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$N = 2$ supersymmetric BKP hierarchy with $SW_{1+\infty}$ symmetries and its multicomponent generalization

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ABSTRACT

In this paper, we define a $N = 2$ supersymmetric BKP(SBKP) hierarchy which is a $N = 2$ extension of the supersymmetric BKP hierarchy and construct its additional symmetries. These additional flows constitute an interesting B type $SW_{1+\infty}$ Lie algebra. Further we generalize the $N = 2$ SBKP hierarchy to a $N = 2$ supersymmetric multi-component BKP hierarchy equipped with a B type $\bigotimes SW_{1+\infty}$ Lie algebra. These studies are the further extension on our previous work published in Nucl. Phys. B in 2015.

Contents

1. Introduction

In the studies of mathematical physics particularly the integrable systems, it is important to construct the symmetries and identify their algebraic structure. In these symmetries, the additional symmetry is an interesting and important one. As we know, Kadomtsev-Petviashvili(KP) hierarchy is an important integrable system which attract more and more attention in mathematical physics. Additional symmetries of the KP hierarchy were introduced by Orlov and Shulman [1] which contain virasoro symmetries which have Virasoro constraints on partition functions of matrix models of string theory under the additional non-isospectral symmetries. There are two important sub-hierarchies of KP hierarchy which are the BKP hierarchy and CKP hierarchy [2–7].

Various supersymmetric extensions [8] of the KP hierarchy particularly the supersymmetric Manin-Radul Kadomtsev-Petviashvili (MR-SKP) hierarchy [9] contains a lot of integrable super solitary equations. Mulase also supersymmetrize the KP hierarchy by constructing a hierarchy and they call the hierarchy the Jacobian SKP hierarchy [10] which has strict Jacobian flows and it preserves the super Riemman surface. The additional symmetries for super hierarchies were constructed in the paper [11] and the additional symmetry of the MR-SKP hierarchy was studied by Stanciu [12]. Later the ghost symmetries, Hamiltonian structures and extensions of the MR-SKP hierarchy were studied as well as reductions of the MR-SKP hierarchy [13,14]. Later as a B type reduction on the MR-SKP hierarchy, the supersymmetric

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BKP (SBKP) hierarchy was constructed in [15]. After that this series of super hierarchies attracts more attentions particularly of our group [16–19] including the studies on their Darboux transformations and symmetries. Bosonic hierarchies and their connection with physical models is interesting in $N=2$ conformal field theories.

In [20], we construct the symmetries of the two-component BKP hierarchy and the D type Drinfeld-Sokolov hierarchy to derive a Block algebra. About the Block algebra, we did a series of works in [21,22]. In [19], we construct the additional symmetries of the supersymmetric BKP(SBKP) hierarchy which constitute a B type $SW_{1+\infty}$ Lie algebra. Further we generalize the SBKP hierarchy to a supersymmetric multi-component BKP (SMBKP) hierarchy equipped with a B type $SW_{1+\infty}\times SW_{1+\infty}$ Lie algebra. As a Bosonic reduction of the S2BKP hierarchy, we defined a new constrained system called the supersymmetric Drinfeld-Sokolov hierarchy of type D which admits a supersymmetric Block type symmetry. Recently in [23], basing on Darboux transformations for the supersymmetric constrained KP(ScKP) hierarchy, we construct a supersymmetric constrained B type KP(ScBKP) and supersymmetric constrained C type KP(ScCKP) hierarchies of Manin-Radul and Jacobian types, and derive Darboux transformations on them. The main topic of this article is the study of additional symmetries of scalar and multicomponent $N=2$ supersymmetric BKP integrable hierarchies. These symmetries are shown to form an infinite-dimensional non-Abelian superloop superalgebra.

This paper is arranged as follows. In the next section we define the $N=2$ supersymmetric BKP hierarchy. In Sections 3, we will give the additional symmetries for the $N=2$ supersymmetric BKP hierarchy. Further we define the $N=2$ multicomponent supersymmetric BKP hierarchy in Sections 4, and in Section 5 we will give the additional symmetries for the $N=2$ multicomponent supersymmetric BKP hierarchy.

2. The $N=2$ supersymmetric BKP hierarchy

In this section, we will define a $N=2$ extension on the supersymmetric BKP system. Let $\mathcal{A}$ be an algebra of smooth functions of a spatial coordinate x and super-derivations $D_\pm=\partial_{\theta_\pm}+\theta_\pm\partial$ with grassmann variable $\theta_\pm$. This algebra $\mathcal{A}$ has the following multiplying rule

$$D_\pm^n f=\sum_{i=0}^{\infty}\begin{bmatrix} n\\ n-i\end{bmatrix}(-1)^{|f|(n-i)}f^{\pm[i]}D_\pm^{n-i}, \tag{2.1}$$

$$\begin{bmatrix} n\\ n-i\end{bmatrix}=\begin{cases}0 & i<0 \ \ or\ \ (n,i)=(0,1)\pmod 2;\\ \begin{pmatrix}[\frac{n}{2}]\\ [\frac{n-i}{2}]\end{pmatrix} & i\ge 0,(n,i)\ne(0,1)\pmod 2.\end{cases} \tag{2.2}$$

Here the value $|f|$ means the super degree of the operator f which shows the operator f is Fermionic or Bosonic. The supersymmetric derivatives satisfy the supersymmetric Leibniz rule

$$D_\pm(ab)=D_\pm(a)b+(-1)^{|a|}aD_\pm(b), \tag{2.3}$$

where a is a homogeneous element of $\mathcal{A}$. Now we will introduce the even and odd time variables $(t_2,t_3^\pm,t_6,t_7^\pm,\cdots)$ and their flows

$$D_{4i-2}=\frac{\partial}{\partial t_{4i-2}},\ \ D_{4i-1}^\pm=\frac{\partial}{\partial t_{4i-1}^\pm}+\sum_{j=1}^{\infty}t_{4j-1}^\pm\frac{\partial}{\partial t_{4i+4j-2}}. \tag{2.4}$$

