Novel $|V_{us}|$ Determination Using Inclusive Strange τ Decay and Lattice Hadronic Vacuum Polarization Functions

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We propose and apply a new approach to determining $|V_{us}|$ using dispersion relations with weight functions having poles at Euclidean (spacelike) momentum which relate strange hadronic τ decay distributions to hadronic vacuum polarization (HVP) functions obtained from lattice quantum chromodynamics. We show examples where spectral integral contributions from the region where experimental data have large errors or do not exist are strongly suppressed but accurate determinations of the relevant lattice HVP combinations remain possible. The resulting $|V_{us}|$ agrees well with determinations from K physics and three-family Cabibbo-Kobayashi-Maskawa unitarity. Advantages of this new approach over the conventional hadronic τ decay determination employing flavor-breaking sum rules are also discussed.

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Introduction.—Precise determinations of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{us}|$ are important in the context of three-family unitarity tests and searches for physics beyond the standard model (SM). The most precise such determination, $|V_{us}| = 0.2253(7)$, is from $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, using the lattice input $f_K/f_\pi = 1.193(3)$ [1–4]. Three-family unitarity and $|V_{ud}| = 0.97417(21)$ [5], similarly, imply $|V_{us}| = 0.2258(9)$, whereas K_{l3}, with lattice $f_+(0)$ input, yields $|V_{us}| = 0.2237(10)$ [6]. It is a long-standing puzzle that conventional flavor-breaking (FB) finite-energy sum rules (FESRs) employing hadronic τ decay data yield much lower $|V_{us}|$, most recently 0.2186 (21) [7] [0.2207(27) when $\Gamma[K_{\mu 2}]$ and dispersive $K\pi$ form factor constraints are incorporated [8]].

The conventional FB FESR implementation employs assumptions for unknown dimension $D = 6$ and 8 operator product expansion (OPE) condensates which turn out to fail self-consistency tests [9]. An alternate implementation, fitting $D > 4$ condensates to data, yields results passing these tests and compatible with determinations from other sources [9]. The resulting error is dominated by uncertainties on the relevant weighted inclusive flavor us spectral integrals and a factor > 2 larger than that of K-decay-based approaches. Improved branching fractions (BFs) used in normalizing low-multiplicity us exclusive-mode Belle and *BABAR* distributions would help, but ~25% errors on higher-multiplicity us "residual mode" contributions [10], involving modes not remeasured at the B factories, preclude a factor of 2 improvement [9,11].

This Letter presents a novel dispersive approach to determining $|V_{us}|$ using inclusive strange hadronic τ decay data, hadronic vacuum polarization (HVP) functions computed on the lattice, and weight functions, $\omega_N(s) = \Pi_{k=1}^N (s + Q_k^2)^{-1}$, $Q_k^2 > 0$, having poles at Euclidean $Q^2 = Q_k^2 > 0$. We show examples of such ω_N which strongly suppress spectral contributions from the high-multiplicity us "residual" region without blowing up errors on the related lattice HVP combinations. The approach yields $|V_{us}|$ in good agreement with K-decay analysis results and three-family CKM unitarity expectations. The lattice error is comparable to the experimental one, and the total error is less than that of the inclusive FB FESR τ decay determination.

New inclusive determination.—The conventional inclusive FB τ decay determination is based on the FESR relation [12,13]

$$\int_0^{s_0} \omega(s)\Delta\rho(s)ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} \omega(s)\Delta\Pi(-s)ds, \quad (1)$$

connecting, for any s_0 and analytic $\omega(s)$, the relevant FB combination, $\Delta\Pi(-s) = \Pi_{us}(-s) - \Pi_{ud}(-s)$, of spin $J = 0$, 1 HVPs and associated spectral function $\Delta\rho(s) = (1/\pi)\mathrm{Im}\Delta\Pi(-s)$. Experimental data are used on the LHS and, for large enough s_0, the OPE on the RHS. In the SM, the differential distribution, $dR_{V/A;ij}/ds$, associated with the flavor $ij = ud$, us vector (V) or axial vector (A) current-induced decay ratio $R_{V/A;ij} = \Gamma[\tau^- \to \nu_\tau \mathrm{hadrons}_{V/A;ij}]/\Gamma[\tau^- \to e^-\bar{\nu}_e\nu_\tau]$, is related to the $J = 0$, 1 spectral functions $\rho^{(J)}_{ij;V/A}(s)$, by [14]

