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New massless and massive infinite derivative gravity in three dimensions

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Abstract

In this paper we will consider the most general quadratic curvature action with infinitely many covariant derivatives of massless gravity in three spacetime dimensions. The action is parity invariant and torsion-free and contains the same off-shell degrees of freedom as the Einstein-Hilbert action in general relativity. In the ultraviolet, with an appropriate choice of the propagator given by the *exponential of an entire function*, the point-like curvature singularity can be smoothened to a Gaussian distribution, while in the infrared the theory reduces to general relativity. We will also show how to embed new massive gravity in ghost-free infinite derivative gravity in Minkowski background as one of the infrared limits. Finally, we will provide the tree-level unitarity conditions for infinite derivative gravity in presence of a cosmological constant in deSitter and Anti-deSitter spacetimes in three dimensions by perturbing the geometries.

1. Introduction

Einstein's theory of general relativity (GR) contains classical and quantum singularities in $3+1$ spacetime dimensions [1,2]. Quadratic curvature gravity in four spacetime dimensions ($4d$) indeed ameliorates the renormalizability issue, but contains massive spin-2 ghosts [3]. A ghost-free theory of quadratic gravity in $4d$ that contains infinitely many covariant derivatives has been constructed in Refs. [4,8–10,12,13,11] and [5,6]. The corresponding action is the most

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general one which is parity invariant and torsion free as long as one is interested in linear perturbations around maximally symmetric spacetimes [6,7]. By construction, the action contains 3 analytic form factors $F_i(\Box) = \sum_n f_{i,n}\Box^n$ with infinitely many covariant derivatives representing the Ricci scalar, the Ricci tensor and the Riemann/Weyl tensor. It was shown that the full action of ghost-free analytical infinite derivative theory of gravity (AIDG) in $4d$ can ameliorate the static Schwarzschild blackhole singularity [5,14–18], [19,20] and also produces rotating non-singular metrics [21] at linear level. The gravitational interaction weakens enough that astrophysical objects with even billions of solar masses may have no singularities, provided certain conditions are met [22,20]. The singularities can also be resolved with extended objects such as static p-branes [23]. At the quantum level, the interaction introduces non-locality [24–29] and it has been argued that the theory would be power-counting renormalizable [9,33,4]. A careful analysis of this issue can be found in [31,32].

It is now wishful to consider whether we could construct ghost-free AIDG in $3d$, in particular we are motivated to study the gravitational action in the UV where the higher derivatives play a significant role around Minkowski and in (A)dS backgrounds. In $3d$, GR itself has some interesting properties which can be captured by either a metric theory of gravity or by a Chern-Simons theory [38–40]. It contains 3 off-shell degrees of freedom but on-shell they do not survive, hence the physical graviton does not propagate. Furthermore, there is an interesting connection between $3d$-gravity in AdS and conformal field theory (CFT) in the boundary [41,42] and in AdS_3 there exists an intriguing non-trivial blackhole solution [43,44]. All these non-trivial features in $3d$ demand further study on the construction of both massless and massive ghost-free AIDG up to quadratic in curvature.

The aim of this paper will be to construct the conditions on the gravitational form factors, which will at least guarantee a linearized ghost-free propagator around Minkowski and (A)dS backgrounds which has massless Einstein-Hilbert gravity as an IR limit. Finding the propagator in (A)dS involves non-trivial computations which we will carry out for the first time in A(dS) for AIDG in $3d$. We will also show how such a construction can yield new massive gravity [45,46] around Minkowski background, see also ref. [47] for further classifications of new massive gravity in the IR. We will restrict ourselves to classical properties and will not consider quantization of gravity in any of the backgrounds.

The paper is organized as follows: In section two the full equations of motion of AIDG in $3d$ are discussed and in section three linearized gravity around Minkowski background and the ghost-free conditions for the propagator are discussed. In section four we have shown how AIDG can resolve point like curvature singularities and in section five, we will discuss how new massive gravity can be treated as an IR limit of ghost-free AIDG. In section 6, we briefly discuss maximally symmetric solutions of this action and in section 7, we consider the conditions for AIDG to be ghost-free in (A)dS backgrounds.[1]

[1] We will use the following conventions:

- $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$
- $a, b, ...$ are abstract indices in 3d, $\mu, \nu, ...$ are coordinate indices in 3d and $i, j, ...$ are purely spatial coordinate indices
- $(a_1, ..., a_n)$ and $[a_1, ..., a_n]$ denote (anti-)symmetrization including a factor $\frac{1}{n!}$
- $c = G_N = \hbar = 1$.

2. The full equations of motion

The $3d$ analogue of AIDG can be constructed in a very similar fashion as in $4d$, see for detailed derivations in $4d$ from the most general ansatz of parity invariant and torsionless setup in [5–7]. The action in $3d$ can be captured by the Ricci scalar and the Ricci tensor, with two form factors:

$$S_{AIDG} = \int d^3x \sqrt{-g} \left[\frac{R}{2} + R F_1(\Box) R + R_{ab} F_2(\Box) R^{ab} \right] \tag{1}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ has a mass dimension of 2. Therefore, $\Box \equiv \Box/M_s^2$, where M_s is a new scale of gravity in $3d$, beyond which the infinite derivative part becomes important, while below M_s GR becomes a viable option.[2]

The two form factors are assumed to be analytic and given by an infinite power series in $\Box$[3]:

$$F_i(\Box) = \sum_n f_{i,n} \Box^n . \tag{2}$$

This is the most general form of a covariant action with terms quadratic in curvature containing infinitely many derivatives with reduces to GR in the IR regime. The equations of motion can be derived in a very similar fashion in the $4d$ case, which was first derived in [53]. The only major difference is that the Weyl tensor is identically zero in three dimensions, hence there are only two quadratic curvature terms in the action. Here we present the $3d$ version of that:

$$\begin{aligned} & G_{ab} + 4G_{ab}F_1(\Box)R + g_{ab}RF_1(\Box)R - 4(\nabla_a\nabla_b - g_{ab}\Box)F_1R - 2\Omega_{1ab} \\ & + g_{ab}\left(\Omega_{1c}^{c} + \overline{\Omega}_1\right) + 4R_a^c F_2(\Box) R_{cb} - g_{ab}R_{cd}F_2(\Box)R^{cd} - 4\nabla_c\nabla_b F_2(\Box) R_a^c \\ & + 2\Box F_2(\Box)R_{ab} + 2g_{ab}\nabla_c\nabla_d F_2(\Box)R^{cd} - 2\Omega_{2ab} + g_{ab}\left(\Omega_{2c}^{c} + \overline{\Omega}_2\right) - 4\Delta_{2ab} = \tau_{ab} \end{aligned} \tag{3}$$

where τ_{ab} is the energy momentum tensor and G_{ab} is the Einstein tensor. We have defined the symmetric tensors[4]

$$\Omega_{1ab} = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \nabla_a R^{(l)} \nabla_b R^{(n-l-1)}, \quad \overline{\Omega}_1 = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)} \tag{4}$$

$$\Omega_{2ab} = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla_a R^{cd(l)} \nabla_b R_{cd}^{(n-l-1)}, \quad \overline{\Omega}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} R^{cd(l)} R_{cd}^{(n-l)} \tag{5}$$

$$\Delta_{2ab} = \frac{1}{2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla_c \left(R_d^{c(l)} \nabla_{(a} R_{b)}^{d(n-l-1)} - \nabla_{(a} R^{cd(l)} R_{b)d}^{(n-l-1)} \right). \tag{6}$$

Obviously, these equations containing all the double sums are very hard to solve exactly. Nevertheless, one can see that constant curvature backgrounds are indeed solutions of this theory, i.e.

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[2] We will suppress writing M_s in order not to clutter our formulae, but while discussing physical situation, we will invoke M_s and then we will take care of mentioning it appropriately.

[3] The f_{in} have to have the dimension $[mass]^{-2-2n}$ according to our conventions. Further note that here we will only consider analytic operators of $\Box$, and not non-analytic operators such as $1/\Box$ [48,52] or $\ln(\Box)$ [49–51].

[4] The notation $A^{(l)}$ is an abbreviation of $\Box^l A$ for any tensor A.

$$R = \text{constant}, \quad R_{ab} = \text{constant} \times g_{ab},$$

see section 6. In $4d$ such solutions exist, in fact non-trivial solutions are conformally flat in asymptotically Minkowski background, see for details [20]. We can also consider the linearized equations of motion: If we only keep the terms linear in curvature we obtain

$$G_{ab} - 4\left(\nabla_a \nabla_b - g_{ab}\Box\right) F_1 R - 4\nabla_c \nabla_b F_2(\Box) R_a^c \\ +2 g_{ab} \nabla_c \nabla_d F_2(\Box) R^{cd} + 2\Box F_2(\Box) R_{ab} = \tau_{ab}. \tag{7}$$

3. Unitarity and propagator around Minkowski background

The linearized limit is particular useful to examine the mechanical properties of the theory, such as tree-level unitarity and the propagator. As usual, we will write the metric as

$$g_{ab} = \eta_{ab} + h_{ab} \tag{8}$$

and treat $h_{ab} \ll 1$ as a small quantity. Since we want to reduce the equations of motion to linear order in h_{ab}, the action should contain only terms up to quadratic order in h_{ab} and, moreover, we expect it to be constructed solely out of h_{ab}, η_{ab} (the Minkowski metric) and ∂_a. The most general action of this kind consists of several terms according to the various index contractions and reads

$$S_{qua} = \frac{1}{4}\int d^3x \sqrt{-g}\left[\frac{1}{2} h_{ab}\Box a(\Box) h^{ab} + h_b^c b(\Box)\partial_c\partial_a h^{ab} + h c(\Box)\partial_a\partial_b h^{ab}\right. \\ \left. + \frac{1}{2} h\Box d(\Box) h + h^{cd}\frac{f(\Box)}{2\Box}\partial_c\partial_d\partial_a\partial_b h^{ab}\right] \tag{9}$$

with analytic functions $a(\Box), \ldots, f(\Box)$ (the exact definitions are merely a convention, of course; extra factors of $\frac{1}{2}$ or $\Box$ are inserted for later convenience). The resulting linearized equations of motion read

$$\frac{1}{2}\Box a(\Box) h_{ab} + b(\Box)\partial_c\partial_{(a} h_{b)}^c + \frac{1}{2} c(\Box)\left(\eta_{ab}\partial_c\partial_d h^{cd} + \partial_a\partial_b h\right) \\ + \frac{1}{2}\Box d(\Box) h\eta_{ab} + \frac{f(\Box)}{2\Box}\partial_a\partial_b\partial_c\partial_d h^{cd} = -\tau_{ab}\,. \tag{10}$$