The supercommutator is defined as $[X,Y]=XY-(-1)^{|X||Y|}YX$, and the bracket has a property as $[X,YZ]=[X,Y]Z+(-1)^{|X||Y|}Y[X,Z]$. Then $D_\pm^2=\frac{1}{2}[D_\pm,D_\pm]=\partial,\ \{D_+,D_-\}=0$. This infinite odd and even flows satisfy a nonabelian Lie superalgebra as

$$\begin{aligned}&[D_{4i-2},D_{4j-2}]=0\,,\quad [D_{4i-2},D^\pm_{4j-1}]=0\,,\quad [D^\pm_{4i-1},D^\pm_{4j-1}]=-2\frac{\partial}{\partial t_{4i+4j-2}}\,,\\ &[D^\pm_{4i-1},D^\mp_{4j-1}]=0,\ \ [D^\pm_{4i-2},D_\pm]=0\,,\quad [D^\pm_{4i-1},D_\pm]=0.\end{aligned} \tag{2.5}$$

For any operator $A=\sum_{i\in\mathbb{Z}}(f_i^{(0)}+f_i^+D^++f_i^-D^-+f_i^{(1)}D^+D^-)\partial^i\in\mathcal{A}$ and homogeneous operators P,Q, the nonnegative projection, negative projection, adjoint operator are respectively defined as

$$A_+=\sum_{i\ge 0}(f_i^{(0)}+f_i^+D_++f_i^-D_-+f_i^{(1)}D_+D_-)\partial^i,\quad A_-=A-A_+, \tag{2.6}$$

$$(PQ)^*=(-1)^{|P||Q|}Q^*P^*,\ \ (P^{-1})^*=(-1)^{|P|}(P^*)^{-1}. \tag{2.7}$$

Particularly for the operator D^k, the adjoint operator is defined as

$$(D_\pm^k)^*=(-1)^{\frac{k(k+1)}{2}}D_\pm^k, \tag{2.8}$$

and $(\partial^k)^*=(-1)^k\partial^k$ and $u^*=u$ for any superfield u.

The Lax operator of the $N=2$ supersymmetric BKP hierarchy has a form as

$$L=\Phi D_-\Phi^{-1},\ \ L^*=-D_-LD_-^{-1}, \tag{2.9}$$

where

$$\Phi = 1 + \sum_{i\ge 1}(a_i^{(0)} + a_i^+ D_+ + a_i^- D_- + a_i^{(1)} D_+ D_-)\partial^{-i}, \tag{2.10}$$

satisfy

$$\Phi^* = D_-\Phi^{-1}D_-^{-1}. \tag{2.11}$$

The $N=2$ supersymmetric BKP hierarchy is defined by the following Lax equations

$$D_{4k-2}L = [(L^{4k-2})_+, L], \qquad D^-_{4k-1}L = [(L^{4k-1})_+, L] - 2L^{4k}, \quad k \ge 1, \tag{2.12}$$

$$D^+_{4k-1}L = [(\Lambda^{4k-1})_+, L], \quad \Lambda = \Phi D_+ \Phi^{-1}, \quad k \ge 1. \tag{2.13}$$

The eq. (2.11) will be called the B type condition of the $N=2$ supersymmetric BKP hierarchy. The $N=2$ supersymmetric BKP hierarchy (2.12) can also be redefined in a form as the Sato equations as follows

$$D_{4k-2}\Phi = -(L^{4k-2})_-\Phi, \quad D^-_{4k-1} = -(L^{4k-1})_-\Phi, \quad D^+_{4k-1}\Phi = -(\Lambda^{4k-1})_-\Phi, \tag{2.14}$$

with $k \ge 1$.

3. Additional symmetries of the N=2 supersymmetric BKP hierarchy

In this section, we will define additional symmetries for the N=2 supersymmetric BKP hierarchy by using the Orlov–Schulman operators. The Orlov–Schulman operators $M_i^\pm$ with auxiliary operator $\mathbb{Q}_\pm$ are constructed in the following dressing structure

$$M_i^\pm = \Phi\Gamma_i^\pm\Phi^{-1}, \quad i = 0, 1; \quad \mathbb{Q}_\pm = \Phi Q_\pm \Phi^{-1},$$

where

$$\Gamma_0^\pm = x + \frac{1}{2}\sum_{k\ge 1}(4k-2)t_{4k-2}D_\pm^{4k-4} + \frac{1}{2}(4k-1)t^\pm_{4k-1}D_\pm^{4k-3} - \frac{1}{2}\sum_{k\ge 1}t^\pm_{4k-1}\partial^{2k-2}Q_\pm + \sum_{i,j\ge 1}(i-j)t^\pm_{4i-1}t^\pm_{4j-1}\partial^{2i+2j-2}, \tag{3.1}$$

$$\Gamma_1^\pm = \theta_\pm + \sum_{k\ge 1}t^\pm_{4k-1}\partial^{2k-1}, \tag{3.2}$$

where $Q_\pm = \partial_{\theta_\pm} - \theta_\pm\partial$. Then we can derive the following lemma.

Lemma 3.1. *The operators $\Gamma_j^\pm$, $Q_\pm$ satisfy*

$$[D_{4i-2} - D_\pm^{4i-2}, \Gamma_j^\pm] = [D^\pm_{4i-1} - D_\pm^{4i-1}, \Gamma_j^\pm] = 0; \quad j = 0, 1, \tag{3.3}$$

$$[D_{4i-2} - D_\pm^{4i-2}, Q_\pm] = [D^\pm_{4i-1} - D_\pm^{4i-1}, Q_\pm] = 0, \tag{3.4}$$

$$[Q_\pm, \Gamma_0^\pm] = -\Gamma_1^\pm, \quad [Q_\pm, \Gamma_1^\pm] = 1, \quad [\partial, \Gamma_0^\pm] = 1. \tag{3.5}$$

