PTEP

Prog. Theor. Exp. Phys. **2014**, 091B01 (6 pages)
DOI: 10.1093/ptep/ptu115

Letter

Chiral anomaly for V-A fields in four- and six-dimensional curved space

Satoshi Yajima*, Kohei Eguchi, Makoto Fukuda, Tomonori Oka, Hideo Taira, and Shinji Yamashita

Department of Physics, Kumamoto University, 2-39-1 Kurokami, Chuo-ku, Kumamoto 860-8555, Japan
*E-mail: yajima@sci.kumamoto-u.ac.jp

Received May 13, 2014; Revised July 15, 2014; Accepted July 20, 2014; Published September 5, 2014

The chiral U(1) anomalies associated with a fermion of spin $\frac{1}{2}$ interacting with nonabelian vector and axial-vector fields in four- and six-dimensional curved space are given in tensorial form.

Subject Index B31

1. *Introduction* The chiral U(1) anomaly has been derived by calculating some Feynman triangle diagrams of fermions in four-dimensional quantum electrodynamics [1,2] and is studied in quantum field theory because it is a fruitful topic. The anomaly is obtained from the chiral transformation of the Euclidean path integral measure for gauge theories with fermions [3,4]. The derivation of the anomalous axial-vector Ward–Takahashi identities in the method has attracted some attention [5]. The anomaly is related to the chiral magnetic effect and topological insulators in condensed matter physics [6–9].

In quantum field theory, several fermionic loop corrections are perturbatively described by a fermion propagator with a background field. The chiral U(1) anomaly in four-dimensional curved space [10] has been obtained using the heat kernel [11], by which the propagator should be given. Some anomalies for chiral fermions interacting with gauge fields in higher even-dimensional curved space can be calculated using this method. The chiral U(1) anomaly in six dimensions has topologically similar features to that in ten dimensions because of the index theorem [12,13].

In $d = 10$, $N = 1$ supergravity [14,15], the Rarita–Schwinger field ψ_μ describes a gravitino by fixing a gauge suitably, and the evaluation of the anomalies requires the heat kernel for a fermion with spin $\frac{3}{2}$ [16–20]. However, by regarding the vector index "μ" of ψ_μ as that of the representation matrices of gauge group SO(10), the heat kernel for a spinor with spin $\frac{3}{2}$ can be expressed by that for a spinor with spin $\frac{1}{2}$. Moreover, the Lagrangian in the supergravity contains not only the minimal interaction between the gravitino and the gravitational fields but also four-gravitino interactions. In the fermionic one-loop diagram, the four-gravitino vertex is connected by two internal and two external lines. Then, by regarding the two fermion external lines as a boson line, the four-gravitino interaction is treated as two-gravitino interactions with bosonic background fields, which are odd-order tensors. The totally antisymmetric tensor of the highest order $2n - 1$ in even $2n$ dimensions is rewritten as the axial-vector by contracting the Levi–Civita symbol with its tensor.

Funded by SCOAP[3]

A fermion on which the projection matrix $(1 \pm \gamma_{2n+1})/2$ acts is expressed by the Weyl spinor with either positive or negative chirality, which corresponds to an eigenstate with right- or left-handed helicity in massless fermions, respectively. In the Dirac operator of the Weyl fermion, the gauge fields are separated into two types of boson by the projection. However, we must note the Hermiticity of the bosonic coupling in the Dirac operator $\not{D}$ because, to evaluate the correctness of the result of the anomaly, it is necessary to calculate by replacing $\not{D}^2$ with $\not{D}\not{D}^\dagger + \not{D}^\dagger\not{D}$; if the Dirac operator is not Hermitian, $\not{D}^\dagger \neq \not{D}$. In contrast, the gauge bosons in $\not{D}$ and $\not{D}^\dagger$ are described by a suitable linear combination of the vector (V) and axial-vector (A) fields. The chiral U(1) anomaly for the Weyl fermion is half of that for the Dirac fermion, up to the sign of the chirality of the Weyl fermion. Therefore, it is simple to begin the derivation of the anomaly with the Hermitian gauge couplings in the Dirac operator. In this article, we consider the chiral U(1) anomalies for the Dirac fermion of spin $\frac{1}{2}$ interacting with Hermitian and nonabelian V-A fields in four- and six-dimensional curved space.