[5] To compute $a(\Box), \ldots, f(\Box)$ in terms of $F_1(\Box)$ and $F_2(\Box)$, we insert the linearized expressions for the curvature quantities

$$R_{ab}^{(1)} = \partial_c\partial_{(a} h_{b)}^c - \frac{1}{2}\partial_a\partial_b h - \frac{1}{2}\Box h_{ab}\,, \tag{11}$$

$$R^{(1)} = \partial_a\partial_b h^{ab} - \Box h\,, \tag{12}$$

into Eq. (7) and compare the coefficients of the different terms. We get [5]:

$$a(\Box) = 1 + 2F_2(\Box)\Box = -b(\Box)\,, \\ c(\Box) = 1 - 8F_1(\Box)\Box - 2F_2(\Box)\Box = -d(\Box)\,, \\ f(\Box) = 8F_1(\Box)\Box + 4F_2(\Box)\Box\,. \tag{13}$$

[5] The definition of τ_{ab} here defers from the previous section by an unimportant numerical factor.

The constant terms correspond to the Einstein-Hilbert contribution, so for $F_1,\ F_2 \to 0$ we recover pure Einstein gravity.

If we wish to demand that the IR limit of the action Eq. (1) is that of Einstein's GR, then similar to the argument provided in Refs. [5–7], we want the equations of motion and hence the propagator to be proportional to the GR-case, so we demand that $f(\Box)$ should be zero. As a result, $a(\Box) = c(\Box)$, and the equations of motion can now be written in momentum space, using

$$h_{ab}(x) = \int d^3k\, e^{ik_\nu x^\nu} h_{ab}(k)\,, \tag{14}$$

and

$$\frac{1}{2} a\left(-k^2\right)\left(-k^2 h_{ab} + 2k_c k_{(a} h^c_{b)} - \eta_{ab} k_c k_d h^{cd} - k_a k_b h + k^2 h \eta_{ab}\right) = -\tau_{ab}\,. \tag{15}$$

To obtain the free propagator, we have to invert the field equations which is not possible directly, because they contain zero modes corresponding to gauge degrees of freedom. An easy way to get rid of the gauge modes is to use spin projection operators [5,54,55], see Appendix A.1 for the details. We arrive at the propagator where the momentum dependent part is given by:

$$\Pi_{AIDG} = \frac{P^2_s}{a\left(-k^2\right)k^2} - \frac{P^0_s}{a\left(-k^2\right)k^2} = \frac{1}{a\left(-k^2\right)}\Pi_{GR}\,. \tag{16}$$

As promised, the propagator is proportional to the GR-propagator and by choosing $a(\Box)$ in a clever way, namely as exponential of an *entire function*, we will not introduce any new pole in the graviton propagator in flat background.[6] Therefore, by going from UV to IR *only* the 3 dynamical degrees of freedom, namely the spin-2 and spin-0 components propagate in a sandwiched propagator, sandwiched between two conserved currents. Otherwise the propagator would have additional poles associated to additional particle excitations. The simplest choice is [5]

$$a(\Box) = e^{-\Box/M_s^2}\,, \tag{17}$$

with a certain mass scale M_s that can be interpreted as the *scale of non-locality*. The choice of sign in $4d$ was obtained by demanding that the Newtonian gravitational potential recovers $1/r$ behavior in the IR, see for details [33]. The negative sign in the exponent also helps the UV properties which we will exhibit below. Since the propagator is suppressed in the high energy regime, there is an indication that the theory may become asymptotically free. It implies from $a(\Box) = c(\Box)$ [5] that the form factors are now constrained in the Minkowski background:

$$2F_1(\Box) + F_2(\Box) = 0\,, \tag{18}$$

$$F_1(\Box) = -\frac{e^{-\Box/M_s^2} - 1}{4\Box}\,, \qquad F_2(\Box) = \frac{e^{-\Box/M_s^2} - 1}{2\Box}\,. \tag{19}$$

Note that in the low-energy-limit $M_s \to \infty$ the $F_i(\Box)$ tend to zero so we get Einstein gravity as expected. There is one more important issue: The second term in Eq. (16) has the wrong sign and therefore indicates the presence of a ghost state. However, this is an example of a *benign ghost* which does not spoil unitarity of the associated quantum theory.[7]

Benign ghosts are a common feature of gauge theories in general.

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6 There actually exist exceptions to this principle, e.g. by using complex conjugate poles, see [34–37].

7 Since we do not quantize the theory in a rigorous way, what is shown here is just *tree-level* unitarity. For a discussion of perturbative unitarity to all orders see [63,64].

It is a well known fact that Einstein gravity in $3d$ has no on-shell propagating degrees of freedom, and since per construction we did not change the number of local excitations, we expect that statement still to be true (the derivation can be found in Appendix A.2). One should however keep in mind that this is only true on-shell; if we do not demand the vacuum field equations to hold we can only remove three degrees of freedom through Eq. (115). The remaining three propagate off-shell as it can be seen in the propagator, see Eq. (16).

Moreover, due to non-locality we conclude that also causality must be violated in the UV regime. In [66] it was argued that those violations can never be detected in any laboratory experiment. We will not go into detail regarding this issue and refer to the discussions in [66,30].

4. Adding a Dirac-delta source

In this section, we wish to show that by adding sources (or more precisely, a point source) will change the behavior of the solutions drastically. Let us briefly recall the situation in Einstein gravity: Due to the local field equations $R_{ab} = 0$ space is always flat everywhere, adding a point source $\tau_{ab} \sim \delta(x^i)$ (where x^i denote the two spatial coordinates) will merely change the behavior of R_{ab} at $x = 0$, leading to a *conical singularity*.

To analyze the problem in AIDG, we wish to work again with the linearized field equations for h_{ab}. In momentum space, we have

$$\tau_{ab}(k) = \int d^3x\, e^{-ikx} \delta^2\left(x^i\right) m \delta_a^0 \delta_b^0 = 2\pi m \delta_a^0 \delta_b^0 \delta\left(k^0\right). \tag{20}$$

Acting on it with the propagator yields

$$\Pi_{ab}^{\ \ cd} \tau_{cd}(k) = h_{ab}(k) = 2\pi m \frac{1}{k^2 a\left(-k^2\right)} \left(\delta_a^0 \delta_b^0 + \eta_{ab}\right) \delta\left(k^0\right), \tag{21}$$

so h_{ab} takes the form takes the simple form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix}$ with

$$\psi = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \delta\left(k^0\right) \frac{2\pi m}{k^2 a\left(-k^2\right)} = \int \frac{d^2k}{(2\pi)^2} e^{ik_i x^i} \frac{m}{k^i k_i a\left(-k^i k_i\right)}. \tag{22}$$

Plugging that into the linearized Ricci tensor Eq. (11) yields

$$R_{ab} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta\psi & 0 \\ 0 & 0 & \Delta\psi \end{pmatrix}, \tag{23}$$

where Δ denotes the two dimensional (purely spatial) Laplacian. $\Delta\psi$ can now be evaluated straight forwardly for our preferred choice $a = e^{k^2/M_s^2}$:

$$\Delta\psi = \partial_i \partial^i \int \frac{d^2k}{(2\pi)^2} e^{ik_i x^i} \frac{m}{k^i k_i} e^{\frac{-k_i k^i}{M_s^2}} = -m \int \frac{d^2k}{(2\pi)^2} e^{ik_i x^i} e^{\frac{-k^i k_i}{M_s^2}} = -\frac{M_s^2 m}{4\pi} e^{-\frac{M_s^2}{4}\left(x_1^2 + x_2^2\right)}. \tag{24}$$

So the Ricci tensor turns out to be a *Gaussian distribution* around the point source. If we take the limit $M_s \to \infty$ the Gaussian turns into a delta distribution and we recover the expected result of pure Einstein gravity. The infinitely many derivatives have the effect of *smearing out* the conical singularity and the Ricci scalar stays finite, in strong analogy with the $4d$ case [5].

5. New massive gravity as an IR limit of AIDG

So far we have analyzed AIDG under the condition that it reduces to Einstein gravity in the limit $M_s \to \infty$. In $3d$, however, there exists another possible low-energy-limit called *New Massive Gravity,* see [45–47].[8]

It consists more or less in Stelle's fourth order theory adapted to three dimensions, with the action

$$S_{NMG} = \int d^3x \sqrt{-g} \left[\frac{R}{2} + \alpha R^2 + \beta R_{ab} R^{ab} \right], \tag{25}$$

(Note that R and R_{ab} here denote the *actual* Ricci tensor and scalar again, not the background quantities.) In contrast to four dimensions, there is a possibility to get rid of the Weyl ghost; namely for the choice $\alpha = -\frac{3}{8}\beta$. More specifically, we will consider the action [45,46]

$$S_{NMG} = \int d^3x \sqrt{-g} \left[-\frac{R}{2} - \frac{3}{8m^2} R^2 + \frac{1}{m^2} R_{ab} R^{ab} \right], \tag{26}$$

where m is a new mass parameter. Notice that we changed the sign of the Einstein-Hilbert term deliberately. The propagator can be straight forwardly evaluated with the help of spin projection operators, and reads

$$\Pi_{NMG} = -\frac{P_s^2}{\left(1 + \frac{2k^2}{m^2}\right) k^2} + \frac{P_s^0}{k^2} = -\Pi_{GR} + \frac{P_s^2}{k^2 + \frac{m^2}{2}}, \tag{27}$$

so we have one additional propagating mode compared to GR which is a spin two tensor with mass squared $\frac{m^2}{2}$. This is the usual Weyl-ghost familiar from Stelle's theory, however, we have reversed its sign here so that it has positive energy. As a result, the GR-part comes with the wrong sign. But this is not a problem since the GR-excitations do not propagate and the theory is still unitary.

