Holography of AdS hairy black holes and Cardy-Verlinde formula

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We discuss some aspects related to holography of anti–de Sitter (AdS) dyonic hairy black holes, which break the conformal symmetry of the boundary. We use counterterms for the scalar field that satisfies mixed boundary conditions to compute the Euclidean action and dual stress tensor. We apply these results to show that the Cardy-Verlinde formula is not satisfied. However, when the magnetic (or electric) charge vanishes, the conformal symmetry is preserved and the entropy of the black hole can be put in the Cardy-Verlinde form. In our analysis, there is no need of adding extra finite counterterms and, in this particular case, we explicitly show that our results match the ones obtained from using the superpotential as a counterterm.

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I. INTRODUCTION

The anti–de Sitter/conformal field theory (AdS-CFT) duality has led to a new method to regularize the Euclidean action of gravity systems [1–3]. This method is useful for studying the thermodynamics of AdS hairy black holes, e.g., [4–12] as well as of the compact objects such as boson stars [13–15], and it was also generalized to asymptotically flat systems [16–22].

Holographically, the mixed boundary conditions obeyed by the scalar field are associated with a multitrace deformation of the dual field theory [23]. Interestingly, the scalar fields can break the conformal symmetry of the AdS boundary [24] such that the trace of the holographic stress tensor of the dual field theory does not vanish. Lu, Pope, and Pang (LPP) have obtained an exact hairy black hole solution with this property in [25]. However, when the magnetic (or electric) charge vanishes, the conformal symmetry is preserved. We use the counterterms for scalar fields in AdS with mixed boundary conditions proposed in [6] to compute the Euclidean action and holographic stress tensor for the LPP solution. Since the variational principle is well defined, there is no need of adding extra finite counterterms as in [26].

An interesting and puzzling (due to its generality, including flat spacetime solutions for which there is no concrete holographic proposal) result of Verlinde [27] is that the entropy of AdS black holes, with a field theory dual, can be put in a form similar to the Cardy formula of two-dimensional CFTs. The main observation of Verlinde is that one can define a Casimir energy in the dual theory as the violation of the Euler identity $E_C = 2(E + pV - TS)$ (in four bulk dimensions), that is the subextensive part of the total energy. For that, one has to consider the thermodynamic system (in our case, the dual field theory) in a finite volume and the total energy as a function of entropy and volume.[1]

Since in two dimensions the modular symmetry is at the basis of the Cardy formula and this symmetry is absent in higher dimensions, it is not clear what is the origin of the Cardy-Verlinde formula. However, it was checked for many examples of black holes, e.g., in AdS [29–31], dS [32–34] and flat spacetime [35,36]. Particularly, Cai has checked a variant of this formula for a family of hairy charged black holes in AdS [29]. In this paper, we use holographic tools to put Cai's proposal on a firm ground. We also explicitly show, though, that the Cardy-Verlinde formula is not satisfied for the dyonic LPP hairy black hole solution when the asymptotic conformal symmetry is broken.

This paper is organized as follows. In the next section, the Reissner-Nordström (RN) black hole in "brane coordinates" and exact hairy black hole solutions relevant to our analysis are briefly presented. In Sec. III, we use holographic methods to obtain the stress tensor of the dual theory corresponding to hairy black holes. Section IV describes the Cardy-Verlinde formula, establishing that it is always satisfied for the hairy black holes when the conformal symmetry of the AdS boundary is not broken.

[1]We emphasize that the analysis presented here is not related in any way with the extended thermodynamics of hairy black holes in [28] where the cosmological constant is associated to the pressure.

Finally, we summarize our findings and present some conclusions in the last section.

II. EXACT HAIRY BLACK HOLE SOLUTIONS

In this section we present the relevant exact hairy black hole solutions. We show that, up to a redefinition of the solution's parameters, the electrically charged LPP solution is equivalent with a R-charged black hole in a theory with only one gauge field turned on.