2. *Heat kernel* The heat kernel $K^{(d)}(x, x')$ for a fermion of spin $\frac{1}{2}$ in d dimensions is defined by

$$\frac{\partial}{\partial t} K^{(d)}(x, x'; t) = -H K^{(d)}(x, x'; t), \tag{1}$$

$$K^{(d)}(x, x'; 0) = \mathbf{1}\, |h(x)|^{-\frac{1}{2}}\, |h(x')|^{-\frac{1}{2}}\, \delta^{(d)}(x, x'), \tag{2}$$

where $\delta^{(d)}(x, x')$ is the d-dimensional invariant δ function, $\mathbf{1}$ is the unit matrix for the spinor, and $h = \det h^a{}_\mu$, in which $h^a{}_\mu$ is a vielbein in curved space. Here H is the second-order differential operator corresponding to the square of the Dirac operator $\not{D}$ for the fermion ψ,

$$\begin{aligned} &H = \not{D}^2 = D_\mu D^\mu + X, \quad \not{D} = \gamma^\mu \nabla_\mu + Y, \quad D_\mu = \nabla_\mu + Q_\mu, \quad Q_\mu = \tfrac{1}{2}\{\gamma_\mu, Y\}, \\ &X = Z - \nabla_\mu Q^\mu - Q_\mu Q^\mu, \quad \nabla_\mu \psi = \partial_\mu \psi + \tfrac{1}{4}\, \omega^{ab}{}_\mu\, \gamma_{ab}\, \psi, \quad \gamma_{a_1 \cdots a_j} = \gamma_{[a_1} \cdots \gamma_{a_j]}, \\ &Z = \tfrac{1}{2}\, \gamma^{\mu\nu}\, [\nabla_\mu, \nabla_\nu] + \gamma^\mu \nabla_\mu Y + Y^2, \qquad [D_\mu, D_\nu]\psi = \Lambda_{\mu\nu}\, \psi, \end{aligned} \tag{3}$$

where $\omega^{ab}{}_\mu$ is Ricci's coefficient of rotation. When in $d = 2n$ dimensions the fermion interacts with vector and axial-vector fields that do not commute, the Dirac operator contains the coupling of these bosons in Y:

$$Y = \gamma^\mu V_\mu + \gamma_{2n+1} \gamma^\mu A_\mu, \quad V_\mu \equiv V^a_\mu T^a, \quad A_\mu \equiv A^a_\mu T^a, \quad \gamma_{2n+1} = i^n \gamma^1 \gamma^2 \cdots \gamma^{2n}. \tag{4}$$

Here T^a is the representation matrix of a gauge group, and V^a_μ (A^a_μ) is purely imaginary (real), because of the Hermiticity of the Dirac operator. The quantities Q_μ, X, and $\Lambda_{\mu\nu}$ are expressed in the following tensorial form:

$$\begin{aligned} Q_\mu &= V_\mu - \gamma_{2n+1}\, \gamma_{\mu\rho} A^\rho, \qquad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu], \\ X &= -\frac{1}{4} R + 2(n-1)\, A_\mu A^\mu - \gamma_{2n+1}\, A^\mu{}_{:\mu} + \gamma^{\mu\nu} \left(\frac{1}{2} F_{\mu\nu} + \frac{2n-3}{2} [A_\mu, A_\nu] \right), \\ \Lambda_{\mu\nu} &= \frac{1}{4}\, \gamma^{\rho\sigma} R_{\rho\sigma\mu\nu} + F_{\mu\nu} - [A_\mu, A_\nu] - 2\, \gamma_{\mu\nu} A_\rho A^\rho + 2\, \gamma_{[\mu|}{}^\rho \{A_{|\nu]}, A_\rho\} \\ &\quad + 2\gamma_{2n+1}\, \gamma_{[\mu|\rho} A^\rho{}_{:|\nu]} - 2\, \gamma_{\mu\nu\rho\sigma} A^\rho A^\sigma, \end{aligned} \tag{5}$$