The other open issue is renormalizability: As Stelle has proved, fourth-order gravity is renormalizable and in three dimensions we still expect that statement to hold. While this is true in principle, there are specific combinations of α and β which destroy the renormalizability, namely exactly those which provide unitarity! Hence, like in four dimensions, we cannot have unitarity and renormalizability at the same time in fourth-order gravity.

Here our aim is to embed NMG in AIDG, and see how it arises in the IR. We now want to construct a theory containing infinitely many derivatives which reduces to NMG in the limit $M_s \to \infty$ and does not change the particle content. Note that we have now two mass parameters in the theory and will assume the hierarchy $m \ll M_s$. As before, we want the propagator be proportional to Π_{NMG}, but suppressed in the UV-limit. The factor of proportionality must not have any zeros and shall therefore be of the form $Ce^{\gamma(-k^2)}$ with C a constant and $\gamma\left(-k^2\right)$ an entire function. The AIDG-action will be again of the form

$$S_{AIDG} = \int d^3x \sqrt{-g} \left[-\frac{R}{2} + R F_1(\Box) R + R_{ab} F_2(\Box) R^{ab} \right], \tag{28}$$

(with the reversed sign in front of Einstein-Hilbert term again), and we demand

[8] A similar embedding of massive gravity into infinite derivative gravity has been done in four dimensions, see [56].

$$F_1(\Box) \to -\frac{3}{8m^2} \quad \text{and} \quad F_1(\Box) \to \frac{1}{m^2}, \tag{29}$$

in the limit $M_s \to \infty$. The relations (13) containing the new sign now, read

$$\begin{aligned} a(\Box) &= -1 + 2F_2(\Box)\Box = -b(\Box), \\ c(\Box) &= -1 - 8F_1(\Box)\Box - 2F_2(\Box)\Box = -d(\Box), \\ f(\Box) &= 8F_1(\Box)\Box + 4F_2(\Box)\Box, \end{aligned} \tag{30}$$

and the propagator (now written in the momentum space) is still given by

$$\Pi_{AIDG} = \frac{P_s^2}{a\left(-k^2\right)k^2} + \frac{P_s^0}{\left(a\left(-k^2\right) - 2c\left(-k^2\right)\right)k^2} = \frac{1}{Ce^{\gamma(-k^2)}}\Pi_{NMG}. \tag{31}$$

We can now read off the relations

$$a\left(-k^2\right) = C\left(1 + \frac{2k^2}{m^2}\right)e^{\gamma(-k^2)} \quad \text{and} \quad a\left(-k^2\right) - 2c\left(-k^2\right) = Ce^{\gamma(-k^2)}, \tag{32}$$

which implies

$$F_1\left(-k^2\right) = \frac{Ce^{\gamma(-k^2)} + 1}{4k^2} + \frac{3}{8}\frac{Ce^{\gamma(-k^2)}}{m^2}, \qquad F_2\left(-k^2\right) = -\frac{Ce^{\gamma(-k^2)} + 1}{2k^2} - \frac{Ce^{\gamma(-k^2)}}{m^2}. \tag{33}$$

We see that analyticity of the F_i requires $C = -1$. For the function $\gamma\left(-k^2\right)$, we can choose the simplest analytic possibility $\gamma = k^2/M_s^2$, with M_s the scale of non-locality. The form factors then take the form

$$F_1\left(-k^2\right) = -\frac{e^{\frac{k^2}{M_s^2}} - 1}{4k^2} - \frac{3}{8}\frac{e^{\frac{k^2}{M_s^2}}}{m^2}, \qquad F_2\left(-k^2\right) = \frac{e^{\frac{k^2}{M_S^2}} - 1}{2k^2} + \frac{e^{\frac{k^2}{M_S^2}}}{m^2}, \tag{34}$$

and we see that the constant terms are given by

$$f_{10} = -\frac{1}{4M_s^2} - \frac{3}{8m^2}, \qquad f_{20} = \frac{1}{2M_s^2} + \frac{1}{m^2}, \tag{35}$$

and fulfill Eq. (29) in the limit when $M_s \to \infty$.

Hence, we have constructed a viable infinite derivative extension of New Massive Gravity which does not alter the particle content. As per construction it is tree-level unitary, renormalizability has to be checked separately and no proof is available yet.

6. Maximally symmetric solutions

In this section we want to go beyond the linearized limit and study *maximally symmetric solutions* of the full field equations. The solutions we find will be important in the consequent chapters about AIDG in (A)dS-background. In a maximally symmetric spacetime, the relations

$$R_{abcd} = \frac{R}{6}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \quad \text{and} \quad R_{ab} = \frac{R}{3}g_{ab}, \tag{36}$$

hold with R constant over the manifold. That implies that every curvature quantity is annihilated by the covariant derivative, so most of the terms in Eq. (3) drop out. From the infinite derivative terms only the zeroth order terms without boxes denoted as f_{10} and f_{20}, contribute. Plugging in the expressions above yields

$$R^2\left(\frac{1}{3}f_{10}+\frac{1}{9}f_{20}\right)-\frac{R}{6}+\Lambda=0\,, \tag{37}$$

which is a quadratic equation, so we will in general have *two* solutions of curvature for a given Λ. A particularly interesting case is $\Lambda=0$: Here we get additionally to $R=0$ also the second solution

$$R=\frac{1}{2f_{10}+\frac{2}{3}f_{20}}\,, \tag{38}$$

so an "effective cosmological constant" is generated by the higher-order terms.

One could ask now: Is it then even necessary to include Λ in the action? The answer is yes, because, as we will see in section 7.4, the form factors and hence also their zero'th coefficients will be constrained to certain values.[9]

So if $\Lambda=0$, we have only one specific numerical value available for the background curvature. To obtain the full variety of backgrounds, we have to include Λ.

The situation is fundamentally different in four dimensions because the Weyl tensor C_{abcd} is not necessarily zero, so we have to include a term $C_{abcd}F_3(\Box)\,C^{abcd}$ in the action. However, C_{abcd} is zero for maximally symmetric spacetimes, so the additional term does not change the calculation. Computing Eq. (37) in a general number of dimensions d using arbitrary f_{i0} gives

$$R^2\left(f_{10}\frac{4-d}{d}+f_{20}\frac{4-d}{d^2}\right)+R\frac{2-d}{2d}+\Lambda=0\,, \tag{39}$$

and one sees that for $d=4$ the quadratic part completely drops out. That means in four dimensions we have always the familiar relation

$$R=4\Lambda\,, \tag{40}$$

and the background geometry is independent of the form factors $F_i(\Box)$. This apparent simplification is absent in three dimensions which leads to some caveats as can be seen in the next chapter.

As a side remark, we see from these results that also the famous BTZ black hole is an exact solution: As shown in detail in Ref. [43,44], the BTZ is just an orbifold of AdS space and hence locally indistinguishable from global AdS. The non-trivial boundary conditions do not cause any problems and the BTZ is a perfectly viable background for AIDG in $3d$. We should point out here that the infinitely many derivatives did not play any role in this section, only the zeroth order terms entered the calculation. The result Eq. (37) is also true in Stelle's fourth order gravity.

7. AIDG in (A)dS(3)

7.1. The perturbations around (A)dS

We would like to derive the linearized equations of motion in a stable (A)dS background.[10]

[9] However, the constraints are not the same as derived in sections 3 and 5 because those results were obtained in Minkowski background with $R=0$.

[10] Recently this analysis was extended to more general backgrounds like conformally flat spacetimes [65].

The whole procedure is in close analogy with Refs. [6,7,57]. The action we will consider here is in 3 dimension, and it is given by:

$$S_{AIDG} = \int d^3x \sqrt{-g} \left[\frac{R}{2} + R F_1(\Box) R + R_{ab} F_2(\Box) R^{ab} - \Lambda \right]. \tag{41}$$

In this paper, we will consider the details of the scalar, vector and tensor decomposition of the quadratic part of the action around (A)dS. For later convenience we will rewrite this action using the *traceless Ricci tensor*

$$S_{ab} = R_{ab} - \frac{1}{3} g_{ab} R,$$

as

$$S_{AIDG} = \int d^3x \sqrt{-g} \left[\frac{R}{2} + R \widehat{F}_1(\Box) R + S_{ab} \widehat{F}_2(\Box) S^{ab} - \Lambda \right]. \tag{42}$$

This amounts just to a trivial redefinition of the F_i's, we obtain:

$$\widehat{F}_1(\Box) = F_1(\Box) + \frac{1}{3} F_2(\Box), \qquad \widehat{F}_2(\Box) = F_2(\Box). \tag{43}$$

To obtain the linearized equations of motion, we have to compute the *second variation*, which is a straight forward but laborious task. We have to replace all the quantities by their second order perturbation (for the details see Appendix A.3) using

$$g_{ab} = \overline{g}_{ab} + h_{ab}, \tag{44}$$

and keep only terms quadratic in h_{ab}. The bars on the background quantities have been omitted for simplicity. The different parts of the action shall be analyzed separately.