Proof. The proof is similar as in [18] as follows

$$[D_{4i-2} - D_\pm^{4i-2}, \Gamma_0^\pm] = \frac{1}{2}(4i-2)D_\pm^{4i-4} - [D_\pm^{4i-2}, x] = 0, \tag{3.6}$$

$$[D^\pm_{4i-1} - D_\pm^{4i-1}, \Gamma_0^\pm] = [\frac{\partial}{\partial t^\pm_{4i-1}} - \sum_{j=1}^{\infty} t^\pm_{4j-1}\frac{\partial}{\partial t_{4i+4j-2}} - D_\pm^{4i-1}, \Gamma_0^\pm] \tag{3.7}$$

$$= \frac{1}{2}(4i-1)D_\pm^{4i-3} - \frac{1}{2}\partial^{2i-2}Q_\pm + 2\sum_{j\ge 1}(i-j)t^\pm_{4j-1}\partial^{2i+2j-2} \tag{3.8}$$

$$-\sum_{j=1}^{\infty}(2i+2j-1)t^\pm_{4j-1}D_\pm^{4i+4j-4} - [D_\pm^{4i-1}, x - \frac{1}{2}\sum_{k\ge 1}(4k-1)t^\pm_{4k-1}D_\pm^{4k-3}]$$

$$= 0, \tag{3.9}$$

in which we used

$$[D_\pm^{4i-1}, x] = \frac{1}{2}(4i-1)D_\pm^{4i-3} - \frac{1}{2}\partial^{2i-2}Q_\pm,$$

and

$$[D_\pm^{4i-1}, \frac{1}{2}\sum_{k\ge 1}(4k-1)t^\pm_{4k-1}D_\pm^{4k-3}] = -\sum_{j\ge 1}(4j-1)t^\pm_{4j-1}D_\pm^{4i+4j-4}. \tag{3.10}$$

Also we can get

$$[Q_\pm, \Gamma_0^\pm] = [Q_\pm, x] - [Q_\pm, \frac{1}{2}\sum_{k\ge 1}t^\pm_{4k-1}\partial^{2k-2}Q_\pm] = -\theta_\pm - \sum_{k\ge 1}t^\pm_{4k-1}\partial^{2k-1} = -\Gamma_1^\pm,$$

and other identities can be proved similarly. □

Then one can get the following lemma.

Lemma 3.2. *The operators $M_j^{\pm}, \mathbb{Q}_{\pm}, L$ satisfy the following identities*

$$[\mathbb{Q}_{\pm}, M_0^{\pm}] = -M_1^{\pm}, \quad [\mathbb{Q}_{\pm}, M_1^{\pm}] = 1, \quad [L^2, M_0^{\pm}] = 1, \tag{3.11}$$

$$D_{4i-2} M_j^{\pm} = [(L^{4i-2})_+, M_j^{\pm}], \quad D_{4i-2}\mathbb{Q}_{\pm} = [(L^{4i-2})_+, \mathbb{Q}_{\pm}], \tag{3.12}$$

$$D_{4i-1}^{-} M_j^{\pm} = [(L^{4i-1})_+, M_j^{\pm}], \quad D_{4i-1}^{-}\mathbb{Q}_{\pm} = [(L^{4i-1})_+, \mathbb{Q}_{\pm}], \tag{3.13}$$

$$D_{4i-1}^{+} M_j^{\pm} = [(\Lambda^{4i-1})_+, M_j^{\pm}], \quad D_{4i-1}^{+}\mathbb{Q}_{\pm} = [(\Lambda^{4i-1})_+, \mathbb{Q}_{\pm}], \quad i \in \mathbb{Z}_+. \tag{3.14}$$

Proof. We will use the following dressing structure

$$\Phi[D_{4i-1}^{\pm} - D_{\pm}^{4i-1}, \Gamma_1^{\pm}]\Phi^{-1} = 0; \tag{3.15}$$

which further lead to

$$[\Phi D_{4i-1}^{\pm}\Phi^{-1} - \Phi D_{\pm}^{4i-1}\Phi^{-1}, M_1^{\pm}] = 0. \tag{3.16}$$

Here we will consider

$$[D_{4i-1}^{+} - (D_{4i-1}^{+}\Phi)\Phi^{-1} + \sum_{j=0}^{\infty} t_{4j-1}^{+}\Phi_{4i+4j-2}\Phi^{-1} - \Lambda^{4i-1}, M_1^{+}] = 0. \tag{3.17}$$

Then we can derive

$$[D_{4i-1}^{+} - (D_{4i-1}^{+}\Phi)\Phi^{-1} - \Lambda^{4i-1}, M_1^{+}] = 0; \tag{3.18}$$

and

$$[D_{4i-1}^{+} - (\Lambda^{4i-1})_+, M_1^{+}] = 0. \tag{3.19}$$

Similarly we can get

$$[D_{4i-1}^{-} - (L^{4i-1})_+, M_1^{-}] = 0. \tag{3.20}$$

The other identities can be proved similarly. □

Now we will introduce the following additional flow operator $B_{mklp}^{\pm}$ defined as

$$B_{mklp}^{\pm} = M_0^{\pm k} M_1^{\pm l}\mathbb{Q}_{\pm}^{p} L^{2m} - (-1)^{pl+m+p+l} L^{2m-1}(\mathbb{Q}_{\pm}^{p}) M_1^{\pm l} M_0^{\pm k} L, \tag{3.21}$$

where $k, m \geq 0; l, p = 0, 1$. This operator will generate the additional symmetry of the $N = 2$ SBKP hierarchy.

Then the following proposition can be derived.

Proposition 3.3. *The operator $B_{mnlp}^{\pm}$ satisfies the following flow equations*

$$D_{4k-2} B_{mnlp}^{\pm} = -[(L^{4k-2})_-, B_{mnlp}^{\pm}], \quad D_{4k-1}^{-} B_{mnlp}^{\pm} = -[(L^{4k-1})_-, B_{mnlp}^{\pm}]. \tag{3.22}$$

$$D_{4k-1}^{+} B_{mnlp}^{\pm} = -[(\Lambda^{4k-1})_-, B_{mnlp}^{\pm}]. \tag{3.23}$$

Proof. The identities can be proved by dressing the following identities by Φ

$$[D_{4k-2} - D_{\pm}^{4k-2}, \Gamma_0^{\pm n}\Gamma_1^{\pm l} Q_{\pm}^{p}\partial^m] = [D_{4k-1}^{\pm} - D_{\pm}^{4k-1}, \Gamma_0^{\pm n}\Gamma_1^{\pm l} Q_{\pm}^{p}\partial^m] = 0. \quad \square \tag{3.24}$$

One need the following lemma to prove that $B_{mnlp}^{\pm}$ satisfies the B type condition.