$$\frac{dR_{ij;V/A}}{ds} = \frac{12\pi^2|V_{ij}|^2 S_{\mathrm{EW}}}{m_\tau^2} \times \left[\omega_\tau(s)\rho^{(0+1)}_{ij;V/A}(s) - \omega_L(y_\tau)\rho^{(0)}_{ij;V/A}(s)\right], \quad (2)$$

where $y_\tau = s/m_\tau^2$, $\omega_\tau(y) = (1-y)^2(1+2y)$, $\omega_L(y) = 2y(1-y)^2$, and S_{EW} is a known short-distance electroweak correction [15,16]. Experimental $dR_{ij;V/A}/ds$ distributions thus determine, up to factors of $|V_{ij}|^2$, combinations of the $\rho^{(J)}_{ij;V/A}$.

The low $|V_{us}|$ noted above results from a conventional implementation [17] of Eq. (1) which employs fixed $s_0 = m_\tau^2$ and $\omega = \omega_\tau$ and assumptions for experimentally unknown $D = 6$ and 8 condensates. With $s_0 = m_\tau^2$ and $\omega = \omega_\tau$, inclusive nonstrange and strange BFs determine the ud and us spectral integrals. Testing $D = 6$ and 8 assumptions by varying s_0 and/or ω, however, yields $|V_{us}|$ with significant unphysical s_0- and ω dependence, motivating an alternate implementation employing variable s_0 and ω which allows a simultaneous fit of $|V_{us}|$ and the $D > 4$ condensates. Significantly larger (now stable) $|V_{us}|$ are found, the conventional implementation results $|V_{us}| = 0.2186(21)$ [7] and 0.2207(27) [8], shifting up to 0.2208 (23) and 0.2231(27) [9], respectively, with the new implementation. us spectral integral uncertainties dominate the error, with current ~25% residual mode contribution errors precluding a competitive determination [9].

Motivated by this limitation, we switch to generalized dispersion relations involving the experimental us $V + A$ inclusive distribution and weights, $\omega_N(s) \equiv \prod_{k=1}^N (s + Q_k^2)^{-1}$, $0 < Q_k^2 < Q_{k+1}^2$, having poles at $s = -Q_k^2$. From Eq. (2), $dR_{us;V+A}/ds$ directly determines $|V_{us}^2|\tilde{\rho}_{us}(s)$, with

$$\tilde{\rho}_{us}(s) \equiv \left(1 + 2\frac{s}{m_\tau^2}\right)\rho^{(1)}_{us;V+A}(s) + \rho^{(0)}_{us;V+A}(s). \quad (3)$$

For $N \geq 3$, the associated HVP combination

$$\tilde{\Pi}_{us} \equiv \left(1 - 2\frac{Q^2}{m_\tau^2}\right)\Pi^{(1)}_{us;V+A}(Q^2) + \Pi^{(0)}_{us;V+A}(Q^2) \quad (4)$$

satisfies the convergent dispersion relation

$$\int_0^\infty \tilde{\rho}_{us}(s)\omega_N(s)ds = \sum_{k=1}^N \underset{s=-Q_k^2}{\mathrm{Res}}[\tilde{\Pi}_{us}(-s)\omega_N(s)] = \sum_{k=1}^N \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}(Q_j^2 - Q_k^2)} \equiv \tilde{F}_{\omega_N}. \quad (5)$$

With $\tilde{\Pi}_{us}(Q_k^2)$ measured on the lattice, $dR_{us;V+A}/ds$ used to fix $s < m_\tau^2$ spectral integral contributions, and $s > m_\tau^2$ contributions approximated using perturbative quantum chromodynamics (pQCD), one has

$$|V_{us}| = \sqrt{\tilde{R}_{us;w_N}/\left(\tilde{F}_{\omega_N} - \int_{m_\tau^2}^\infty \tilde{\rho}^{\mathrm{pQCD}}_{us}(s)\omega_N(s)ds\right)}, \quad (6)$$

where $\tilde{R}_{w_N} \equiv (m_\tau^2/12\pi^2 S_{\mathrm{EW}})\int_0^{m_\tau^2}[1/(1-y_\tau)^2][dR_{us;V+A}(s)/ds]\omega_N(s)ds$.

