HOMFLY polynomials in representation $[3, 1]$ for 3-strand braids

A. Mironov,[a,b,c,d] **A. Morozov,**[b,c,d] **An. Morozov**[c,d,e] **and A. Sleptsov**[b,c,d,e]

[a] *Theory Department, Lebedev Physics Institute,*
 Leninsky prospekt 53, Moscow 119991 Russia

[b] *ITEP,*
 Bol. Cheremushkinskaya 25, Moscow 117218, Russia

[c] *Institute for Information Transmission Problems,*
 Bolshoy Karetny per. 19, build. 1, Moscow 127051, Russia

[d] *National Research Nuclear University MEPhI,*
 Kashirskoe shosse 31, Moscow 115409, Russia

[e] *Laboratory of Quantum Topology, Chelyabinsk State University,*
 Bratiev Kashirinyh st. 129, Chelyabinsk 454001, Russia

 E-mail: mironov@lpi.ru, morozov@itep.ru, andrey.morozov@itep.ru, sleptsov@itep.ru

ABSTRACT: This paper is a new step in the project of systematic description of colored knot polynomials started in [1]. In this paper, we managed to explicitly find the *inclusive* Racah matrix, i.e. the whole set of mixing matrices in channels $R^{\otimes 3} \longrightarrow Q$ with all possible Q, for $R = [3, 1]$. The calculation is made possible by the use of a newly-developed efficient highest-weight method, still it remains tedious. The result allows one to evaluate and investigate $[3, 1]$-colored polynomials for arbitrary 3-strand knots, and this confirms many previous conjectures on various factorizations, universality, and differential expansions. We consider in some detail the next-to-twist-knots three-strand family $(n, -1 \,|\, 1, -1)$ and deduce its colored HOMFLY. Also confirmed and clarified is the eigenvalue hypothesis for the Racah matrices, which promises to provide a shortcut to generic formulas for arbitrary representations.

Contents

1 Introduction

Colored knot/link polynomials [2–9] are characteristics of knots and links, which factorize into products for composite knots. In this sense, the knot polynomials for the *prime* knots play the role of *prime* numbers, only in the world of knots, and the most challenging is the question if they are indeed *prime* or there is some additional "more elementary" structure, which allows one to reconstruct them from some simpler constituents. Another source of interest to the colored knot/link polynomials is that they are the Wilson loop averages in Chern-Simons theory [10, 11], the simplest of all Yang-Mills theories, and they can be exactly calculated *non-perturbatively*, because they are *polynomials* in the variable $q = \exp(g^{-2})$. This provides a very non-trivial check for our emerging understanding of non-perturbative methods in quantum field and string theory. Both these reasons make evaluation of colored knot polynomials extremely important problem, since it is the first step towards understanding their properties. Until recently, it was an unachievable task, but development of theoretical methods in [12]–[103], combined with the current computer power makes it nearly realistic.

This explains our reasons to make a try, and it is already partly successful. At the previous stages, the calculus for totally symmetric and antisymmetric representations was successfully developed [64–66], then extended in [74, 91, 92, 96, 97, 100] to representation $[2, 1]$, and, in the present paper, we report a new achievement: results for 3-strand polynomials in representation $[3, 1]$. The next big challenge is the first two-hook representation $[4, 2]$, this requires a new serious theoretical advance, but now it seems within reach. The powerful method needed for that purpose will be described in detail in a separate publication, here we just mention it in general description of our approach.

It involves the following steps:

1. Define the representation R by a highest weight in the space $[1]^{\otimes R}$.

2. Find the highest weight of representation $Y \in R \otimes R$.

3. Distinguish between Y belonging to the symmetric and antisymmetric squares of R. If a given representation Y appears in these both, we treat Y_+ and Y_- as different representations: they can have there own multiplicities, but they are never summed. The reason for this separation of $Y_\pm$ is that the knot polynomials depend on the eigenvalues of quantum $\mathcal{R}$-matrices: these are different (in sign) for Y_+ and Y_-.

4. Find highest weights of representations $Q \in Y \otimes R$ and $Q \in R \otimes Y$. These weights $h_Q^l(Y)$ and $h_Q^r(Y)$ (superscripts stand for "left" and "right") are related by the Racah (mixing) matrix $\mathcal{U}_Q$:

$$h_Q^r(Y) = \sum_{Y' \in R \otimes R} \mathcal{U}_Q(Y, Y')\, h_Q^l(Y') \tag{1.1}$$

In the case of non-trivial multiplicities, $\mathcal{U}$ is defined modulo rotations in the multiplicity spaces of Q, Y and Y', but these rotations leave $\mathcal{R}$-matrix intact, and therefore do not affect knot polynomials. These rotations, however, can be essential for the eigenvalue hypothesis, expressing the entries of $\mathcal{U}$ through those of $\mathcal{R}$, then this freedom should be somehow fixed.

5. Evaluate the reduced (normalized to unknot) knot/link polynomial for a knot represented by a closure of the 3-strand braid $B^{(m_1,n_1|m_2,n_2|...)}$ by [59–62]

$$H_R^{(m_1,n_1|m_2,n_2|...)} = \sum_{Q \in R^{\otimes 3}} \frac{D_Q}{D_R} \cdot \text{Tr}_Q \left\{ \mathcal{R}_Q^{m_1} \mathcal{U}_Q \mathcal{R}_Q^{n_1} \mathcal{U}_Q^{\dagger} \mathcal{R}_Q^{m_2} \mathcal{U}_Q \mathcal{R}_Q^{n_2} \mathcal{U}_Q^{\dagger} \cdots \right\} \qquad (1.2)$$

In the following picture $m_1 = 0, n_1 = -2, m_2 = 2, n_2 = -1, m_3 = 3$:
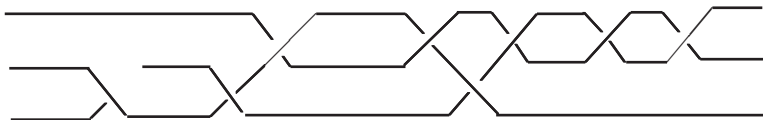
6. Examine properties of the mixing matrices (say, the eigenvalue hypothesis) and the knot polynomials (say, various factorization properties, differential expansions, recursions with the change of R etc).

All this sounds simple, but is quite difficult in practice. We comment on steps $\mathbf{1} - \mathbf{4}$ in section 2 and, more specifically, 3, provide some results of step $\mathbf{5}$ in sections 4–5 and describe some checks from step $\mathbf{6}$ in section 6. We end with a brief conclusion.

Throughout the paper we use the notation

$$\{x\} \equiv x - \frac{1}{x}, \qquad D_n \equiv \frac{\{Aq^n\}}{\{q\}}, \qquad [n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (1.3)$$

Let us also note that we use throughout the text the term "k-strand knot/link" which implies the knot/link whose braid representation with minimal number of strands is k-strand.

2 Comments on the highest weight calculus

One of the crucial decisions that makes calculations doable is to extract the mixing matrices from the highest weights. We already described the basis of this technique in [100], nowadays step $\mathbf{2}$ from the above list is provided by a fast working computer program, which finds highest weight of $Y \in R_1 \otimes R_2$, the same program is used at step $\mathbf{4}$. We now briefly describe the new aspects, which were not clear enough at the time of [100], other details can be found in that paper.

We distinguish between elementary and advanced levels of the method: most calculations for [3, 1] were performed at the former level, but further work on higher representations can be hard without going to the latter one.

2.1 Elementary level

● Embed all representations in tensor powers $[1]^{\otimes M}$ of the fundamental representation and parameterize elements of the Verma modules by number sequences. For example, for the fundamental representation $M = 1$, it has the highest weight (0) and its other elements are $(1), (2), \ldots$. They are generated from (0) by action of the lowering Chevalley operators $\hat{T}_{-a} : (a-1) \longrightarrow (a)$. Similarly, for the symmetric representation $[r]$ the highest weight is a sequence of $M = r$ zeroes $(00\ldots0)$ and elements of the Verma module are q-symmetric

linear combinations $(10\ldots0) + q^{-1}(01\ldots0) + \ldots + q^{1-r}(00\ldots1)$ and so on. The first antisymmetric representation $[1,1]$ has the highest vector $(10) - q(01)$. The action of $\hat{T}_{-a}$ for $M > 1$ is defined by comultiplication.

• Elements of the Verma modules are generated from the highest weights by action of the Chevalley generators $\hat{T}_{-a}$ and their *ordered* products $\hat{T}_{-A} = \prod_i \hat{T}_{-a_i}$. The sets of ordered sequences A are *not* Young diagrams (like they were in the case of Heisenberg or Virasoro Fock modules), they can seem to be exponentially growing with the level $|A| = \sum_i a_i$, but this is not true if one takes into account the conditions

$$\hat{T}_{-a}\hat{T}_{-b} = \hat{T}_{-b}\hat{T}_{-a}, \qquad \forall\, a, b : |a - b| < 2 \tag{2.1}$$

and the Serre relations

$$\hat{T}_{-a}^2 \hat{T}_{-a-1} + \hat{T}_{-a-1}\hat{T}_{-a}^2 = (q + q^{-1})\,\hat{T}_{-a}\hat{T}_{-a-1}\hat{T}_{-a}, \qquad \forall\, a \tag{2.2}$$

guaranteeing the PBW property for all simple algebras with allowed Dynkin diagrams. In fact, for the sl_N algebras the sets A are closer to the $3d$ Young diagrams (plane partitions), especially, if one does not restrict N: the Verma module for (continuous) gl_∞ is naturally similar to that of the double-affine $\mathrm{DIM}(gl_1)$ [104]. In practice, imposing (2.1) and (2.2) is indeed important for $R = [3,1]$ to make the problem solvable for finite time. One can alternatively build the Verma modules with the help of all Borel generators, not Chevalley, but we actually used the latter way.

• Highest weights are the vectors in $[1]^{\otimes M}$ annihilated by all rising operators $\hat{T}_a$, they are parameterized by the Young diagrams of size M.

• The highest weight V_R is a linear combination of sequences which contain definite amounts of zeroes, units, twos etc. These quantities are directly dictated by the Young diagram: for $R = \{r_1 \geq r_2 \geq \ldots\}$ the number of zeroes is r_1, the number of units is r_2 and so on. For example, $V_{[5,1,1,1]}$ is a combination of sequences with $\#(0) = 5$, $\#(1) = 1$, $\#(2) = 1$, $\#(3) = 1$. We call collection of these entries the *type* t_R of representation R, e.g. $t_{[5,1,1,1]} = \{00000123\}$. The highest weight V_R is a certain linear combination of these sequences with different orderings.

• The highest weight V_Y of representation $Y \in R_1 \otimes R_2$ can be obtained by acting with the lowering operators on the tensor product of the highest weights of R_1 and R_2:

$$V_Y \in \oplus\Big(\hat{T}_{-A}V_{R_1} \otimes \hat{T}_{-B}V_{R_2}\Big), \tag{2.3}$$

The question is how to choose A and B.

• The first criterion is simple: one can look at the difference of *types* of Y and R_1 and R_2. For example, for $[5,1,1,1] \in [3,1] \otimes [3,1]$ one should compare $t_{[5,1,1,1]} = \{00000123\}$ with $t_{[3,1]} \cup t_{[3,1]} = \{0001\} \cup \{0001\} = \{00000011\}$. To get the former from the latter, one should apply $\hat{T}_{-1}\hat{T}_{-2}\hat{T}_{-2}\hat{T}_{-3}$, which means that together $A \cup B = \{1, 2, 2, 3\}$. Applying this criterion, one restricts the set of indices $\{a_i, b_i\}$ in the pair (A, B), and (2.1)+(2.2) reduce the ordering freedom. Still, there are many terms of this type, differing by the positions of different operators, and they can enter with arbitrary coefficients.

- To fix the coefficients, we apply the second criterion: the highest weight condition

$$T_a V_Y = 0 \quad \forall a \tag{2.4}$$

If there are no multiplicities, i.e. Y appears in $R_1 \otimes R_2$ exactly once, the coefficients are defined unambiguously modulo a common normalization factor.

- Find the highest weights of representations $Q \in Y \otimes R$ and $Q \in R \otimes Y$. If Q has multiplicities, then the highest weights can be chosen in an arbitrary way. This freedom affects the form of the Racah/mixing matrix, but not the answer for the knot polynomial.