Lemma 3.4. *The operators $M_i^{\pm}$ satisfy the following conjugate identities,*

$$M_i^{\pm *} = (-1)^i D_- L^{-1} M_i^{\pm} L D_-^{-1}, \quad \mathbb{Q}_{\pm}^{*} = -D_- L^{-1}\mathbb{Q}_{\pm} L D_-^{-1}. \tag{3.25}$$

Proof. We can get

$$\Phi^* = D_-\Phi^{-1}D_-^{-1}, \quad \Gamma_i^{\pm *} = (-1)^i\Gamma_i^{\pm}, \quad Q_{\pm}^{*} = -Q_{\pm}, \tag{3.26}$$

and do the following calculations

$$M_i^{\pm *} = \Phi^{*-1}\Gamma_i^{\pm *}\Phi^* = (-1)^i D_-\Phi D_-^{-1}\Gamma_i^{\pm} D_-\Phi^{-1}D_-^{-1} = (-1)^i D_-\Phi D_-^{-1}\Phi^{-1} M_i^{\pm}\Phi D_-\Phi^{-1}D_-^{-1},$$

to lead to the first identity of this lemma. The other identities can be proved similarly. □

We can further get the following proposition basing on the Lemma 3.4 above.

Proposition 3.5. *The operator $B^{\pm}_{mklp}$ satisfies a B type condition, namely*

$$B^{\pm *}_{mklp} = -D_- B^{\pm}_{mklp} D_-^{-1}. \tag{3.27}$$

Proof. Using the Proposition 3.4, the following calculation

$$\begin{aligned} B^{\pm *}_{mklp} &= (M_0^{\pm k} M_1^{\pm l} \mathbb{Q}_\pm^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} (\mathbb{Q}_\pm^p) M_1^{\pm l} M_0^{\pm k} L)^* \\ &= (-1)^{pl} L^{2m*} (\mathbb{Q}_\pm^p)^* M_1^{\pm l*} M_0^{\pm k*} + (-1)^{m+p+l} L^* M_0^{\pm k*} M_1^{\pm l*} (\mathbb{Q}_\pm^p)^* L^{2m-1*} \\ &= (-1)^{pl+m+p+l} D_- L^{2m-1} \mathbb{Q}_\pm^p M_1^{\pm l} M_0^{\pm k} L D_-^{-1} - D_- M_0^{\pm k} M_1^{\pm l} \mathbb{Q}_\pm^p L^{2m} D_-^{-1} \\ &= -D_- (M_0^{\pm k} M_1^{\pm l} \mathbb{Q}_\pm^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} \mathbb{Q}_\pm^p M_1^{\pm l} M_0^{\pm k} L) D_-^{-1}, \end{aligned}$$

will lead to the proof of this proposition. □

Basing on the above proposition, it is time to define additional flows of the $N = 2$ supersymmetric BKP hierarchy as follows

$$D^{\pm}_{mklp} L = [-(B^{\pm}_{mklp})_-, L], \quad k, m \geq 0; l, p = 0, 1. \tag{3.28}$$

Proposition 3.6. *The flows (3.28) commute with the original flows of the $N = 2$ supersymmetric BKP hierarchy as*

$$\left[D^{\pm}_{mnlp}, D_{4i-2}\right] = \left[D^{\pm}_{mnlp}, D^{\pm}_{4i-1}\right] = 0, \qquad m, n \geq 0; l, p = 0, 1, \quad i \in \mathbb{Z}_+, \tag{3.29}$$

which holds in the sense of acting on Φ.

Proof. The part of the proposition can be checked as

$$\begin{aligned} \left[D^{\pm}_{mnlp}, D^{\pm}_{4i-1}\right] \Phi &= D^{\pm}_{mnlp} D^{\pm}_{4i-1} \Phi - (-1)^{(l+p(4i-1)k} D^{\pm}_{4i-1} D^{\pm}_{mnlp} \Phi \\ &= (-1)^{(l+p)(4i-1)} [(L^{4i-1})_-, (B^{\pm}_{mnlp})_-] \Phi + [(B^{\pm}_{mnlp})_-, L^{4i-1}]_- \Phi & (3.30) \\ &+ (-1)^{(l+p)(4i-1)} [(L^{4i-1})_+, B^{\pm}_{mnlp}]_- \Phi & (3.31) \\ &= 0. \end{aligned}$$

The case $\left[D^{\pm}_{mnlp}, D_{4i-2}\right] = 0$ can be proved in a similar way. □

That tells us that the additional flows of the $N = 2$ supersymmetric BKP hierarchy are symmetries of the original flows and their algebraic structure can be shown in the following proposition.