where $R_{\alpha\beta\mu\nu}$ stands for the curvature tensor, and the colon (:) represents the Riemannian covariant differentiation ∇_μ. The completely antisymmetric product $\gamma_{\mu\nu\rho\sigma}$ of γ matrices in the last term of $\Lambda_{\mu\nu}$ is rewritten as $-\epsilon_{\mu\nu\rho\sigma}\gamma_5$ and $-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma\kappa\lambda}\gamma_7\gamma^{\kappa\lambda}$ in four and six dimensions, respectively.

The differential equation (1) of the heat kernel for the fermion interacting with the general boson fields is not strictly solvable. Therefore, the heat kernel is usually calculated by using De Witt's ansatz [21], which automatically satisfies (2),

$$K^{(2n)}(x, x'; t) \sim \frac{\Delta^{1/2}(x, x')}{(4\pi t)^n} \exp\left(\frac{\sigma(x, x')}{2t}\right) \sum_{q=0}^{\infty} a_q(x, x')\, t^q, \tag{6}$$

where $\sigma(x, x')$ and $\Delta(x, x')$ are half of the square of the geodesic distance and the Van Vleck–Morette determinant between x and x', respectively, and $a_q(x, x')$ are bispinors. Note that the coincidence limit of a_0 is $\lim_{x' \to x} a_0(x, x') \equiv [a_0](x) = \mathbf{1}$, and that the metric tensor in curved space is $g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab}$ with $\eta_{ab} = -\delta_{ab}$ in flat tangent space.

3. *Chiral U(1) anomaly* The formal expression of the chiral U(1) anomaly $\mathcal{A}^{(2n)}$ in $2n$ dimensions is derived from the path integral measure [3,4] using the coincidence limit of the bispinor a_n of the heat kernel,

$$\nabla_\mu \langle \bar{\psi}(x) \gamma^\mu \gamma_{2n+1} \psi(x) \rangle = \mathcal{A}^{(2n)}(x) = \frac{2i}{(4\pi)^n} \mathrm{Tr}\left(\gamma_{2n+1} [a_n](x)\right), \tag{7}$$

where Tr runs over both indices of the γ matrices and representation matrices T^a of the gauge group. The concrete form of $[a_n]$ is given as follows [22]:

$$[a_2] = \frac{1}{12}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \frac{1}{6}X_{!\mu}{}^{\mu} + \frac{1}{2}\left(\frac{1}{6}R + X\right)^2 + \cdots, \tag{8}$$

$$\begin{aligned}
[a_3] = {} & \frac{1}{60}\left(-\frac{1}{3}\left(X_{!\mu}{}^{\mu}{}_{\nu}{}^{\nu} + X_{!\mu\nu}{}^{\nu\mu} + X_{!\mu\nu}{}^{\mu\nu}\right) - \frac{1}{3}\Lambda_{\mu\nu}{}^{!\nu}\Lambda^{\mu\rho}{}_{!\rho} - \frac{4}{3}\Lambda_{\mu\nu!\rho}\Lambda^{\mu\nu!\rho} - 4\Lambda^{\mu\nu}\Lambda_{\mu\rho}{}^{!\rho}{}_{\nu}\right. \\
& \left. - \frac{10}{3}R^{\mu\nu}\Lambda_{\mu\rho}\Lambda_{\nu}{}^{\rho} + R^{\mu\nu\rho\sigma}\Lambda_{\mu\nu}\Lambda_{\rho\sigma} - 6\Lambda_{\mu}{}^{\nu}\Lambda_{\nu}{}^{\rho}\Lambda_{\rho}{}^{\mu}\right) \\
& + \frac{1}{6}\left(\frac{1}{6}R + X\right)\left(-\frac{1}{2}\Lambda_{\mu\nu}\Lambda^{\mu\nu} - X_{!\mu}{}^{\mu} - \frac{1}{5}R_{:\mu}{}^{\mu} + \frac{1}{30}R_{\mu\nu}R^{\mu\nu} - \frac{1}{30}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \\
& - \frac{1}{12}\left(\frac{1}{6}R + X\right)_{!\rho}\left(\frac{1}{6}R + X\right)^{!\rho} - \frac{1}{6}\left(\frac{1}{6}R + X\right)^3 + \cdots,
\end{aligned} \tag{9}$$