A. R-charged black holes

We are interested in a truncated version of $D=4, N=8$ gauged supergravity and use the conventions of [37]

$$I[g_{\mu\nu},\vec{\phi},A_i]=\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\sqrt{-g}\left[R-\frac{1}{2}(\partial\vec{\phi})^2-\frac{1}{4}\sum_i^4 X_i^{-2}F_i^2+\frac{1}{l^2}\sum_{i<j}^4 X_iX_j\right] \tag{1}$$

with $G=c=1$. The three scalar fields are written in the compact form $\vec{\phi}=(\phi_1,\phi_2,\phi_3)$; $\{A_i\}$ are the four 1-form gauge potentials, and the four $\{X_i\}$, which satisfy the constraint $X_1X_2X_3X_4=1$, denote

$$X_i=e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}},\qquad \vec{a}_i\cdot\vec{a}_j=4\delta_{ij}-1. \tag{2}$$

A convenient choice [38] for the constants $\vec{a}_i$ is

$$\vec{a}_1=(1,1,1),\qquad \vec{a}_2=(1,-1,-1),\qquad \vec{a}_3=(-1,1,-1),\qquad \vec{a}_4=(-1,-1,1) \tag{3}$$

with which the action and potential can be rewritten as

$$I[g_{\mu\nu},\vec{\phi},A_i]=\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\sqrt{-g}\left[R-\frac{1}{2}(\partial\vec{\phi})^2-\frac{1}{4}\sum_i e^{\vec{a}_i\cdot\vec{\phi}}F_i^2-V(\vec{\phi})\right] \tag{4}$$

$$V(\vec{\phi})=-\frac{2}{l^2}(\cosh\phi_1+\cosh\phi_2+\cosh\phi_3). \tag{5}$$

The equations of motion consistent with this action are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T^{\phi}_{\mu\nu}+T^{\rm EM}_{\mu\nu}, \tag{6}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\vec{\phi})=\frac{\partial V(\vec{\phi})}{\partial\vec{\phi}}+\frac{1}{4}\sum_i^4\frac{\partial(e^{\vec{a}_i\cdot\vec{\phi}})}{\partial\vec{\phi}}F_i^2 \tag{7}$$

where the energy-momentum tensors are

$$T^{\phi}_{\mu\nu}=\frac{1}{2}(\partial_\mu\vec{\phi})\cdot(\partial_\nu\vec{\phi})-\frac{1}{2}g_{\mu\nu}\left[\frac{1}{2}(\partial\vec{\phi})^2+V(\vec{\phi})\right],$$
$$T^{\rm EM}_{\mu\nu}=\frac{1}{2}\sum_i^4 e^{\vec{a}_i\cdot\vec{\phi}}\left[(F_{\mu\lambda})_i(F_\nu^{\lambda})_i-\frac{1}{4}g_{\mu\nu}F_i^2\right]. \tag{8}$$

There exists an exact static charged hairy black hole solution [39] for this theory. The metric has the following form:

$$ds^2=-\left(\prod_{i=1}^4 H_i\right)^{-1/2}fdt^2+\left(\prod_{i=1}^4 H_i\right)^{1/2}(f^{-1}d\rho^2+\rho^2d\Omega_2^2) \tag{9}$$

where the angular line element is $d\Omega_2^2\equiv d\theta^2+\sin^2\theta d\varphi^2$ and the functions f and H_i are

$$f(\rho)=1-\frac{\eta}{\rho}+\frac{\rho^2}{l^2}\prod_{i=1}^4 H_i(\rho),\qquad H_i(\rho)=1+\frac{q_i}{\rho}. \tag{10}$$

The gauge potentials and the quantities $\{X_i\}$ are given by

$$A_i=-(1-H_i^{-1})\frac{\sqrt{q_i(q_i+\eta)}}{q_i}dt=-\frac{\sqrt{q_i(q_i+\eta)}}{\rho+q_i}dt,$$
$$X_i=\left(\prod_{j=1}^4 H_j\right)^{1/4}H_i^{-1} \tag{11}$$

and the scalar fields are

$$\phi_1(\rho)=\frac{1}{2}\ln\left(\frac{H_1H_2}{H_3H_4}\right),\qquad \phi_2(\rho)=\frac{1}{2}\ln\left(\frac{H_1H_3}{H_2H_4}\right),$$
$$\phi_3(\rho)=\frac{1}{2}\ln\left(\frac{H_1H_4}{H_2H_3}\right). \tag{12}$$

1. RN black hole truncation

If we fix $q_1=q_2=q_3=q_4\equiv q$, then, the gauge fields become $F_1=F_2=F_3=F_4\equiv F$ and the scalar fields vanish, $\phi_1=\phi_2=\phi_3=0$. The action reduces to

$$I=\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\sqrt{-g}\left(R-F^2+\frac{6}{l^2}\right). \tag{13}$$

The metric becomes

$$ds^2=-\frac{f}{H^2}dt^2+H^2\left(\frac{d\rho^2}{f}+\rho^2d\Omega_2^2\right),$$
$$f(\rho)=1-\frac{\eta}{\rho}+\frac{\rho^2}{l^2}H(\rho)^4,\qquad H(\rho)=1+\frac{q}{\rho} \tag{14}$$

and the gauge potential is

$$A = -\frac{\sqrt{q(q+\eta)}}{\rho+q}dt. \tag{15}$$

This is nothing else than RN black hole solution in the so-called brane coordinates.