Proposition 3.7. *The algebra of additional symmetries of the $N = 2$ SBKP hierarchy given by eq. (3.28) is isomorphic to the $N = 2$ super Lie algebra $SW_{1+\infty}$.*

Proof. The isomorphism is given by

$$z \mapsto \partial, \quad \xi \mapsto Q_+ + \Gamma_1^+ \partial, \quad \eta \mapsto Q_- + \Gamma_1^- \partial, \tag{3.32}$$

$$\partial_z \mapsto \Gamma_0^{\pm}, \quad \partial_\xi \mapsto \Gamma_1^+, \quad \partial_\eta \mapsto \Gamma_1^-, \tag{3.33}$$

which further lead to

$$z \mapsto L^2, \quad \xi \mapsto \mathbb{Q}_+ + M_1^+ L^2, \quad \eta \mapsto \mathbb{Q}_- + M_1^- L^2, \tag{3.34}$$

$$\partial_z \mapsto M_0^{\pm}, \quad \partial_\xi \mapsto M_1^+, \quad \partial_\eta \mapsto M_1^-. \tag{3.35}$$

The above construction keeps ξ, η commuting with z, ξ anti-commute with η. □

4. The $N = 2$ multicomponent supersymmetric BKP hierarchy

Basing on the above construction of single component $N = 2$ supersymmetric BKP hierarchy, we will define a $N = 2$ multicomponent supersymmetric BKP system in this section. Let $\hat{\mathcal{A}}$ be assumed as an algebra of smooth matrix-valued functions of a spatial coordinate x, grassmann variables $\theta_\pm$ and their super-derivation denoted as $D_\pm = \partial_{\theta_\pm} + \theta_\pm \partial$. We introduce the even and odd time variables $(t_{2,\alpha}, t^{\pm}_{3,\alpha}, t_{6,\alpha}, t^{\pm}_{7,\alpha}, \cdots)$ with $1 \leq \alpha \leq s$ and the following definition of even and odd flows

$$D_{4i-2,\alpha} = \frac{\partial}{\partial t_{4i-2,\alpha}}, \quad D^{\pm}_{4i-1,\alpha} = \frac{\partial}{\partial t^{\pm}_{4i-1,\alpha}} + \sum_{j=1}^{\infty} t^{\pm}_{4j-1,\alpha} \frac{\partial}{\partial t_{4i+4j-2,\alpha}}. \tag{4.1}$$

This family of infinite odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$[D_{4i-2,\alpha}, D_{4j-2,\beta}] = 0\,, \quad [D_{4i-2,\alpha}, D^{\pm}_{4j-1,\beta}] = 0\,, \quad [D^{\pm}_{4i-1,\alpha}, D^{\pm}_{4j-1,\beta}] = -2\frac{\partial}{\partial t_{4i+4j-2,\alpha}}\delta_{\alpha,\beta}\,,$$
$$[D^{\pm}_{4i-1,\alpha}, D^{\mp}_{4j-1,\beta}] = 0, \;\; [D_{4i-2,\alpha}, D_{\pm}] = 0\,, \quad [D^{\pm}_{4i-1,\alpha}, D_{\pm}] = 0. \tag{4.2}$$

The rules of conjugation $u^* = u^T$ holds true for any superfield. Here u^T means the transpose of the matrix-valued superfield u. The Lax operator of the $N = 2$ multicomponent supersymmetric BKP hierarchy has a form as

$$L = \Phi D_- \Phi^{-1}, \;\; L^* = -D_- L D_-^{-1}, \;\; R_\alpha = \Phi E_\alpha \Phi^{-1}, \tag{4.3}$$

where

$$\Phi = 1 + \sum_{i\geq 1}(a_i^{(0)} + a_i^+ D_+ + a_i^- D_- + a_i^{(1)} D_+ D_-)\partial^{-i}, \tag{4.4}$$

satisfy

$$\Phi^* = D_- \Phi^{-1} D_-^{-1}, \tag{4.5}$$

and E_α is the matrix with element at the position of α-th row and α-th column being 1 and other ones being zeroes. The $N = 2$ multicomponent supersymmetric BKP hierarchy can be defined by the following Lax equations

$$D_{4k-2,\alpha} L = [(L^{4k-2} R_\alpha)_+, L], \qquad D^-_{4k-1,\alpha} L = [(L^{4k-1} R_\alpha)_+, L] - 2L^{4k} R_\alpha, \tag{4.6}$$
$$D^+_{4k-1,\alpha} L = [(\Lambda^{4k-1} R_\alpha)_+, L], \;\; \Lambda = \Phi D_+ \Phi^{-1}, \quad k \geq 1. \tag{4.7}$$

We can call the eq. (4.5) the B type condition of the $N = 2$ multicomponent supersymmetric BKP hierarchy. The $N = 2$ multicomponent supersymmetric BKP hierarchy (4.6) can also be redefined as the following Sato equations

$$D_{4k-2,\alpha} \Phi = -(L^{4k-2} R_\alpha)_- \Phi, \quad D^-_{4k-1,\alpha} = -(L^{4k-1} R_\alpha)_- \Phi, \tag{4.8}$$
$$D^+_{4k-1,\alpha} \Phi = -(\Lambda^{4k-1} R_\alpha)_- \Phi, \tag{4.9}$$

with $k \geq 1$.

With the above preparation, it is time to construct additional symmetries for the $N = 2$ multicomponent supersymmetric BKP hierarchy in the next section.

5. Additional symmetries of the N=2 supersymmetric BKP hierarchy

The additional symmetries for the $N = 2$ s-component supersymmetric BKP hierarchy can be defined by using the Orlov–Schulman operators. The Orlov–Schulman operators $M_i^{\pm}$ and auxiliary operator $\mathbb{Q}_{\pm}$ can be constructed in the following dressing structure

$$M^{\pm}_{i,\alpha} = \Phi \Gamma^{\pm}_{i,\alpha} \Phi^{-1}, \quad i = 0, 1; \; 1 \leq \alpha \leq s; \; \hat{\mathbb{Q}}_{\pm} = \Phi Q_{\pm} \Phi^{-1},$$

where

$$\Gamma^{\pm}_{0,\alpha} = x E_\alpha + \frac{1}{2}\sum_{k\geq 1}(4k-2)t_{4k-2,\alpha} E_\alpha D_{\pm}^{4k-4} + \frac{1}{2}(4k-1)t^{\pm}_{4k-1,\alpha} E_\alpha D_{\pm}^{4k-3}$$
$$-\frac{1}{2}\sum_{k\geq 1} t^{\pm}_{4k-1,\alpha} E_\alpha \partial^{2k-2} Q_{\pm} + \sum_{i,j\geq 1}(i-j)t^{\pm}_{4i-1,\alpha} t^{\pm}_{4j-1,\alpha} E_\alpha \partial^{2i+2j-2}, \tag{5.1}$$
$$\Gamma^{\pm}_{1,\alpha} = \theta_{\pm} E_\alpha + \sum_{k\geq 1} t^{\pm}_{4k-1,\alpha} E_\alpha \partial^{2k-1}, \tag{5.2}$$

where $Q_{\pm} = \partial_{\theta_{\pm}} - \theta_{\pm}\partial$.