Choosing uniform pole spacing Δ, ω_N can be characterized by Δ, N, and the pole-interval midpoint, $C = (Q_1^2 + Q_N^2)/2$. With large enough N, and all Q_k^2 below ~1 GeV^2, spectral integral contributions from $s > m_\tau^2$ and the higher-s, larger-error part of the experimental distribution can be strongly suppressed. Increasing N lowers the error of the LHS in Eq. (5) but increases the relative RHS error. With results insensitive to modest changes of Δ, we fix $\Delta = 0.2/(N-1)$ GeV^2, ensuring ω_N with the same C but different N have poles spanning the same Q^2 range. C and N are varied to minimize the error on $|V_{us}|$.

We employ the following us spectral input: $K_{\mu 2}$ or $\tau \to K\nu_\tau$ [7] for K pole contributions, unit-normalized Belle or *BABAR* distributions for $K\pi$ [18,19], $K^-\pi^+\pi^-$ [20], $\bar{K}^0\pi^-\pi^0$ [21] and $\bar{K}\bar{K}K$ [22,23], the most recent Heavy Flavor Averaging Group (HFLAV) BFs [7], and 1999 ALEPH results [10], modified for current BFs, for the residual mode distribution. Multiplication of a unit-normalized distribution by the ratio of corresponding exclusive mode to electron BFs converts that distribution to the corresponding contribution to $dR_{us;V+A}(s)/ds$. The dispersively constrained $K\pi$ BFs of Ref. [8] (ACLP) provide an alternate $K\pi$ normalization. In what follows, we illustrate the lattice approach using the HFLAV $K\pi$ normalization. Alternate results using the ACLP normalization are given in Ref. [24].

Lattice calculation method.—We compute the two-point functions of the flavor us V and A currents, $J^\mu_{V/A}(\vec{x}, t) = J^\mu_{V/A}(x)$, $\mu = x, y, z, t$, via

$$C^{\mu\nu}_{V/A}(t) = \sum_{\vec{x}} \langle J^{\nu}_{V/A}(\vec{x},t)[J^{\mu}_{V/A}(0,0)]^{\dagger}\rangle. \qquad (7)$$

The continuum spin $J = 0$, 1, us HVPs, $\Pi^{(J)}_{us;V/A} \equiv \Pi^{(J)}_{V/A}$, are related to the two-point functions by

$$\sum_x e^{iqx}\langle J^{\mu}_{V/A}(x)[J^{\nu}_{V/A}(0)]^{\dagger}\rangle$$
$$= (q^2 g^{\mu\nu} - q^{\mu}q^{\nu})\Pi^{(1)}_{V/A}(q^2) + q^{\mu}q^{\nu}\Pi^{(0)}_{V/A}(q^2), \quad (8)$$

up to finite volume (FV) and discretization corrections. With $C^{(1)}_{V/A}(t) = \frac{1}{3}\sum_{k=x,y,z} C^{kk}_{V/A}(t)$, $C^{(0)}_{V/A}(t) = C^{tt}_{V/A}(t)$, the analogous $J = 0$, 1 parts of $C^{\mu\nu}_{V/A}$ are [25,26]

$$\Pi^{(J)}_{V/A}(Q^2) - \Pi^{(J)}_{V/A}(0) = \sum_t K(q,t)C^{(J)}_{V/A}(t), \qquad (9)$$

with $K(q,t) = (\cos qt - 1/\hat{q}^2) + \frac{1}{2}t^2$ and $\hat{q}$ the lattice momentum, $\hat{q}_\mu = 2\sin q_\mu/2$. FV corrections to this infinite volume result are discussed below. We use lattice HVPs measured on the near-physical quark mass, $2+1$ flavor $48^3 \times 96$ and $64^3 \times 128$ Möbius domain wall fermion ensembles of the RBC and UKQCD collaborations [27], employing all-mode averaging (AMA) [28,29] to reduce costs. Slight u, d, s mass mistunings are corrected by measuring the HVPs with partially quenched (PQ) physical valence quark masses [27], also using AMA.