2. One-charge hairy black hole truncation

Now, we are interested in the special consistent truncation obtained after making $q_1 = q \neq 0$ and $q_2 = q_3 = q_4 = 0$. In this case, there are three identical scalar fields that can be redefined as $\phi_1 = \phi_2 = \phi_3 \equiv -\frac{\phi(\rho)}{\sqrt{3}}$. Under these considerations, the theory reduces to the action

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}F^2 + \frac{6}{l^2}\cosh\left(\frac{\phi}{\sqrt{3}}\right)\right] \tag{16}$$

and the hairy black hole solution is

$$ds^2 = -\frac{f}{\sqrt{H}}dt^2 + \sqrt{H}\left(\frac{d\rho^2}{f} + \rho^2 d\Omega_2^2\right),$$
$$A = -\frac{\sqrt{q(q+\eta)}}{\rho+q}dt, \qquad \phi = -\frac{\sqrt{3}}{2}\ln H \tag{17}$$

where

$$f(\rho) = 1 - \frac{\eta}{\rho} + \frac{\rho^2}{l^2}\left(1 + \frac{q}{\rho}\right), \qquad H(\rho) = 1 + \frac{q}{\rho}. \tag{18}$$

B. Dyonic LPP black hole

In this section we follow [25]. We start with the action

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}F^2 + \frac{6}{l^2}\cosh\left(\frac{1}{\sqrt{3}}\phi\right)\right] \tag{19}$$

where $(\partial\phi)^2 \equiv g^{\mu\nu}\partial_\mu\partial_\nu\phi$, $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The exact static dyonic hairy black hole solution was presented in [25], and it is

$$ds^2 = -\frac{f dt^2}{\sqrt{H_1 H_2}} + \sqrt{H_1 H_2}\left(\frac{d\rho^2}{f} + \rho^2 d\Omega_2^2\right), \tag{20}$$

$$A = \frac{\sqrt{2}(1-\beta_1)f_0}{\sqrt{\beta_1\gamma_2}H_1}dt + \frac{2\sqrt{2\beta_2\gamma_1}\mu}{\gamma_2}\cos\theta\, d\varphi,$$
$$\phi = \frac{\sqrt{3}}{2}\ln\left(\frac{H_2}{H_1}\right) \tag{21}$$

where

$$f(\rho) = f_0(\rho) + \frac{\rho^2}{l^2}H_1(\rho)H_2(\rho), \qquad f_0(\rho) = 1 - \frac{2\mu}{\rho} \tag{22}$$

and

$$H_i(\rho) = \gamma_i^{-1}[1 - 2\beta_i f_0(\rho) + \beta_1\beta_2 f_0(\rho)^2],$$
$$\gamma_i \equiv 1 - 2\beta_i + \beta_1\beta_2 \tag{23}$$

for $i = 1, 2$. The quantities β_1, β_2, and μ are the integration constants of the solution, related to the mass, and the electric and magnetic charges.

Notice that, if we fix $\beta_2 = 0$, we recover the one-charge hairy black hole truncation from Sec. II A 2, that is, the solution (17), provided the following identification between the parameters:

$$\mu = \frac{\eta}{2}, \qquad \beta_1 = \frac{q}{2(\eta+q)}. \tag{24}$$

III. HOLOGRAPHY OF HAIRY BLACK HOLES

In this section, we use the method of [6] that provides a well-defined variational principle and counterterms that regularize the action to obtain the dual stress tensor of AdS hairy black holes presented in the previous section. The advantage of this method is that it can be applied to hairy black holes [40–42] in extended supergravity models when the superpotential is complex. For the electrically charged family, we prove that our results match the ones obtained by using the superpotential as a counterterm [4].