Then one can get the following lemma.

Lemma 5.1. *The operators $\Gamma^{\pm}_{j,\alpha}$, $Q_{\pm}$ can be proved to satisfy*

$$[D_{4i-2,\alpha} - D_{\pm}^{4i-2} E_\alpha, \Gamma^{\pm}_{j,\beta}] = [D^{\pm}_{4i-1,\alpha} - D_{\pm}^{4i-1} E_\alpha, \Gamma^{\pm}_{j,\beta}] = 0; \;\; j = 0, 1, \tag{5.3}$$
$$[D_{4i-2,\alpha} - D_{\pm}^{4i-2} E_\alpha, Q_{\pm}] = [D^{\pm}_{4i-1,\alpha} - D_{\pm}^{4i-1} E_\alpha, Q_{\pm}] = 0, \tag{5.4}$$
$$[Q_{\pm}, \Gamma^{\pm}_{0,\alpha}] = -\Gamma^{\pm}_{1,\alpha}, \quad [Q_{\pm}, \Gamma^{\pm}_{1,\alpha}] = E_\alpha, \quad [\partial, \Gamma^{\pm}_{0,\alpha}] = E_\alpha. \tag{5.5}$$

Proof. The proof can be proved as follows

$$[D_{4i-2,\alpha}-D_{\pm}^{4i-2}E_\alpha,\Gamma_{0,\beta}^{\pm}]=\frac{1}{2}(4i-2)D_{\pm}^{4i-4}E_\alpha\delta_{\alpha,\beta}-[D_{\pm}^{4i-2},x]E_\alpha\delta_{\alpha,\beta}=0,$$

$$[D^{\pm}_{4i-1,\alpha}-D_{\pm}^{4i-1}E_\alpha,\Gamma_{0,\beta}^{\pm}]=[\frac{\partial}{\partial t^{\pm}_{4i-1,\alpha}}-\sum_{j=1}^{\infty}t^{\pm}_{4j-1,\alpha}\frac{\partial}{\partial t_{4i+4j-2,\alpha}}-D_{\pm}^{4i-1}E_\alpha,\Gamma_{0,\beta}^{\pm}] \tag{5.6}$$

$$=\frac{1}{2}(4i-1)D_{\pm}^{4i-3,\alpha}E_\alpha\delta_{\alpha,\beta}-\frac{1}{2}\partial^{2i-2}Q_{\pm}E_\alpha\delta_{\alpha,\beta}+2\sum_{j\geq1}(i-j)t^{\pm}_{4j-1,\alpha}E_\alpha\delta_{\alpha,\beta}\partial^{2i+2j-2} \tag{5.7}$$

$$-\sum_{j=1}^{\infty}(2i+2j-1)t^{\pm}_{4j-1,\alpha}E_\alpha\delta_{\alpha,\beta}D_{\pm}^{4i+4j-4}-E_\alpha\delta_{\alpha,\beta}[D_{\pm}^{4i-1},x-\frac{1}{2}\sum_{k\geq1}(4k-1)t^{\pm}_{4k-1,\alpha}D_{\pm}^{4k-3}]$$

$$=0, \tag{5.8}$$

in which we used

$$[D_{\pm}^{4i-1},x]=\frac{1}{2}(4i-1)D_{\pm}^{4i-3}-\frac{1}{2}\partial^{2i-2}Q_{\pm},$$

and

$$[D_{\pm}^{4i-1},\frac{1}{2}\sum_{k\geq1}(4k-1)t^{\pm}_{4k-1,\alpha}D_{\pm}^{4k-3}]=-\sum_{j\geq1}(4j-1)t^{\pm}_{4j-1,\alpha}D_{\pm}^{4i+4j-4}. \tag{5.9}$$

Also we do get

$$[Q_{\pm},\Gamma_{0,\beta}^{\pm}]=[Q_{\pm},xE_\beta]-[Q_{\pm},\frac{1}{2}\sum_{k\geq1}t^{\pm}_{4k-1,\beta}E_\beta\partial^{2k-2}Q_{\pm}]=-\Gamma^{\pm}_{1,\beta}.$$

For $\Gamma^{\pm}_{1,\alpha}$, $Q_{\pm}$, and other identities can be proved similarly. □

Then one can get the following lemma using dressing structures.

Lemma 5.2. *The operators $M^{\pm}_{j,\alpha}$, $\hat{\mathbb{Q}}_{\pm}$, Ł satisfy*

$$[\hat{\mathbb{Q}}_{\pm},M^{\pm}_{0,\alpha}]=-M^{\pm}_{1,\alpha},\quad[\hat{\mathbb{Q}}_{\pm},M^{\pm}_{1,\alpha}]=E_\alpha,\quad[Ł^2,M^{\pm}_{0,\alpha}]=E_\alpha, \tag{5.10}$$

$$D_{4i-2,\alpha}M^{\pm}_{j,\beta}=[(Ł^{4i-2}R_\alpha)_+,M^{\pm}_{j,\beta}],\quad D_{4i-2,\alpha}\hat{\mathbb{Q}}_{\pm}=[(Ł^{4i-2}R_\alpha)_+,\hat{\mathbb{Q}}_{\pm}], \tag{5.11}$$

$$D^{-}_{4i-1,\alpha}M^{\pm}_{j,\beta}=[(Ł^{4i-1}R_\alpha)_+,M^{\pm}_{j,\beta}],\quad D^{-}_{4i-1,\alpha}\hat{\mathbb{Q}}_{\pm}=[(Ł^{4i-1}R_\alpha)_+,\hat{\mathbb{Q}}_{\pm}], \tag{5.12}$$

$$D^{+}_{4i-1,\alpha}M^{\pm}_{j,\beta}=[(\Lambda^{4i-1}R_\alpha)_+,M^{\pm}_{j,\beta}],\quad D^{+}_{4i-1,\alpha}\hat{\mathbb{Q}}_{\pm}=[(\Lambda^{4i-1}R_\alpha)_+,\hat{\mathbb{Q}}_{\pm}],\ \ i\in\mathbb{Z}_+. \tag{5.13}$$

