In the case of the axial gauge the divergent self-energy does not depend on the field, while in the case of the Coulomb gauge it does. One can subtract the self-energy by introducing the counter term, $\delta^a_{\rm self}$, in the definition of the operator $\hat{\mathcal{O}}$, and the self-energy does not appear in the final expression of the covariant path integral form.

In proposing the operators (4.17) and (5.25), we have taken into account the fact that the Polyakov loop operator has the same physical effect of introducing a color charge density. In fact, we have guessed the forms of the operators (4.17) and (5.25) from the second and third terms in the Hamiltonians (A.6) and (B.17). It is also very important and interesting to investigate the theoretical aspects of the non-Abelian gauge theory at finite temperature based on the present work in addition to giving the complete physical interpretation of the operator.

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Appendix A. Axial gauge

In this appendix we present some details of the calculations and formulae necessary in order to derive the results in the text.

A.1. Hamiltonian in the axial gauge

The Lagrangian $\mathcal{L}(A_k^a, \psi)_{\rm axial}$ is obtained by imposing the axial gauge $A_3^a = 0$ and eliminating A_0^a by using the constraint (2.5) in the Lagrangian (2.1). Straightforward calculations yield

$$\begin{aligned}\mathcal{L}(A_k^a, \psi)_{\rm axial} &= \frac{1}{2}\dot{A}_k^a (M_{kl}\dot{A}_l)^a - \dot{A}_k^a (D_k \mathcal{D}^{-1} J_F^0)^a + \frac{1}{2} J_F^{a0} (\mathcal{D}^{-1} J_F^0)^a \\ &\quad - \frac{1}{4} F_{kl}^a F_{kl}^a - \frac{1}{2}\partial_3 A_k^a \partial_3 A_k^a + \bar{\psi} i\gamma^0 \dot{\psi} + \bar{\psi} i \gamma^k D_k \psi + \bar{\psi} i\gamma^3 \partial_3 \psi, \end{aligned} \tag{A.1}$$

where we have used the definition for $(M)_{kl}^{ab}$, (2.8). In deriving (A.1), we have formally performed the partial integration with respect to D_k and $\mathcal{D}$. The partial integration with respect to D_k accompanies a minus sign, while that with respect to $\mathcal{D}$ does not. And we also note that the partial integration with respect to $(M)_{kl}^{ab}, (M^{-1})_{kl}^{ab}$ does not accompany a minus sign.

As mentioned in Sect. 2, the Hamiltonian (2.12),

$$\begin{aligned}\mathcal{H}_{\rm axial} &= \Pi_A^{ak}\dot{A}_k^a + \Pi_\psi \dot{\psi} - \mathcal{L}(A_k, \psi)_{\rm axial} \\ &= \frac{1}{2}\Pi_A^{ak}(M_{kl}^{-1}\Pi_A^l)^a + \Pi_A^{ak}(M_{kl}^{-1} D_l \mathcal{D}^{-1} J_F^0)^a \\ &\quad + \frac{1}{2}(D_k \mathcal{D}^{-1} J_F^0)^a (M_{kl}^{-1} D_l \mathcal{D}^{-1} J_F^0)^a - \frac{1}{2} J_F^{a0}(\mathcal{D}^{-1} J_F^0)^a \\ &\quad + \frac{1}{4} F_{kl}^a F_{kl}^a + \frac{1}{2}\partial_3 A_k^a \partial_3 A_k^a - \bar{\psi} i\gamma^k D_k \psi - \bar{\psi} i \gamma^3 \partial_3 \psi, \end{aligned} \tag{A.2}$$

can be rewritten into another form, which is obtained by using the explicit form of $(M^{-1})_{kl}^{ab}$ for the second and the third terms in Eq. (A.2) after the partial integration with respect to M_{kl}^{-1}, D_k, and $\mathcal{D}^{-1}$. These two terms are written as

$$\begin{aligned}&- (\mathcal{D}^{-1} D_l M_{lk}^{-1}\Pi_A^k)^a J_F^{a0} - \frac{1}{2} J_F^{a0} (\mathcal{D}^{-1} D_k M_{kl}^{-1} D_l \mathcal{D}^{-1} J_F^0)^a \\ &= \Pi_A^{ak}\Big(D_k (\partial_3)^{-2} J_F^0\Big)^a - \frac{1}{2} J_F^{a0}\Big(\big((\partial_3)^{-2}\delta^{ab} - (\mathcal{D}^{-1})^{ab}\big) J_F^{b0}\Big), \end{aligned} \tag{A.3}$$

where we have used the relations

$$(\mathcal{D}^{-1} D_l M_{lk}^{-1})^{ab} = (\partial_3)^{-2} (D_k)^{ab} \tag{A.4}$$

and

$$\Big(\mathcal{D}^{-1} D_k M_{kl}^{-1} D_l \mathcal{D}^{-1}\Big)^{ab} = (\partial_3)^{-2}\delta^{ab} - (\mathcal{D}^{-1})^{ab}, \tag{A.5}$$

which can easily be shown by using the explicit form of $(M^{-1})_{kl}^{ab}$ given by Eq. (2.10). Finally, we have performed the partial integration with respect to D_k and $(\partial_3)^{-2}$ in order to obtain the result (A.3). Then, we find that

$$\begin{aligned}\mathcal{H}_{\rm axial} &= \frac{1}{2}\Pi_A^{ak}(M_{kl}^{-1}\Pi_A^l)^a + \Pi_A^{ak}(D_k(\partial_3)^{-2} J_F^0)^a - \frac{1}{2} J_F^{a0} (\partial_3)^{-2} J_F^{a0} \\ &\quad + \frac{1}{4} F_{kl}^a F_{kl}^a + \frac{1}{2}\partial_3 A_k^a \partial_3 A_k^a - \bar{\psi} i\gamma^k D_k \psi - \bar{\psi} i\gamma^3 \partial_3 \psi. \end{aligned} \tag{A.6}$$

Note that the third term in the first line in Eq. (A.6) is well known to be the self-energy of the point color charge densities of a fermion in the axial gauge. The self-energy manifestly appears in this form of the Hamiltonian.