A. Electrically charged hairy black hole

For the one-charge hairy black hole solution presented in Sec. II A 2, the scalar field has the following falloff

$$\phi(r) = \frac{A}{r} + \frac{B}{r^2} + O(r^{-3}), \quad A = -\frac{\sqrt{3}q}{2}, \quad B = \frac{\sqrt{3}q^2}{8} \tag{25}$$

where $r = \rho H^{\frac{1}{4}}$ is the canonical coordinate. The boundary conditions for the scalar field are given by the function $W(A)$ that satisfies $B = dW(A)/dA$. It is straightforward to verify that $B = \sqrt{3}A^2/6$ and $W(A) = \sqrt{3}A^3/18$. Following [6,24,43,44], we point out that this result corresponds to the case in which the conformal symmetry on the boundary is preserved and so the regularized Euclidean action [6,45] is[2]

[2]The coefficient of the ϕ^3 counterterm is obtained from the general counterterm (35) evaluated by using the integrability condition $W(A)$ for the solution (17) such that it can be easily compared with the counterterm provided by the superpotential (30).

$$I^E = I^E_{\rm bulk} - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h^E}K + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h^E}\left(\frac{2}{l}+\frac{l\mathcal{R}}{2}\right) + \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h^E}\left(\frac{\phi^2}{2l}+\frac{\sqrt{3}}{18l}\phi^3\right) \quad (26)$$

where

$$I^E_{\rm bulk} = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g^E}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}F^2 + \frac{6}{l^2}\cosh\left(\frac{1}{\sqrt{3}}\phi\right)\right]. \quad (27)$$

Here, we use a foliation with hypersurfaces $\rho = \text{const}$ and the corresponding induced metric h_{ab}. The quantities K and $\mathcal{R}$ in Eq. (26) are the trace of the extrinsic curvature K_{ab} and the Ricci scalar on the boundary, respectively.

For the solution being considered, we have that the energy, electric charge, and chemical potential have the following expressions:

$$E = \frac{1}{2}\left(\eta + \frac{q}{2}\right), \qquad Q = \frac{1}{16\pi}\oint_{S^2} e^{-\sqrt{3}\phi}\star F = \frac{\sqrt{q(q+\eta)}}{4}, \qquad \Psi = \frac{4Q}{q+\rho_+}. \quad (28)$$

These quantities are helpful, because, after a lengthy computation of the Euclidean action, we can verify the quantum-statistical relation

$$F = I^E T = -(TS + Q\Psi) + \frac{1}{2}\eta + \frac{1}{4}q = E - TS - Q\Psi. \quad (29)$$

Notice that there is no need of extra finite terms as in [26].

For completeness, we emphasize that we can also use the superpotential $\mathcal{W}$ as a counterterm [4] to get the same result. By using this method, the only gravitational counterterm required is

$$I_{\rm counterterm} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[\mathcal{W}(\phi) + \frac{l\mathcal{R}}{2}\right] \quad (30)$$

as can be explicitly compared with the asymptotic expansion of the counterterms in (26).

The quasilocal stress tensor [46] for the action in Eq. (26), including the boundary term for the scalar field [6], is

$$\tau_{ab} \equiv \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h^{ab}} = -\frac{1}{8\pi}\left[K_{ab} - \left(K - \frac{2}{l}\right)h_{ab} - lG_{ab}\right] - \frac{h_{ab}}{16\pi}\left[\frac{\phi^2}{2l} + \frac{W(A)}{lA^3}\phi^3\right] \quad (31)$$

where $\mathcal{R}_{ab}$ is the Ricci tensor on the boundary and $G_{ab} = \mathcal{R}_{ab} - \frac{1}{2}h_{ab}\mathcal{R}$. The regularized dual stress tensor is related to the quasilocal stress tensor by the conformal transformation [47]

$$\langle\tau^{\rm dual}_{ab}\rangle = \lim_{\rho\to\infty}\frac{\rho}{l}\tau_{ab} \quad (32)$$

and its components are

$$\langle\tau^{\rm dual}_{tt}\rangle = \frac{1}{8\pi l^2}\left(\eta + \frac{q}{2}\right) = \frac{E}{4\pi l^2}, \qquad \langle\tau^{\rm dual}_{\theta\theta}\rangle = \frac{E}{8\pi}, \qquad \langle\tau^{\rm dual}_{\phi\phi}\rangle = \sin^2\theta\langle\tau^{\rm dual}_{\theta\theta}\rangle. \quad (33)$$

This stress tensor is covariantly conserved and its trace vanishes, $\langle\tau^{\rm dual}\rangle = 0$, as expected from the fact that the boundary conditions preserve the conformal symmetry.