7.1.1. Einstein-Hilbert part of the action including Λ

The pure Einstein-Hilbert part of the action from Eq. (42) becomes

$$\begin{aligned} S_{EH} \simeq \int d^3x \sqrt{-g} \left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right) \Bigg[\frac{1}{2} \left(R_{ab} + \delta R_{ab} + \delta^2 R_{ab} \right) \\ \times \left(g^{ab} - h^{ab} + h^{ac} h_c^b \right) - \Lambda \Bigg], \end{aligned} \tag{45}$$

where δR_{ab} and $\delta^2 R_{ab}$ are the first and second order variations of the Ricci tensor, defined in Eq. (120). After a lengthy calculation outlined in Appendix A.4, collecting all the quadratic terms yields

$$\begin{aligned} \delta^2 S_{EH} = \int d^3x \sqrt{-g} \Bigg[\frac{1}{8} h^{ab} \Box h_{ab} - \frac{1}{8} h \Box h + \frac{1}{4} h \nabla_a \nabla_b h^{ab} + \\ \frac{1}{4} \nabla_a h^{ab} \nabla_c h_b^c + \frac{1}{48} h^2 R - \frac{1}{12} h_{ab} h^{ab} R - \Lambda \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4} \right) \Bigg]. \end{aligned} \tag{46}$$

For future purposes we shall write this quantity as

$$\delta^2 S_{EH} \equiv \frac{1}{2} \int d^3x \sqrt{-g} \delta_0, \tag{47}$$

with

$$\delta_0 = \delta^2 R + \frac{h}{2} \delta R + \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4} \right) (R - 2\Lambda). \tag{48}$$

7.1.2. Terms containing $\widehat{F}_1(\Box)$

The part of the action containing the Ricci scalar reads:

$$S_R \simeq \int d^3x \sqrt{-g} \left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right) \left(R + \delta R + \delta^2 R\right) \\ \left(\widehat{F}_1(\Box) + \delta \widehat{F}_1(\Box) + \delta^2 \widehat{F}_1(\Box)\right) \left(R + \delta R + \delta^2 R\right). \tag{49}$$

Again, collecting all terms quadratic in h_{ab} results in

$$\delta^2 S_R = \int d^3x \sqrt{-g} \Bigg[R \widehat{F}_1 (\Box) \delta^2 R + R \delta \widehat{F}_1 (\Box) \delta R + R \delta^2 \widehat{F}_1 (\Box) R + \widehat{F}_1 (\Box) \delta R \\ + \delta R \delta \widehat{F}_1 (\Box) R + \delta^2 R \widehat{F}_1 (\Box) R + \frac{h}{2} \left(R \widehat{F}_1 (\Box) \delta R + R \delta \widehat{F}_1 (\Box) R + \delta R \widehat{F}_1 (\Box) R\right) \\ + \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right) R \widehat{F}_1 (\Box) R \Bigg]. \tag{50}$$

The variation of $\Box$ acting on a scalar is given by

$$\delta (\Box) \varphi = \left(-h^{ab} \nabla_a \partial_b - g^{ab} \delta \Gamma^c_{ab} \partial_c\right) \varphi , \tag{51}$$

where $\delta \Gamma^c_{ab}$ denotes the variation of the Christoffel symbol (see Appendix A.3). We conclude that the constant background curvature R is annihilated by all variations, $\delta^i \widehat{F}_1 (\Box)$, but the zeroth coefficient $\widehat{f}_{10}$ in the expansion of $\widehat{F}_1 (\Box) = \sum_{n=0}^{\infty} \widehat{f}_{1n} \Box^n$ survives. A reorganization of the terms now yields

$$\delta^2 S_R = \int d^3x \sqrt{-g} \Bigg[\left(h \delta R + \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right) R + 2\delta^2 R\right) \widehat{f}_{10} R + \delta R \widehat{F}_1 (\Box) \delta R \\ + \frac{h}{2} R \left(\widehat{F}_1 (\Box) - \widehat{f}_{10}\right) \delta R + R \delta \widehat{F}_1 (\Box) \delta R \Bigg]. \tag{52}$$

It can be further simplified by using Eq. (48), and we arrive at

$$\delta^2 S_R = \int d^3x \sqrt{-g} \Bigg[2 \widehat{f}_{10} R \delta_0 + 2 \widehat{f}_{10} R \left(2\Lambda - \frac{R}{2}\right) \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right) + \delta R \widehat{F}_1 (\Box) \delta R \\ + \frac{h}{2} R \left(\widehat{F}_1 (\Box) - \widehat{f}_{10}\right) \delta R + R \delta \widehat{F}_1 (\Box) \delta R \Bigg]. \tag{53}$$

[11] In the second line, the two terms can be seen to cancel away. First, the variation in the last term has to appear at the extreme left, otherwise the term becomes a total derivative. Next, by expanding the power series in both of the terms gives:

$$\sum_{n=1}^{\infty} \widehat{f}_{1n} R \int d^3x \sqrt{-g} \left(\frac{h}{2} \Box + \delta (\Box)\right) \Box^{n-1} \delta R , \tag{54}$$

---

[11] In four dimensions, the second term vanishes due to the background constraint $\Lambda = \frac{R}{4}$.

and by using Eq. (51) again, we find

$$\begin{aligned}
&\sum_{n=1}^{\infty}\widehat{f}_{1n}R\int d^3x\sqrt{-g}\left(\frac{h}{2}\Box-h^{ab}\nabla_a\partial_b-g^{ab}\delta\Gamma^c_{ab}\partial_c\right)\Box^{n-1}\delta R\\
=&\sum_{n=1}^{\infty}\widehat{f}_{1n}R\int d^3x\sqrt{-g}\left(\frac{h}{2}\Box-h^{ab}\nabla_a\partial_b-\nabla_a h^{ab}\partial_b+\nabla_a h\partial^a\right)\Box^{n-1}\delta R\\
=&\sum_{n=1}^{\infty}\widehat{f}_{1n}R\int d^3x\sqrt{-g}\left(\frac{h}{2}\Box^n\delta R-\nabla_a\left(h^{ab}\partial_b\Box^{n-1}\delta R\right)-h\nabla_a\partial^a\Box^{n-1}\delta R\right)\\
=&\sum_{n=1}^{\infty}\widehat{f}_{1n}R\int d^3x\sqrt{-g}\left(-\nabla_a\left(h^{ab}\partial_b\Box^{n-1}\delta R\right)\right),
\end{aligned}\tag{55}$$

which is a total derivative and therefore vanishes. Hence the final result is given by:

$$\delta^2 S_R=\int d^3x\sqrt{-g}\left[2\widehat{f}_{10}R\delta_0+2\widehat{f}_{10}R\left(2\Lambda-\frac{R}{2}\right)\left(\frac{h^2}{8}-\frac{h_{ab}h^{ab}}{4}\right)+\delta R\widehat{F}_1(\Box)\delta R\right].\tag{56}$$

7.1.3. Terms containing $\widehat{F}_2(\Box)$

The last variation is particularly simple because the traceless Ricci tensor vanishes for maximally symmetric spacetimes, so the two variations have to act on both S_{ab} to produce a non-zero result. We have

$$\begin{aligned}
\delta^2 S_S&=\int d^3x\sqrt{-g}\left(\delta R_{ab}-\frac{1}{3}g_{ab}\delta R\right)\widehat{F}_2(\Box)\left(\delta R^{ab}-\frac{1}{3}g^{ab}\delta R\right)\\
&=\int d^3x\sqrt{-g}\left[\delta R_{ab}\widehat{F}_2(\Box)\delta R^{ab}-\frac{1}{3}\delta R\widehat{F}_2(\Box)\delta R\right],
\end{aligned}\tag{57}$$

so that we can write the complete variation as

$$\begin{aligned}
\delta^2 S=\int d^3x\sqrt{-g}\Bigg[&\left(\frac{1}{2}+2\widehat{f}_{10}R\right)\delta_0+\widehat{f}_{10}R(4\Lambda-R)\left(\frac{h^2}{8}-\frac{h_{ab}h^{ab}}{4}\right)\\
&+\delta R F_1(\Box)\delta R+\delta R_{ab}F_2(\Box)\delta R^{ab}\Bigg],
\end{aligned}\tag{58}$$

using the unhatted $F_i(\Box)$ again. This expression can be simplified considerably by inserting the relation Eq. (37) between the cosmological constant and the background curvature. Some of the terms of higher order in R cancel away and we are left with

$$\delta^2 S=\int d^3x\sqrt{-g}\left[\left(\frac{1}{2}+\widehat{f}_{10}R\right)\widetilde{\delta}_0+\delta R F_1(\Box)\delta R+\delta R_{ab}F_2(\Box)\delta R^{ab}\right],\tag{59}$$

where $\widetilde{\delta}_0$ is defined as

$$\widetilde{\delta}_0=\frac{1}{4}h^{ab}\Box h_{ab}-\frac{1}{4}h\Box h+\frac{1}{2}h\nabla_a\nabla_b h^{ab}+\frac{1}{2}\nabla_a h^{ab}\nabla_c h^c_b-\frac{R}{12}h_{ab}h^{ab}.\tag{60}$$

Note that $\widetilde{\delta}_0$ is exactly what we have had obtained in pure Einstein gravity: Eq. (37) would then reduce to $\Lambda=6R$ and from Eq. (46) we would obtain

$$\delta^2 S_{EH} \equiv \frac{1}{2}\int d^3x\,\sqrt{-g}\widetilde{\delta}_0. \tag{61}$$

So the terms generated by the non-trivial relation Eq. (37) cancel away and Eq. (59) has the same form as in 4d, apart from the fact that the Weyl term is absent.

7.2. Scalar, vector and tensor decompositions of the metric perturbations

To proceed further, we have to decompose the metric perturbation into the different spin states [6,7]. However, there is one big difference: In an AdS background we cannot go to Fourier space globally, hence the group theoretic arguments outlined in Appendix A.1 do not work straight forwardly anymore. Instead, we will follow the procedure outlined in [58,6,7], see also Refs. [59–62] where the authors have found the graviton propagator in dS in 4 dimensions. Let us define the metric perturbation as

$$h_{ab} = h^{\perp}_{ab} + \nabla_a A^{\perp}_b + \nabla_b A^{\perp}_a + \nabla_a\nabla_b B - g_{ab}\phi\,, \tag{62}$$

where the tensor part obeys the transverse and traceless condition

$$\nabla^a h^{\perp}_{ab} = h^{\perp} = 0,$$

and so does the vector part $\nabla^a A^{\perp}_a = 0$. This decomposition corresponds exactly to the one which was done in flat space using the spin projection operators, i.e. $h^{\perp}_{ab}$ corresponds to P^2_s, $A^{\perp}_a$ corresponds to P^1_w, B corresponds to P^0_w and ϕ to P^0_s. Since we do not want to increase the number of degrees of freedom as compared to Einstein's gravity, we have to demand that $A^{\perp}_a$ and B drop out of the quadratic action Eq. (59). This we will show explicitly below. Then, $h^{\perp}_{ab}$ and ϕ will correspond exactly to the 3 off-shell propagating degrees of freedom. Let us start with decomposing the variation of the Ricci tensor δR_{ab} which appears in the higher-derivative terms. δR can then be obtained by a simple contraction.