- The mixing matrices are unitary if the highest weights are normalized. Technically simplest is the norm

$$\left\| \sum_{a_1,\ldots,a_M} c_{a_1 \ldots a_M}(q) \cdot \{a_1, \ldots, a_M\} \right\|^2 = \sum_{a_1,\ldots,a_M} \left(c_{a_1 \ldots a_M}(q) \right)^2 \tag{2.5}$$

- The issue of multiplicities does not matter for the knot polynomials, because $\mathcal{R}$-matrix is diagonal in representation Y, so the basis in the multi-dimensional space of highest vectors can be chosen arbitrarily. There is, however, one important exception: for $R_1 = R_2 = R$ one should distinguish between $Y_\pm$ belonging to q-symmetric and q-antisymmetric squares of R, because the corresponding eigenvalues of $\mathcal{R}$ have different signs and do not coincide. This is not a simple task, but there is a simple solution: if one rotates the unitary mixing matrix $\mathcal{U}_Y$ into *symmetric* (simply symmetric, with no reference to q-symmetry), it automatically separates Y_+ from Y_-.

2.2 Advanced level

The method of the previous subsection provides highest weight vectors V_Y as linear combinations of integer-number sequences from $[1]^{\otimes M}$, which often contain enormously many terms (not exponentially growing, but still too much). The mixing matrices define linear dependencies between such combinations and they are difficult to find even on powerful computers, despite it is just a linear algebra problem.

The following procedure helps to tame these linear combinations by noting that they are made from certain standard pieces so that one can combine substantially smaller number pieces rather than the original number sequences:

- Parametrization by the number sequences can be converted into the one by polynomials of M auxiliary x-variables by the rule $(a_1, a_2, \ldots a_M) \longleftrightarrow x_1^{a_1} x_2^{a_2} \ldots x_M^{a_M}$.

- The highest weights of totally antisymmetric representations $[1^r]$ are totally q-antisymmetric polynomials in x, i.e. certain quantum deformations of the Vandermonde determinants $\Delta^{(r)} = \Delta_{1 \ldots r} = \prod_{i<j}^r (x_i - x_j)$. The point now is that the highest weights of arbitrary representations R are expressed through the same determinants.

• The Young diagram $R = \{r_1 \geq r_2 \geq \ldots\}$ has r_i as the heights of its columns. The lengths of its lines are similar parameters of the transposed diagram $R' = \{r_1' \geq r_2' \geq \ldots\}$. These can be associated with totally antisymmetric representations $[1^{r_i'}]$, and the highest weight $V_R(x)$ in x representation is just a linear combination of $\prod_i \Delta^{(r_i')}$:

$$V_R = \sum_{\sigma \in S_{|R|}} C_\sigma \cdot \Delta^{(r_1')}_{\sigma_1 \ldots \sigma_{r_1'}} \cdot \Delta^{(r_2')}_{\sigma_{r_1'+1} \ldots \sigma_{r_1'+r_2'}} \cdot \ldots \tag{2.6}$$

The sum goes over different distributions of $|R|$ variables x_i between different Vandermonde determinants, and the coefficients C_σ are determined by the highest weight condition (that the sum is annihilated by all raising operators $\hat{T}_a$).

• The Vandermonde decomposition is highly ambiguous, because there are many linear relations between products of the Vandermonde determinants of a given type R', but this ambiguity does not change the highest weight itself. Of course, in the case of non-trivial multiplicities, there are several linear independent solutions to the highest weight condition.

• These expressions for the highest weights can be straightforwardly quantized (q-deformed) keeping the highest weights. We do not explain the quantization procedure in the present text: it is a separate long story of its own interest.

3 Specification to the case of $R = [3, 1]$

3.1 Decomposition of the square $[3, 1]^{\otimes 2}$ and highest weights

The square of $[3, 1] \otimes [3, 1]$ contains 11 different representations, two of them twice, but we can distinguish between them, because they belong two symmetric and antisymmetric squares, thus, for our purposes, there are 13 different representations, all with unit multiplicities (true multiplicities occur for the first time in the square of $R = [4, 2]$):

$$[3, 1]^{\otimes 2} = [6, 2] + [6, 1, 1] + [5, 3] + 2 \cdot [5, 2, 1] + [5, 1, 1, 1] + [4, 4] + 2 \cdot [4, 3, 1] + [4, 2, 2] + [4, 2, 1, 1] + [3, 3, 2] + [3, 3, 1, 1] \tag{3.1}$$

In what follows, we mark representations by indices $\pm$ depending on their belonging to symmetric or antisymmetric squares. The corresponding $\mathcal{R}$-matrix eigenvalues are:

$$\lambda_{[6,2]} = q^{14}, \quad \lambda_{[6,1,1]} = -q^{12}, \quad \lambda_{[5,3]} = -q^{10}, \quad \lambda_{[5,2,1]_\pm} = \pm q^7, \quad \lambda_{[5,1,1,1]} = q^4, \quad \lambda_{[4,4]} = q^8,$$

$$\lambda_{[4,3,1]_\pm} = \pm q^4, \quad \lambda_{[4,2,2]} = q^2, \quad \lambda_{[4,2,1,1]} = -1, \quad \lambda_{[3,3,2]} = -1, \quad \lambda_{[3,3,1,1]} = q^{-2} \tag{3.2}$$

They should all be additionally divided by $q^{4\varkappa_R} A^{|R|} = q^8 A^4$ to provide the knot polynomials in the topological framing.

If $[3, 1] \otimes [3, 1]$ is represented by $\Delta_{x_1, x_2} \otimes \Delta_{x_5, x_6} = \Delta_{12} \Delta_{56}$, then the highest weights of the emerging representations are (we take them unreduced to simplify both formulas and actual calculations: normalization of the intermediate representations Y does not affect the answers for mixing matrices):

$$V_{[6,2]_+} = \Delta_{12} \Delta_{56}$$
$$V_{[6,1,1]_-} = \Delta_{125} - \Delta_{126} = \Delta_{256} - \Delta_{156}$$

$$V_{[5,3]_-} = \Delta_{12}\Big(\Delta_{37}\Delta_{56} + \Delta_{47}\Delta_{56} + \Delta_{38}\Delta_{56} + \Delta_{48}\Delta_{56}\Big)$$

$$V_{[5,2,1]_+} = 4\Big(\Delta_{127}\Delta_{56} + \Delta_{128}\Delta_{56} + \Delta_{12}\Delta_{356} + \Delta_{12}\Delta_{456}\Big)$$
$$- 3\Big(\Delta_{123}\Delta_{56} + \Delta_{124}\Delta_{56} + \Delta_{12}\Delta_{567} + \Delta_{12}\Delta_{568}\Big)$$

$$V_{[5,2,1]_-} = 4\Big(\Delta_{127}\Delta_{56} + \Delta_{128}\Delta_{56} - \Delta_{12}\Delta_{356} - \Delta_{12}\Delta_{456}\Big)$$
$$- \Big(\Delta_{123}\Delta_{56} + \Delta_{124}\Delta_{56} - \Delta_{12}\Delta_{567} - \Delta_{12}\Delta_{568}\Big)$$

$$V_{[5,1,1,1]_+} = 4\Delta_{1256} + \Big(\Delta_{1235} + \Delta_{1245} - \Delta_{1236} - \Delta_{1246}\Big) - \Big(\Delta_{1567} + \Delta_{1568} - \Delta_{2567} - \Delta_{2568}\Big)$$

$$V_{[4,4]_-} = \Delta_{12}\Big(\Delta_{38}\Delta_{47}\Delta_{56} + \Delta_{37}\Delta_{48}\Delta_{56}\Big)$$

$$V_{[4,3,1]_+} = \Delta_{12}\Delta_{356}\Delta_{47} - \Delta_{127}\Delta_{38}\Delta_{56} + \ (3 \leftrightarrow 4) + (7 \leftrightarrow 8)$$

$$V_{[4,3,1]_-} = \Delta_{12}\Delta_{356}\Delta_{47} + \Delta_{127}\Delta_{38}\Delta_{56} - \Delta_{125}\Delta_{67}\Delta_{38} + \Delta_{126}\Delta_{57}\Delta_{38} - \Delta_{125}\Delta_{36}\Delta_{47} + \Delta_{126}\Delta_{35}\Delta_{47}$$
$$+ \ (3 \leftrightarrow 4) + (7 \leftrightarrow 8)$$

$$V_{[4,2,2]_+} = -8\Big(\Delta_{127}\Delta_{356} + \Delta_{127}\Delta_{456} + \Delta_{128}\Delta_{356} + \Delta_{128}\Delta_{456}\Big)$$
$$+ \Big(\Delta_{123}\Delta_{567} + \Delta_{124}\Delta_{567} + \Delta_{123}\Delta_{568} + \Delta_{124}\Delta_{568}\Big)+$$
$$+ 4\Big(\Delta_{123}\Delta_{456} + \Delta_{124}\Delta_{356} + \Delta_{127}\Delta_{568} + \Delta_{128}\Delta_{567}\Big)$$

$$V_{[4,2,1,1]} = 2\Delta_{1256}\Delta_{37} - \Big(\Delta_{1237}\Delta_{56} + \Delta_{12}\Delta_{3567}\Big)$$
$$+ 2\Big(\Delta_{1257}\Delta_{38} - \Delta_{1267}\Delta_{38} + \Delta_{1356}\Delta_{47} - \Delta_{2356}\Delta_{47}\Big)+$$
$$+ \Big(\Delta_{1257}\Delta_{36} - \Delta_{1267}\Delta_{35} + \Delta_{1356}\Delta_{27} - \Delta_{2356}\Delta_{17}\Big) + \ (3 \leftrightarrow 4) + (7 \leftrightarrow 8)$$

$$V_{[3,3,2]} = \Delta_{125}\Delta_{367}\Delta_{48} - \Delta_{126}\Delta_{357}\Delta_{48} - 3 \cdot \Delta_{127}\Delta_{356}\Delta_{48} + \ (3 \leftrightarrow 4) + (7 \leftrightarrow 8)$$

$$V_{[3,3,1,1]} = 3\Big(\Delta_{1237}\Delta_{48}\Delta_{56} + \Delta_{12}\Delta_{3567}\Delta_{48}\Big) - 6\Delta_{1256}\Delta_{37}\Delta_{48}-$$
$$- 5\Big(\Delta_{1257}\Delta_{36}\Delta_{48} - \Delta_{1267}\Delta_{35}\Delta_{48} + \Delta_{1356}\Delta_{27}\Delta_{48} - \Delta_{2356}\Delta_{17}\Delta_{48}\Big)$$
$$+ \ (3 \leftrightarrow 4) + (7 \leftrightarrow 8) \tag{3.3}$$

The symmetrizations in $(3 \leftrightarrow 4)$ and $(7 \leftrightarrow 8)$ are done independently, i.e. each explicitly written term is substituted by the *four* ones. As already mentioned, there are many different ways to represent the r.h.s. because there are many linear relations between Δ-products of a given type and the choice of basis is somewhat arbitrary. Also we do not provide here the exact definitions of q-deformed Δ's, thus these formulas are mostly for demonstrative purposes. They, however, can be directly used for $q = 1$, i.e. for studying the *classical* Racah matrices, which is by itself quite a non-trivial problem for non-symmetric representations.