Proof. The following dressing structure

$$\Psi[D^{\pm}_{4i-1,\alpha}-D_{\pm}^{4i-1}E_\alpha,\Gamma^{\pm}_{1,\beta}]\Psi^{-1}=0; \tag{5.14}$$

will lead to the identity

$$[\Psi D^{\pm}_{4i-1,\alpha}\Psi^{-1}-\Psi D_{\pm}^{4i-1}E_\alpha\Psi^{-1},M^{\pm}_{1,\beta}]=0. \tag{5.15}$$

Here we focus on

$$[D^{+}_{4i-1,\alpha}-(D^{+}_{4i-1,\alpha}\Psi)\Psi^{-1}+\sum_{j=0}^{\infty}t^{+}_{4j-1,\alpha}\Psi_{4i+4j-2}\Psi^{-1}-\Lambda^{4i-1}R_\alpha,M^{+}_{1,\beta}]=0.$$

Then we can derive

$$[D^{+}_{4i-1,\alpha}-(D^{+}_{4i-1,\alpha}\Psi)\Psi^{-1}-\Lambda^{4i-1}R_\alpha,M^{+}_{1,\beta}]=0; \tag{5.16}$$

and

$$[D^{+}_{4i-1,\alpha}-(\Lambda^{4i-1}R_\alpha)_+,M^{+}_{1,\beta}]=0. \tag{5.17}$$

Similarly we can get

$$[D^{-}_{4i-1,\alpha}-(Ł^{4i-1}R_\alpha)_+,M^{-}_{1,\beta}]=0, \tag{5.18}$$

and the other identities can be proved similarly. □

To define the additional flows with B type condition, we need to introduce the following operator $B^{\pm\alpha\beta\gamma}_{mklp}$ defined as

$$B^{\pm\alpha\beta\gamma}_{mklp} = M^{\pm k}_{0,\alpha} M^{\pm l}_{1,\beta} \hat{\mathbb{Q}}^{p}_{\pm} Ł^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} Ł^{2m-1} (\hat{\mathbb{Q}}^{p}_{\pm}) M^{\pm l}_{1,\beta} M^{\pm k}_{0,\alpha} L, \tag{5.19}$$

where $k, m \geq 0; l, p = 0, 1; 1 \leq \alpha, \beta, \gamma, \rho \leq s$. This operator can be the generator of the additional flows of the $N = 2$ multicomponent SBKP hierarchy.

Then we can get the following proposition.

Proposition 5.3. *The operator $B^{\pm\alpha\beta\gamma}_{mklp}$ satisfies the following flow equations*

$$D_{4k-2,\rho} B^{\pm\alpha\beta\gamma}_{mklp} = -[(Ł^{4k-2} R_{\rho})_{-}, B^{\pm\alpha\beta\gamma}_{mklp}], \;\; D^{-}_{4k-1,\rho} B^{\pm\alpha\beta\gamma}_{mklp} = -[(Ł^{4k-1} R_{\rho})_{-}, B^{\pm\alpha\beta\gamma}_{mklp}], \tag{5.20}$$

$$D^{+}_{4k-1,\rho} B^{\pm\alpha\beta\gamma}_{mklp} = -[(\Lambda^{4k-1} R_{\rho})_{-}, B^{\pm\alpha\beta\gamma}_{mklp}]. \tag{5.21}$$

Proof. The lemma can be proved by dressing the following identities using Ψ

$$[D_{4k-2,\rho} - D^{4k-2}_{\pm} E_{\rho}, \Gamma^{\pm n}_{0,\alpha} \Gamma^{\pm l}_{1,\beta} Q^{p}_{\pm} E_{\gamma} \partial^{m}] = [D^{\pm}_{4k-1,\rho} - D^{4k-1}_{\pm} E_{\rho}, E_{\gamma} \Gamma^{\pm n}_{0,\alpha} \Gamma^{\pm l}_{1,\beta} Q^{p}_{\pm} \partial^{m}] = 0. \quad \square \tag{5.22}$$

We need the following lemma to prove that $B^{\pm\alpha\beta\gamma}_{mklp}$ satisfies the B type condition.

Lemma 5.4. *The operators $M^{\pm}_{i}$ satisfy the following conjugate property*

$$M^{\pm *}_{i,\alpha} = (-1)^{i} D_{-} Ł^{-1} M^{\pm}_{i,\alpha} L D^{-1}_{-}, \;\; \hat{\mathbb{Q}}^{*}_{\pm} = -D_{-} Ł^{-1} \hat{\mathbb{Q}}_{\pm} L D^{-1}_{-}. \tag{5.23}$$

Proof. Using

$$\Psi^{*} = D_{-} \Psi^{-1} D^{-1}_{-}, \;\; \Gamma^{\pm *}_{i,\alpha} = (-1)^{i} \Gamma^{\pm}_{i,\alpha}, \;\; Q^{*}_{\pm} = -Q_{\pm}, \tag{5.24}$$

the following calculations

$$M^{\pm *}_{i,\alpha} = \Psi^{*-1} \Gamma^{\pm *}_{i,\alpha} \Psi^{*} = (-1)^{i} D_{-} \Psi D^{-1}_{-} \Gamma^{\pm}_{i,\alpha} D_{-} \Psi^{-1} D^{-1}_{-} = (-1)^{i} D_{-} \Psi D^{-1}_{-} \Psi^{-1} M^{\pm}_{i,\alpha} \Psi D_{-} \Psi^{-1} D^{-1}_{-},$$

will lead to the first identity of this lemma. The other identities can be proved in a similar way. $\square$

One can check the following proposition holds basing on the Lemma 5.4 above.