7.2.1. Decomposition of δR_{ab}

We note that the content of this subsection is entirely geometrical, without referring to any particular theory.

- We start with the vector mode $A^{\perp}_b$; inserting Eq. (62) into Eq. (119) (see Appendix A.3) and contracting with δ^c_a yields

$$\delta R^a{}_b\left(A^{\perp}\right) = -\frac{R}{12}\left(\nabla_b A^a + \nabla^a A_b\right) + \frac{1}{2}\Big(\nabla_c\nabla^a\nabla_b A^c + \nabla_c\nabla^a\nabla^c A_b - \Box\nabla_b A^a - \Box\nabla^a A_b + \nabla_b\Box A^a + \nabla_b\nabla^c\nabla^a A_c\Big), \tag{63}$$

already using $\nabla_a A^a = 0$. With the help of the Riemann tensor substitution and the commutation relation Eq. (127) in Appendix A.5, we can rewrite this as

$$\begin{aligned}\delta R^a{}_b\left(A^{\perp}\right) &= -\frac{R}{12}\left(\nabla_b A^a + \nabla^a A_b\right) + \frac{1}{2}\Big(\nabla^a\nabla_c\nabla_b A^c + R_c{}^a{}_b{}^d\nabla_d A^c \\ &\quad + R_c{}^{ac}{}_d\nabla_b A^d + R^{ac}{}_{bd}\nabla_c A^d - \frac{R}{3}\left(\nabla_b A^a + \nabla^a A_b\right) + R^{ca}{}_{cd}\nabla_b A^d\Big) \\ &= -\frac{R}{12}\left(\nabla_b A^a + \nabla^a A_b\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\bigg(\frac{R}{3}\nabla^a A_b+\frac{R}{6}\nabla^a A_b+\frac{R}{3}\nabla_b A^a-\frac{R}{6}\nabla_b A^a-\frac{R}{3}\left(\nabla_b A^a+\nabla^a A_b\right)\\
&+\frac{R}{3}\nabla_b A^a\bigg)\\
=\,&0\,,
\end{aligned}
\tag{64}
$$

which is zero as desired.

- For the scalar mode B, we again insert Eq. (62) into Eq. (119) and contract with δ^c_a to get

$$
\begin{aligned}
\delta R^a{}_b\,(B)=&-\frac{R}{12}\left(\nabla^a\nabla_b B+\delta^a_b\Box B\right)+\frac{1}{2}\big(\nabla_c\nabla^a\nabla^c\nabla_b B+\Box\nabla^a\nabla_b B-\nabla_b\nabla^a\Box B\\
&+\nabla_b\nabla^c\nabla^a\nabla_c B\big)\,.
\end{aligned}
\tag{65}
$$

Exchanging ∇^a and ∇^c in the first and the last term of the second parenthesis yields

$$
\begin{aligned}
\delta R^a{}_b\,(B)&=-\frac{R}{12}\left(\nabla^a\nabla_b B+\delta^a_b\Box B\right)+\frac{1}{2}\left(R^{ac}{}_b{}^d\nabla_c\nabla_d B+R^{ca}{}_c{}^d\nabla_b\nabla_d B\right)\\
&=-\frac{R}{12}\left(\nabla^a\nabla_b B+\delta^a_b\Box B\right)+\frac{1}{2}\left(\frac{R}{6}\delta^a_b\Box B-\frac{R}{6}\nabla^a\nabla_b B+\frac{R}{3}\nabla^a\nabla_b B\right)\\
&=0\,.
\end{aligned}
\tag{66}
$$

- For the remaining two modes we expect a non-zero result: The scalar ϕ inserted into (119) yields

$$
\delta R^{ab}{}_{cd}\,(\phi)=\frac{R}{6}\delta^{ab}_{cd}\phi-\frac{1}{2}\left(\nabla_c\nabla^b\delta^a_d-\nabla_c\nabla^a\delta^b_d-\nabla_d\nabla^b\delta^a_c+\nabla_d\nabla^a\delta^b_c\right)\phi. \tag{67}
$$

It is now practical to define the traceless differential operator

$$
D^a_b=\nabla_b\nabla^a-\delta^a_b\frac{\Box}{3}. \tag{68}
$$

Using D^a_b the equation above can be rewritten as

$$
\delta R^{ab}{}_{cd}\,(\phi)=\frac{1}{2}\left(D^a_c\delta^b_d+D^b_d\delta^a_c-D^b_c\delta^a_d-D^a_d\delta^b_c\right)\phi+\frac{2\Box+R}{6}\delta^{ab}_{cd}\phi\,, \tag{69}
$$

and

$$
\delta R^a{}_b\,(\phi)=\left(\frac{1}{2}D^a_b+\frac{R+2\Box}{3}\delta^a_b\right)\phi\quad\text{and}\quad\delta R\,(\phi)=(2\Box+R)\,\phi\,, \tag{70}
$$

follow straight forwardly.

- Similarly, the tensor mode gives

$$
\begin{aligned}
\delta R^{ab}{}_{cd}=&\frac{R}{12}\left(\delta^a_d h^{\perp b}_c-\delta^a_c h^{\perp b}_d+\delta^b_c h^{\perp a}_d-\delta^b_d h^{\perp a}_c\right)+\\
&\frac{1}{2}\left(\nabla_c\nabla^b h^{\perp a}_d-\nabla_c\nabla^a h^{\perp b}_d-\nabla_d\nabla^b h^{\perp a}_c+\nabla_d\nabla^a h^{\perp b}_c\right)\,,
\end{aligned}
\tag{71}
$$

and after contracting with δ^c_a, and using the Riemann tensor substitution

$$
\left[\nabla^c,\nabla_a\right]h^{\perp}_{cb}=R^{cd}{}_{ca}h^{\perp}_{db}+R_b{}^{dc}{}_a h^{\perp}_{cd}\,, \tag{72}
$$

we obtain

$$\delta R^a{}_b\left(h^\perp\right) = -\frac{R}{12}h_b^{\perp a} - \frac{1}{2}\Box h_b^{\perp a} + \frac{1}{2}\nabla^c\nabla^a h_{cb}^\perp = -\frac{1}{2}\left(\Box - \frac{R}{3}\right)h_b^{\perp a}\,, \quad \text{and}$$
$$\delta R\left(h^\perp\right) = 0\,. \tag{73}$$

7.2.2. Decomposition of the Einstein-Hilbert part

Of course we want to find the same non-vanishing degrees of freedom also in the pure GR-part. Though, it is a well-known result that linearized Einstein gravity does not contain any longitudinal excitations, we will verify it here explicitly.

- By inserting Eq. (62) into $\delta^2 S_{EH}$, we obtain for the vector mode

$$\begin{aligned}\widetilde{\delta}_0\left(A_a^\perp\right) =& \frac{1}{4}\left(\nabla_a A_b + \nabla_b A_a\right)\Box\left(\nabla^a A^b + \nabla^b A^a\right) + \frac{1}{2}\left(\Box A^b + \nabla_a\nabla^b A^a\right)\\ &\times\left(\Box A_b + \nabla_a\nabla_b A^a\right) - \frac{R}{12}\left(\nabla_a A_b + \nabla_b A_a\right)\left(\nabla^a A^b + \nabla^b A^a\right).\end{aligned} \tag{74}$$

$\widetilde{\delta}_0$ is an integrand, so we can perform partial integration, moreover utilizing the commutation relations Eqs. (127), (129), we obtain:

$$\widetilde{\delta}_0\left(A_a^\perp\right) = -\frac{1}{2}A_b\nabla_a\Box\left(\nabla^a A^b + \nabla^b A^a\right) + \frac{1}{2}A^b\left(\Box + \frac{R}{3}\right)^2 A_b + \frac{R}{6}A^b\left(\Box + \frac{R}{3}\right)A_b. \tag{75}$$

The first term can be further simplified using Eqs. (128), (129):

$$\begin{aligned}\widetilde{\delta}_0\left(A_a^\perp\right) =& -\frac{1}{2}A_b\left(\Box + \frac{2R}{3}\right)\left(\Box + \frac{R}{3}\right)A^b + \frac{1}{2}A^b\left(\Box + \frac{R}{3}\right)^2 A_b\\ &+ \frac{R}{6}A^b\left(\Box + \frac{R}{3}\right)A_b = 0\,.\end{aligned} \tag{76}$$

- For the scalar mode B, we obtain similarly

$$\begin{aligned}\widetilde{\delta}_0(B) =& \frac{1}{4}\nabla_a\nabla_b B\Box\nabla^a\nabla^b B - \frac{1}{4}\Box B\Box^2 B + \frac{1}{2}\Box B\nabla_a\nabla_b\nabla^a\nabla^b B +\\ &\frac{1}{2}\Box\nabla_a B\Box\nabla^a B - \frac{1}{12}R\nabla_a\nabla_b B\nabla^a\nabla^b B\\ =& B\left(\frac{1}{4}\nabla_b\left(\Box + \frac{2R}{3}\right)\Box\nabla^b - \frac{R}{12}\Box^2 + \frac{1}{4}\Box^3 +\right.\\ &\left.\frac{1}{2}\Box\nabla_a R^a{}_b\nabla^b - \frac{1}{2}\left(\Box + \frac{R}{3}\right)\nabla_a\Box\nabla^a - \frac{1}{12}R\left(\Box + \frac{R}{3}\right)\Box\right)B\\ =& B\left(-\frac{1}{4}\left(\Box + \frac{R}{3}\right)^2\Box + \frac{R}{6}\left(\Box + \frac{R}{3}\right)\Box + \frac{1}{4}\Box^3 - \frac{1}{36}R^2\Box\right)B\\ =& 0\,.\end{aligned} \tag{77}$$