where the exclamation mark (!) represents the modified covariant differentiation D_μ, and some terms without a γ matrix are omitted from (8) and (9). After a straightforward calculation, the tensorial form of the anomalies in four and six dimensions is obtained as

$$\begin{aligned}
\mathcal{A}^{(4)} = {} & \frac{i}{8\pi^2}\mathrm{tr}\left[\epsilon_{\alpha\beta\gamma\delta}\left(-\frac{1}{48}R^{\alpha\beta}{}_{\rho\sigma}R^{\gamma\delta\rho\sigma} - \frac{1}{2}F^{\alpha\beta}F^{\gamma\delta} - \frac{2}{3}A^{\alpha:\beta}A^{\gamma:\delta} - \frac{2}{3}[A^\alpha, A^\beta]F^{\gamma\delta}\right)\right. \\
& \left. + \left(-\frac{2}{3}A_{\nu:\mu}{}^{\mu} + \frac{1}{3}RA_\nu + \frac{8}{3}A_\nu A_\mu A^\mu\right)^{:\nu}\right],
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathcal{A}^{(6)} = {} & \frac{i}{4\pi^3}\mathrm{tr}\left[-\frac{i}{8}\epsilon_{\alpha\beta\gamma\delta\kappa\lambda}\left(\frac{1}{48}R^{\alpha\beta}{}_{\rho\sigma}R^{\gamma\delta\rho\sigma} + \frac{1}{6}F^{\alpha\beta}F^{\gamma\delta}\right)F^{\kappa\lambda}\right. \\
& + \left(\frac{1}{180}\left(A^{\mu}{}_{:\mu\nu}{}^{\nu\rho} + A^{\mu}{}_{:\mu\nu}{}^{\rho\nu} + A^{\mu}{}_{:\mu}{}^{\rho\nu}{}_{\nu}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}\left(-F^{\mu\nu}F_{\mu\nu}A^{\rho}+\{F^{\mu\nu},F^{\rho}{}_{\nu}\}A_{\mu}\right)-\frac{1}{72}RA_{\mu}{}^{:\mu\rho}\\
&+\frac{1}{120}\left(-R^{:\rho}A_{\mu}{}^{:\mu}+R_{:\mu}A^{\rho:\mu}+R_{:\mu}A^{\mu:\rho}\right)-\frac{1}{90}R^{\rho\mu}A_{\mu:\nu}{}^{\nu}\\
&+\frac{2}{45}R_{\mu\nu}A^{\mu:\nu\rho}+\frac{1}{30}\left(-R^{\rho\mu:\nu}+R^{\mu\nu:\rho}\right)A_{\nu:\mu}-\frac{1}{36}R^{\mu\nu\lambda\rho}A_{\mu:\lambda\nu}\\
&+\frac{1}{36}R_{\mu\nu}R^{\mu\lambda\nu\rho}A_{\lambda}-\frac{1}{180}R^{\mu\nu\kappa\rho}R_{\mu\nu\kappa\lambda}A^{\lambda}+\frac{1}{40}R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda}A^{\rho}\\
&+\frac{1}{90}R_{\mu\nu}R^{\mu\rho}A^{\nu}-\frac{1}{72}RR^{\mu\rho}A_{\mu}+\frac{1}{288}R^{2}A^{\rho}\Big)_{:\rho}-\frac{i}{12}\epsilon_{\alpha\beta\gamma\delta\kappa\lambda}\left(F^{\alpha\beta}A^{\gamma:\delta}A^{\kappa}\right)^{:\lambda}\\
&+\Bigg(\frac{11}{15}A_{\mu}A^{\mu}A^{\rho:\nu}{}_{\nu}+\frac{4}{3}A_{\mu}A^{\mu}A_{\nu}{}^{:\nu\rho}-\frac{19}{15}A_{\mu}A^{\mu}A_{\nu}{}^{:\rho\nu}\\