Proposition 5.5. *The operator $B^{\pm\alpha\beta\gamma}_{mklp}$ satisfies a B type condition, namely*

$$B^{\pm\alpha\beta\gamma *}_{mklp} = -D_{-} B^{\pm\alpha\beta\gamma}_{mklp} D^{-1}_{-}. \tag{5.25}$$

Proof. The following calculation

$$\begin{aligned}
B^{\pm\alpha\beta\gamma *}_{mklp} &= (M^{\pm k}_{0,\alpha} M^{\pm l}_{1,\beta} \hat{\mathbb{Q}}^{p}_{\pm} Ł^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} Ł^{2m-1} (\hat{\mathbb{Q}}^{p}_{\pm}) M^{\pm l}_{1,\beta} M^{\pm k}_{0,\alpha} L)^{*} \\
&= (-1)^{pl} R^{*}_{\gamma} Ł^{2m*} (\hat{\mathbb{Q}}^{p}_{\pm})^{*} M^{\pm l *}_{1,\beta} M^{\pm k *}_{0,\alpha} + (-1)^{m+p+l} Ł^{*} M^{\pm k *}_{0,\alpha} M^{\pm l *}_{1,\beta} (\hat{\mathbb{Q}}^{p}_{\pm})^{*} Ł^{2m-1*} R^{*}_{\gamma} \\
&= (-1)^{pl+m+p+l} D_{-} R_{\gamma} Ł^{2m-1} \hat{\mathbb{Q}}^{p}_{\pm} M^{\pm l}_{1,\beta} M^{\pm k}_{0,\alpha} L D^{-1}_{-} - D_{-} M^{\pm k}_{0,\alpha} M^{\pm l}_{1,\beta} \hat{\mathbb{Q}}^{p}_{\pm} Ł^{2m} R_{\gamma} D^{-1}_{-} \\
&= -D_{-} (M^{\pm k}_{0,\alpha} M^{\pm l}_{1,\beta} \hat{\mathbb{Q}}^{p}_{\pm} Ł^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} Ł^{2m-1} \hat{\mathbb{Q}}^{p}_{\pm} M^{\pm l}_{1,\beta} M^{\pm k}_{0,\alpha} L) D^{-1}_{-},
\end{aligned}$$

will lead to this proposition. $\square$

Basing on above proposition, it is reasonable to define additional flows of the N=2 supersymmetric BKP hierarchy as

$$D^{\pm\alpha\beta\gamma}_{mklp} L = [-(B^{\pm\alpha\beta\gamma}_{mklp})_{-}, L], \;\; k, m \geq 0; l, p = 0, 1; 1 \leq \alpha, \beta, \gamma \leq s. \tag{5.26}$$

Proposition 5.6. *The flows (5.26) commute with the original flows of the $N = 2$ multicomponent supersymmetric BKP hierarchy as*

$$\left[D^{\pm\alpha\beta\gamma}_{mnlp}, D_{4i-2,\rho}\right] = \left[D^{\pm\alpha\beta\gamma}_{mnlp}, D^{\pm}_{4i-1,\rho}\right] = 0, \qquad m, n \geq 0; l, p = 0, 1, \;\; k = 4i-2, 4i-1, \; i \in \mathbb{Z}_{+}, \tag{5.27}$$

by acting on Ψ.

Proof. The proposition can be checked as when $k = 4i - 1$

$$\left[D_{mnlp}^{\pm\alpha\beta\gamma}, D_{k,\rho}^{\pm}\right]\Psi = D_{mnlp}^{\pm\alpha\beta\gamma} D_{k,\rho}^{\pm}\Psi - (-1)^{(l+p)k} D_{k,\rho}^{\pm} D_{mnlp}^{\pm\alpha\beta\gamma}\Psi$$

$$= (-1)^{(l+p)k}[(Ł^k R_\rho)_-, (B_{mnlp}^{\pm\alpha\beta\gamma})_-]\Psi + [(B_{mnlp}^{\pm\alpha\beta\gamma})_-, Ł^k R_\rho]_-\Psi \tag{5.28}$$

$$+ (-1)^{(l+p)k}[(Ł^k R_\rho)_+, B_{mnlp}^{\pm\alpha\beta\gamma}]_-\Psi \tag{5.29}$$

$$= 0.$$

The case when $k = 4i - 2$ can be proved similarly. □

That shows that the additional flows of the $N = 2$ multicomponent supersymmetric BKP hierarchy are symmetries of the original flows whose algebraic structure can be shown in the following proposition.

Proposition 5.7. *The algebra of additional symmetries of the $N = 2$ multicomponent SBKP hierarchy given by eq. (5.26) is isomorphic to the Lie algebra $N = 2 \bigotimes SW_{1+\infty}$.*

Proof. The isomorphism can be given by

$$z \mapsto \partial, \quad \xi E_\beta \mapsto Q_+ E_\beta + \Gamma_{1,\beta}^+ \partial, \quad \eta E_\beta \mapsto Q_- E_\beta + \Gamma_{1,\beta}^- \partial, \tag{5.30}$$

$$\partial_z E_\beta \mapsto \Gamma_{0,\beta}^{\pm}, \quad \partial_\xi E_\beta \mapsto \Gamma_{1,\beta}^{+}, \quad \partial_\eta E_\beta \mapsto \Gamma_{1,\beta}^{-}, \tag{5.31}$$

which further lead to

$$z \mapsto Ł^2, \quad \xi E_\beta \mapsto \hat{\mathbb{Q}}_+ R_\beta + M_{1,\beta}^+ Ł^2, \; \eta E_\beta \mapsto \hat{\mathbb{Q}}_- R_\beta + M_{1,\beta}^- Ł^2, \tag{5.32}$$

$$\partial_z E_\beta \mapsto M_{0,\beta}^{\pm}, \quad \partial_\xi E_\beta \mapsto M_{1,\beta}^{+}, \quad \partial_\eta E_\beta \mapsto M_{1,\beta}^{-}, \tag{5.33}$$

where keeps ξ, η commuting with z, ξ anti-commute with η. □

Declaration of competing interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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