- The non-zero modes can also be evaluated straight forwardly resulting in

$$\widetilde{\delta}_0(\phi) = \frac{3}{4}\phi\Box\phi - \frac{9}{4}\phi\Box\phi + \frac{3}{2}\phi\Box\phi + \frac{1}{2}\nabla_a\phi\nabla^a\phi - \frac{R}{4}\phi^2 = -\frac{1}{4}\phi\left(2\Box + R\right)\phi\,, \tag{78}$$

and

$$\widetilde{\delta}_0\left(h^{\perp}\right)=\frac{1}{4}h^{\perp}_{ab}\left(\Box-\frac{R}{3}\right)h^{\perp ab}\,. \tag{79}$$

7.3. Propagator of AIDG in (A)dS(3)

Finally we wish to obtain the propagators for the two remaining modes $h^{\perp}_{ab}$ and ϕ. The first question is, if those two modes really decouple from each other, i.e. that we can write the final quadratic action as a sum of the two separate actions. We know this to be true for the GR-part and in the F_1-term no $h^{\perp}_{ab}$ can survive. Hence, the only suspicious term is the F_2-term, but here we can show straight forwardly that no coupling occurs: (see also Ref. [7]). The question is now: do any terms survive in the combination

$$\begin{aligned}&\int d^3x\,\sqrt{-g}\,\delta R_{ab}\left(\phi\right)F_2\left(\Box\right)\delta R^{ab}\left(h^{\perp}\right)\\&=-\int d^3x\,\sqrt{-g}\,\phi\left(\frac{1}{2}D_{ab}+\frac{R+2\Box}{3}g_{ab}\right)F_2\left(\Box\right)\frac{1}{2}\left(\Box-\frac{R}{3}\right)h^{\perp ab}\,.\end{aligned} \tag{80}$$

The metric g_{ab} can be commuted through to annihilate $h^{\perp ab}$, so the only potentially problematic term is of the form

$$\nabla_a\nabla_b F_2\left(\Box\right)\left(\Box-\frac{R}{3}\right)h^{\perp ab}.$$

By expanding $F_2\left(\Box\right)$ in its power series $F_2=\sum_{n=0}^{\infty}f_{2n}\Box^n$ and then using the commutation relation Eq. (128) iteratively, we can commute through ∇_b all the way till it annihilates $h^{\perp ab}$. Hence, we have shown that the physical fields decouple nicely and we can turn now to the evaluation of the propagators. To start with the scalar mode, we use the expressions

$$\begin{aligned}\delta R_{ab}\left(\phi\right)&=\left(\frac{1}{2}D_{ab}+\frac{R+2\Box}{3}g_{ab}\right)\phi\,,\\\delta R\left(\phi\right)&=\left(2\Box+R\right)\phi\,,\\\widetilde{\delta}_0\left(\phi\right)&=-\frac{1}{4}\phi\left(2\Box+R\right)\phi\,,\end{aligned} \tag{81}$$

derived above to write

$$\begin{aligned}S_{qua}\left(\phi\right)=\int d^3x\,\sqrt{-g}\Bigg[&-\left(\frac{1}{8}+\frac{1}{2}\widehat{f}_{10}R\right)\phi\left(2\Box+R\right)\phi\\&+\phi\left(2\Box+R\right)F_1\left(\Box\right)\left(2\Box+R\right)\phi\\&+\phi\left(\frac{1}{2}D_{ab}+\frac{R+2\Box}{3}g_{ab}\right)F_2\left(\Box\right)\left(\frac{1}{2}D^{ab}+\frac{R+2\Box}{3}g^{ab}\right)\phi\Bigg].\end{aligned} \tag{82}$$

The last term still allows for some simplification. After expanding $F_2\left(\Box\right)$ the commutation relation Eq. (130) can be used to obtain:

$$\int d^3x\,\sqrt{-g}\left[\phi D_{ab}F_2\left(\Box\right)D^{ab}\phi\right]=\int d^3x\,\sqrt{-g}\left[\phi F_2\left(\Box+R\right)\left(\frac{2\Box+R}{3}\right)\Box\phi\right]. \tag{83}$$

Now, putting everything together and using the tracelessness of D_{ab}, see Eq. (68), yields the final result

$$S_{qua}(\phi) = \int d^3x \sqrt{-g}\, \phi \Bigg[-\left(\frac{1}{8} + \frac{1}{2}\widehat{f}_{10} R\right) + F_1(\Box)(2\Box + R) + \frac{1}{3} F_2(\Box)(2\Box + R) + \frac{1}{12} F_2(\Box + R)\Box \Bigg] (2\Box + R)\,\phi. \tag{84}$$

The tensor mode is a lot simpler to handle because of its transverse-traceless property. Using the expressions Eq. (73), (79) the final action results in

$$S_{qua}\left(h^{\perp}\right) = \int d^3x \sqrt{-g}\, \frac{1}{8} h^{\perp}_{ab} \left[\left(\Box - \frac{R}{3}\right)\left(1 + 4\widehat{f}_{10} R\right) + 2F_2(\Box)\left(\Box - \frac{R}{3}\right)^2 \right] h^{\perp ab}. \tag{85}$$

The tensor and scalar part of the propagators can now be given by:

$$\Pi(\phi) = \frac{P_s^0}{\left[1 + 4\widehat{f}_{10} R - 16 F_1(\Box)\left(\Box + \frac{R}{2}\right) - \frac{16}{3} F_2(\Box)\left(\Box + \frac{R}{2}\right) - \frac{2}{3} F_2(\Box + R)\Box\right]\left(\Box + \frac{R}{2}\right)}, \tag{86}$$

and

$$\Pi\left(h^{\perp}\right) = -\frac{P_s^2}{\left[1 + 4\widehat{f}_{10} R + 2F_2(\Box)\left(\Box - \frac{R}{3}\right)\right]\left(\Box - \frac{R}{3}\right)}. \tag{87}$$

An important issue here is the normalization of the propagators: Since we want to take the Minkowski limit $R \to 0$ in the next section, the propagators have to be normalized correctly. From Eq. (16) and Eq. (13) we see that the first constant term in the denominator of $\Pi\left(h^{\perp}\right)$ should be 1, hence we chose that as our normalization condition and removed a factor of $\frac{1}{8}$ from both propagators. For the scalar part we had to add an additional factor of $\frac{1}{2}$ which is contained in the spin projection operator.

7.4. Discussions, comparisons and IR limits

We turn now to the interpretation of the results obtained in Eqs. (86), (87). As a nice cross-check we can take the limit $R \to 0$, which should of course reproduce the propagator in Minkowski space. We get

$$\Pi_{\Lambda=0}\left(h^{\perp}\right) = -\frac{1}{\Box\left(1 + 2F_2(\Box)\Box\right)} = -\frac{1}{a(\Box)\Box}, \tag{88}$$

by using the relations Eq. (13) which is the desired result. For the scalar part we have to add an additional factor of $\frac{1}{2}$ which is contained in the spin projection operator but after that we get

$$\Pi_{\Lambda=0}(\phi) = -\frac{1}{\Box\left(-1 + 16F_1(\Box)\Box + 6F_2(\Box)\Box\right)} = -\frac{1}{\left(a(\Box) - 2c(\Box)\right)\Box}, \tag{89}$$

as expected.

Another instructive limit is to take is $F_i(\Box) \to 0$, in Eqs. (86), (87), i.e. turning off the infinite derivative terms. This would leave the graviton off-shell propagator in the (A)ds background:

$$\Pi = \frac{P_s^2}{-\Box + \frac{R}{3}} - \frac{P_s^0}{-\Box - \frac{R}{2}}\,, \tag{90}$$

where we see that the graviton acquires a non-vanishing mass due to the spacetime curvature.

The last question remains is what is the most natural choice for the form factors $F_i(\Box)$? In flat space we required the propagator to be proportional to the GR-propagator and this should also be our goal here. By comparing Eqs. (86), (87) with Eq. (90), we obtain the necessary constraint on the form factors:

$$F_1(\Box) = -\frac{1}{8}F_2(\Box)\frac{\left(\Box - \frac{R}{3}\right)}{\left(\Box + \frac{R}{2}\right)} - \frac{1}{24}F_2(\Box + R)\frac{\Box}{\left(\Box + \frac{R}{2}\right)} - \frac{1}{3}F_2(\Box)\,. \tag{91}$$

In the Minkowski limit, when $R = 0$, we obtain exactly $2F_1(\Box) + F_2(\Box) = 0$, see Eq. (18).

Now, back into the (A)ds, the function of proportionality which we shall call $a(\Box)$ in accordance with the chapter 3 (the treatment in the Minkowski space) is given by

$$a(\Box) = 1 + 4\widehat{f}_{10}R + 2F_2(\Box)\left(\Box - \frac{R}{3}\right). \tag{92}$$

For not introducing any new zeros in the propagator, $a(\Box)$ has to be an exponential of an entire function, i.e. $Ce^{\gamma(\Box)}$ where $\gamma(\Box)$ is an entire function and $C \neq 0$ a constant. However, a simple choice of $a(\Box) = e^{-\frac{\Box}{M_s^2}}$ is not viable anymore, because $F_1(\Box)$ and $F_2(\Box)$ will not be analytic then: If we solve Eq. (92) for $F_2(\Box)$ we get

$$F_2(\Box) = \frac{Ce^{\gamma(\Box)} - 1 - 4\widehat{f}_{10}R}{2\left(\Box - \frac{R}{3}\right)}\,. \tag{93}$$