B. Dyonic hairy black hole

For the LPP solution from Sec. II B, the scalar field falls off as $\phi(r) = \frac{A}{r} + \frac{B}{r^2} + O(r^{-3})$, where

$$A = \frac{2\sqrt{3}(\beta_2-\beta_1)(1-\beta_1\beta_2)}{\gamma_1\gamma_2}\mu, \qquad B = \frac{2\sqrt{3}(\beta_2-\beta_1)[4\beta_1\beta_2(\gamma_1+\gamma_2) - (1-\beta_1\beta_2)^2(\beta_1+\beta_2)]}{\gamma_1\gamma_2}\mu^2. \quad (34)$$

In general, these boundary conditions for the scalar field do not preserve the isometries of AdS at the boundary [6,24], unless $\beta_1 = 0$ or $\beta_2 = 0$. Since the conformal symmetry is broken, there are subtleties with a correct definition of the black hole energy that are discussed in great detail in [44,48–50] and we do not repeat those issues here.

We are going to keep the discussion general and obtain the dual stress tensor as a function of W that will be sufficient for the next section. Following [6], the counterterm for the scalar field with mixed boundary conditions [with the integrability condition $B = W'(A)$ imposed] is

$$I^E_\phi = \frac{1}{16\pi}\int d^3x\sqrt{h^E}\left[\frac{\phi^2}{2l} + \frac{W(A)}{lA^3}\phi^3\right]. \quad (35)$$

The electric and magnetic conserved charges are

$$Q = \frac{\mu\sqrt{\beta_1\gamma_2}}{\gamma_1\sqrt{2}}, \qquad P = \frac{\mu\sqrt{\beta_2\gamma_1}}{\gamma_2\sqrt{2}} \quad (36)$$

and their conjugate electric and magnetic potentials have the following expressions:

$$\Psi = \sqrt{\frac{2}{\beta_1\gamma_2}}\left[1 - \beta_1 - \frac{1-\beta_1 f_0(\rho_+)}{H_1(\rho_+)}\right], \quad (37)$$

$$\Upsilon = \sqrt{\frac{2}{\beta_2\gamma_1}}\left[1-\beta_2-\frac{1-\beta_2 f_0(\rho_+)}{H_2(\rho_+)}\right], \tag{38}$$

where ρ_+ is the coordinate of the outer horizon of the black hole, $f(\rho_+)=0$, from Eq. (22). These expressions are going to be helpful for the following results.

The components of the regularized dual stress tensor and its nonvanishing trace are

$$\langle\tau_{tt}^{\rm dual}\rangle = \frac{M}{4\pi l^2}+\frac{1}{16\pi l^4}\left(W-\frac{AB}{3}\right),$$
$$\langle\tau_{\theta\theta}^{\rm dual}\rangle = \frac{M}{8\pi}-\frac{1}{16\pi l^2}\left(W-\frac{AB}{3}\right), \qquad \langle\tau_{\phi\phi}^{\rm dual}\rangle = \sin^2\theta\langle\tau_{\theta\theta}^{\rm dual}\rangle, \tag{39}$$

$$\langle\tau^{\rm dual}\rangle = -\frac{3}{16\pi l^4}\left(W-\frac{AB}{3}\right). \tag{40}$$

In these expressions, we have used M as a new parameter

$$M=\frac{1}{2}\left(\eta+\frac{q}{2}\right). \tag{41}$$

By computing the Euclidean action, including the counterterm in Eq. (35), we obtain the quantum-statistical relation

$$F = I^E T = E - TS - Q\Psi - P\Upsilon \tag{42}$$

where

$$E = M + \frac{1}{4l^2}\left(W-\frac{AB}{3}\right) \tag{43}$$

is conserved energy E [46], read from the τ_{tt} component of the dual stress tensor (39).

In general, the energy receives a correction from the scalar field. One can explicitly check that, if $B\sim A^2$, the trace of the stress tensor $\langle\tau^{\rm dual}\rangle=0$, the conformal symmetry is restored and $E=M$.

IV. CARDY-VERLINDE FORMULA

In this section, we use the previous holographic results to put on a firm ground the Cai's proposal [29] for a consistent Cardy-Verlinde formula for charged hairy black holes.