A.2. Trace formula in the axial gauge

In order to perform the integration $\mathcal{D}\Pi_A^{ak}$, we complete the square with respect to Π_A^{ak} in the exponent of Eq. (3.5),

$$
\begin{aligned}
& i\Pi_A^{ak}\dot{A}_k^a + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\text{axial}} \\
= & -\frac{1}{2}\Big(\Pi_A^{ak} - i(M_{kl}\dot{A}_l)^a + (D_k\mathcal{D}^{-1}J_F^0)^a\Big)(M^{-1})_{km}^{ab}\Big(\Pi_A^{bm} - i(M_{mn}\dot{A}_n)^b + (D_m\mathcal{D}^{-1}J_F^0)^b\Big) \\
& - \frac{1}{2}\dot{A}_k^a(M_{kl}\dot{A}_l)^a - i\dot{A}_k^a(D_k\mathcal{D}^{-1}J_F^0)^a + \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a - \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}\partial_3A_k^a\partial_3A_k^a \\
& + i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3\partial_3\psi.
\end{aligned}
\tag{A.7}
$$

The first term in the third line of Eq. (A.7) can be written as

$$
-\frac{1}{2}\dot{A}_k^a(M_{kl}\dot{A}_l)^a = -\frac{1}{2}\dot{A}_k^a\dot{A}_k^a - \frac{1}{2}(D_k\dot{A}_k)^a(\mathcal{D}^{-1}D_l\dot{A}_l)^a \tag{A.8}
$$

by using the explicit form for $(M)_{kl}^{ab}$, (2.8), and by performing the partial integration for D_k. Then, we obtain Eq. (3.7). If we expand the exponent of the Gaussian integral (3.8), we have, after performing the partial integration with respect to $\mathcal{D}$,

$$
\begin{aligned}
& -\frac{1}{2}\Big(iA_\tau^a - (\mathcal{D}^{-1})^{ac}\big(i(D_k\dot{A}_k)^c + J_F^{c0}\big)\Big)(\mathcal{D})^{ad}\Big(iA_\tau^d - (\mathcal{D}^{-1})^{de}\big(i(D_l\dot{A}_l)^e + J_F^{e0}\big)\Big) \\
= & -\frac{1}{2}(D_kA_\tau)^a(D_kA_\tau)^a - \frac{1}{2}\partial_3A_\tau^a\partial_3A_\tau^a + iA_\tau^aJ_F^{a0} + \dot{A}_k^a(D_kA_\tau)^a \\
& + \frac{1}{2}(D_k\dot{A}_k)^a(\mathcal{D}^{-1}D_l\dot{A}_l)^a + i\dot{A}_k^a(D_k\mathcal{D}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a.
\end{aligned}
\tag{A.9}
$$

We observe that some of the terms in the second line of the right-hand side of Eq. (A.7) and the right-hand side of Eq. (A.9) cancel each other to yield the result (3.9).

A.3. Polyakov loop in the axial gauge

In order to perform the integration $\mathcal{D}\Pi_A^{ak}$, we need to complete the square with respect to Π_A^{ak} in the exponent of Eq. (4.12), which is given by

$$
\begin{aligned}
& -\frac{1}{2}\Big(\Pi_A^{ak} - (M)_{kl}^{ab}\big(i\dot{A}_l^b - g(D_l)^{bd}(\partial_3)^{-2}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (D_k\mathcal{D}^{-1}J_F^0)^b\Big)(M^{-1})_{km}^{ag} \\
& \times\Big(\Pi_A^{gm} - (M)_{mn}^{gc}\big(i\dot{A}_n^c - g(D_n)^{cf}(\partial_3)^{-2}(\tilde{T}^f)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (D_m\mathcal{D}^{-1}J_F^0)^g\Big).
\end{aligned}
\tag{A.10}
$$

There are new terms containing the gauge coupling g, due to introducing the operator (4.11), which do not appear in Eq. (A.7). According to the perfect square with respect to Π_A^{ak}, we should add the following terms:

$$
\begin{aligned}
& -g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\partial_3)^{-2}(D_k\Pi_A^k)^a\big) + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\partial_3)^{-2}D_kM_{kl}\dot{A}_l\big)^a \\
& + g(M)_{kl}^{ab}(D_l)^{bd}(\partial_3)^{-2}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((M^{-1})_{km}^{ag}(D_m\mathcal{D}^{-1}J_F^0)^g\big) \\
& -\frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(D_kM_{kl}D_l)^{ab}(\partial_3)^{-2}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big),
\end{aligned}
\tag{A.11}
$$

where we have performed the partial integration with respect to $(\partial_3)^{-2}$, D_k, and $(M)_{kl}^{ab}$. They can be rewritten into simple forms by using the explicit form of M_{kl}^{ab} given by Eq. (2.8). It is easy to check that

$$
\begin{aligned}
(D_l M_{lk})^{ab} &= (D_l)^{ac}\big(\delta^{cb}\delta_{lk} - (D_l\mathcal{D}^{-1}D_k)^{cb}\big) = (D_k)^{ab} - (D_l^2\mathcal{D}^{-1}D_k)^{ab} \\
&= (D_k)^{ab} - \big((\mathcal{D} - (\partial_3)^2)\mathcal{D}^{-1}D_k\big)^{ab} = (\partial_3)^2(\mathcal{D}^{-1}D_k)^{ab}
\end{aligned}
\tag{A.12}
$$

and

$$
\begin{aligned}
(D_k M_{kl} D_l)^{ab} &= (D_k M_{kl})^{ac}(D_l)^{cb} \overset{\text{(A.12)}}{=} (\partial_3)^2(\mathcal{D}^{-1}D_l)^{ac}(D_l)^{cb} \\
&= (\partial_3)^2\big(\mathcal{D}^{-1}(\mathcal{D} - (\partial_3)^2)\big)^{ab} = (D_k^2)^{ab} - (D_k^2\mathcal{D}^{-1}D_l^2)^{ab} \\
&= (\partial_3)^2\delta^{ab} - (\partial_3)^2(\mathcal{D}^{-1})^{ab}(\partial_3)^2.
\end{aligned}
\tag{A.13}
$$