3.2 Representations in the cube $[3, 1]^{\otimes 3}$

The representation content of the cube is now

$$[3, 1] \otimes \Big([3, 1] \otimes [3, 1]\Big) =$$
$$= [3, 1] \otimes \Big([6, 2] + [6, 1, 1] + [5, 3] + 2 \cdot [5, 2, 1] + [5, 1, 1, 1] + [4, 4] + 2 \cdot [4, 3, 1] + [4, 2, 2] + [4, 2, 1, 1]$$
$$+ [3, 3, 2] + [3, 3, 1, 1]\Big) =$$
$$= \Big([9, 3] + [9, 2, 1] + [8, 4] + [8, 3, 1] + [8, 3, 1] + [8, 2, 2] + [8, 2, 1, 1] + [7, 5] + 2 \cdot [7, 4, 1] + 2 \cdot [7, 3, 2]$$
$$+ [7, 2, 2, 1] + [7, 3, 1, 1] + [6, 5, 1] + [6, 4, 2] + [6, 4, 1, 1] + [6, 3, 3] + [6, 3, 2, 1]\Big)+$$

$$+ \Big([9,2,1] + [9,1,1,1] + [8,3,1] + [8,2,2] + 2 \cdot [8,2,1,1] + [8,1,1,1,1] + [7,4,1] + [7,3,2]$$
$$+ 2 \cdot [7,3,1,1] + [7,2,2,1] + [7,2,1,1,1] + [6,4,2] + [6,4,1,1] + [6,3,2,1] + [6,3,1,1,1] \Big) +$$
$$+ \Big([8,4] + [8,3,1] + [7,5] + 2 \cdot [7,4,1] + [7,3,2] + [7,3,1,1] + [6,6] + 2 \cdot [6,5,1] + 2 \cdot [6,4,2] + [6,4,1,1]$$
$$+ [6,3,3] + + [6,3,2,1] + [5,5,2] + [5,5,1,1] + [5,4,3] + [5,4,2,1] + [5,3,3,1] \Big) +$$
$$+ 2 \cdot \Big([8,3,1] + [8,2,2] + [8,2,1,1] + [7,4,1] + 2 \cdot [7,3,2] + 2 \cdot [7,3,1,1] + 2 \cdot [7,2,2,1] + [7,2,1,1,1]$$
$$+ [6,5,1] + + 2 \cdot [6,4,2] + 2 \cdot [6,4,1,1] + [6,3,3] + 3 \cdot [6,3,2,1] + [6,3,1,1,1] + [6,2,2,2] + [6,2,2,1,1]$$
$$+ [5,5,2] + [5,5,1,1] + + [5,4,3] + 2 \cdot [5,4,2,1] + [5,4,1,1,1] + [5,3,2,2] + [5,3,3,1] + [5,3,2,1,1] \Big) +$$
$$+ \Big([8,2,1,1] + [8,1,1,1,1] + [7,3,1,1] + [7,2,2,1] + 2 \cdot [7,2,1,1,1] + [7,1,1,1,1,1] + [6,4,1,1]$$
$$+ [6,3,2,1] + + 2 \cdot [6,3,1,1,1] + [6,2,2,1,1] + [6,2,1,1,1,1] + [5,4,2,1] + [5,4,1,1,1] + [5,3,2,1,1]$$
$$+ [5,3,1,1,1,1] \Big) + + \Big([7,5] + [7,4,1] + [6,5,1] + [6,4,2] + [6,4,1,1] + [5,5,2] + [5,4,3] + [5,4,2,1]$$
$$+ [4,4,3,1] \Big) + + 2 \cdot \Big([7,4,1] + [7,3,2] + [7,3,1,1] + [6,5,1] + 2 \cdot [6,4,2] + 2 \cdot [6,4,1,1] + [6,3,3]$$
$$+ 2 \cdot [6,3,2,1] + [6,3,1,1,1] + + [5,5,2] + [5,5,1,1] + 2 \cdot [5,4,3] + 3 \cdot [5,4,2,1] + [5,4,1,1,1]$$
$$+ [5,3,2,2] + 2 \cdot [5,3,3,1] + [5,3,2,1,1] + + [4,4,4] + [4,4,2,2] + 2 \cdot [4,4,3,1] + [4,4,2,1,1]$$
$$+ [4,3,3,2] + [4,3,3,1,1] \Big) + \Big([7,3,2] + [7,2,2,1] + [6,4,2] + [6,3,3] + 2 \cdot [6,3,2,1] + [6,2,2,2]$$
$$+ [6,2,2,1,1] + [5,5,2] + [5,4,3] + 2 \cdot [5,4,2,1] + + 2 \cdot [5,3,2,2] + [5,3,3,1] + [5,3,2,1,1]$$
$$+ [5,2,2,2,1] + [4,4,2,2] + [4,4,3,1] + [4,4,2,1,1] + [4,3,3,2] + [4,3,2,2,1] \Big) + \Big([7,3,1,1]$$
$$+ [7,2,2,1] + [7,2,1,1,1] + [6,4,1,1] + 2 \cdot [6,3,2,1] + 2 \cdot [6,3,1,1,1] + [6,2,2,2] + 2 \cdot [6,2,2,1,1]$$
$$+ [6,2,1,1,1,1] + [5,5,1,1] + 2 \cdot [5,4,2,1] + 2 \cdot [5,4,1,1,1] + [5,3,3,1] + [5,3,2,2] + 3 \cdot [5,3,2,1,1]$$
$$+ [5,3,1,1,1,1] + [5,2,2,2,1] + [5,2,2,1,1,1] + [4,4,3,1] + [4,4,2,2] + 2 \cdot [4,4,2,1,1] + [4,4,1,1,1,1]$$
$$+ [4,3,3,1,1] + + [4,3,2,2,1] + [4,3,2,1,1,1] \Big) + \Big([6,4,2] + [6,3,3] + [6,3,2,1] + [5,4,3] + [5,4,2,1]$$
$$+ 2 \cdot [5,3,3,1] + [5,3,2,2] + [5,3,2,1,1] + [4,4,3,1] + [4,4,2,2] + 2 \cdot [4,3,3,2] + [4,3,3,1,1]$$
$$+ [4,3,2,2,1] + [3,3,3,3] + [3,3,3,2,1] \Big) + + \Big([6,4,1,1] + [6,3,2,1] + [6,3,1,1,1] + [5,4,2,1]$$
$$+ [5,4,1,1,1] + [5,3,3,1] + [5,3,2,2] + 2 \cdot [5,3,2,1,1] + [5,3,1,1,1,1] + + [4,4,3,1] + [4,4,2,1,1]$$
$$+ [4,3,3,2] + 2 \cdot [4,3,3,1,1] + [4,3,2,2,1] + [4,3,2,1,1,1] + [3,3,3,2,1] + [3,3,3,1,1,1] \Big) \qquad (3.4)$$

and the highest weights are too numerous to be listed here, see [105].

Each block of lines here is associated with the corresponding intermediate representation: $[6,2]$, $[6,1,1]$, ... from (3.1), and most representations appear in several blocks in the table below. The number of times the representation Q enters (3.4) is given in the first column of the table, and it is the size of the mixing matrix $\mathcal{U}_Q$ which we need to calculate. Clearly, there are quite a few (40) matrices of non-unit size, some are quite big. Only 26 of them (up to size 6) can be found from the eigenvalue hypothesis of [72, 73] and its recent generalization in [101], all the rest had to be calculated by the methods of section2 (actually we did so also for the matrices of sizes 5 and 6 to double check the eigenvalue hypothesis).

– 8 –

matrix size	Q	number of matrices
1	[9,3], [9,1,1,1], [7,1,1,1,1,1], [6,6], [5,2,2,1,1,1], [4,4,1,1,1,1], [3,3,3,3], [3,3,3,1,1,1]	8
2	[9,2,1], [8,4], [8,1,1,1,1], [6,2,1,1,1,1], [5,2,2,2,1], [4,4,4], [4,3,2,1,1,1], [3,3,3,2,1]	8
3	[7,5], [5,3,1,1,1,1]	2
4	[8,2,2], [6,2,2,2], [4,3,2,2,1]	3
5	[4,4,2,2]	1
6	[8,3,1], [8,2,1,1], [7,2,1,1,1], [6,2,2,1,1], [5,5,1,1], [4,4,2,1,1], [4,3,3,2], [4,3,3,1,1]	8
7	[5,5,2]	1
8	[6,5,1], [6,3,3], [5,4,1,1,1]	3
9	[7,2,2,1], [5,3,2,2], [4,4,3,1]	3
10	[7,4,1], [6,3,1,1,1], [5,4,3]	3
11	[7,3,2]	1
12	[7,3,1,1], [5,3,3,1], [5,3,2,1,1]	3
15	[6,4,2], [6,4,1,1]	2
19	[5,4,2,1]	1
20	[6,3,2,1]	1

$$(3.5)$$

4 Knot polynomials in representation [3, 1]

4.1 2-strand (torus) knots and links

Among the knots and links having 3-strand braid representation there are many composites consisting of 2-strand torus components. The reduced HOMFLY polynomials for the composites are just products of those for the constituents and, for the sake of completeness, we complement (1.2) by its simple 2-strand counterpart:

$$H_R^{(n)} = \sum_{Y \in R \otimes R} \frac{d_Y}{d_R} \cdot \left(\frac{\epsilon_Y q^{\varkappa_Y}}{q^{4\varkappa_R} A^{|R|}} \right)^n \tag{4.1}$$

with Casimir eigenvalue $\varkappa_Y = \sum_{(i,j) \in Y} (i - j)$, quantum dimension

$$d_Y = \dim_q(Y) = \text{Schur}_Y \left(p_k = \{A^k\}/\{q^k\} \right) = \prod_{(i,j) \in Y} \frac{\{Aq^{i-j}\}}{\{q^{1+\text{arm}(i,j)+\text{leg}(i,j)}\}}, \tag{4.2}$$

and $\epsilon_Y = \pm 1$ depending on whether Y belongs to the symmetric or antisymmetric square of R.

4.2 Abundance of 3-strand prime knots in Rolfsen table

As to the *prime* knots given by 3-strand braids, their place in the entire set is clear from the following table, where the minimal number of strands is shown for up to 10 crossings. Explicitly listed are the 3-strand braid representations of knots, and just the number of strands is given for everything else. The non-arborescent knots are boldfaced.

knot	braid index
3_1	2
4_1	$(1, -1, 1, -1)$
5_1	2
5_2	$(3, 1, -1, 1)$
6_1	4
6_2	$(3, -1, 1, -1)$
6_3	$(2, -1, 1, -2)$
7_1	2
7_2	4
7_3	$(5, 1, -1, 1)$
7_4	4
7_5	$(4, 1, -1, 2)$
7_6	4
7_7	4
8_1	5
8_2	$5, -1, 1, -1$
8_3	5
8_4	4
8_5	$3, -1, 3, -1$
8_6	4
8_7	$4, -1, 1, -2$
8_8	4
8_9	$3, -1, 1, -3$
8_{10}	$3, -1, 2, -2$
8_{11}	4
8_{12}	5
8_{13}	4
8_{14}	4
8_{15}	4
8_{16}	$2, -1, 2, -1, 1, -1$
8_{17}	$2, -1, 1, -1, 1, -2$
$\mathbf{8_{18}}$	$(1, -1)^4$
8_{19}	$3, 1, 3, 1$
8_{20}	$3, -1, -3, -1$
8_{21}	$3, 1, -2, 2$

9_1	2
9_2	5
9_3	$7, 1, -1, 1$
9_4	4
9_5	5
9_6	$6, 1, -1, 2$
9_7	4
9_8	5
9_9	$5, 1, -1, 3$
9_{10}	4
9_{11}	4
9_{12}	5
9_{13}	4
9_{14}	5
9_{15}	5
9_{16}	$4, 2, -1, 3$
9_{17}	4
9_{18}	4
9_{19}	5
9_{20}	4
9_{21}	5
9_{22}	4
9_{23}	4
9_{24}	4
9_{25}	5
9_{26}	4
9_{27}	4
9_{28}	4
9_{29}	4
9_{30}	4
9_{31}	4
9_{32}	4
9_{33}	4
$\mathbf{9_{34}}$	4
9_{35}	5
9_{36}	4
9_{37}	5
9_{38}	4
$\mathbf{9_{39}}$	5
$\mathbf{9_{40}}$	4
$\mathbf{9_{41}}$	5
9_{42}	4
9_{43}	4
9_{44}	4
9_{45}	4
9_{46}	4
$\mathbf{9_{47}}$	4
9_{48}	4
$\mathbf{9_{49}}$	4