If we expand the exponential we see that we must have

$$C = 1 + 4\widehat{f}_{10}R\,, \tag{94}$$

otherwise we produce a term proportional to $\frac{1}{\left(\Box - \frac{R}{3}\right)}$. Moreover, $\gamma(\Box)$ has to contain a factor $\left(\Box - \frac{R}{3}\right)$ to cancel the denominator, so we arrive at

$$a(\Box) = \left(1 + 4\widehat{f}_{10}R\right)e^{\left(\Box - \frac{R}{3}\right)\tau(\Box)} \tag{95}$$

with some entire function $\tau(\Box)$. Now, solving Eq. (91) for $F_1(\Box)$ yields

$$\begin{aligned} F_1(\Box) = &-\frac{1}{16}\frac{\left(1 + 4\widehat{f}_{10}R\right)\left(e^{\left(\Box - \frac{R}{3}\right)\tau(\Box)} - 1\right)}{\Box + \frac{R}{2}} \\ &-\frac{1}{48}\frac{\left(1 + 4\widehat{f}_{10}R\right)\left(e^{\left(\Box + \frac{2R}{3}\right)\tau(\Box + R)} - 1\right)\Box}{\left(\Box + \frac{R}{2}\right)\left(\Box + \frac{2R}{3}\right)} \\ &-\frac{1}{6}\frac{\left(1 + 4\widehat{f}_{10}R\right)\left(e^{\left(\Box - \frac{R}{3}\right)\tau(\Box)} - 1\right)}{\Box - \frac{R}{3}} \end{aligned} \tag{96}$$

and again analyticity demands that we cancel the denominators. We see that $\tau(\Box)$ has to contain the factors $\Box + \frac{R}{2}$ and $\Box - \frac{R}{2}$, hence the simplest choice for $a(\Box)$ is

$$a(\Box) = \left(1 + 4\widehat{f}_{10}R\right) e^{-\frac{\left(\Box+\frac{R}{2}\right)\left(\Box-\frac{R}{2}\right)\left(\Box-\frac{R}{3}\right)}{M_s^6}}, \tag{97}$$

then the $F_i(\Box)$ will be perfectly analytic:

$$\begin{aligned} F_1(\Box) = &-\left(1 + 4\widehat{f}_{10}R\right)\left(\frac{e^{-\frac{\left(\Box+\frac{R}{2}\right)\left(\Box-\frac{R}{2}\right)\left(\Box-\frac{R}{3}\right)}{M_s^6}} - 1}{16\left(\Box + \frac{R}{2}\right)} + \right. \\ &\left. \frac{\left(e^{-\frac{\left(\Box+\frac{3R}{2}\right)\left(\Box+\frac{R}{2}\right)\left(\Box+\frac{2R}{3}\right)}{M_s^6}} - 1\right)\Box}{48\left(\Box + \frac{2R}{3}\right)\left(\Box + \frac{R}{2}\right)} + \frac{e^{-\frac{\left(\Box+\frac{R}{2}\right)\left(\Box-\frac{R}{2}\right)\left(\Box-\frac{R}{3}\right)}{M_s^6}} - 1}{6\left(\Box - \frac{R}{3}\right)}\right), \end{aligned} \tag{98}$$

and

$$F_2(\Box) = \frac{\left(1 + 4\widehat{f}_{10}R\right)\left(e^{-\frac{\left(\Box+\frac{R}{2}\right)\left(\Box-\frac{R}{2}\right)\left(\Box-\frac{R}{3}\right)}{M_s^6}} - 1\right)}{2\left(\Box - \frac{R}{3}\right)}. \tag{99}$$

There is one last point to consider: Note that we have treated $\widehat{f}_{10}$ as an independent variable so far, however, it is supposed to be the zeroth order coefficient of $\widehat{F}_1(\Box)$. With the help of Eqs. (43), (98), we obtain:

$$\begin{aligned} \widehat{F}_1(\Box) = &-\left(1 + 4\widehat{f}_{10}R\right) \\ &\times \left(\frac{e^{-\frac{\left(\Box+\frac{R}{2}\right)\left(\Box-\frac{R}{2}\right)\left(\Box-\frac{R}{3}\right)}{M_s^6}} - 1}{16\left(\Box + \frac{R}{2}\right)} + \frac{\left(e^{-\frac{\left(\Box+\frac{3R}{2}\right)\left(\Box+\frac{R}{2}\right)\left(\Box+\frac{2R}{3}\right)}{M_s^6}} - 1\right)\Box}{48\left(\Box + \frac{2R}{3}\right)\left(\Box + \frac{R}{2}\right)}\right). \end{aligned} \tag{100}$$

We can now extract the zeroth order term from above, and obtain

$$\widehat{f}_{10} = -\left(1 + 4\widehat{f}_{10}R\right)\frac{e^{-\frac{R^3}{12M_s^6}} - 1}{8R}, \tag{101}$$

with the solution

$$\widehat{f}_{10} = \frac{1}{4R}\frac{1 - e^{-\frac{R^3}{12M_s^6}}}{1 + e^{-\frac{R^3}{12M_s^6}}}. \tag{102}$$

We should point out that the form factors depend explicitly on the background curvature. If one takes the limit $R \to 0$ in the above expressions, we will reduce $F_1(\Box)$ and $F_2(\Box)$ to

$$F_1(\Box) = -\frac{e^{-\Box^3/M_s^6} - 1}{4\Box}, \qquad F_2(\Box) = \frac{e^{-\Box^3/M_s^6} - 1}{2\Box} \tag{103}$$

which defers from the flat space case in the sense that we have different powers of $\frac{\Box}{M_s^2}$, see Eq. (19). If we want to have a smooth Minkowski limit we have to replace $a(\Box)$ in Eq. (17) by

$$a'(\Box) = e^{-\Box^3/M_s^6}. \tag{104}$$

By the above choice, the desirable properties of the theory like tree level unitarity will remain unchanged by this modification.

8. Conclusion

This paper provides the ghost free conditions for parity invariant and torsion free AIDG in $3d$. At first we determined the full equations of motion and deduced the linearized limit in flat space in complete analogy to the $4d$ case without introducing any new degrees of freedom. As expected, we also found exact maximally symmetric solutions which can serve as background solutions for linearization. With considerable algebraic effort, it was possible to construct a well defined linearized theory around those (A)dS-backgrounds. We also considered New Massive Gravity as a low-energy limit instead of GR, and succeeded in constructing an AIDG action around new massive gravity around the Minkowski background.

The main highlights of the paper are following. First of all we have shown that the vacuum solution of AIDG in 3 dimensions respects the BTZ blackhole solution in AdS. However, adding a point source generates a non-trivial, non-singular solution. The solution so far has been obtained only around the Minkowski background. Second important result is that we have derived two main equations in this paper containing the scalar and the graviton propagators for AIDG action in (A)dS in 3 dimensions, see Eqs. (86), (87). These have been obtained by perturbing the action up to quadratic in metric potential, i.e. $\mathcal{O}(h^2)$ around (A)dS background in 3 dimensions. We have discussed various consistency checks, such as our results of the propagators match the expectations around the Minkowski background. We have also verified that the propagator reduces to that of Einstein gravity in 3 dimensions around the (A)dS background when we take the appropriate limit $\Box/M_s^2 \to 0$, or $F_i(\Box) \to 0$. We have also provided an example of the analytic form factors $F_1(\Box)$ and $F_2(\Box)$ around (A)dS backgrounds.

There are still some open questions remain. First, we have not proven that the maximally symmetric spacetimes are really the only vacuum solutions. If that is the case, it would be natural to assume that AIDG in the vacuum, as GR, is a topological field theory. Since it does not seem to be a Chern-Simons-theory (at least there is no natural connection) yet, it is an interesting open problem to classify it as some other topological field theory. Furthermore, one could add a boundary and try to find the dual conformal field theory, if it exists, to provide a new realization of the holographic principle. All in all, AIDG in three dimensions has shown many interesting features which make it worth studying these aspects further.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

A.1. Inverting the field equations

We can write the linearized field equations in the form $\left(\Pi^{-1}\right)_{ab}^{\ \ cd} h_{cd} = \kappa\tau_{ab}$ with the linear operator

$$\left(\Pi^{-1}\right)_{ab}^{\ \ cd} = k^2 a\left(-k^2\right)\delta_a^{(c}\delta_b^{d)} + 2b\left(-k^2\right)k^{(c}k_{(a}\eta_{b)}^{d)} + c\left(-k^2\right)\left(\eta_{ab}k^c k^d + k_a k_b \eta^{cd}\right) + k^2 d\left(-k^2\right)\eta_{ab}\eta^{cd} + \frac{f\left(-k^2\right)}{k^2}k_a k_b k^c k^d. \tag{105}$$

One can significantly simplify Π^{-1} by using invariance properties: Eq. (105) is constructed solely out of η_{ab} and k^a, hence the little group of k^a commutes with it. If we take k^a to be time-like, then the little group is SO(2).[12] By Schur's lemma, every operator that commutes with all elements of a group in one of its irreducible representations, has to be proportional to the identity operator. The symmetric-tensor-representation of SO(2) is decomposable into four irreducible representations: one with spin two (2 degrees of freedom), one with spin one (2 dof) and two scalars (1 dof each).[13] It is now useful to define the so-called *spin projection operators* which project on these four subspaces [55,54]:

$$\begin{aligned}
P_s^2 &= \frac{1}{2}\left(\theta_{ac}\theta_{bd} + \theta_{ad}\theta_{bc}\right) - \frac{1}{2}\theta_{ab}\theta_{cd}\,,\\
P_w^1 &= \frac{1}{2}\left(\theta_{ac}\omega_{bd} + \theta_{ad}\omega_{bc} + \theta_{bc}\omega_{ad} + \theta_{bd}\omega_{ac}\right)\,,\\
P_s^0 &= \frac{1}{2}\theta_{ab}\theta_{cd}\,,\\
P_w^0 &= \omega_{ab}\omega_{cd}\,,
\end{aligned} \tag{106}$$

with

$$\theta_{ab} = \eta_{ab} - \frac{k_a k_b}{k^2} \ \text{ and } \ \omega_{ab} = \frac{k_a k_b}{k^2}. \tag{107}$$

It is easy to verify that they are all orthogonal and satisfy:

$$P_s^2 + P_w^1 + P_s^0 + P_w^0 = 1 \ \text{ and } \ P_a^i P_b^j = \delta^{ij}\delta_{ab}. \tag{108}$$

[12] Later it will turn out that k^a is actually light-like, however, SO(2) is more useful to decompose the eoms than the little group of a light-like vector, ISO(1).

[13] In three dimensions, the notion of *spin* should be regarded with care: Usually, spin refers to representations of SO(3) which are important for massive particles in four dimensions. The representations of SO(2) we use here are the same as for massless particles in 4d which we classify according to their *helicity*.