Then, Eq. (A.11) becomes

$$
\begin{aligned}
&-g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\partial_3)^{-2}(D_k\Pi_A^k)^a\big) + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\partial_3)^{-2}D_k M_{kl}\dot{A}_l\big)^a \\
&+ g(M)_{kl}^{ab}(D_l)^{bd}(\partial_3)^{-2}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((M^{-1})_{km}^{ag}(D_m\mathcal{D}^{-1}J_F^0)^g\big) \\
&- \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(D_k M_{kl} D_l)^{ab}(\partial_3)^{-2}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
={}& -g\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big((\partial_3)^{-2}\big((D_k\Pi_A^k)^a + J_F^0)^a\Big) \\
&+ g\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big(\mathcal{D}^{-1}J_F^0\big)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}D_k\dot{A}_k)^a \\
&- \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
&+ \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big),
\end{aligned}
\tag{A.14}
$$

where the first term, aside from the minus sign, in the right-hand side is nothing but the exponent of the operator (4.11). Therefore, the exponent of the right-hand side of Eq. (4.12) is written into the quadratic form with respect to Π_A^{ak} as

$$
\begin{aligned}
&g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\partial_3)^{-2}\big((D_k\Pi_A^k)^a + J_F^{a0}\big)\Big) + i\Pi_A^{ak}\dot{A}_k^a + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\text{axial}} \\
={}& -\frac{1}{2}\Big(\Pi_A^{ak} - (M)_{kl}^{ab}\big(i\dot{A}_l^b - g(D_l)^{bd}(\partial_3)^{-2}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (D_k\mathcal{D}^{-1}J_F^0)^b\Big)(M^{-1})_{km}^{ag} \\
&\times\Big(\Pi_A^{gm} - (M)_{mn}^{gc}\big(i\dot{A}_n^c - g(D_n)^{cf}(\partial_3)^{-2}(\tilde{T}^f)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (D_m\mathcal{D}^{-1}J_F^0)^g\Big) \\
&- \frac{1}{2}\dot{A}_k^a(M_{kl}\dot{A}_l)^a - i\dot{A}_k^a(D_k\mathcal{D}^{-1}J_F^0)^a + \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a - \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}\partial_3A_k^a\partial_3A_k^a \\
&+ i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3\partial_3\psi \\
&+ g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J_F^0)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}D_k\dot{A}_k)^a \\
&- \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\partial_3)^{-2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
&+ \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big).
\end{aligned}
\tag{A.15}
$$

We see that Eq. (4.13) is obtained after the integration $\mathcal{D}\Pi_A^{ak}$. Let us note that the first term in the fourth line of Eq. (A.15) is rewritten by Eq. (A.8). If we expand the exponent of the Gaussian integral

(4.14), we have

$$
\begin{aligned}
&-\frac{1}{2}\Big(iA^a_\tau-(\mathcal{D}^{-1})^{ac}\big(i(D_k\dot{A}_k)^c+J^{c0}_F+g(\tilde{T}^c)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\\
&\times(\mathcal{D})^{ad}\Big(iA^d_\tau-(\mathcal{D}^{-1})^{de}\big(i(D_l\dot{A}_l)^e+J^{e0}_F+g(\tilde{T}^e)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\\
=&-\frac{1}{2}(D_kA_\tau)^a(D_kA_\tau)^a-\frac{1}{2}\partial_3A^a_\tau\partial_3A^a_\tau+iA^a_\tau J^{a0}_F+\dot{A}^a_k(D_kA_\tau)^a\\
&+\frac{1}{2}(D_k\dot{A}_k)^a(\mathcal{D}^{-1}D_l\dot{A}_l)^a+i\dot{A}^a_k(D_k\mathcal{D}^{-1}J^0_F)^a-\frac{1}{2}J^{a0}_F(\mathcal{D}^{-1}J^0_F)^a\\
&+igA^a_\tau(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\\
&-g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J^0_F)^a-ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}D_k\dot{A}_k)^a\\
&-\frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big).
\end{aligned}
\tag{A.16}
$$

We find that some of the terms in Eqs. (A.15) and (A.16) cancel each other, and we obtain the result (4.15). The fifth line in Eq. (A.16) corresponds to the Polyakov loop, as it should. Note that the last term in Eq. (A.16) is canceled with the last term in Eq. (A.15). As for the term with the two delta functions, only the seventh line in Eq. (A.15) is left, which is the divergent self-energy of the point color charge densities owing to introducing the operator (4.11), as discussed in the text.

Appendix B. Coulomb gauge

We present the formulae and relations in order to obtain the results of Sect. 5.

B.1. *Hamiltonian in the Coulomb gauge*

Let us recall the operators defined in Eqs. (5.4) and (5.5), from which we obtain new operators given by Eqs. (5.10) and (5.11). These are proved by using $\partial_l^2=\Delta-(\partial_3)^2$:

$$
\begin{aligned}
(\tilde{C}'_k)^{ab}&\equiv(\delta_{kl}-\partial_k\Delta^{-1}\partial_l)(\tilde{C}_l)^{ab}=(\delta_{kl}-\partial_k\Delta^{-1}\partial_l)\big((D_l)^{ab}-\partial_l(\partial_3)^{-1}(D_3)^{ab}\big)\\
&=(D_k)^{ab}-\partial_k\Delta^{-1}\partial\cdot(D)^{ab},
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
(C'_k)^{ab}&\equiv(C_l)^{ab}(\delta_{lk}-\partial_l\Delta^{-1}\partial_k)=\big((D_l)^{ab}-(D_3)^{ab}(\partial_3)^{-1}\partial_l\big)(\delta_{lk}-\partial_l\Delta^{-1}\partial_k)\\
&=(D_k)^{ab}-(D)^{ab}\cdot\partial\Delta^{-1}\partial_k,
\end{aligned}
\tag{B.2}
$$

where we have used the notation $\partial_i(D_i)^{ab}=\partial\cdot(D)^{ab}$, which is also $(D)^{ab}\cdot\partial$ due to the Coulomb gauge, $\partial_iA^{ai}=0$.

We also find the useful formulae given by

$$
\begin{aligned}
(C_k\tilde{C}'_k)^{ab}&=\big((D_k)^{ac}-(D_3)^{ac}(\partial_3)^{-1}\partial_k\big)\big((D_k)^{cb}-\partial_k\Delta^{-1}\partial\cdot(D)^{cb}\big)\\
&=(D_k^2)^{ab}+(D_3^2)^{ab}-(D_j\partial_j\Delta^{-1}\partial_iD_i)^{ab}\\
&=\mathcal{D}^{ab}-(\tilde{\Delta})^{ab},
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
(C'_k\tilde{C}_k)^{ab}&=\big((D_k)^{ac}-(D)^{ac}\cdot\partial\Delta^{-1}\partial_k\big)\big((D_k)^{cb}-\partial_l(\partial_3)^{-1}(D_3)^{ac}\big)\\
&=(D_k^2)^{ab}+(D_3^2)^{ab}-(D_i\partial_i\Delta^{-1}\partial_jD_j)^{ab}\\
&=\mathcal{D}^{ab}-(\tilde{\Delta})^{ab},
\end{aligned}
\tag{B.4}
$$

where $\tilde{\Delta}$ is defined by $D_i\partial_i\Delta^{-1}\partial_jD_j$, whose inverse operator is given by Eq. (5.12).