$\tilde{F}_{\omega_N}$ in Eq. (5) can be decomposed into four contributions, $\tilde{F}^{(J)}_{V/A;\omega_N}$, labeled by the spin $J = 0$ or 1, and current type, V or A. $\tilde{F}^{(J)}_{V/A;\omega_N} = \lim_{t\to\infty} L^{(J)}_{V/A;\omega_N}(t)$, where $L^{(J)}_{V/A;\omega_N}(t) = \sum_{l=-t}^{t}\omega^{(J)}_N(l)C^{(J)}_{V/A}(l)$. From Eqs. (9) and (4)

$$\omega^{(1)}_N = \sum_{k=1}^{N} K\left(\sqrt{Q_k^2}, t\right)\left(1 - \frac{2Q_k^2}{m_\tau^2}\right)\underset{s=-Q_k^2}{\mathrm{Res}}[\omega_N(s)],$$
$$\omega^{(0)}_N = \sum_{k=1}^{N} K\left(\sqrt{Q_k^2}, t\right)\underset{s=-Q_k^2}{\mathrm{Res}}[\omega_N(s)]. \qquad (10)$$

With finite lattice temporal extent, finite-time effects may exist. Increasing N increases the level of cancellation and relative weight of large-t contributions on the RHS of Eq. (5). The restrictions $0.1\ \mathrm{GeV}^2 < C < 1\ \mathrm{GeV}^2$ and $N \le 5$, chosen to strongly suppress large-t contributions, allow us to avoid modeling the large-t behavior. Figure 1 shows, as an example, the large-t plateaus of the partial sums $L^{(J)}_{V/A;\omega}(t)$, obtained in all four channels, for $N = 4$, $C = 0.5\ \mathrm{GeV}^2$ on the $48^3 \times 96$ ensemble.

The upper panel of Fig. 2 shows the relative sizes of the four C-dependent lattice contributions, $V^{(J)}$, $A^{(J)}$, for $N = 4$. The lower panel, similarly, shows the relative sizes of different contributions to the weighted us spectral integrals. $K\pi$ denotes the sum of $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions, $pQCD$ the contribution from $s > m_\tau^2$, evaluated using the five-loop-truncated pQCD form [30,31]. Varying C (and N) varies the level of suppression of the pQCD and higher-multiplicity contributions, the relative size of K and $K\pi$ contributions, and hence the level of "inclusiveness" of the analysis. The stability of $|V_{us}|$ under such variations provides additional systematic cross-checks.

Analysis and results.—The $A^{(0)}$ channel produces the largest RHS contribution to Eq. (5). On the LHS, the K pole dominates $\rho^{(0)}_{us;A}(s)$, with continuum contributions doubly chirally suppressed. Estimated LHS continuum $A^{(0)}$ contributions, obtained using sum-rule $K(1460)$ and $K(1830)$

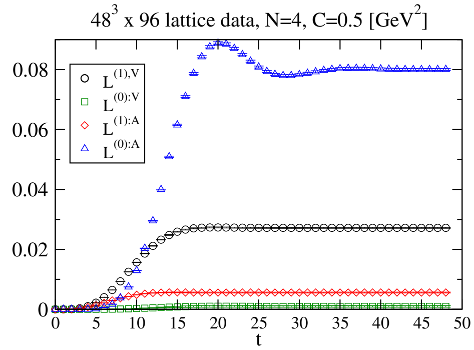

FIG. 1. Partial sum of the residues $L^{(J)}_{V/A;\omega_N}(t)$.

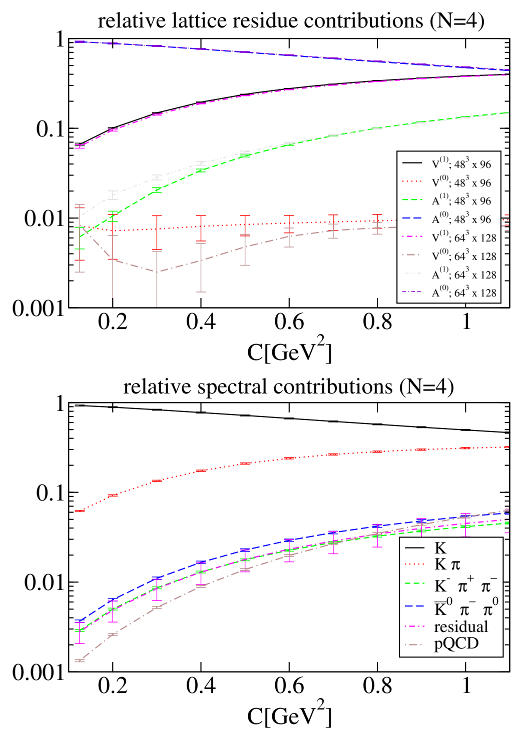

FIG. 2. Relative contributions of $J = 0$, 1, V and A channels to the sum of lattice residues (upper panel), and different (semi-) exclusive modes to the weighted us spectral integrals (lower panel), as a function of C, for $N = 4$.