&-\frac{1}{10}\{A^{\rho},A^{\mu}\}A^{\nu}{}_{:\mu\nu}+\frac{3}{10}\{A^{\rho},A^{\mu}\}A_{\mu:\nu}{}^{\nu}+\frac{1}{30}\{A^{\rho},A^{\mu}\}A^{\nu}{}_{:\nu\mu}\\
&+\frac{1}{3}\{A^{\mu},A^{\nu}\}A_{\mu:\nu}{}^{\rho}+\frac{1}{15}\{A^{\mu},A^{\nu}\}A^{\rho}{}_{:\mu\nu}-\frac{11}{30}\{A^{\mu},A^{\nu}\}A_{\mu}{}^{:\rho}{}_{\nu}\\
&-\frac{1}{15}\{A^{\mu:\rho},A_{\mu:\nu}\}A^{\nu}-\frac{1}{2}\{A_{\mu:\nu},A^{\nu:\rho}\}A^{\mu}-\frac{1}{30}\{A^{\rho:\mu},A_{\mu:\nu}\}A^{\nu}\\
&+\frac{4}{5}\{A^{\rho:\mu},A_{\nu:\mu}\}A^{\nu}-\frac{1}{10}\{A^{\mu}{}_{:\mu},A^{\nu:\rho}\}A_{\nu}+\frac{1}{15}\{A^{\mu}{}_{:\mu},A^{\rho:\nu}\}A_{\nu}\\
&+\frac{1}{15}A^{\mu}{}_{:\mu}A^{\nu}{}_{:\nu}A^{\rho}+\frac{2}{5}A^{\mu:\nu}A_{\mu:\nu}A^{\rho}-\frac{2}{15}A^{\mu:\nu}A_{\nu:\mu}A^{\rho}\\
&-\frac{49}{30}F^{\nu\rho}[A_{\mu}A^{\mu},A_{\nu}]-\frac{2}{5}F_{\mu\nu}\{A^{\mu}A^{\nu},A^{\rho}\}-\frac{2}{15}F_{\mu\nu}A^{\mu}A^{\rho}A^{\nu}\\
&+\frac{29}{45}R^{\rho\mu}A_{\mu}A_{\nu}A^{\nu}-\frac{1}{45}R^{\mu\nu}A_{\mu}A_{\nu}A^{\rho}-\frac{1}{18}RA_{\mu}A^{\mu}A^{\rho}\Bigg)_{:\rho}-\frac{i}{20}\epsilon_{\alpha\beta\gamma\delta\kappa\lambda}\left(A^{\alpha}A^{\beta}A^{\gamma}A^{\delta:\kappa}\right)^{:\lambda}\\
&+\left(\frac{1}{5}A_{\mu}A^{\mu}A_{\nu}A^{\nu}A^{\rho}-\frac{1}{3}A_{\mu}A_{\nu}A^{\mu}A^{\nu}A^{\rho}+\frac{2}{5}A_{\mu}A_{\nu}A^{\nu}A^{\mu}A^{\rho}\right)_{:\rho}\Bigg],
\end{aligned}
\tag{11}
$$

where tr runs only over the internal group indices. We note that the terms of three degrees of A_{μ} in (11) can be rewritten using the identities of the curvature tensor and the properties of matrices, as follows:

$$
\begin{aligned}
\mathrm{tr}\left[(A_{\lambda}A^{\lambda}A^{\rho})^{:\nu}{}_{\nu}-(A_{\lambda}A^{\lambda}A^{\nu})^{:\rho}{}_{\nu}\right]_{:\rho}&=0,\\
R^{\lambda\alpha\beta\rho}\,\mathrm{tr}\left[\{A_{\alpha},A_{\beta}\}A_{\lambda}\right]_{:\rho}&=0,\\
\mathrm{tr}\left[A_{\lambda}A^{\lambda}(A_{\nu}{}^{:\rho\nu}-A_{\nu}{}^{:\nu\rho})-R^{\nu\rho}A_{\nu}A_{\lambda}A^{\lambda}\right]_{:\rho}&=0,
\end{aligned}
\tag{12}
$$

although these terms may naively appear with an ambiguous constant factor. Using similar identities, the anomaly terms may give equivalent expressions, though those seem to be different forms.