The Lagrangian $\mathcal{L}(A_k^a, \psi)_{\rm Coul}$ is obtained by imposing the Coulomb gauge $\partial_i A^{ai} = 0$ and eliminating A_0^a by using the constraint (5.3) in the Lagrangian (2.1), and after straightforward calculations we have

$$\mathcal{L}(A_k^a, \psi)_{\rm Coul} = \frac{1}{2}\dot{A}_k^a (N_{kl}\dot{A}_l)^a - \dot{A}_k^a (\tilde{C}_k \mathcal{D}^{-1} J_F^0)^a + \frac{1}{2} J_F^{a0} (\mathcal{D}^{-1} J_F^0)^a$$
$$- \frac{1}{4} F_{kl}^a F_{kl}^a - \frac{1}{2} F_{3k}^a F_{3k}^a + \bar{\psi} i\gamma^0 \dot{\psi} + \bar{\psi} i\gamma^k D_k \psi + \bar{\psi} i\gamma^3 D_3 \psi, \qquad \text{(B.5)}$$

where we have used the definition (5.7). In deriving (B.5), we have formally performed the partial integration with respect to $\tilde{C}_k$ and $\mathcal{D}$. The partial integration with respect to $\tilde{C}_k$ gives a new operator C_k, accompanying a minus sign, and vice versa. Incidentally, the partial integration with respect to $(N)_{kl}^{ab}, (N^{-1})_{kl}^{ab}$ does not accompany a minus sign.

Let us show that $(N^{-1})_{kl}^{ab}$ is given by Eq. (5.9). To this end, we first introduce a new operator defined by

$$(N')_{km}^{ab} \equiv (\delta_{kl} - \partial_k \Delta^{-1} \partial_l)(N)_{lm}^{ab}. \qquad \text{(B.6)}$$

$(N)_{km}^{ab}$ is defined by Eq. (5.7). Since it is easy to show that

$$(\delta_{kl} - \partial_k \Delta^{-1} \partial_l)(\delta_{lm} + \partial_l (\partial_3)^{-2} \partial_m) = \delta_{km}, \qquad \text{(B.7)}$$

Eq. (B.6) becomes

$$(N')_{km}^{ab} = \delta_{km}\delta^{ab} - (\tilde{C}'_k \mathcal{D}^{-1} C_m)^{ab}, \qquad \text{(B.8)}$$

where we have used the definition (B.1). From Eq. (B.6), we obtain that

$$(N^{-1})_{kl}^{ab} = (N^{-1\prime})_{km}^{ab} (\delta_{ml} - \partial_m \Delta^{-1} \partial_l). \qquad \text{(B.9)}$$

The explicit form of $(N^{-1'})_{kl}^{ab}$ is easily found to be

$$(N^{-1'})_{kl}^{ab} = \partial_{kl}\delta^{ab} + (\tilde{C}'_k \tilde{\Delta}^{-1} C_l)^{ab}. \qquad \text{(B.10)}$$

This is, by using the relation (B.3), because

$$\begin{aligned} &(N')_{kl}^{ac} (N^{-1\prime})_{lm}^{cb} \\ &= \big(\delta_{kl}\delta^{ac} - (\tilde{C}'_k \mathcal{D}^{-1} C_l)^{ac}\big)\big(\partial_{lm}\delta^{cb} + (\tilde{C}'_l \tilde{\Delta}^{-1} C_m)^{cb}\big) \\ &= \delta_{km}\delta^{ab} + (\tilde{C}'_k \tilde{\Delta}^{-1} C_m)^{ab} - (\tilde{C}'_k \mathcal{D}^{-1} C_m)^{ab} - (\tilde{C}'_k \mathcal{D}^{-1} C_l \tilde{C}'_l \tilde{\Delta}^{-1} C_m)^{ab} \\ &= \delta_{km}\delta^{ab} + (\tilde{C}'_k \tilde{\Delta}^{-1} C_m)^{ab} - (\tilde{C}'_k \mathcal{D}^{-1} C_m)^{ab} - (\tilde{C}'_k \mathcal{D}^{-1} (\mathcal{D} - \tilde{\Delta}) \tilde{\Delta}^{-1} C_m)^{ab} \\ &= \delta_{km}\delta^{ab}. \end{aligned} \qquad \text{(B.11)}$$

Then, Eq. (B.9) is evaluated as

$$\begin{aligned} (N^{-1})_{kl}^{ab} &= \big(\delta_{km}\delta^{ab} + (\tilde{C}'_k \tilde{\Delta}^{-1} C_m)^{ab}\big)(\delta_{ml} - \partial_m \Delta^{-1} \partial_l) \\ &= (\delta_{kl} - \partial_k \Delta^{-1} \partial_l)\delta^{ab} + (\tilde{C}'_k \tilde{\Delta}^{-1} C'_l)^{ab}, \end{aligned} \qquad \text{(B.12)}$$

where we have used the definition (B.2) for C'_l. We have finished proving Eq. (5.9).

As in the case for the axial gauge, the Hamiltonian (5.15),

$$\begin{aligned}\mathcal{H}_{\mathrm{Coul}} &= \Pi_A^{ak}\dot{A}_k^a + \Pi_\psi\dot{\psi} - \mathcal{L}(A_k,\psi)_{\mathrm{Coul}}\\ &= \frac{1}{2}\Pi_A^{ak}(N_{kl}^{-1}\Pi_A^l)^a + \Pi_A^{ak}(N_{kl}^{-1}\tilde{C}_l\mathcal{D}^{-1}J_F^0)^a\\ &\quad + \frac{1}{2}(\tilde{C}_k\mathcal{D}^{-1}J_F^0)(N_{kl}^{-1}\tilde{C}_l\mathcal{D}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a\\ &\quad + \frac{1}{4}F_{kl}^aF_{kl}^a + \frac{1}{2}F_{3k}^aF_{3k}^a - \bar{\psi}i\gamma^kD_k\psi - \bar{\psi}i\gamma^3D_3\psi, \end{aligned} \tag{B.13}$$

can be rewritten into another form. The second and the third terms in Eq. (B.13) after the partial integration with respect to N_{kl}^{-1}, $\tilde{C}_k$, and $\mathcal{D}^{-1}$ are rewritten as