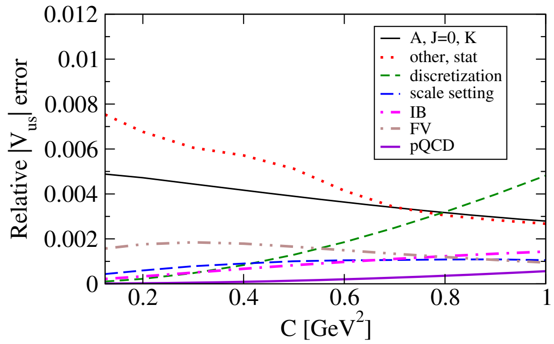

FIG. 3. Lattice $|V_{us}|$ error contributions for $N = 4$.

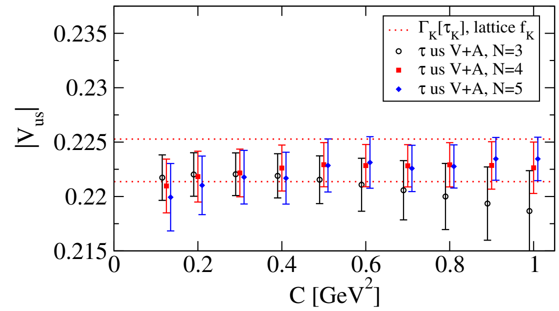

FIG. 4. $|V_{us}|$ vs C for $N = 3, 4, 5$. $N = 3, 5$ results are shifted horizontally for presentational clarity and statistical and systematic errors added in quadrature. The determination using $\gamma[\tau_K]$ and lattice f_K is shown for comparison.

decay constant results [32], are numerically negligible for the ω_N we employ. An "exclusive" $A^{(0)}$ analysis relating $\tilde{F}^{A^{(0)}}_{w_N}$ to the K-pole contribution $\tilde{R}^K_{w_N} = \gamma_K \omega_N(m_K^2)$ is, therefore, possible, with $\gamma_K = 2|V_{us}|^2 f_K^2$ obtained from either $K_{\mu 2}$ or $\Gamma[\tau \to K\nu_\tau]$. Because the simulations underlying $\tilde{F}^{A^{(0)}}_{w_N}$ are isospin symmetric, we correct γ_K for leading-order electromagnetic (EM) and strong isospin-breaking (IB) effects [4,8]. With PDG τ lifetime [6] and HFLAV $\tau \to K\nu_\tau$ BF [7] input, $\gamma_K[\tau_K] = 0.0012061(167)_{\rm exp}(13)_{\rm IB}$ GeV^2. $\gamma_K[\tau_K]$ is employed in our main, fully τ-based analysis. The more precise result $\gamma_K[K_{\mu 2}] = 0.0012347(29)_{\rm exp}(22)_{\rm IB}$ [6] from $\Gamma[K_{\mu 2}]$ can also be used if SM dominance is assumed. Exclusive analysis $|V_{us}|$ results are independent of C for $C < 1$ GeV^2 (confirming tiny continuum $A^{(0)}$ contributions) and agree with the results, $|V_{\rm us}| = 0.2233(15)_{\rm exp}(12)_{\rm th}$ and $0.2260(3)_{\rm exp}(12)_{\rm th}$, obtained using $|V_{us}| = \sqrt{\gamma_K/(2f_K^2)}$, the isospin-symmetric lattice result $F_K \equiv \sqrt{2} f_K = 0.15551(83)$ GeV [27] and $\gamma_K = \gamma_K[\tau_K]$ and $\gamma_K[K_{\mu 2}]$, respectively. See Ref. [24] for further details.