10_1	6
10_2	$7, -1, 1, -1$
10_3	6
10_4	5
10_5	$6, -1, 1, -2$
10_6	4
10_7	5
10_8	4
10_9	$5, -1, 1, -3$
10_{10}	5
10_{11}	5
10_{12}	4
10_{13}	6
10_{14}	4
10_{15}	4
10_{16}	5
10_{17}	$4, -1, 1, -4$
10_{18}	5
10_{19}	4
10_{20}	5
10_{21}	4
10_{22}	4
10_{23}	4
10_{24}	5
10_{25}	4
10_{26}	4
10_{27}	4
10_{28}	5
10_{29}	5
10_{30}	5
10_{31}	5
10_{32}	4
10_{33}	5
10_{34}	5
10_{35}	6
10_{36}	5
10_{37}	5
10_{38}	5
10_{39}	4
10_{40}	4
10_{41}	5
10_{42}	5
10_{43}	5
10_{44}	5
10_{45}	5
10_{46}	$5, -1, 3, -1$
10_{47}	$5, -1, 2, -2$
10_{48}	$4, -2, 1, -3$
10_{49}	4
10_{50}	4
10_{51}	4
10_{52}	4
10_{53}	5
10_{54}	4
10_{55}	5

10_{56}	4	$\mathbf{10_{111}}$	4
10_{57}	4	$\mathbf{10_{112}}$	$3, -1, 1, -1, 1, -1, 1, -1$
10_{58}	6	$\mathbf{10_{113}}$	4
10_{59}	5	$\mathbf{10_{114}}$	4
10_{60}	5	$\mathbf{10_{115}}$	5
10_{61}	4	$\mathbf{10_{116}}$	$2, -1, 2, -1, 1, -1, 1, -1$
10_{62}	$4, -1, 3, -2$	$\mathbf{10_{117}}$	4
10_{63}	5	$\mathbf{10_{118}}$	$2, -1, 1, -1, 1, -2, 1, -1$
10_{64}	$3, -1, 3, -3$	$\mathbf{10_{119}}$	4
10_{65}	4	$\mathbf{10_{120}}$	5
10_{66}	4	$\mathbf{10_{121}}$	4
10_{67}	5	$\mathbf{10_{122}}$	4
10_{68}	5	$\mathbf{10_{123}}$	$(1, -1)^5$
10_{69}	5	10_{124}	$5, 1, 3, 1$
10_{70}	5	10_{125}	$5, -1, -3, -1$
10_{71}	5	10_{126}	$5, 1, -3, 1$
10_{72}	4	10_{127}	$5, 1, -2, 2$
10_{73}	5	10_{128}	4
10_{74}	5	10_{129}	4
10_{75}	5	10_{130}	4
10_{76}	4	10_{131}	4
10_{77}	4	10_{132}	4
10_{78}	5	10_{133}	4
10_{79}	$3, -2, 2, -3$	10_{134}	4
10_{80}	4	10_{135}	4
10_{81}	5	10_{136}	4
10_{82}	$4, -1, 1, -1, 1, -2$	10_{137}	5
10_{83}	4	10_{138}	5
10_{84}	4	10_{139}	$4, 1, 3, 2$
10_{85}	$4, -1, 2, -1, 1, -1$	10_{140}	4
10_{86}	4	10_{141}	$4, -1, -3, -2$
10_{87}	4	10_{142}	4
10_{88}	5	10_{143}	$4, 1, -3, 2$
10_{89}	5	10_{144}	4
10_{90}	4	10_{145}	4
10_{91}	$3, -1, 1, -2, 1, -2$	10_{146}	4
10_{92}	4	10_{147}	4
10_{93}	4	10_{148}	$4, 1, -2, 1, -1, 1$
10_{94}	$3, -1, 2, -2, 1, -1$	10_{149}	$4, 1, -1, 1, -1, 2$
10_{95}	4	10_{150}	4
10_{96}	5	10_{151}	4
10_{97}	5	10_{152}	$3, 2, 2, 3$
10_{98}	4	10_{153}	4
10_{99}	$2, -1, 2, -2, 1, -2$	10_{154}	4
$\mathbf{10_{100}}$	$3, -1, 2, -1, 2, -1$	$\mathbf{10_{155}}$	$3, 1, -2, 1, -2, 1$
$\mathbf{10_{101}}$	5	$\mathbf{10_{156}}$	4
$\mathbf{10_{102}}$	4	$\mathbf{10_{157}}$	$3, 2, -1, 1, -1, 2$
$\mathbf{10_{103}}$	4	$\mathbf{10_{158}}$	4
$\mathbf{10_{104}}$	$3, -2, 1, -1, 1, -2$	$\mathbf{10_{159}}$	$3, 1, -1, 1, -2, 2$
$\mathbf{10_{105}}$	5	$\mathbf{10_{160}}$	4
$\mathbf{10_{106}}$	$3, -1, 1, -1, 2, -2$	$\mathbf{10_{161}}$	$3, 1, -1, 1, 2, 2$
$\mathbf{10_{107}}$	5	$\mathbf{10_{162}}$	4
$\mathbf{10_{108}}$	4	$\mathbf{10_{163}}$	4
$\mathbf{10_{109}}$	$2, -1, 1, -2, 2, -2$	$\mathbf{10_{164}}$	4
$\mathbf{10_{110}}$	5	$\mathbf{10_{165}}$	4

4.3 Examples

Explicit expressions for the knot polynomials are quite lengthy, we give just three examples in the two simplest and one more complicated case:

Trefoil knot $3_1 = (m_1 = 1, n_1 = 1, m_2 = 1, n_2 = 1)$ also known as the torus knot $T[3,2]$: $H^{3_1}_{[3,1]} = q^{12}A^{16} + \left(-q^{30} - q^{28} - q^{26} - q^{24} - q^{20} - q^{18} - q^{16} - q^{14}\right)A^{14}q^{-12} + \left(q^{34} + 2q^{30} + 2q^{28} + q^{26} + 3q^{24} + 2q^{22} + 2q^{20} + 4q^{18} + q^{16} + 2q^{14} + q^{12} + q^{10} + q^8 + q^6\right)A^{12}q^{-12} + \left(-q^{34} - q^{32} - q^{30} - 3q^{28} - q^{26} - 3q^{24} - 4q^{22} - q^{20} - 4q^{18} - 2q^{16} - q^{14} - 4q^{12} - 2q^{10} - q^8 - 2q^6 - 1\right)A^{10}q^{-12} + \left(q^{32} + 2q^{28} + q^{26} + 2q^{22} + q^{20} + q^{18} + 2q^{16} + 2q^{12} + q^{10} + q^6 + q^4 + 1\right)A^8 q^{-12}$

figure-eight knot $4_1 = (m_1 = 1, n_1 - 1, m_2 = 1, n_2 = -1)$: $H^{4_1}_{[3,1]} = q^8 A^8 + \left(-q^{32} - q^{28} - q^{26} + q^{24} - q^{22} - q^{16}\right)A^6 q^{-18} + \left(q^{36} - q^{34} + 3q^{32} + q^{30} - 2q^{28} + 4q^{26} - 2q^{24} - q^{22} + 5q^{20} - q^{18} + q^{16} + 2q^{14} - q^{12} + q^8\right)A^4 q^{-18} + \left(-2q^{36} - 5q^{30} + 5q^{28} - q^{26} - 8q^{24} + 7q^{22} - 6q^{20} - 5q^{18} + 9q^{16} - 6q^{14} - 2q^{12} + 3q^{10} - 5q^8 + q^4 - q^2\right)A^2 q^{-18} + \left(q^{36} + 2q^{34} - 3q^{32} + 4q^{30} + 2q^{28} - 7q^{26} + 12q^{24} - q^{22} - 8q^{20} + 15q^{18} - 8q^{16} - q^{14} + 12q^{12} - 7q^{10} + 2q^8 + 4q^6 - 3q^4 + 2q^2 + 1\right)q^{-18} + \left(-q^{34} + q^{32} - 5q^{28} + 3q^{26} - 2q^{24} - 6q^{22} + 9q^{20} - 5q^{18} - 6q^{16} + 7q^{14} - 8q^{12} - q^{10} + 5q^8 - 5q^6 - 2q^{-18}A^{-2}\right) + \left(q^{28} - q^{24} + 2q^{22} + q^{20} - q^{18} + 5q^{16} - q^{14} - 2q^{12} + 4q^{10} - 2q^8 + q^6 + 3q^4 - q^2 + 1\right)q^{-18}A^{-4} + \left(-q^{20} - q^{14} + q^{12} - q^{10} - q^8 - q^4\right)q^{-18}A^{-6} + q^{-8}A^{-8}$

Knot 10_{161} (non-arborescent, thick, but with relatively short $H_{[31]}$): $H^{10_{161}}_{[3,1]} = \left(q^{50} - q^{48} - q^{46} + 3q^{44} - 4q^{42} + 2q^{40} + 5q^{38} - 9q^{36} + 5q^{34} + 4q^{32} - 9q^{30} + 9q^{28} - 2q^{26} - 6q^{24} + 6q^{22} - 2q^{20} + q^{16} - q^{14}\right)A^{40} + \left(2q^{52} - q^{50} + 3q^{46} - 4q^{44} + 2q^{42} + 4q^{40} - 10q^{38} + 8q^{36} + q^{34} - 14q^{32} + 16q^{30} - 8q^{28} - 6q^{26} + 16q^{24} - 11q^{22} + 6q^{18} - 5q^{16} + 2q^{14} + 3q^{12} - q^{10} - q^8 + q^6 + 1\right)A^{38} + \left(-1 - q^{62} + 2q^{54} + 6q^{48} - 9q^{46} + 9q^{38} - 7q^{34} + 8q^{42} - q^{58} - 5q^{22} + 2q^{20} + q^{26} + 2q^{44} - 3q^{50} - 13q^{40} + q^{-2} - 7q^{16} - 4q^{52} + 3q^{36} + 6q^{24} + 7q^{32} - q^{-10} + q^{12} - 2q^{56} + 3q^8 + 3q^{30} - q^2 - 2q^6 + 3q^{18} - 5q^{28} - 2q^{10} + q^{14}\right)A^{36} + \left(7 + 4q^{54} + 2q^{48} - 2q^{46} - 7q^{38} + 3q^{34} + 2q^{60} - 5q^{22} - 13q^{20} - 21q^{26} + 3q^{-6} + q^{50} + 2q^{40} + q^{66} - 9q^{-2} - 7q^{16} - 2q^{52} - 8q^4 + 15q^{24} - 16q^{32} - 2q^{-14} + q^{-10} + 9q^{12} - 7q^8 + 6q^{30} + q^{-4} - 3q^2 + 12q^6 + 16q^{18} - 4q^{-8} + 3q^{28} - 6q^{10} - 8q^{14}\right)A^{34} + \left(-8 + q^{-26} - q^{64} + q^{62} - q^{54} - 4q^{48} + 3q^{46} - 7q^{38} + 6q^{34} - 8q^{42} - 14q^{22} + 13q^{20} + q^{26} + q^{44} - 7q^{-6} + 4q^{50} + 12q^{40} + 9q^{-2} - 7q^{16} - q^{52} + 5q^{36} + 10q^4 + 10q^{24} - 10q^{32} + 4q^{-14} - 5q^{-12} + 4q^{-10} - 15q^{12} + q^{56} + 4q^8 + 13q^{30} + 2q^{-16} + 2q^2 - 14q^6 - 4q^{18} + 4q^{-8} - 6q^{28} + q^{-22} + 2q^{-20} - 3q^{-18} + 7q^{10} + 14q^{14}\right)A^{32} + \left(5 + q^{-30} + q^{64} - 5q^{54} - 2q^{48} + 8q^{46} - 4q^{34} - q^{-32} + 4q^{42} - 3q^{60} + 2q^{58} - 4q^{22} + 28q^{20} + 25q^{26} - 4q^{44} + 2q^{-6} - 4q^{50} - q^{40} - q^{66} + 3q^{-2} + 5q^{16} + 6q^{52} + 7q^{36} + 3q^4 - 9q^{24} + 12q^{32} + 7q^{-14} + q^{-12} - 2q^{-10} - 7q^{12} + q^{56} + 9q^8 + 3q^{-24} + 2q^{30} + 6q^{-4} - 3q^{-16} + 4q^2 + 2q^6 - 15q^{18} + 4q^{-8} - 13q^{28} - 4q^{-22} + q^{-20} - q^{-28} + 3q^{-18} + 7q^{10} + 17q^{14}\right)A^{30} + \left(-31 - q^{-30} - 3q^{-26} + q^{54} + q^{48} - 5q^{46} + 20q^{38} - 2q^{34} - 4q^{42} + q^{60} + 26q^{22} + 22q^{20} + 30q^{26} + 9q^{44} - 20q^{-6} + 3q^{50} - 6q^{40} + 7q^{-2} + 34q^{16} - 22q^{36} + 9q^4 + q^{-34} - 55q^{24} + 28q^{32} - 3q^{-14} - 8q^{-12} + 2q^{-10} - 40q^{12} + 11q^8 - q^{-24} - 32q^{30} + 8q^{-4} + 15q^2 - 34q^6 - 53q^{18} - q^{-36} + 4q^{-8} + 8q^{28} + 2q^{-22} - 3q^{-20} + 2q^{-28} - q^{-18} + 21q^{10} + 6q^{14}\right)A^{28} + \left(13 - q^{-30} + q^{-26} - 3q^{54} - 7q^{48} + 3q^{46} - 11q^{38} + 4q^{34} + q^{-32} - 8q^{42} - q^{60} - q^{58} + 21q^{22} - 47q^{20} - 25q^{26} - 6q^{44} + 12q^{-6} + 5q^{40} + 8q^{-2} + 17q^{16} - 2q^{52} - 3q^{36} + 8q^4 - q^{-34} - q^{24} - 16q^{32} - 4q^{-14} + 9q^{-12} - 10q^{-10} + 24q^{12} - 36q^8 - 2q^{-24} - 7q^{30} - 23q^{-4} - q^{-16} - 28q^2 + 18q^6 + 17q^{18} + 17q^{28} + 2q^{-22} - 2q^{-20} - q^{-28} + 5q^{10} - 46q^{14}\right)A^{26} + \left(17 + q^{-30} + q^{-26} + 2q^{54} + 5q^{48} + q^{38} - 6q^{34} + 10q^{42} + q^{60} - 14q^{22} - 3q^{20} - 3q^{26} - q^{44} + 10q^{-6} - q^{50} - 5q^{40} - 19q^{-2} - 31q^{16} + q^{52} + 11q^{36} - 22q^4 + 25q^{24} + 5q^{32} - 2q^{-14} + 3q^{-12} + 6q^{-10} + 25q^{12} + 15q^8 + q^{-24} + 8q^{30} + 8q^{-4} + 2q^{-16} + 11q^2 + 17q^6 + 30q^{18} + q^{-36} - 10q^{-8} - 5q^{28} - q^{-22} + 2q^{-20} + q^{-18} - 31q^{10} + 15q^{14}\right)A^{24}$