$$\begin{aligned}&-(\mathcal{D}^{-1}C_lN_{lk}^{-1}\Pi_A^k)^aJ_F^{a0} - \frac{1}{2}J_F^{0a}(\mathcal{D}^{-1}C_kN_{kl}^{-1}\tilde{C}_l\mathcal{D}^{-1}J_F^0)^a\\ &= \Pi_A^{ak}\big(\tilde{C}_k'\tilde{\Delta}^{-1}J_F^0\big)^a - \frac{1}{2}J_F^{a0}\Big(\big((\tilde{\Delta}^{-1})^{ab} - (\mathcal{D}^{-1})^{ab}\big)J_F^{b0}\Big),\end{aligned} \tag{B.14}$$

where we have used the relations

$$(\mathcal{D}^{-1}C_lN_{lk}^{-1})^{ab} = (\tilde{\Delta}^{-1}C_k')^{ab} \tag{B.15}$$

and

$$\Big(\mathcal{D}^{-1}C_kN_{kl}^{-1}\tilde{C}_l\mathcal{D}^{-1}\Big)^{ab} = (\tilde{\Delta}^{-1})^{ab} - (\mathcal{D}^{-1})^{ab}. \tag{B.16}$$

The above relations can be shown by using Eqs. (B.3) and (B.4) and the explicit form of $(N^{-1})_{kl}^{ab}$ given by Eq. (B.12). Finally, we have performed the partial integration with respect to $\tilde{\Delta}^{-1}$ and C_k' in order to arrive at Eq. (B.14). Note that the partial integration with respect to C_k' yields a new operator $\tilde{C}_k'$, accompanying a minus sign, and vice versa, while the partial integration of $\tilde{\Delta}^{-1}$ does not have a minus sign. Then, we find that

$$\begin{aligned}\mathcal{H}_{\mathrm{Coul}} &= \frac{1}{2}\Pi_A^{ak}(N_{kl}^{-1}\Pi_A^l)^a + \Pi_A^{ak}(\tilde{C}_k'\tilde{\Delta}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\tilde{\Delta}^{-1}J_F^0)^a\\ &\quad + \frac{1}{4}F_{kl}^aF_{kl}^a + \frac{1}{2}F_{3k}^aF_{3k}^a - \bar{\psi}i\gamma^kD_k\psi - \bar{\psi}i\gamma^3D_3\psi.\end{aligned} \tag{B.17}$$

Note that the third term in the first line of Eq. (B.17) is well known to be the self-energy of the point color charge densities of a fermion in the Coulomb gauge. Unlike the case for the axial gauge, it depends on the gauge field through the operator $\tilde{\Delta}^{-1}$. We also point out the correspondence of the Hamiltonians (A.6) and (B.17) such as $M^{-1} \leftrightarrow N^{-1}$, $D_k \leftrightarrow \tilde{C}_k'$, and $(\partial_3)^{-2} \leftrightarrow \tilde{\Delta}^{-1}$, though the explicit forms for the operators are quite different.

B.2. Trace formula in the Coulomb gauge

We complete the square with respect to Π_A^{ak} in the exponent of Eq. (5.17) in order to perform the integration $\mathcal{D}\Pi_A^{ak}$,

$$\begin{aligned}&i\Pi_A^{ak}\dot{A}_k^a + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\mathrm{Coul}}\\ &= -\frac{1}{2}\Big(\Pi_A^{ak} - i(N_{kl}\dot{A}_l)^a + (\tilde{C}_k\mathcal{D}^{-1}J_F^0)^a\Big)(N^{-1})_{km}^{ac}\Big(\Pi_A^{cm} - i(N_{mn}\dot{A}_n)^c + (\tilde{C}_m\mathcal{D}^{-1}J_F^0)^c\Big)\end{aligned}$$

$$
-\frac{1}{2}\dot{A}_k^a(N_{kl}\dot{A}_l)^a - i\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}J_F^0)^a + \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a - \frac{1}{4}F_{kl}^aF_{kl}^a - \frac{1}{2}F_{3k}^aF_{3k}^a
+ i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3D_3\psi. \tag{B.18}
$$

The first term in the third line of Eq. (B.18) is rewritten as

$$
-\frac{1}{2}\dot{A}_k^a(N_{kl}\dot{A}_l)^a = -\frac{1}{2}\dot{A}_k^a\dot{A}_k^a - \frac{1}{2}\big((\partial_3)^{-1}\partial_k\dot{A}_k^a\big)\big((\partial_3)^{-1}\partial_l\dot{A}_l^a\big) + \frac{1}{2}\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}C_l\dot{A}_l)^a, \tag{B.19}
$$

where we have used the explicit form for $(N)_{kl}^{ab}$, Eq. (5.7), and performed the partial integration with respect to $(\partial_3)^{-1}$ and ∂_k.

The expansion of the exponent of the Gaussian integral (5.19) yields

$$
\begin{aligned}
&-\frac{1}{2}\Big(iA_\tau^a - (\mathcal{D}^{-1})^{ac}\big(i(C_k\dot{A}_k)^c + J_F^{c0}\big)\Big)(\mathcal{D})^{ad}\Big(iA_\tau^d - (\mathcal{D}^{-1})^{de}\big(i(C_l\dot{A}_l)^e + J_F^{e0}\big)\Big)\\
&= -\frac{1}{2}(D_iA_\tau)^a(D_iA_\tau)^a + iA_\tau^aJ_F^{a0} + \dot{A}_k^a(D_kA_\tau)^a - \frac{1}{2}\dot{A}_l^a(\tilde{C}_l\mathcal{D}^{-1}C_k\dot{A}_k)^a\\
&\quad + i\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a.
\end{aligned} \tag{B.20}
$$

Note that the third and the fourth terms in Eq. (B.20) come from one of the terms obtained by expanding the exponent of Eq. (B.20),

$$
-A_\tau^a\big(C_k\dot{A}_k\big)^a = \big(D_kA_\tau\big)^a\dot{A}_k^a - \big(D_3A_\tau\big)^a\big((\partial_3)^{-1}\partial_k\dot{A}_k^a\big), \tag{B.21}
$$

where we have used Eq. (5.4), and the partial integration with respect to D_k has been done. We see that some of the terms in Eqs. (B.18) and (B.20) cancel each other to yield the result (5.20) in the text.