For clarity, we briefly present the Cardy-Verlinde formula and define the relevant physical quantities. First, to simplify the notation, even if the dual stress tensor is an operator, we are going to denominate it as $\tau_{ab}^{\rm dual}$ instead of $\langle\tau_{ab}^{\rm dual}\rangle$. What we have computed is the stress tensor of the dual CFT in the UV regime, namely, at the boundary of AdS. However, in the context of Cardy-Verlinde formula, the CFT is living on a sphere of finite radius R. Consequently, a conformal transformation

$$\gamma_{ab}dx^a dx^b = \frac{R^2}{l^2}[-dt^2+l^2(d\theta^2+\sin^2\theta d\varphi^2)] \tag{44}$$

together with a redefinition of the time coordinate $t=\frac{l}{R}\tau$, put the metric in the required form for the Cardy-Verlinde (CV) formula,

$$\gamma_{ab}^{CV}dx'^a dx'^b = -d\tau^2+R^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{45}$$

Once the holographic stress tensor compatible with this metric is computed, we obtain the pressure and energy density of the dual field theory and verify explicitly the Cardy-Verlinde formula [27]

$$S=\pi R\sqrt{E_C(2E-E_C)}, \qquad E_C = 2(E+pV-TS), \tag{46}$$

where E_C is the Casimir energy, and its extension for charged hairy black holes.

We would like to emphasize that the Casimir energy in Eq. (46) as defined by Verlinde is not the usual Casimir energy that is computed in the context of AdS-CFT in [3]. For the four-dimensional hairy black hole solutions analyzed in this section, the Casimir energy defined by Verlinde does not vanish even if the dual theory is three dimensional.

A. RN black hole

As an warm-up exercise, let us first consider the RN black hole in AdS spacetime. The Cardy-Verlinde formula is still valid, but with a small change in the energy expression, as it was presented in [29]. The novelty of this section is that we provide a physical interpretation of this result, by relating it to Ruffini's irreducible mass [51].

The RN black hole solution in canonical coordinates is

$$ds^2 = -f(r)dt^2+f(r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2),$$
$$f(r) = 1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{r^2}{l^2}. \tag{47}$$

Now, as a consequence of having changed $t\to\frac{l}{R}\tau$, we have that some thermodynamic quantities require a corresponding rescaling. The thermodynamic quantities for this solution are

$$E=\frac{l}{R}M=\frac{l}{2R}\left(r_+ + \frac{r_+^3}{l^2}+\frac{Q^2}{r_+}\right), \qquad \Psi=\frac{Ql}{Rr_+},$$
$$S=\pi r_+^2 \qquad T=\frac{l}{4R\pi}\left(\frac{3r_+}{l^2}+\frac{1}{r_+}-\frac{Q^2}{r_+^3}\right) \tag{48}$$

where r_+ is the coordinate of the outer horizon, $f(r_+)=0$. The thermodynamic volume of the system is $V=4\pi R^2$ and the first law of the dual field theory is

$$dE = TdS - pdV + \Psi dQ,$$
$$p \equiv -\left(\frac{\partial E}{\partial V}\right)_{Q,S} = -\left(\frac{\partial E}{\partial R}\right)_{Q,r_+}\left(\frac{\partial R}{\partial V}\right) = \frac{E}{2V}. \tag{49}$$

It was observed in [29] that a modified Cardy-Verlinde is valid in this case, namely,

$$S = \pi R\sqrt{E_C[2(E-E_Q)-E_C]} = \pi r_+^2 \tag{50}$$

where

$$E_C = 2(E+pV-TS-Q\Psi) = \frac{lr_+}{R}, \quad E_Q = \frac{\Psi Q}{2} = \frac{lQ^2}{2Rr_+}. \tag{51}$$

Interestingly, E_Q is nothing else than Ruffini's energy of the electromagnetic field that can be extracted from the black hole [51], that is,

$$E_Q \equiv \frac{1}{8\pi}\int_{\Sigma_t}\xi^\mu T^{\mathrm{EM}}_{\mu\nu}d\Sigma^\nu = \frac{1}{8\pi}\int d^3x\sqrt{h}\xi^\mu n^\nu T^{\mathrm{EM}}_{\mu\nu}$$
$$= -\frac{Q^2}{2r}\bigg|_{r_+}^{\infty} = \frac{Q^2}{2r_+} \tag{52}$$

where n^μ is the normal on the surface Σ_t (defined by $t = \mathrm{const}$), $\xi_\mu = \partial/\partial t$ is the Killing vector, and

$$\sqrt{h} = \frac{r^2\sin^2\theta}{\sqrt{f(r)}}, \qquad T_{00} = \frac{f(r)Q^2}{r^4}. \tag{53}$$