Clearly, in this form they are not too informative. A list, suitable for further studies is provided in the form of a txt-file at site [105]. Much more informative are expressions for the evolution families, of which we provide a couple of simple examples in the next section, and spectacularly elegant *properties* of these complicated expressions in the later sections.

5 Knot polynomials from evolution

5.1 $(m_1, n_1 | m_2, n_2)$-evolution

Studying the knot polynomials for *families* of knots is the most natural and effective approach, see [1, 56, 83, 86, 100, 102] for motivation and examples. The simplest of this kind is the *evolution* method of [56] and [83], for the three-strand braids $(m_1, n_1 | m_2, n_2 | \ldots)$ it studies the dependence on parameters m_i and n_i.

The simplest family (m, n) fully consists of *composites* of the 2-strand knots, this means that their reduced polynomials are products of those for the constituents, which in this case are the 2-strand torus knots/links with the HOMFLY in arbitrary representation given in section 4.1:

$$H_R^{(m,n)} = H_R^{(m)} \cdot H_R^{(n)} \tag{5.1}$$

Since knots/links $(m_1, n_1 | m_2, 0) = (m_1 + m_2, n_1)$, the same remains true for three-parametric families.

The first non-trivial family is $(m_1, n_1 | m_2, n_2)$. According to the general rules of the evolution method

$$H_R^{(m_1, n_1 | m_2, n_2)} = \sum_{a_1, a_2, a_3, a_4 = 1}^{N_R} \mathfrak{h}_R^{a_1 a_2 a_3 a_4} \lambda_{a_1}^{m_1} \lambda_{a_2}^{n_1} \lambda_{a_3}^{m_2} \lambda_{a_4}^{n_2} \tag{5.2}$$

where N_R is the number of different representations in the square $R \otimes R$ (if the same representation appears in the symmetric and antisymmetric squares, it contributes twice) and λ_a are the corresponding $\mathcal{R}$-matrix eigenvalues, $\lambda_Q = \epsilon_Q q^{\varkappa_Q - 4\kappa_R} A^{-|R|}$ (the dependence on R is fixed by the topological framing). Thus, within the evolution method, the colored HOMFLY polynomials for our family are described by the 4-rank tensors h_R.

As a simple illustration, in the fundamental representation $N_{[1]} = 2$ $(Q = [2]_+, [1,1]_-)$ and

$$\mathfrak{h}_{[1]}^{abcd} = \frac{\{Aq\}\{A/q\}}{[2]^4 \{q\}^2} \cdot \begin{cases} \frac{\{Aq^3\} + 2\{Aq\}}{\{A/q\}} & \text{for } abcd = 1111 \\ 1 & \text{for } abcd = 1112, 1121, 1211, 2111 \text{ and } 1222, 2122, 2212, 2221 \\ -1 & \text{for } abcd = 1122, 1221, 2211, 2112 \\ [3] & \text{for } abcd = 1212, 2121 \\ \frac{\{A/q^3\} + 2\{A/q\}}{\{Aq\}} & \text{for } abcd = 2222 \end{cases} \tag{5.3}$$

look rather elegant.

A lot of this structure survives in the first non-symmetric representation, where $N_{[2]} = 3$ $(Q = [4]_+, [3,1]_-, [2,2]_+)$ and

$$\mathfrak{h}_{[2]}^{abcd} = \frac{\{Aq^3\}\{Aq^2\}\{A/q\}}{\{q\}^4[4]^3[3]^3[2]^2}$$

$$\cdot\left\{\begin{array}{ll}
\frac{[2]^4}{[4][3]\{A/q\}} \cdot \left(\{Aq^6\}\{Aq^7\} + \{Aq^3\}\left(2\{Aq^8\} + 5[3]\{Aq^4\} + \{A\}\right)\right) & \text{for } abcd = 1111 \\[6pt]
2\frac{[3]^4[4]^2}{\{Aq^3\}}\{Aq^2\}^2 + \frac{[2]^4[3]^4}{[4]}\left(2\frac{[4]}{[2]}(\{A\} - 2\{Aq^2\}) + [2]\{Aq\} + 2\{Aq^6\} - \{A/q^4\}\right) & \text{for } abcd = 2222 \\[6pt]
\frac{\{A\}}{[4]^3[3][2]\{Aq^3\}\{Aq^2\}} \cdot \left(\{A\}^2 + [2]\{Aq^2\}\left(2\{Aq\} + 2\{A/q\} + \{A/q^3\}\right)\right) & \text{for } abcd = 3333 \\[12pt]
[2]^4 \cdot \left(2[2]\{Aq^3\} + \{Aq^6\}\right) & \text{for } abcd = 1112, 1121, 1211, 2111 \\[4pt]
[2]^4 \cdot [3]\left(-[2]\{Aq^3\} + \{A\}\right) & \text{for } abcd = 1122, 1221, 2211, 2112 \\[4pt]
[2]^4 \cdot [3]\left([2]\{Aq^7\} + \{Aq^4\} + [2]^2\{A\}\right) & \text{for } abcd = 1212, 2121 \\[4pt]
[2]^4 \cdot [3]^2\left(2\{Aq^2\} - 2\{A\} + \{A/q^2\}\right) & \text{for } abcd = 1222, 2122, 2212, 2221 \\[12pt]
\frac{\{A\}}{[3]} \cdot [2]^3 & \text{for } abcd = 1113, 1131, 1311, 3111 \\[4pt]
\frac{\{A\}}{[3]} \cdot [4][2]^2 & \text{for } abcd = 1133, 1331, 3311, 3113 \\[4pt]
\frac{\{A\}}{[3]} \cdot [5][4]^2[2] & \text{for } abcd = 1313, 3131 \\[4pt]
\frac{\{A\}}{[3]} \cdot [4]^2[2] & \text{for } abcd = 1333, 3133, 3313, 3331 \\[12pt]
\frac{[3]^2[2]^2\{A\}}{\{Aq^3\}}\left(\{Aq^4\} - 2\{Aq^2\} + 3\{A\} - \{A/q^2\} + \{A/q^4\}\right) & \text{for } abcd = 2223, 2232, 2322, 3222 \\[4pt]
\frac{[4][3][2]\{A\}}{\{Aq^3\}}\left(\{Aq^2\} - 2\{A\} - \{A/q^4\}\right) & \text{for } abcd = 2233, 2332, 3322, 3223 \\[4pt]
\frac{[4]^2[3]\{A\}}{\{Aq^3\}}\left(2\{A\} + \{A/q^2\} + \{A/q^4\}\right) & \text{for } abcd = 2323, 3232 \\[4pt]
\frac{[4]^2\{A\}}{\{Aq^3\}}\left(2\{A\} + \{A/q^2\} + \{A/q^4\}\right) & \text{for } abcd = 2333, 3233, 3323, 3332 \\[12pt]
-[2]^3\{A\} & \text{for } abcd = 1123, 1231, 2311, 3112 \\
 & \phantom{\text{for } abcd =} \text{and } 1132, 1321, 3211, 2113 \\[4pt]
[2]^3[5]\{A\} & \text{for } abcd = 1213, 2131, 1312, 3121 \\[4pt]
-[2]^2[6]\{A\} & \text{for } abcd = 1223, 2231, 2312, 3122 \\
 & \phantom{\text{for } abcd =} \text{and } 3221, 2213, 2132, 1322 \\[4pt]
[2]^3[3]\{A\} & \text{for } abcd = 1232, 2321, 3212, 2123 \\[4pt]
-[2]^2[4]\{A\} & \text{for } abcd = 1233, 2331, 3312, 3123 \\
 & \phantom{\text{for } abcd =} \text{and } 2133, 1332, 3321, 3213 \\[4pt]
[2]\{A\} & \text{for } abcd = 1323, 3231, 2313, 3132
\end{array}\right.$$

5.2 The family $(m, -1 \,|\, 1, -1)$

5.2.1 Generalities

What we can provide at the current stage is the answer for the simplest *one*-parametric family $(m_1, n_1 | m_2, n_2) = (n, -1 \,|\, 1, -1)$. For the low odd values of n it includes:

$$
\begin{array}{ll|ll}
(-1, -1 \,|\, 1, -1) & \text{unknot} & (1, -1 \,|\, 1, -1) & 4_1 \\
(-3, -1 \,|\, 1, -1) & 5_2 & (3, -1 \,|\, 1, -1) & 6_2 \\
(-5, -1 \,|\, 1, -1) & 7_3 & (5, -1 \,|\, 1, -1) & 8_2 \\
(-7, -1 \,|\, 1, -1) & 9_3 & (7, -1 \,|\, 1, -1) & 10_2 \\
\multicolumn{4}{c}{\cdots}
\end{array}
\tag{5.4}
$$

The family $(n, -1 \,|\, 1, -1)$ describes the simple subfamily of pretzel knots $(n, \bar{2}, 1)$ at odd n (see [95]), while for even n we get quite interesting two-component links. This is next to the twist knots series in the Rolfsen table. The twist knots $(\ldots, 9_2, 7_2, 5_2, 3_1, 4_1, 6_1, 8_1, 10_1, \ldots)$ are currently the main source of intuition about colored the knot polynomials [63–66, 83], and consideration of the next family is both natural and important.