B.3. Polyakov loop in the Coulomb gauge

We need to complete the square with respect to Π_A^{ak} in the exponent of Eq. (5.27) in order to perform the integration $\mathcal{D}\Pi_A^{ak}$. It is given by

$$
\begin{aligned}
&-\frac{1}{2}\Big(\Pi_A^{ak} - (N)_{kl}^{ab}\big(i\dot{A}_l^b - g(\tilde{C}_l'\tilde{\Delta}^{-1})^{bd}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (\tilde{C}_k\mathcal{D}^{-1}J_F^0)^b\Big)(N^{-1})_{km}^{ag}\\
&\times\Big(\Pi_A^{gm} - (N)_{mn}^{gc}\big(i\dot{A}_n^c - g(\tilde{C}_n'\tilde{\Delta}^{-1})^{cf}(\tilde{T}^f)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (\tilde{C}_m\mathcal{D}^{-1}J_F^0)^g\Big).
\end{aligned} \tag{B.22}
$$

New terms appear with the gauge coupling g because of introducing the operator (5.25), which we do not have in Eq. (B.18). According to the perfect square with respect to Π_A^{ak}, we should add the following terms:

$$
\begin{aligned}
&-g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big(\tilde{\Delta}^{-1}C_k'\Pi_A^k\big)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\tilde{\Delta}^{-1}C_k'N_{kl}\dot{A}_l)^a\\
&+ g(N)_{kl}^{ab}(\tilde{C}_l'\tilde{\Delta}^{-1})^{bd}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((N^{-1})_{km}^{ag}(\tilde{C}_k\mathcal{D}^{-1}J_F^0)^g\big)\\
&-\frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}(\partial_3)^{-2}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\tilde{\Delta}^{-1}C_k'N_{kl}\tilde{C}_l'\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big),
\end{aligned} \tag{B.23}
$$

where we have performed the partial integration with respect to $\tilde{\Delta}^{-1}$, $\tilde{C}_k'$, and $(N)_{kl}^{ab}$. They can be rewritten into simple forms by using the explicit form of N_{kl}^{ab}. With the help of Eqs. (B.2), (B.3), (B.4), and (B.7), we obtain that

$$
(\tilde{\Delta}^{-1}C_k'N_{kl})^{ab} = (\tilde{\Delta}^{-1}C_k')^{ac}\big(\delta^{cb}(\delta_{kl} + \partial_k(\partial_3)^{-1}\partial_l) - (\tilde{C}_k\mathcal{D}^{-1}C_l)^{cb}\big)
$$

$$
\begin{aligned}
&= (\tilde{\Delta}^{-1})^{ac}(C_l)^{cb} - (\tilde{\Delta}^{-1})^{ac}(C'_k\tilde{C}_k)^{cd}(\mathcal{D}^{-1}C_l)^{db} \\
&= (\tilde{\Delta}^{-1})^{ac}(C_l)^{cb} - (\tilde{\Delta}^{-1})^{ac}(\mathcal{D}-\tilde{\Delta})^{cd}(\mathcal{D}^{-1}C_l)^{db} \\
&= (\mathcal{D}^{-1}C_l)^{ab} \qquad \text{(B.24)}
\end{aligned}
$$

and

$$
\begin{aligned}
(\tilde{\Delta}^{-1}C'_kN_{kl}\tilde{C}'_l\tilde{\Delta}^{-1})^{ab} &= (\tilde{\Delta}^{-1}C'_kN_{kl})^{ac}(\tilde{C}'_l\tilde{\Delta}^{-1})^{cb} \overset{\text{(B.24)}}{=} (\mathcal{D}^{-1}C_l)^{ac}(\tilde{C}'_l\tilde{\Delta}^{-1})^{cb} \\
&= (\mathcal{D}^{-1})^{ac}(C_l\tilde{C}'_l)^{cd}(\tilde{\Delta}^{-1})^{db} = (\mathcal{D}^{-1})^{ac}(\mathcal{D}-\tilde{\Delta})^{cd}(\tilde{\Delta}^{-1})^{db} \\
&= (\tilde{\Delta}^{-1})^{ab} - (\mathcal{D}^{-1})^{ab}. \qquad \text{(B.25)}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&- g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big(\tilde{\Delta}^{-1}C'_k\Pi^k_A\big)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\tilde{\Delta}^{-1}C'_kN_{kl}\dot{A}_l)^a \\
&+ g(N)^{ab}_{kl}(\tilde{C}'_l\tilde{\Delta}^{-1})^{bd}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((N^{-1})^{ag}_{km}(\tilde{C}_k\mathcal{D}^{-1}J^0_F)^g)\big) \\
&- \frac{g^2}{2}\big((\tilde{T}^a)_{\alpha\alpha}(\partial_3)^{-2}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\big((\tilde{\Delta}^{-1}C'_kN_{kl}\tilde{C}'_l\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
= &-g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\tilde{\Delta}^{-1})^{ab}\big((C'_k\Pi^k_A)^a + J^{0b}_F\big)\Big) \\
&+ g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J^0_F)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}C_k\dot{A}^a_k) \\
&- \frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
&+ \frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big), \qquad \text{(B.26)}
\end{aligned}
$$

where the first term, aside from the minus sign, in the right-hand side is the exponent of operator (5.25). Hence, the exponent of the right-hand side of Eq. (5.27) is written into the quadratic form with respect to Π^{ak}_A as

$$
\begin{aligned}
&g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\Big((\tilde{\Delta}^{-1})^{ab}\big((C'_k\Pi^k_A)^b + J^{b0}_F + \delta^b_{\text{self}}\big)\Big) + i\Pi^{ak}_A\dot{A}^a_k + i\Pi_\psi\dot{\psi} - \mathcal{H}_{\text{Coul}} \\
= &-\frac{1}{2}\Big(\Pi^{ak}_A - (N)^{ab}_{kl}\big(i\dot{A}^b_l - g(\tilde{C}'_l\tilde{\Delta}^{-1})^{bd}(\tilde{T}^d)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (\tilde{C}_k\mathcal{D}^{-1}J^0_F)^b\Big)(N^{-1})^{ag}_{km} \\
&\times\Big(\Pi^{gm}_A - (N)^{gc}_{mn}\big(i\dot{A}^c_n - g(\tilde{C}'_n\tilde{\Delta}^{-1})^{cf}(\tilde{T}^f)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) + (\tilde{C}_m\mathcal{D}^{-1}J^0_F)^g\Big) \\
&- \frac{1}{2}\dot{A}^a_k(N_{kl}\dot{A}_l)^a - i\dot{A}^a_k(\tilde{C}_k\mathcal{D}^{-1}J^0_F)^a + \frac{1}{2}J^{a0}_F(\mathcal{D}^{-1}J^0_F)^a - \frac{1}{4}F^a_{kl}F^a_{kl} - \frac{1}{2}F^a_{3k}F^a_{3k} \\
&+ i\Pi_\psi\dot{\psi} + \bar{\psi}i\gamma^kD_k\psi + \bar{\psi}i\gamma^3D_3\psi \\
&+ g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\tilde{\Delta}^{-1})^{ab}\delta^b_{\text{self}} \\
&+ g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J^0_F)^a + ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}C_k\dot{A}_k)^a \\
&- \frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\tilde{\Delta}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big) \\
&+ \frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big). \qquad \text{(B.27)}
\end{aligned}
$$