4. *Discussion* In this article, the first calculation of the chiral U(1) anomaly in the nonabelian V-A model in four- and six-dimensional curved space was performed. The covariant gauge anomaly G for the left-handed Weyl fermion is also derived by a similar method:

$$
D_{\mu}\langle\bar{\psi}(x)\gamma^{\mu}\frac{1-\gamma_{2n+1}}{2}T^{a}\psi(x)\rangle=G^{(2n)}=-\frac{i}{(4\pi)^{n}}\,\mathrm{Tr}\left(T^{a}\gamma_{2n+1}[a_{n}](x)\right). \tag{13}
$$

The general form of the gauge anomaly in the model in four-dimensional flat space is well known [24–33], and the chiral U(1) anomaly is easily given from the gauge anomaly. However, in curved space, a new term appears containing the Riemann curvature tensor $R^{ab}{}_{\mu\nu}$ with the axial-vector A_μ.

The leading terms of the chiral U(1) anomaly in any even-dimensional curved space are known to be expressed by a combination of the Dirac genus by the curvature tensor $R^{ab}{}_{\mu\nu}$ and the Chern character by the field strength $F_{\mu\nu}$ of the vector gauge field [34]. The anomaly $\mathcal{A}$ can be rewritten as the total derivative $\nabla_\mu C^\mu$ because of the index theorem when $\{\not{D}, \gamma_{2n+1}\} = 0$ [3–5]. As is well known, the anomaly provides explicit chirality-breaking terms for the gauge-invariant and general covariant current. Although the modified current $J^\mu \equiv \bar{\psi}\gamma^\mu\gamma_{2n+1}\psi - C^\mu$ satisfies $\nabla_\mu J^\mu = 0$, the current J^μ does not preserve these symmetries. If the phase factor of the anomalous Jacobian from the functional measure of the path integral leads the θ vacuum in the presence of instantons, the zero-mode sector of the measure behaves abnormally. However, the anomaly terms containing the axial-vector field do not affect the relationship between the instanton and the θ vacuum because the terms are the divergence of the covariant quantities, and the Pontryagin index of the terms vanishes. In supergravity in higher even dimensions, the gravitino and gaugino may interact with odd-order tensors constituted by the fermions. The chiral U(1) anomaly in the model has similar topological properties. One may retain some of the leading terms of the anomaly in a model if one wants to explain physical phenomena. Note that then breakdown of the chiral and/or other symmetries appears.

In (10) and (11), the matrix T^a is not restricted. When T^a is traceless, all the terms containing $F_{\mu\nu}$ and A_μ, which are reduced to a linear T^a, disappear. If T^a is abelian, the trace operation yields the dimension number factor of the matrix; a commutator vanishes, and an anticommutator of the two fields becomes twice their product. In the special case that only A_μ is abelian in (10), $\mathcal{A}^{(4)}$ corresponds to the chiral U(1) anomaly in curved space with torsion, which is the three-order antisymmetric tensor and is rewritten by the dual axial-vector in four dimensions [23]. Then, the term containing $\epsilon_{\alpha\beta\gamma\delta}\,\mathrm{tr}([A^\alpha, A^\beta]F^{\gamma\delta})$ in (10) would disappear. (Note that the dual torsion tensor in dimensions higher than six is an antisymmetric tensor of order larger than three.) When only T^a in (13) is a unit matrix, the gauge anomaly agrees with our resultant form (10), up to twice the dimension number of T^a. Then, there is no term containing A_μ of the same degree as the spatial dimensions because of the property of matrices in the trace formula and the contraction of the Levi–Civita symbol with the product of A_μ. In contrast, the gauge anomaly should add terms containing the nonzero commutator of T^a and A_μ to the U(1) anomaly.