Since the family under consideration belongs to the pretzel knots, the coefficients $h_{R,Y}(A, q)$ in the general evolution formula,

$$H_R^{(n, -1 \,|\, 1, -1)}(A, q) = \sum_{Y \in R^{\otimes 2}} \left(\frac{\epsilon_Y q^{\varkappa_Y}}{q^{4\varkappa_R}A^{|R|}}\right)^n h_{R,Y}(A, q) \tag{5.5}$$

are directly described in terms of the Racah matrices [95, eq. (45)], [97, eqs. (38)-(39)], which are known for all (anti)symmetric representations [95, 106], as well as for representation [2, 1] (see [91, 97]):

$$
\begin{aligned}
H_R^{(n,-1\,|\,1,-1)}(A,q) = H_R^{Pr(n,1,\bar{2})}(A,q) &= d_R^2 \sum_{X \in R \otimes \bar{R}} \frac{1}{\sqrt{d_X}} \Big(ST^nS^\dagger\Big)_{\emptyset X} \Big(STS^\dagger\Big)_{\emptyset X} \Big(\bar{S}^\dagger \bar{T}^2 \bar{S}\Big)_{\emptyset X} \\
&= d_R^2 \sum_{Y \in R^{\otimes 2}} \left(\frac{\epsilon_Y q^{\varkappa_Y}}{q^{4\varkappa_R} A^{|R|}}\right)^n \cdot \underbrace{\left(S_{\emptyset Y} \sum_{X \in R \otimes \bar{R}} \frac{S_{YX}^\dagger}{\sqrt{d_X}} \Big(STS^\dagger\Big)_{\emptyset X} \Big(\bar{S}^\dagger \bar{T}^2 \bar{S}\Big)_{\emptyset X}\right)}_{h_{R,Y}(A,q)}
\end{aligned} \tag{5.6}
$$

where S is the Racah matrix that relates the cases of differently placed brackets in the map of the product: $R \otimes R \otimes \bar{R} \to R$, while $\bar{S}$ for $R \otimes \bar{R} \otimes R \to R$, and T is the diagonal matrix with entries being eigenvalues $\frac{\epsilon_Y q^{\varkappa_Y}}{q^{4\varkappa_R} A^{|R|}}$. Note that $S_{\emptyset X} = \sqrt{d_X}/d_R$ and $\bar{S}_{\emptyset X} = \sqrt{d_X}/d_R$.

Thus, we list below the coefficients $h_{R,Y}(A,q)$ for the (anti)symmetric representations and representation [2, 1] and calculate for the [3, 1] case from the 3-strand representation of this paper, since the proper Racah matrices are not known yet.

After summation over X or from the coefficients $\mathfrak{h}_R^{a_1 a_2 a_3 a_4}$ of (5.2) after weighted summation (averaging) over three indices $a_{2,3,4}$, formula (5.6) can be rewritten in the form of corrections to the 2-strand formula (4.1), in terms of the A-independent coefficients $C_i^{R,Y}(q)$:

$$
\begin{aligned}
H_R^{(n,-1\,|\,1,-1)}(A,q) = (Aq^{\mu_R})^{2|R|} \cdot \underbrace{\sum_{Y \in R \otimes R} \frac{d_Y}{d_R} \cdot \left(\frac{\epsilon_Y q^{\varkappa_Y}}{q^{4\varkappa_R} A^{|R|}}\right)^n}_{H_R^{(n)}} \cdot \\
\cdot \left(1 + \sum_{i=1}^{|R|} (-)^i\, C_i^{R,Y}(q) \cdot \left(\frac{\{q\}}{A^2 q^{2\mu_R}}\right)^i\right)
\end{aligned} \tag{5.7}
$$

where $\mu_R = r - s$ for the single hook representations $R = [r, 1^{s-1}]$.

5.2.2 Symmetric representations $R = [r]$

For symmetric $R = [r]$ the sum in (5.5) goes over the representations $[2r - a, a]$, and one can use formulas from [95, 106] to write manifestly

$$
h_{[r],[2r-a,a]}(A,q) = (-1)^{a+1} \sum_{k=0}^{r} \bar{d}_k \mathfrak{a}_{ka} \cdot \left(\sum_{i=0}^{r} \mathfrak{a}_{ki} q^{i^2+i}\right) \left(\sum_{j=0}^{r} \bar{\mathfrak{a}}_{kj} q^{2j^2-2j} A^{2j}\right) \tag{5.8}
$$

where $\bar{d}_k$ is the quantum dimension of the representations arising in the decomposition $[r] \otimes \overline{[r]} = \oplus_{k=0}^{r} [2k, k^{N-2}]$:

$$
\bar{d}_k = D_{2k-1} \cdot \left(\prod_{j=0}^{k-2} \frac{D_j}{[j+2]}\right)^2 \cdot D_{-1} \tag{5.9}
$$

and

$$\mathfrak{a}_{km} = (-1)^{r+k}[2m+1]\frac{G(r-m)}{G(r+k+1)} \cdot \frac{\left([k]![m]!\right)^2 [r-k]! \, [r-m]!}{[r+k+1]! \, [r+m+1]!} \times \tag{5.10}$$

$$\times \sum_{j=\max(r+m,r+k)}^{\min(r+k+m,2r)} \frac{(-1)^j \, [j+1]!}{[2r-j]! \left([j-r-k]! \, [j-r-m]! \, [r+k+m-j]!\right)^2} \cdot \frac{G(j+1)}{G(j-r-m)}$$

$$\bar{\mathfrak{a}}_{km} = \frac{(-1)^{r+k+m} D_{2m-1} G(m)^2}{G(r+k+1) \, G(r+m+1)} \cdot \frac{\left([k]![m]!\right)^2 [r-k]! \, [r-m]!}{[r+k+1]! \, [r+m+1]!}$$

$$\times \sum_{j=\max(r+m,r+k)}^{\min(r+k+m,2r)} \frac{(-1)^j \, [j+1]!}{[2r-j]! \left([j-r-k]! \, [j-r-m]! \, [r+k+m-j]!\right)^2} \cdot \frac{G(j+1)}{G(r+k+m-j)}$$

with

$$G(n) = \frac{1}{[n]!} \prod_{i=-1}^{n-2} D_i = \frac{(A/q;q)_n}{(q;q)_n} \tag{5.11}$$

where we used the symmetric q-Pochhammer symbol $(A;q)_n = \prod_{j=0}^{n-1}\{Aq^j\}$. At $A = q^N$, $G(n)$ becomes the q-binomial $\begin{pmatrix} N+n-2 \\ n \end{pmatrix}_q$.

From (5.8) one can also read off the coefficients $C_i(r,a)$ in representation of $h(A,q)$ in form (5.7)

$$h_{[r],[2r-a,a]}(A,q) = \left(q^{r-1}A\right)^{2r} \cdot \frac{d_{[2r-a,a]}}{d_{[r]}} \cdot \left(1 + \sum_{i=1}^{r}(-)^i q^{\frac{i(i+1)}{2}}[i]! \, C_i(r,a) \left(\frac{\{q\}}{(q^{r-1}A)^2}\right)^i\right) \tag{5.12}$$

In particular, one can realize the following properties of these coefficients:

- They have a symmetry $i \longrightarrow r-i$,

$$C_{r-i}(r,a|q) = (-)^a \cdot q^{-2ar+a(a-1)} \cdot C_i(r,a|q^{-1}) \tag{5.13}$$

  involving also the change $q \longrightarrow q^{-1}$ and rescaling.

- They satisfy simple sum rules in lines:

$$\begin{array}{ll}
\sum_{i=0}^{r} C_i(r,r) = \prod_{j=1}^{r}(1-q^{-2j}) & \sum_{i=0}^{r}(-)^i C_i(r,r) = \prod_{j=1}^{r}(1+q^{-2j}) = q^{-\frac{r(r-1)}{2}}\prod_{j=1}^{r}\frac{[2j]}{[j]} \\
\sum_{i=0}^{r} C_i(r,r-1) = 2\prod_{j=2}^{r}(1-q^{-2j}) & \sum_{i=0}^{r}(-)^i C_i(r,r-1) = 0 \\
\sum_{i=0}^{r} C_i(r,r-2) = (2+[2]^2)\prod_{j=3}^{r}(1-q^{-2j}) & \sum_{i=0}^{r}(-)^i C_i(r,r-2) = -q^{-\frac{(r-2)(r+3)}{2}}\prod_{j=2}^{r}\frac{[2j]}{[j]} \\
\sum_{i=0}^{r} C_i(r,r-3) = 2(1+[3]^2)\prod_{j=4}^{r}(1-q^{-2j}) & \sum_{i=0}^{r}(-)^i C_i(r,r-3) = 0 \\
\cdots & \cdots
\end{array} \tag{5.14}$$

  or, in general,

$$\sum_{i=0}^{r} C_i(r,r-b) = q^{-\frac{(r-b)(r+b+1)}{2}}\{q\}^{r-b}\frac{[r]!}{[b]!} \cdot \sum_{i=0}^{b}\left(\frac{[b]!}{[j]![b-j]!}\right)^2$$

$$\sum_{i=0}^{r}(-)^i C_i(r,r-b) = (-)^{\frac{b}{2}} q^{-\frac{(r-b)(r+b+1)}{2}}\frac{[2r]!! \, [b]!}{[r]! \, ([b]!!)^2} \cdot \begin{cases} 0 \text{ for odd } b \\ 1 \text{ for even } b \end{cases} \tag{5.15}$$

5.2.3 Totally antisymmetric representations $R = [1^r]$

As usual (rank-level duality [56, 58, 63]), for antisymmetric representations

$$H^{(n,-1\,|\,1,-1)}_{[1^r]}(A,q) = H^{(n,-1\,|\,1,-1)}_{[r]}(A,q^{-1}) \tag{5.16}$$

5.2.4 Representation $R = [2,1]$

In this case, the evolution coefficients can be calculated both from formula (5.6) and by the evolution method from several known knot polynomials [83]. Let us use the second possibility here, since only this approach can be used in the $[3,1]$ case.

For $[2,1]^{\otimes 2}$ the eigenvalues of $\mathcal{R}$-matrix are

$$\lambda_{[4,2]} = \frac{q^5}{A^3}, \qquad \lambda_{[4,1,1]} = -\frac{q^3}{A^3}, \qquad \lambda_{[3,3]} = -\frac{q^3}{A^3}, \qquad \lambda_{[3,2,1]_\pm} = \pm\frac{1}{A^3},$$

$$\lambda_{[3,1,1,1]} = \frac{1}{q^3 A^3}, \qquad \lambda_{[2,2,2]} = \frac{1}{q^3 A^3}, \qquad \lambda_{[2,2,1,1]} = -\frac{1}{q^5 A^3}$$

and therefore the m-evolution is defined as follows:

$$H^{(n,-1\,|\,1,-1)}_{[2,1]} = \tag{5.17}$$

$$= \frac{h_{[4,2]}q^{5n} + \left(h_{[4,1,1]} + h_{[3,3]}\right)(-q^3)^n + h_{[3,2,1]_+} + h_{[3,2,1]_-}(-)^n + \left(h_{[3,1,1,1]} + h_{[2,2,2]}\right)(q^{-3})^n + h_{[2,2,1,1]}(-q^{-5})^n}{A^{3n}}$$