We observe that the integration $\mathcal{D}\Pi_A^{ak}$ yields Eq. (5.28). Note that the first term in the fourth line of Eq. (B.27) is rewritten by Eq. (B.19). As explained in the text, the counter term, the sixth line of Eq. (B.27), exactly cancels the divergent self-energy, the eighth line of Eq. (B.27).

If we expand the exponent of the Gaussian integral (5.29), we have

$$
\begin{aligned}
&-\frac{1}{2}\Big(iA_\tau^a - (\mathcal{D}^{-1})^{ac}\big(i(C_k\dot{A}_k)^c + J_F^{c0} + g(\tilde{T}^c)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\\
&\times(\mathcal{D})^{ad}\Big(iA_\tau^d - (\mathcal{D}^{-1})^{de}\big(i(C_l\dot{A}_l)^e + J_F^{e0} + g(\tilde{T}^e)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big)\Big)\\
=&-\frac{1}{2}(D_iA_\tau)^a(D_iA_\tau)^a + iA_\tau^aJ_F^{a0} - A_\tau^a(C_k\dot{A}_k)^a\\
&-\frac{1}{2}\dot{A}_l^a(\tilde{C}_l\mathcal{D}^{-1}C_k\dot{A}_k)^a + i\dot{A}_k^a(\tilde{C}_k\mathcal{D}^{-1}J_F^0)^a - \frac{1}{2}J_F^{a0}(\mathcal{D}^{-1}J_F^0)^a\\
&+igA_\tau^a(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\\
&-g(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}J_F^0)^a - ig(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)(\mathcal{D}^{-1}C_k\dot{A}_k)^a\\
&-\frac{g^2}{2}(\tilde{T}^a)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big((\mathcal{D}^{-1})^{ab}(\tilde{T}^b)_{\alpha\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\big).
\end{aligned}
\tag{B.28}
$$

The third term in the first line of the right-hand side of Eq. (B.28) can be written by

$$
-A_\tau^a(C_k\dot{A}_k)^a = \dot{A}_k^a(D_kA_\tau)^a - (D_3A_\tau)^a((\partial_3)^{-1}\partial_k\dot{A}_k^a),
\tag{B.29}
$$

where we have performed the partial integration with respect to D_3. We observe that some of the terms in Eqs. (B.27) and (B.28) cancel each other to produce Eq. (5.30).

Appendix C. Evaluation of determinants

C.1. Axial gauge

Let us prove Eq. (3.11). We evaluate $\det(M)$, where M is defined by Eq. (2.8). To this end, we write M as

$$
(M)_{kl}^{ab} = \begin{pmatrix} \delta^{ab} - (D_1\mathcal{D}^{-1}D_1)^{ab} & -(D_1\mathcal{D}^{-1}D_2)^{ab} \\ -(D_2\mathcal{D}^{-1}D_1)^{ab} & \delta^{ab} - (D_2\mathcal{D}^{-1}D_2)^{ab} \end{pmatrix} \qquad (a,b = 1,\ldots,N^2-1).
\tag{C.1}
$$

Introducing the relation given by

$$
\det\begin{pmatrix} 0 & \mathbb{I}_{N^2-1} \\ \mathbb{I}_{N^2-1} & 0 \end{pmatrix} = (-1)^{N^2-1},
\tag{C.2}
$$

where $\mathbb{I}_{N^2-1}$ is the $(N^2-1)\times(N^2-1)$ unit matrix, we have

$$
\begin{aligned}
(-1)^{N^2-1}\det(M) &= \det\left[\begin{pmatrix} 0 & \mathbb{I}_{N^2-1} \\ \mathbb{I}_{N^2-1} & 0 \end{pmatrix} M\right]\\
&= \det\begin{pmatrix} -D_2\mathcal{D}D_1 & \mathbb{I}_{N^2-1} - D_2\mathcal{D}^{-1}D_2 \\ \mathbb{I}_{N^2-1} - D_1\mathcal{D}^{-1}D_1 & -D_1\mathcal{D}^{-1}D_2 \end{pmatrix}.
\end{aligned}
\tag{C.3}
$$

With help of the formula

$$
\det\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ D & \mathbb{I} \end{pmatrix}\begin{pmatrix} \mathbb{I} & A^{-1}C \\ 0 & B - DA^{-1}C \end{pmatrix} = \det(A)\times\det(B - DA^{-1}C),
\tag{C.4}
$$

Eq. (C.3) becomes

$$
\begin{aligned}
&\det(-D_2\mathcal{D}D_1)\det\Big(-D_1\mathcal{D}^{-1}D_2-(\mathbb{I}_{N^2-1}-D_1\mathcal{D}^{-1}D_1)(-D_2\mathcal{D}^{-1}D_1)^{-1}(\mathbb{I}_{N^2-1}-D_2\mathcal{D}^{-1}D_2)\Big)\\
&=\det\Big(D_2\mathcal{D}^{-1}(D_1)^2\mathcal{D}^{-1}D_2-(D_2\mathcal{D}^{-1}D_1)(\mathbb{I}_{N^2-1}-D_1\mathcal{D}^{-1}D_1)(D_1^{-1}\mathcal{D}D_2^{-1})(\mathbb{I}_{N^2-1}-D_2\mathcal{D}^{-1}D_2)\Big)\\
&=\det\Big(D_2\mathcal{D}^{-1}(D_1)^2\mathcal{D}^{-1}D_2-\mathbb{I}_{N^2-1}+D_2\mathcal{D}^{-1}D_2+D_2\mathcal{D}^{-1}(D_1)^2D_2^{-1}-D_2\mathcal{D}^{-1}(D_1)^2\mathcal{D}^{-1}D_2\Big)\\
&=(-1)^{N^2-1}\det(D_2^{-1}D_2)\det\Big(\mathbb{I}_{N^2-1}-D_2\mathcal{D}^{-1}D_2-D_2\mathcal{D}^{-1}(D_1)^2D_2^{-1}\Big)\\
&=(-1)^{N^2-1}\det\Big(D_2^{-1}\big(\mathbb{I}_{N^2-1}-D_2\mathcal{D}^{-1}D_2-D_2\mathcal{D}^{-1}(D_1)^2D_2^{-1}\big)D_2\Big)\\
&=(-1)^{N^2-1}\det(\mathcal{D}^{-1}(\partial_3)^2),
\end{aligned}
\tag{C.5}
$$