The anomaly $\mathcal{A}_\nu$ expresses quantum breaking by the general coordinate transformation,

$$D^\mu \langle T_{\mu\nu}(x)\rangle = \mathcal{A}_\nu^{(2n)} = \frac{1}{2}\,\mathrm{Tr}\left[\gamma_{2n+1}\left(2\left[a_{n!\nu}\right] - \left[a_n\right]_{!\nu}\right)(x)\right]. \tag{14}$$

The tensorial form of the anomaly is also a total derivative because of the structure of the gravitational anomalies. The last part of it contains the covariant derivative of the U(1) anomaly. However, the explicit form of the anomaly in the model of this article has not been shown yet because the term obtained from $[a_{n!\nu}]$ should be of a nontrivial derivative form. A clarification of the tensorial form of the anomaly would be of interest in order to estimate the validity of the models.

Funding

Open Access funding: SCOAP[3].

References

[1] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[2] J. S. Bell and R. Jachiw, Nuovo Cim. A **60**, 47 (1969).
[3] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).
[4] K. Fujikawa, Phys. Rev. D **21**, 2848 (1980).
[5] K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies* (Oxford University Press, New York, 2004).
[6] D. T. Son and N. Yamamoto, Phys. Rev. Lett. **109**, 181602 (2012).
[7] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. **109**, 162001 (2012).
[8] O. F. Dayi and M. Elbistan, arXiv:1402.4727 [hep-th].
[9] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[10] T. Kimura, Prog. Theor. Phys. **42**, 1191 (1969).
[11] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[12] M. F. Atiyah and I. M. Singer, Ann. Math. **87**, 485 (1968).
[13] M. F. Atiyah and G. B. Segal, Ann. Math. **87**, 531 (1968).
[14] A. H. Chamseddine, Nucl. Phys. B **185**, 403 (1981).
[15] E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, Nucl. Phys. B **195**, 97 (1982).
[16] N. K. Nielsen, Nucl. Phys. B **140**, 499 (1978).
[17] R. E. Kallosh, Nucl. Phys. B **141**, 141 (1978).
[18] H. Hata and T. Kugo, Nucl. Phys. B **158**, 357 (1979).
[19] R. Endo and T. Kimura, Prog. Theor. Phys. **63**, 683 (1980).
[20] R. Endo and M. Takao, Phys. Lett. B **161**, 155 (1985).
[21] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[22] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975).
[23] S. Yajima and T. Kimura, Prog. Theor. Phys. **74**, 866 (1985).
[24] A. P. Balachandran, G. Marmo, V. P. Nair, and C. G. Trahern, Phys. Rev. D **25**, 2713 (1982).
[25] M. B. Einhorn and D. R. T. Jones, Phys. Rev. D **29**, 331 (1984).
[26] D. W. McKay and B.-L. Young, Phys. Rev. D **28**, 1039 (1983).
[27] S.-K. Hu, B.-L. Young, and D. W. McKay, Phys. Rev. D **30**, 836 (1984).
[28] A. Andrianov and L. Bonora, Nucl. Phys. B **233**, 232(1984).
[29] A. Andrianov and L. Bonora, Nucl. Phys. B **233**, 247 (1984).
[30] R. E. Gamboa-Saravi, M. A. Muschietti, F. A. Schaposnik, and J. E. Solomin, Phys. Lett. B **138**, 145 (1984).
[31] R. E. Gamboa-Saravi, M. A. Muschietti, F. A. Schaposnik, and J. E. Solomin, Phys. Lett. B **233**, 247 (1984).
[32] R. Delbourgo and P. Jaris, J. Phys. G **10**, 591 (1984).
[33] K. Fujikawa, Phys. Rev. D **31**, 341 (1985).
[34] L. Alvarez-Gaumé and P. Ginsparg, Ann. Phys. **161**, 423 (1985).