Note that two pairs of representations have the same eigenvalues, thus their contributions are not separated. The two representations $[3,2,1]_\pm$ contribute equivalently to knot polynomials: for odd n only the difference $h_{[3,2,1]_+} - h_{[3,2,1]_-}$ is seen, but link polynomials with even n separate these two contributions. The other two sums are inseparable within the $(n,-1\,|\,1,-1)$ series, we, however, suggest a plausible decomposition, which can be checked in the analysis of richer evolution patterns. In fact, *after* proposing these decompositions we derived them using (5.6) and manifest formulas for the Racah matrices from [97] (notice an additional sign factors that have to be added in this case in accordance with [97, s.7] and [102, s.2.7]). Unfortunately, at the moment, these Racah matrices are not known for $[3,1]$ case yet and, hence, such a method is unavailable in that case. Finally, the values of the coefficients are:

$$h_{[4,2]} = \frac{d_{[4,2]}}{d_{[2,1]}} \cdot \left(A^6 - q \cdot (q^4 + q^2 + 2 + q^{-4})\{q\}A^4 + q^4(q^4 + q^2 + 3 + q^{-2} + q^{-4})\{q\}^2 A^2 - q^5[3]\{q\}^3\right) \tag{5.18}$$

$$h_{[4,1,1]} + h_{[3,3]} = \frac{d_{[4,1,1]}}{d_{[2,1]}} \cdot \left(A^6 - q \cdot (q^4 + 1 + q^{-4})\{q\}A^4 - q^2(q^4 + 1 + q^{-4})\{q\}^2 A^2 + q^3[3]\{q\}^3\right) +$$

$$+ \frac{d_{[3,3]}}{d_{[2,1]}} \cdot \left(A^6 - q \cdot (q^4 + q^2 + 2 - q^{-6})\{q\}A^4 + q^2(q^6 + q^4 + 2q^2 - q^{-4})\{q\}^2 A^2 + q^3[3]\{q\}^3\right)$$

$$\frac{1}{2}\left(h_{[3,2,1]_+} + h_{[3,2,1]_-}\right) = \frac{d_{[3,2,1]}}{d_{[2,1]}}\left(A^6 - \frac{[3][4]}{[2]}\{q\}^2 A^4 - [3]\{q\}^2 A^2\right)$$

$$\frac{1}{2}\left(h_{[3,2,1]_+} - h_{[3,2,1]_-}\right) = -\frac{d_{[3,2,1]}}{d_{[2,1]}} \cdot [3]\{q\}^3\left(A^2(q^2 + q^{-2}) + 1\right)$$

$$h_{[3,1,1,1]} + h_{[2,2,2]} = \frac{d_{[3,1,1,1]}}{d_{[2,1]}} \cdot \left(A^6 + q^{-1}(q^4 + 1 + q^{-4})\{q\}A^4 - q^{-2}(q^4 + 1 + q^{-4})\{q\}^2 A^2 - q^{-3}[3]\{q\}^3\right) +$$

$$+ \frac{d_{[2,2,2]}}{d_{[2,1]}} \cdot \left(A^6 + q^{-1}(q^{-4} + q^{-2} + 2 - q^6)\{q\}A^4 + q^{-2}(q^{-6} + q^{-4} + 2q^{-2} - q^4)\{q\}^2 A^2 - q^{-3}[3]\{q\}^3\right)$$

$$h_{[2,2,1,1]} = \frac{d_{[2,2,1,1]}}{d_{[2,1]}} \cdot \left(A^6 + q^{-1}(q^4 + 2 + q^{-2} + q^{-4})\{q\}A^4 + q^{-4}(q^4 + q^2 + 3 + q^{-2} + q^{-4})\{q\}^2 A^2 + q^{-5}[3]\{q\}^3\right)$$

The diagram $[2,1]$ is symmetric, this explains the simple form of the coefficients and their symmetries. Clearly, for the transposed diagrams one has $h_{[2,2,1,1]}(q) = h_{[4,2]}(-q^{-1})$, $h_{[3,1,1,1]}(q) = h_{[4,1,1]}(-q^{-1})$, $h_{[2,2,2]}(q) = h_{[3,3]}(-q^{-1})$ and $h_{[3,2,1]_\pm}(q) = h_{[3,2,1]_\pm}(-q^{-1})$. Note that the sum and difference of $h_{[321]_\pm}$ have different parities of powers in q: this is because the corresponding eigenvalue differs in this power from all the others and there is a need to compensate for the difference between odd and even m for knots and links.

5.2.5 Representation $R = [3,1]$

For $[3,1]^{\otimes 2}$ the $\mathcal{R}$-matrix eigenvalues were already listed in (3.2). In the topological framing, they are:

$$\lambda_{[6,2]} = \frac{q^{14}}{q^8 A^4}, \qquad \lambda_{[6,1,1]} = -\frac{q^{12}}{q^8 A^4}, \qquad \lambda_{[5,3]} = -\frac{q^{10}}{q^8 A^4}, \qquad \lambda_{[4,4]} = \frac{q^8}{q^8 A^4}, \qquad \lambda_{[5,2,1]_\pm} = \pm\frac{q^7}{q^8 A^4},$$

$$\lambda_{[5,1,1,1]} = \frac{q^4}{q^8 A^4}, \qquad \lambda_{[4,3,1]_\pm} = \pm\frac{q^4}{q^8 A^4}, \qquad \lambda_{[4,2,2]} = \frac{q^2}{q^8 A^4}, \qquad \lambda_{[4,2,1,1]} = -\frac{1}{q^8 A^4}, \qquad \lambda_{[3,3,2]} = -\frac{1}{q^8 A^4},$$

$$\lambda_{[3,3,1,1]} = \frac{q^{-2}}{q^8 A^4},$$

and

$$H_{[3,1]}^{(n,-1\,|\,1,-1)} = (A^4 q^8)^{-n} \cdot \left\{ h_{[6,2]} q^{14n} + h_{[6,1,1]}(-q^{12})^n + h_{[5,3]}(-q^{10})^n + h_{[44]} q^{8n} \right. \tag{5.19}$$

$$+ h_{[5,2,1]_+} q^{7n} + h_{[5,2,1]_-}(-q^7)^n + + \left(h_{[5,1,1,1]} + h_{[4,3,1]_+} \right) q^{4n}$$

$$\left. + h_{[4,3,1]_-}(-q^4)^n + h_{[4,2,2]} q^{2n} + \left(h_{[4,2,1,1]} + h_{[3,3,2]} \right) \cdot (-1)^n + h_{[3,3,1,1]} q^{-2n} \right\}$$

Our evaluation of $[3,1]$-colored HOMFLY, together with additional arguments already used in section 5.2.4, allows one to find/conjecture all these coefficients $h^{(\cdot,-1\,|\,1,-1)}$:

$$h_{[6,2]} = \frac{d_{[6,2]}}{d_{[3,1]}} \left\{ (qA)^8 - \{q\}(qA)^6(q^3 + q^{-3})(q^4 + q^2 + 2 + q^{-2}) \right.$$

$$+ [2]\{q\}^2(qA)^4(q^{11} + 3q^7 + 2q^5 + 2q^3 + 3q + 2q^{-1} + q^{-5}) -$$

$$\left. - [2]^2\{q\}^3(qA)^2(q^{11} + q^9 + q^7 + 3q^5 + q) + q^8[4][2]\{q\}^4 \right\}$$

$$h_{[6,1,1]} = \frac{d_{[6,1,1]}}{d_{[3,1]}} \left\{ (qA)^8 - [4]\{q\}(qA)^6(q^4 - q^2 + 1 + q^{-2}) \right.$$

$$- [2]\{q\}^2(qA)^4(q^9 - q^7 - q^3 - q - q^{-1} - q^{-5}) +$$

$$\left. + [2]^2\{q\}^3(qA)^2(q^9 + q^7 + 2q^3 - q + q^{-1}) - q^6[4][2]\{q\}^4 \right\}$$

$$h_{[5,3]} = \frac{d_{[5,3]}}{d_{[3,1]}} \left\{ (qA)^8 - [2]\{q\}(qA)^6(q^6 + 2q^2 + q^{-4} - q^{-6}) \right.$$

$$+ [2]\{q\}^2(qA)^4(q^{11} + 2q^7 + 2q^5 + q + q^{-1} - 2q^{-3} - q^{-9}) -$$

$$\left. - [2]^2\{q\}^3(qA)^2(q^9 - q^7 + q^5 - 2q + q^{-1} - q^{-3}) - q^4[4][2]\{q\}^4 \right\}$$

$$h_{[4,4]} = \frac{d_{[4,4]}}{d_{[3,1]}} \left\{ (qA)^8 - [4]\{q\}(qA)^6(q^4 + 1 - q^{-4}) + [4]\{q\}^2(qA)^4(q^9 + q^5 + q^3 - q - q^{-1} - q^{-5} + q^{-9}) + \right.$$

$$\left. + [2][4]\{q\}^3(qA)^2(q^5 + q^{-1} - q^{-3}) + [2][4]\{q\}^4 q^2 \right\}$$

$$\frac{1}{2}\left(h_{[5,2,1]_+} + h_{[5,2,1]_-} \right) = \frac{d_{[5,2,1]}}{d_{[3,1]}} \left\{ (qA)^8 - \left(q^7 + q^3 + q + q^{-3} - q^{-7} \right)\{q\}(qA)^6 \right.$$

$$+ \left(q^6 - 2q^{-2} - 2q^{-4} - q^{-6} - 2q^{-8} - q^{-10} \right)\{q\}^2(qA)^4 +$$

$$+q^{-1}[2][4]\{q\}^3(qA)^2\Big\}$$

$$\frac{1}{2}\Big(h_{[5,2,1]_+} - h_{[5,2,1]_-}\Big) = -\frac{d_{[5,2,1]}}{d_{[3,1]}}\cdot[2][4]\{q\}^3$$

$$\cdot\Big\{q(q^3+q^{-3})(qA)^4 + (qA)^2\Big(-q^4+q^2+1-q^{-2}+q^{-4}\Big) - q\{q\}\Big\}$$

$$h_{[5,1,1,1]} + h_{[4,3,1]_+} \overset{?}{=} \frac{d_{[5,1,1,1]}}{d_{[3,1]}}\Big\{(qA)^8 + q^{-1}[2][4]\{q\}^3(qA)^6 - [2]^2[4]\{q\}^2(qA)^4(q - q^{-1} + q^{-5})$$

$$+q^{-2}[2]^2[4]\{q\}^5(qA)^2 + q^{-2}[2][4]\{q\}^4\Big\} +$$

$$+\frac{d_{[4,3,1]}}{d_{[3,1]}}\Big\{(qA)^8 - [4]\{q\}^2(qA)^6(q^3 + q + q^{-3}) + [4]\{q\}^2(qA)^4(q^5 - q^3 - 1 - q^{-5} + q^{-9})$$

$$+[2][4]\{q\}^4(qA)^2(1 + q^{-2} + q^{-6}) + q^{-2}[4][2]\{q\}^4\Big\}$$

$$h_{[4,3,1]_-} = \frac{d_{[4,3,1]}}{d_{[3,1]}}\Big\{(qA)^8 - [4]\{q\}^2(qA)^6(q^3 + q^{-1} + q^{-3})$$

$$-[4]\{q\}^2(qA)^4(q^5 - q^3 + q + 2q^{-1} - 2q^{-3} + q^{-5} - q^{-9}) -$$

$$-[4]\{q\}^3(qA)^2(q^2 + 1 - q^{-2} + 2q^{-4} - q^{-8}) - [2][4]\{q\}^4q^{-2}\Big\}$$

$$h_{[4,2,2]} = \frac{d_{[4,2,2]}}{d_{[3,1]}}\Big\{(qA)^8 - (q^3 + q^{-3})\{q\}(qA)^6(q^4 - q^{-2} - q^{-4})$$

$$+[2]\{q\}^2(qA)^4(q^5 - q^3 - 2q^{-3} + q^{-5} - q^{-7} + q^{-11}) +$$

$$+[2]\{q\}^3(qA)^2(q^2 + 1 - q^{-2} - 3q^{-6} - q^{-8} - q^{-10} - q^{-12}) + q^{-4}[4][2]\{q\}^4\Big\}$$

$$h_{[4,2,1,1]} + h_{[3,3,2]} \overset{?}{=} \frac{d_{[4,2,1,1]}}{d_{[3,1]}}\Big\{(qA)^8 + \frac{q^6 - q^4 + 1}{q^4}[4]\{q\}(qA)^6 - \frac{(q^6 - q^4 + 1)\{q\}}{q^7}[4][2]\{q\}^2(qA)^4$$

$$-\frac{q^8 + q^6 - q^4 + q^2 + 1}{q^{10}}[4]\{q\}^3(qA)^2 - q^{-6}[4][2]\{q\}^4\Big\} +$$

$$+\frac{d_{[3,3,2]}}{d_{[3,1]}}\Big\{(qA)^8 - \frac{q^8 - q^6 + 1}{q^4}[4]\{q\}(qA)^6$$

$$-\frac{q^8 - q^6 - 1}{q^8}[4][2]\{q\}^2(qA)^4 - \frac{q^{10} + q^8 - q^6 - q^2 - 1}{q^{12}}[4]\{q\}^3(qA)^2 - q^{-6}[4][2]\{q\}^4\Big\}$$

$$h_{[3,3,1,1]} = \frac{d_{[3,1,1,1]}}{d_{[3,1]}}\Big\{(qA)^8$$

$$+[2]\{q\}(qA)^6(q^4 - q^2 + 1 + q^{-2} + q^{-6}) + [2]\{q\}^2(qA)^4(q + q^{-3} + 2q^{-5} + q^{-7} + q^{-9} + q^{-11}) +$$

$$+[2]\{q\}^3(qA)^2\cdot q^{-2}(1 + q^{-2} + q^{-6})(1 + q^{-4} + q^{-6}) + q^{-8}[4][2]\{q\}^4\Big\} \tag{5.20}$$

where the questions marks means conjectural decompositions of the sums. As a corollary of these formulas

$$\frac{1}{2}\Big(h_{[4,3,1]_+} + h_{[4,3,1]_-}\Big) \overset{?}{=} \frac{d_{[4,3,1]}}{d_{[3,1]}}\Big\{(qA)^8 - [4]\{q\}^2(qA)^6(q^3 + q^{-1} + q^{-3}) - [4]\{q\}^2(qA)^4$$

$$\times(q + q^{-1} - q^{-3} + q^{-5} - q^{-9}) - q^{-4}[4]\{q\}^3(qA)^2\Big\}$$

$$\frac{1}{2}\Big(h_{[4,3,1]_+} - h_{[4,3,1]_-}\Big) \overset{?}{=} \frac{d_{[4,3,1]}}{d_{[3,1]}}\cdot[4]\{q\}^3\cdot\Big\{q(q^3 + q^{-3})(qA)^4 + (qA)^2$$

$$\times(q^2 + 1 - q^{-2} + q^{-4} - q^{-8}) + q^{-2}[2]\{q\}\Big\} \tag{5.21}$$

6 Generic properties of $H_{[3,1]}$

6.1 Special and Alexander polynomials at $q = 1$ or $A = 1$

According to [56, 57, 63, 67–70, 82],

$$H_R(A) = H_1^{|R|}(A) \quad \text{at} \quad q = 1 \;\; \forall \text{ representations } R \tag{6.1}$$

and

$$H_R(q) = H_1\left(q^{|R|}\right) \quad \text{at} \quad A = 1 \;\; \forall \text{ singlehook representations } R = [r, 1^{s-1}] \tag{6.2}$$

This holds for all our 3-strand $H_{[3,1]}$.