For the fully inclusive analysis, statistical and systematic uncertainties are reduced by employing $2f_K^2\omega_N(m_K^2)$, with measured f_K, for the K pole $A^{(0)}$ channel contribution. The residual, continuum $A^{(0)}$ contributions are compatible with zero within errors, as anticipated above. IB corrections, beyond those applied to γ_K, are numerically relevant only for $K\pi$. We account for (i) π^0-η mixing, (ii) EM effects, and (iii) IB in the phase space factor, with π^0-η mixing numerically dominant, evaluating these corrections, and their uncertainties, from the results presented in Ref. [8]. A 2% uncertainty, estimated using results from a study of duality violations in the $SU(3)_F$-related flavor ud channels [33], is assigned to pQCD contributions. Because our analysis is optimized for ω_N strongly suppressing higher-multiplicity and $s > m_\tau^2$ contributions, such an uncertainty plays a negligible role in our final error.

Several systematic uncertainties enter the lattice computation. With an assumed continuum extrapolation linear in a^2 but only two lattice spacings, $\mathcal{O}(a^4)$ discretization uncertainties must be estimated. For the ω_N we employ, the two ensembles yield $\tilde{F}_{\omega_N}$ differing by less than (typically significantly less than) 10%, compatible with $\sim Ca^2$ or smaller $\mathcal{O}(a^2)$ errors. Anticipating a further $\sim Ca^2$ reduction of $\mathcal{O}(a^4)$ relative to $\mathcal{O}(a^2)$ corrections, we estimate residual $\mathcal{O}(a^4)$ continuum extrapolation uncertainties to be $\sim 0.1 Ca_f^2$, with $a_f^{-1} = 2.36$ GeV [27] the smaller of the two lattice spacings. We also take into account the lattice scale setting uncertainty. The dominant FV effect is expected to come from $K\pi$ loop contributions in the $V^{(1)}$ channel, which we estimate using a lattice regularized version of finite-volume chiral perturbation theory (ChPT). It is known, from Ref. [34], that one-loop ChPT for HVPs involving the light u, d quarks yields a good semiquantitative representation of observed FV effects [35]; we thus expect it to also work well for the flavor us case considered here, where FV effects involving the heavier s quark should be suppressed relative to those in the purely light u, d quark sector. The result of our one-loop ChPT estimate is a 1% FV correction. We thus assign a 1% FV uncertainty to our $V^{(1)}$ channel contributions [37]. Regarding the impact of the slight u, d, s sea-quark mass mistunings on the PQ results, the shift from slightly mistuned unitary to PQ shifted-valence-mass results for $\tilde{F}(\omega_N)$ corresponds to shifts in $|V_{us}|$ of $< 0.4\%$ for both ensembles. With masses and decay constants typically much less sensitive to sea-quark mass shifts than to the

TABLE I. Sample relative spectral integral contributions.

Contribution	Value [%]			
[N, C(GeV^2)]	[3, 0.3]	[3, 1]	[4, 0.7]	[5, 0.9]
K	65.5	30.9	61.7	66.9
$K\pi$	21.4	28.6	26.4	25.2
$K^-\pi^+\pi^-$	2.4	5.6	2.8	2.1
$\bar{K}^0\pi^-\pi^0$	3.1	7.3	3.6	2.7
Residual	2.7	6.8	2.9	2.1
pQCD	4.9	20.8	2.7	1.1

TABLE II. Error budget for the inclusive $|V_{us}|$ determination.

Contribution		Relative error (%)			
	[N, C(GeV^2)]	[3, 0.3]	[3, 1]	[4, 0.7]	[5, 0.9]
Theory	f_K	0.37	0.20	0.34	0.36
	Others, stat.	0.41	0.19	0.34	0.41
	Discretization	0.10	0.80	0.25	0.27
	Scale setting	0.09	0.08	0.11	0.11
	IB	0.10	0.21	0.11	0.10
	FV	0.10	0.04	0.13	0.18
	pQCD	0.05	0.26	0.03	0.01
	Total	0.59	0.91	0.58	0.65
Experiment	K	0.48	0.27	0.44	0.47
	$K\pi$	0.20	0.32	0.23	0.22
	$K^-\pi^+\pi^-$	0.06	0.16	0.06	0.05
	$\bar{K}^0\pi^-\pi^0$	0.03	0.09	0.03	0.03
	Residual	0.41	1.35	0.41	0.28
	Total	0.66	1.43	0.65	0.59
Combined	Total	0.88	1.70	0.87	0.88

same valence-quark mass shifts, we expect sea-mass PQ effects to be at the sub-$\sim$0.1% level, and hence negligible on the scale of the other errors in the analysis.