P_s^2 corresponds to the transverse and traceless degrees of freedom, P_w^1 to the longitudinal and traceless ones, P_s^0 represents the transverse trace part and P_w^0 the scalar which is neither transverse nor traceless. The operator Eq. (105) has to be proportional to unity in each subspace and must not mix subspaces of different spin, however, it could potentially mix P_s^0 with P_w^0. Fortunately, that does not happen in our case and we can write

$$\left(\Pi^{-1}\right)_{ab}^{\ \ cd} = A P_s^2 + B P_w^1 + C P_s^0 + D P_w^0 \,. \tag{109}$$

Determining the coefficients is straight-forward, one gets

$$\Pi^{-1} = k^2 a P_s^2 + k^2 (a - 2c)\, P_s^0 , \tag{110}$$

so the longitudinal parts P_w^1 and P_w^0 simply dropped out. That implies that the energy-momentum-tensor τ_{ab} also must not have any longitudinal modes, it has to be conserved: $k^a \tau_{ab} = 0$. So if we just remove the longitudinal degrees of freedom from our solution space, the equations of motion can be inverted and the resulting propagator is

$$\Pi_{AIDG} = \frac{P_s^2}{ak^2} + \frac{P_s^0}{(a-2c)\, k^2} \,. \tag{111}$$

If we want Π_{AIDG} to be proportional to the Π_{GR} such that no additional particles are introduced we demand $a = c$ (or equivalently $f = 0$) to obtain

$$\Pi_{AIDG} = \frac{1}{k^2 a \left(-k^2\right)} \left(P_s^2 - P_s^0\right) . \tag{112}$$

A.2. Degrees of freedom in Einstein gravity

It can be shown easily that Eq. (15) with $\tau_{ab} = 0$ imply $k^2 = 0$. After removing the k^2-terms we see that both $k^a h_{ab}$ and h have to be zero, i.e. h_{ab} has to be transverse and traceless. We can expand h_{ab} in a light-like basis using k_a and additionally a second light-like vector l_a and an orthogonal space-like vector e_a as basis vectors. The Minkowski metric then takes the form

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{113}$$

and h_{ab} can be written as

$$h_{ab} = \alpha(k)\, k_a k_b + 2\beta(k)\, k_{(a} l_{b)} + \gamma(k)\, l_a l_b + 2\delta(k)\, k_{(a} e_{b)} + 2\phi(k)\, l_{(a} e_{b)} + \lambda(k)\, e_a e_b \,, \tag{114}$$

with k-dependent coefficients. Transverse traceless now means that $\beta = \gamma = \phi = \lambda = 0$. The remaining coefficients are gauge degrees of freedom and can be removed by a gauge transformation

$$h_{ab} \to h_{ab} + k_a v_b + v_a k_b \,, \tag{115}$$

with some vector v_a.

A.3. Perturbations

Here we summarize the expressions for the perturbations up to second order of all relevant geometrical quantities; background quantities are indicated with a bar. The basic definition is

$$g_{ab} = \overline{g}_{ab} + h_{ab}, \tag{116}$$

raising and lowering indices is always done using $\overline{g}_{ab}$. It follows

$$g^{ab} \approx \overline{g}^{ab} - h^{ab} + h^{ac}h_c^b, \quad \sqrt{-g} \approx \sqrt{-\overline{g}}\left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right), \tag{117}$$

$$\Gamma^a_{bc} \approx \overline{\Gamma}^a_{bc} + \delta\Gamma^a_{bc}, \quad \delta\Gamma^a_{bc} = \frac{1}{2}\left(\overline{\nabla}_b h^a_c + \overline{\nabla}_c h^a_b - \overline{\nabla}^a h_{bc}\right), \tag{118}$$

$$\begin{aligned}\delta R^{ab}{}_{cd} = &\frac{\overline{R}}{12}\left(\delta^a_d h^b_c - \delta^a_c h^b_d + \delta^b_c h^a_d - \delta^b_d h^a_c\right)\\ &+ \frac{1}{2}\left(\bar{\nabla}_c\bar{\nabla}^b h^a_d - \overline{\nabla}_c\overline{\nabla}^a h^b_d - \overline{\nabla}_d\overline{\nabla}^b h^a_c + \overline{\nabla}_d\overline{\nabla}^a h^b_c\right),\end{aligned} \tag{119}$$

$$\begin{aligned}&R_{ab} \approx \overline{R}_{ab} + \delta R_{ab} + \delta^2 R_{ab}, \quad \delta R_{ab} = \overline{\nabla}_c \delta\Gamma^c_{ab} - \overline{\nabla}_b\delta\Gamma^c_{ac},\\ &\delta^2 R_{ab} = \delta\Gamma^c_{dc}\delta\Gamma^d_{ab} - \delta\Gamma^c_{db}\delta\Gamma^d_{ca},\end{aligned} \tag{120}$$

$$R \approx \overline{R} + \delta R, \quad \delta R = -h^{ab}\overline{R}_{ab} + \overline{g}^{ab}\left(\overline{\nabla}_c\delta\Gamma^c_{ab} - \overline{\nabla}_b\delta\Gamma^c_{ac}\right). \tag{121}$$

The order of expansion in h_{ab} is either first or second, depending on what we need to vary the action.

A.4. Quadratic action for Einstein gravity

Obtaining the second variation of S_{EH} in a curved background is a straight forward, but laborious task. We start by expanding every quantity up to second order:

$$\begin{aligned}S_{EH} = &\int d^3x\,\sqrt{-g}\left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right)\\ &\times\left[\frac{1}{2}\left(R_{ab} + \delta R_{ab} + \delta^2 R_{ab}\right)\left(g^{ab} - h^{ab} + h^{ac}h^b_c\right) - \Lambda\right].\end{aligned} \tag{122}$$

Collecting all the quadratic terms yields

$$\begin{aligned}\delta^2 S_{EH} = &\int d^3x\,\sqrt{-g}\left[\frac{1}{2}R_{ab}h^{ac}h^b_c - \frac{1}{2}\delta R_{ab}h^{ab} + \frac{1}{2}\delta^2 R_{ab}g^{ab}\right.\\ &\left.-\frac{h}{4}R_{ab}h^{ab} + \frac{h}{4}\delta R_{ab}g^{ab} + \left(\frac{h^2}{8} - \frac{h_{ab}h^{ab}}{4}\right)\left(\frac{R}{2} - \Lambda\right)\right].\end{aligned} \tag{123}$$

Plugging in the perturbations results in

$$\begin{aligned}\delta^2 S_{EH} = &\int d^3x\,\sqrt{-g}\left[\frac{1}{6}h^{ab}h_{ab} - \frac{1}{4}\left(\nabla_c\nabla_a h^c_b - \nabla_c\nabla_b h^c_a - \Box h_{ab} + \nabla_a\nabla_b h\right)h^{ab}\right.\\ &+\frac{1}{4}\nabla_b h\left(\nabla_a h^{ab} - \frac{1}{2}\nabla^b h\right) - \frac{1}{8}\left(\nabla_b h^a_c + \nabla_c h^a_b - \nabla^a h_{bc}\right)\end{aligned}$$

$$\times\left(\nabla_a h^{bc}+\nabla^c h_a^b-\nabla^b h_a^c\right)$$
$$-\frac{1}{12}h^2R+\frac{h}{4}\left(\nabla_a\nabla_b h^{ab}-\Box h\right)+\left(\frac{h^2}{8}-\frac{h_{ab}h^{ab}}{4}\right)\left(\frac{R}{2}-\Lambda\right)\Bigg]$$
$$=\int d^3x\sqrt{-g}\Bigg[\frac{1}{8}h^{ab}\Box h_{ab}-\frac{1}{8}h\Box h+\frac{1}{4}h\nabla_a\nabla_b h^{ab}$$
$$-\frac{1}{4}h^{ab}\nabla_c\nabla_a h_b^c-\frac{R}{48}h^2+\frac{R}{24}h_{ab}h^{ab}-\Lambda\left(\frac{h^2}{8}-\frac{h_{ab}h^{ab}}{4}\right)\Bigg]. \tag{124}$$

The fourth term can be further modified

$$-\frac{1}{4}h^{ab}\nabla_c\nabla_a h_b^c=\frac{1}{4}\nabla_a h^{ab}\nabla_c h_b^c-\frac{1}{4}h^{ab}R^d{}_{bac}h_d^c-\frac{1}{4}h^{ab}R^c{}_{dca}h_b^d$$
$$=\frac{1}{4}\nabla_a h^{ab}\nabla_c h_b^c-\frac{R}{24}h_{ab}h^{ab}+\frac{R}{24}h^2-\frac{R}{12}h_{ab}h^{ab}\,, \tag{125}$$

such that the final result becomes

$$\delta^2 S_{EH}=\int d^3x\sqrt{-g}\Bigg[\frac{1}{8}h^{ab}\Box h_{ab}-\frac{1}{8}h\Box h+\frac{1}{4}h\nabla_a\nabla_b h^{ab}$$
$$+\frac{1}{4}\nabla_a h^{ab}\nabla_c h_b^c+\frac{R}{48}h^2-\frac{R}{12}h_{ab}h^{ab}-\Lambda\left(\frac{h^2}{8}-\frac{h_{ab}h^{ab}}{4}\right)\Bigg]. \tag{126}$$

A.5. Commutation relations

We list here some useful commutation relations for differential operators which hold on maximally symmetric backgrounds:

$$\nabla_a\Box t^a=\left(\Box+\frac{R}{3}\right)\nabla_a t^a\,, \tag{127}$$

for a generic vector t^a,

$$\nabla_a\Box t^{ab}=\left(\Box+\frac{2R}{3}\right)\nabla_a t^{ab}-\frac{R}{3}\nabla^b t_a^a\,, \tag{128}$$

for symmetric tensors t^{ab}, and

$$\nabla_a\nabla^b A^a=\frac{R}{3}A^b\,, \tag{129}$$

for transverse vectors A^a. In general,

$$\nabla_a\nabla_b\Box^n D^{ab}\phi=(\Box+R)^n\left(\frac{2\Box+R}{3}\right)\Box\phi\,, \tag{130}$$

holds for the operator D_{ab} defined in section 5. All of those relations can be derived by straight forward Riemann tensor substitution.

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