This definition for E_Q [51] is rather in the spirit of the definition of the quasilocal charges [46] of Brown and York with respect to the corresponding Killing vector and it is not the purely electromagnetic energy obtained by the usual integration of the energy density outside the horizon,

$$\int_{\Sigma_t} n^\mu T^{\mathrm{EM}}_{\mu\nu}d\Sigma^\nu = \frac{1}{2}\int_{r_+}^{\infty}\sqrt{g_{rr}}\frac{Q^2}{r^2}dr. \tag{54}$$

B. Cardy-Verlinde formula for hairy black holes

The scalar field that represents degrees of freedom living outside the horizon is going to also get subtracted from the total energy that enters in the Cardy-Verlinde formula. However, a direct computation—as in the case of an RN black hole—produces a divergent result and, at this point, it is not clear to us how it can be regularized. We are going to consider an alternative approach proposed by Cai [29] and provide a holographic interpretation of this proposal using the dual stress tensor to get the pressure and energy density.

The hairy black hole metric (17) is written in the so-called brane coordinate system. The interesting observation made by Cai in [29] is that, to get a consistent Cardy-Verlinde formula for hairy black holes, one should use a proper internal energy obtained by subtracting the mass of supersymmetric background from the total energy.

For the hairy electrically charged solution from Sec. II A 2, the rescaled thermodynamic quantities (obviously, not the entropy and electric charge) are

$$E = \frac{l}{2R}\left(\eta+\frac{q}{2}\right), \quad T = \frac{1}{4R}\frac{3\rho_+^2+2q\rho_++l^2}{\pi l\sqrt{\rho_+(\rho_++q)}},$$
$$S = \pi\rho_+^2\sqrt{1+\frac{q}{\rho_+}}, \quad Q = \frac{\sqrt{q(q+\eta)}}{4}, \quad \Psi = \frac{4Ql}{(q+\rho_+)R}. \tag{55}$$

Following the method of Cai, we can identify the energy E_q by doing $\eta = 0$,

$$E_q = \lim_{\eta\to 0}E = \frac{ql}{4R} \tag{56}$$

and the pressure is obtained by the expression

$$\bar{p} \equiv \frac{E-E_q}{2V} = \frac{\eta l}{16\pi R^3}. \tag{57}$$

The Casimir energy, $\bar{E}_C$, is computed as usual, though we also use the pressure from which the background contribution was extracted, that is,

$$\bar{E}_C = 2(E+\bar{p}V-TS-\Psi Q) = \frac{l\rho_+}{R} \tag{58}$$

and the entropy, then, can be put in the following form:

$$S = \pi R\sqrt{\bar{E}_C[2(E-E_q)-\bar{E}_C]} \tag{59}$$

which is the Cardy-Verlinde formula for hairy black holes.

Consider now the general metric that describes the dyonic black hole (20) and the rescaled holographic stress tensor that corresponds to a theory associated with the $(2+1)$-dimensional metric, $\gamma^{\mathrm{CV}}_{ab}$, (45):

$$\tau^{\mathrm{CV}}_{\tau\tau} = \frac{l}{4\pi R^3}\left[M+\frac{1}{4l^2}\left(W-\frac{AB}{3}\right)\right],$$
$$\frac{\tau^{\mathrm{CV}}_{\varphi\varphi}}{\sin^2\theta} = \tau^{\mathrm{CV}}_{\theta\theta} = \frac{l}{8\pi R}\left[M-\frac{1}{2l^2}\left(W-\frac{AB}{3}\right)\right]. \tag{60}$$

As we have already shown in Sec. III B, for general mixed boundary conditions for the scalar field, the dyonic LPP black hole does not preserve the isometries of AdS at the boundary. Putting the stress tensor in the form of a thermal gas,

$$\tau_{ab}^{\mathrm{CV}} = (\epsilon + p)u_a u_b + p\gamma_{ab}, \tag{61}$$

we can read the pressure and energy density as

$$\epsilon = \frac{1}{V}\frac{l}{R}\left[M + \frac{1}{4l^2}\left(W - \frac{AB}{3}\right)\right],$$
$$p = \frac{1}{2V}\frac{l}{R}\left[M - \frac{1}{2l^2}\left(W - \frac{AB}{3}\right)\right], \tag{62}$$

where, again, the volume is $V = 4\pi R^2$. For a general boundary condition, the trace of the stress tensor does not vanish and the energy receives a correction from the scalar field. As expected, the Cardy-Verlinde formula is not valid in this case.