Figure 3 shows the C dependence of relative, nondata, inclusive analysis error contributions. K labels the f_K-induced $A^{(0)}$ uncertainty, *other* that induced by the statistical error on the sum of $V^{(1)}$, $V^{(0)}$, $A^{(1)}$, and tiny continuum $A^{(0)}$ channel contributions. The statistical error dominates for low C, the discretization error for large C.

Figure 4 shows our $|V_{us}|$ results. These agree well for different N, and $C < 1\ \mathrm{GeV}^2$. The slight trend toward lower central values for weights less strongly suppressing high-s spectral contributions ($N = 3$ and higher C) suggests the residual mode distribution may be somewhat underestimated due to missing higher-multiplicity contributions. Such missing high-s strength would also lower the $|V_{us}|$ obtained from FB FESR analyses. Table I lists relative spectral integral contributions for selected ω_N. Note the significantly larger (6.8% and 21%) residual mode and pQCD contributions for $N = 3$ and $C = 1\ \mathrm{GeV}^2$. Restricting C to $< 1\ \mathrm{GeV}^2$ keeps these from growing further and helps control higher-order discretization errors. The error budget for various sample weight choices is summarized in Table II.

Our optimal inclusive determination is obtained for $N = 4$, $C = 0.7\ \mathrm{GeV}^2$, where residual mode and pQCD contributions are highly suppressed, and yields results

$$|V_{us}| = \begin{cases} 0.2228(15)_{\mathrm{exp}}(13)_{\mathrm{th}}, & \text{for } \gamma_K[\tau_K] \\ 0.2245(11)_{\mathrm{exp}}(13)_{\mathrm{th}}, & \text{for } \gamma_K[K_{\mu 2}], \end{cases} \tag{11}$$

consistent with determinations from K physics and three-family unitarity. Theoretical (lattice) errors are comparable to experimental ones, and combined errors improve on those of the corresponding inclusive FB FESR determinations. A comparison to the results of other determinations is given in Fig. 5.

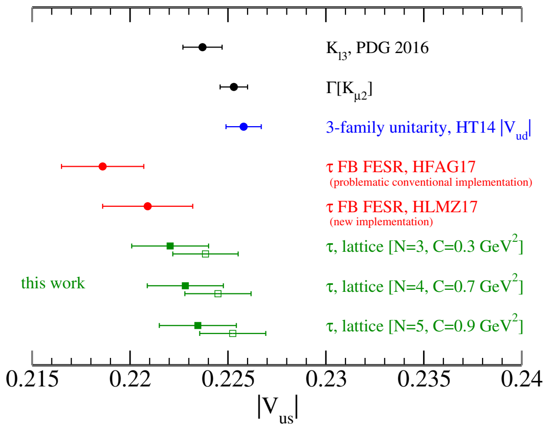

FIG. 5. Our $|V_{us}|$ determinations [inclusive, $\gamma_K[\tau_K]$-based exclusive (filled square) and $\gamma_K[K_{\mu 2}]$-based exclusive (empty square)] c.f. results from other sources.

Conclusion and discussion.—We have presented a novel method for determining $|V_{us}|$ using inclusive strange hadronic τ decay data. Key advantages over the related FB FESR approach employing the same us data are (i) the use of systematically improvable precision lattice data in place of the OPE, and (ii) the existence of weight functions that more effectively suppress spectral contributions from the larger-error, high-s region without blowing up theory errors. The results provide not only the most accurate inclusive τ decay sum rule determination of $|V_{us}|$ but also evidence that high-s region systematic errors may be underestimated in the alternate FB FESR approach. The combined experimental uncertainty can be further reduced through improvements to the experimental $\tau \to K, K\pi$ BFs, whereas the largest of the current theoretical errors, that due to lattice statistics, is improvable by straightforward lattice computational effort. Such future improvements will help constrain the flavor dependence of any new physics contributions present in hadronic tau decays, contributions expected to be present at some level if the apparent violation of lepton flavor universality seen recently in semileptonic $b \to c$ decays involving τ persists.

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