where we have used $\mathcal{D}=(D_1)^2+(D_2)^2+(\partial_3)^2$ in the last line in Eq. (C.5). Hence, we obtain that

$$
\det(M)=\det\big(\mathcal{D}^{-1}(\partial_3)^2\big),
\tag{C.6}
$$

which is Eq. (3.11).

C.2. Coulomb gauge

Instead of evaluating $\det(N)$ directly, let us consider the determinant defined by

$$
\det\left(\begin{pmatrix}0 & \mathbb{I}_{N^2-1}\\ \mathbb{I}_{N^2-1} & 0\end{pmatrix}N'\right),
\tag{C.7}
$$

where N' is given by Eq. (B.8) and is written as

$$
(N')^{ab}_{kl}=\begin{pmatrix}\mathbb{I}_{N^2-1}-\tilde{C}'_1\mathcal{D}^{-1}C_1 & -\tilde{C}'_1\mathcal{D}^{-1}C_2\\ -\tilde{C}'_2\mathcal{D}^{-1}C_1 & \mathbb{I}_{N^2-1}-\tilde{C}'_2\mathcal{D}^{-1}C_2\end{pmatrix}.
\tag{C.8}
$$

Equation (C.7) becomes, by using the formula (C.4),

$$
\begin{aligned}
(\text{C.7})&=\det\begin{pmatrix}-\tilde{C}'_2\mathcal{D}^{-1}C_1 & \mathbb{I}_{N^2-1}-\tilde{C}'_2\mathcal{D}^{-1}C_2\\ \mathbb{I}_{N^2-1}-\tilde{C}'_1\mathcal{D}^{-1}C_1 & -\tilde{C}'_1\mathcal{D}^{-1}C_2\end{pmatrix}\\
&=\det\Big(-\tilde{C}'_2\mathcal{D}^{-1}C_1\Big)\\
&\quad\times\det\Big(-\tilde{C}'_1\mathcal{D}^{-1}C_2-(\mathbb{I}_{N^2-1}-\tilde{C}'_1\mathcal{D}^{-1}C_1)(-\tilde{C}'_2\mathcal{D}^{-1}C_1)^{-1}(\mathbb{I}_{N^2-1}-\tilde{C}'_2\mathcal{D}^{-1}C_2)\Big)\\
&=\det\Big(-\mathbb{I}_{N^2-1}+\tilde{C}'_2\mathcal{D}^{-1}C_2+\tilde{C}'_2\mathcal{D}^{-1}C_1\tilde{C}'_1(\tilde{C}'_2)^{-1}\Big)\\
&=(-1)^{N^2-1}\det(\tilde{C}'^{-1}_2\tilde{C}'_2)\det\Big(\mathbb{I}_{N^2-1}-\tilde{C}'_2\mathcal{D}^{-1}C_2-\tilde{C}'_2\mathcal{D}^{-1}C_1\tilde{C}'_1(\tilde{C}'_2)^{-1}\Big)\\
&=(-1)^{N^2-1}\det(\mathbb{I}_{N^2-1}-\mathcal{D}^{-1}C_2\tilde{C}'_2-\mathcal{D}^{-1}C_1\tilde{C}'_1)\\
&=(-1)^{N^2-1}\det(\mathcal{D}^{-1})\det(\mathcal{D}-C_2\tilde{C}'_2-C_1\tilde{C}'_1)\\
&=(-1)^{N^2-1}\det(\mathcal{D}^{-1})\det(\mathcal{D}-C_k\tilde{C}'_k)\\
&=(-1)^{N^2-1}\det(\mathcal{D}^{-1})\det(\tilde{\Delta})\\
&=(-1)^{N^2-1}\det(\mathcal{D}^{-1})\det(\Delta^{-1})(\det(\partial_iD_i))^2,
\end{aligned}
\tag{C.9}
$$

where we have used Eq. (B.3) and $\tilde{\Delta} = \partial \cdot D\Delta^{-1}\partial \cdot D$. Hence, we obtain that

$$\det\left(\begin{pmatrix} 0 & \mathbb{I}_{N^2-1} \\ \mathbb{I}_{N^2-1} & 0 \end{pmatrix} N'\right) = (-1)^{N^2-1}\det(\mathcal{D}^{-1})\det(\Delta^{-1})(\det(\partial_i D_i))^2. \tag{C.10}$$

On the other hand, it is easy to see that

$$\det\big(\delta_{kl} - \partial_k\Delta^{-1}\partial_l\big) = \det\begin{pmatrix} 1-\Delta^{-1}(\partial_1)^2 & -\Delta^{-1}\partial_1\partial_2 \\ -\Delta^{-1}\partial_2\partial_1 & 1-\Delta^{-1}(\partial_2)^2 \end{pmatrix} = \det(\Delta^{-1}(\partial_3)^2), \tag{C.11}$$

so that we have

$$\begin{aligned}\det\left(\begin{pmatrix} 0 & \mathbb{I}_{N^2-1} \\ \mathbb{I}_{N^2-1} & 0 \end{pmatrix} N'\right) &= (-1)^{N^2-1}\det(N') \overset{(B.6)}{=} (-1)^{N^2-1}\det\Big(\big(\delta_{kl} - \partial_k\Delta^{-1}\partial_l\big)N\Big) \\ &= (-1)^{N^2-1}\det(\Delta^{-1}(\partial_3)^2)\det(N).\end{aligned} \tag{C.12}$$

From Eqs. (C.10) and (C.12), we obtain that

$$\det(N) = \frac{\det(\mathcal{D}^{-1})(\det(\partial_i D_i))^2}{\det((\partial_3)^2)}. \tag{C.13}$$

We have thus proved Eq. (5.22).

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