However, when the magnetic (or the electric) charge vanishes, e.g., $\beta_2 = 0$ ($\beta_1 = 0$) as in Sec. III A, we have that $W - \frac{AB}{3} = 0$ and now the trace of the stress tensor vanishes. This explicitly proves that, for this particular case, the conformal symmetry in the boundary is recovered, the pressure and energy density become

$$\epsilon = \tau_{\tau\tau}^{\mathrm{CV}} = \frac{l}{2VR}\left(\eta + \frac{q}{2}\right), \qquad p = \frac{\tau_{\theta\theta}^{\mathrm{CV}}}{R^2} = \frac{l}{4VR}\left(\eta + \frac{q}{2}\right), \tag{63}$$

and so we obtain the stress tensor of a perfect gas of massless particles with energy and pressure

$$E = \frac{l}{2R}\left(\eta + \frac{q}{2}\right), \qquad p = \frac{E}{2V}. \tag{64}$$

Now, we have all the holographic ingredients to check Cai's proposal. Under this consideration, we can separate the energy in three different parts

$$E = E_E + \frac{1}{2}\bar{E}_C + E_q, \qquad E_q = \frac{ql}{4R},$$
$$E_E = E - E_q - \frac{1}{2}\bar{E}_C = \frac{1}{2}\frac{\rho_+^2(\rho_+ + q)}{lR}. \tag{65}$$

As in the original work of Verlinde [27], we interpret the term E_E as the purely extensive part of the energy and $\bar{E}_C$ as the Casimir energy (the subextensive part). Additionally, we have that E_q is the energy of the supersymmetric background when $\eta = 0$, as proposed by Cai in [29]. A straightforward computation proves that the Cardy-Verlinde formula (59) is satisfied for hairy black hole solutions that are asymptotically locally AdS.

V. CONCLUSIONS

We have analyzed the holography of a dyonic LPP AdS black hole for which the conformal symmetry on the boundary is broken. We have shown that, up to a redefinition of black hole parameters, the electrically charged LPP solution (when the magnetic charge vanishes) matches an R-charged black hole truncation with only one gauge field turned on for which the conformal symmetry of the boundary is preserved. This can be explicitly checked by computing the trace of the dual stress tensor, see Eq. (40). One important observation is that, when the conformal symmetry is broken, there is an extra contribution of the scalar field to the black hole's mass. Therefore, we have a nice setup where to explicitly verify some holographic properties of hairy black holes when the conformal symmetry can be broken or not, depending of the existence of electric and magnetic charges.

Particularly, we have checked the validity of the Cardy-Verlinde formula for the LPP black hole solutions. In [29], it was shown that a variant of this formula is also satisfied for hairy black holes, particularly the R-charged black holes. In Sec. IV, we have provided a holographic computation of the stress tensor that supports this proposal. In the case of RN black hole, we also gave a physical interpretation of the energy used in the Cardy-Verlinde formula by relating it to the irreducible mass of Ruffini [51]. In the presence of the scalar field, we did not succeed to compute the regularized contribution to be able to compare the two relevant quantities. However, we have used an alternative method put forward by Cai in [29]. In the brane coordinate system, the relevant contributions to the energy can be separated and it is possible to subtract the mass of the (supersymmetric) background with $\eta = 0$ (64). For the RN black hole presented in Sec. II A 1, the solution with $\eta = 0$ is the extremal black hole and one has to do a background subtraction with respect to the zero temperature state. One obvious problem with this proposal, when applied to hairy black holes, is that the background is a naked singularity, but we know that in string theory this solution can be interpreted as a system of giant gravitons [52] and so everything fits nicely in this context.

When the conformal symmetry is preserved, the entropy of the hairy black hole can be put in the Cardy-Verlinde form, otherwise this result is spoiled. We have checked the robust proposal of Cai [29] using a direct holographic computation of the pressure from the dual stress tensor and tried to apply the same reasoning to the dyonic solution. In this case, the energy has an extra contribution from the scalar field when the trace of the dual stress tensor does not vanish. However, even when the contribution of the hairy degrees of freedom that live outside the horizon is subtracted from the energy, the result does not match the black hole entropy. We did not succeed to modify the

Cardy-Verlinde formula for the dyonic LPP black hole in a similar way as it was done in [29]. Intuitively, since the conformal symmetry is broken in the boundary, it could very well be that such a formula does not exist in this case.

For the non-AdS black holes for which the Cardy-Verlinde formula is valid (and for which there is no concrete holographic proposal), there still exists a conformal mapping between the non-AdS geometries and the $AdS_3 \times S^q$ spacetimes [53]. A general proposal for explaining the universality of black hole entropy based on conformal symmetry can be found in [54,55].

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