6.2 Factorization at roots of unity, $q^8 = 1$

According to [99],

$$H_{[3,1]} = H_{[4]} \quad \text{at} \quad q^8 = 1 \tag{6.3}$$

This is true in all our examples and this provides a new serious support to the factorization conjectures of [99].

6.3 Symmetric Jones polynomials

It is well-known that the representation $R = [3, 1]$ reduces to $R = [2]$ for the Lie algebra sl_2. Therefore, if we put $A = q^2$, the HOMFLY polynomial $H_{[3,1]}$ reduces to the Jones polynomial in the first symmetric representation $J_{[2]}$:

$$H_{[3,1]} = J_{[2]} \quad \text{at} \quad A = q^2. \tag{6.4}$$

Since the symmetric Jones polynomials are known for many knots, in particular, for all knots from the Rolfsen table, one can use them to check our results.

6.4 Universality and adjoint representation at $A = q^4$

By the general rule (rank-level duality [56, 58, 63]), for the transposition of Young diagram

$$H_{R^{tr}}(A, q) = H_R(A, -q^{-1}) \tag{6.5}$$

(an additional inversion $A \to -A^{-1}$ provides a mirror knot instead of transposing the representation).

Thus, together with $H_{[3,1]}$ we simultaneously know

$$H_{[2,1,1]}(A, q) = H_{[3,1]}(A, -q^{-1}) \tag{6.6}$$

For the particular case of sl_4 algebra, $[2, 1, 1]$ is the adjoint representation and the adjoint HOMFLY polynomial satisfies [101, 107] the *universality* hypothesis [108], unifying them with the adjoint polynomials for other groups, including the much simpler adjoint Kauffman polynomials (they are simpler because the adjoint representation of so_N is just $[1, 1]$ for all N, while it is an N-dependent $[2, q^{N-2}]$ for sl_N). This allows one to compare our $H_{[3,1]}$ at $A = q^4$ with the universal formulas from [101]. They match in all examples that we looked at.

6.5 Loop expansion and Vassiliev invariants

The HOMFLY polynomials have an interesting well-defined expansion when $\hbar \to 1$, $A = \exp\left(\frac{N\hbar}{2}\right)$, $q = \exp\left(\frac{\hbar}{2}\right)$. This expansion is known as loop expansion, and the coefficients are the celebrated Vassiliev invariants [109–111]. In the Vassiliev approach, the HOMFLY polynomial can be written as [114]

$$H_R^\mathcal{K}(A = e^{\frac{N\hbar}{2}} | q = e^{\frac{\hbar}{2}}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{N}_i} r_{i,j}^{(R)} v_{i,j}^\mathcal{K} \qquad (6.7)$$

where $r_{i,j}^{(R)}$ are the polynomials of degree $|i|$ in N corresponding to the trivalent diagrams [112–114], and $\mathcal{N}_i$ is the dimension of the vector space formed by the trivalent diagrams. Here $v_{i,j}^\mathcal{K}$ are finite type knot invariants introduced by V.Vassiliev. Thus, what stands in (6.7) is the double series in powers of $\hbar$ and N, such that the degree of $\hbar$ exceeds or equals to the degree of N. The polynomials $r_{i,j}^{(R)}$ are known up to degree 6 in arbitrary representation R, whereas the Vassiliev invariants up to order 6 are known for all knots with number of crossings less than 15. Thus, it provides a lot of explicit checks. To be more concrete, we consider one example of the knot 10_{161} from section 4.3.

As an illustration, we look at the first terms of the expansion (6.7) in the case of $R = [3, 1]$ and for the knot 10_{161} mentioned in section 4.3. Taking the knot-independent trivalent diagrams from [89] and the lowest Vassiliev invariants for 10_{161} from [115],

$$\begin{aligned}
r_{2,1} &= -N^2 - N + 4 & v_{2,1}^{10_{161}} &= 28 \\
r_{3,1} &= \frac{1}{2} N \left(N^2 + N - 4 \right) & v_{3,1}^{10_{161}} &= -144 \\
r_{4,1} &= \left(-N^2 - N + 4 \right)^2 & v_{4,1}^{10_{161}} &= 392 \\
r_{4,2} &= -\frac{1}{4} N^2 \left(N^2 + N - 4 \right) & v_{4,2}^{10_{161}} &= \frac{2882}{3} \\
r_{4,3} &= \frac{1}{4} N^4 + \frac{3}{4} N^3 + \frac{11}{2} N^2 + 4N - 32 & v_{4,3}^{10_{161}} &= \frac{430}{3}
\end{aligned} \qquad (6.8)$$

we obtain the first terms of the $\hbar$-expansion (6.7):

$$\begin{aligned}
H_{[3,1]}^{10_{161}} = {}& 1 - 28 \left(N^2 + N - 4 \right) \hbar^2 - 72N \left(N^2 + N - 4 \right) \hbar^3 \\
& + \left(\frac{5056}{3} - 995N^2 - \frac{7688N}{3} + \frac{1954N^3}{3} + \frac{563N^4}{3} \right) \hbar^4 + O\left(\hbar^5 \right)
\end{aligned}$$

Comparing with the coefficients of the $\hbar$ expansion of (6.8), we find a complete agreement.

At the present moment, we know the trivalent diagrams explicitly only up to degree 6 (they are available at [89] up to degree 4). The corresponding Vassiliev invariants are fully determined by the HOMFLY polynomials in representations $R = [1]$ and $R = [2]$. To extract new Vassiliev invariants from more interesting higher symmetric, $[2, 1]$- and the newly-calculated $[3, 1]$-colored HOMFLY, more complicated diagrams are needed.

6.6 Genus expansion and Hurwitz τ-function

Another interesting expansion of the HOMFLY polynomials is the large N, or *genus* expansion, which relates knot theory with the Hurwitz enumeration problem. In this expansion, one expands the knot polynomials in powers of $\hbar$ ($q = e^{\hbar}$) similarly to the previous subsection, but now A, i.e. $\hbar \cdot N$ rather that N is fixed. It turns out that the Ooguri-Vafa partition functions of the colored HOMFLY polynomials [47] is equal to certain generating functions of the Hurwitz numbers [78–80, 89], hence, we also call this expansion Hurwitz, and the corresponding generalization of the KP/Toda τ-functions (a general solution to the AMM/EO topological recursion [116–119]) is called Hurwitz τ-function.

The genus expansion for colored HOMFLY is also interesting, because it separates the representation- and knot-dependencies:

$$H_R^{\mathcal{K}}(q, A) = \left(\sigma_{[1]}^{\mathcal{K}}\right)^{|R|} \cdot \exp\left(\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2} \, S_{\Delta}^{\mathcal{K}}\big(A^2, \hbar^2\big) \, \varphi_R(\Delta)\right) \tag{6.9}$$

(in the leading order, this reproduces (6.1), also known as exponential growth property [82]). The sum here goes over all Young diagrams Δ, as usual, $|\Delta|$ and $l(\Delta)$ denote respectively the number of boxes and of lines in the diagram, $\varphi_R(\Delta)$ are the symmetric group characters: they do not depend on the knot and are common for all Hurwitz τ-functions, [78–80]. Dependent on the knot are the general *special* polynomials in the free energy expansion

$$S_{\Delta}^{\mathcal{K}}(A^2, \hbar^2) = \sum_{g \geq 0} \frac{\hbar^{2g}}{\left(\sigma_{[1]}^{\mathcal{K}}\right)^{2g}} \cdot \sigma_{\Delta}^{\mathcal{K}}(g) \tag{6.10}$$

where $\sigma_{\Delta}^{\mathcal{K}}(g)$ are knot-dependent polynomials in A, presumably related by the AMM/EO topological recursion for a knot-dependent spectral curve.

For us of importance is that, since $\varphi_R(\Delta)$'s are non-diagonal, one and the same special polynomial $\sigma_{\Delta}^{\mathcal{K}}(n)$ affects the HOMFLY polynomial in different representations. Since the lowest polynomials were already extracted from study of the symmetric representations, we can now use them to test our new results for $R = [3, 1]$. Indeed, since $\varphi_{[3]}([2, 1]) = 3 \neq 0$ in

$$\log \frac{H_R^{\mathcal{K}}(q, A)}{\left(\sigma_{[1]}^{\mathcal{K}}(0)\right)^{|R|}} = \left(\frac{\hbar}{\sigma_{[1]}}\right) \cdot \sigma_{[2]}^{\mathcal{K}}(1)\varphi_R([2]) + \left(\frac{\hbar}{\sigma_{[1]}}\right)^2 \cdot \left(\sigma_{[1]}^{\mathcal{K}}(2)\varphi_R([1]) + \sigma_{[1,1]}^{\mathcal{K}}(2)\varphi_R([1, 1]) + \sigma_{[3]}^{\mathcal{K}}(2)\varphi_R([3])\right) +$$

$$+ \left(\frac{\hbar}{\sigma_{[1]}}\right)^3 \cdot \left(\sigma_{[2]}^{\mathcal{K}}(3)\varphi_R([2]) + \sigma_{[2,1]}^{\mathcal{K}}(3)\varphi_R([2, 1]) + \sigma_{[4]}^{\mathcal{K}}(3)\varphi_R([4])\right) + O(\hbar^4) \tag{6.11}$$

one can compute the first special polynomials $\sigma_{\Delta}^{\mathcal{K}}(g)$ with the help of only symmetric representations $R = [1], [2], [3], [4]$ using [72, 73].

Let us briefly consider our sample example of knot 10_{161}. In this case,

$$\sigma_{[1]}^{10_{161}}(0) = -A^6(A^4 + A^2 - 3)$$
$$\sigma_{[2]}^{10_{161}}(1) = 2A^{12}(A - 1)(A + 1)\left(11 A^6 + 8 A^4 + 6 A^2 - 39\right)$$
$$\sigma_{[1]}^{10_{161}}(2) = 4A^{24}\left(A^4 + A^2 - 3\right)^3\left(A^4 + A^2 - 9\right)$$
$$\sigma_{[1,1]}^{10_{161}}(2) = -2A^{24}\left(8 A^{16} - 185 A^{14} + 686 A^{12} - 608 A^{10} + 142 A^8 - 1072 A^6 + 824 A^4 + 1419 A^2 - 1242\right)$$

$$\sigma_{[3]}^{10_{161}}(2) = -2A^{24}\,(A-1)^2\,(A+1)^2\left(30\,A^{12} - 319\,A^{10} + 890\,A^8 - 731\,A^6 + 618\,A^4 - 355\,A^2 - 348\right)$$

$$\sigma_{[2]}^{10_{161}}(3) = \frac{4A^{36}}{3}\left(A-1\right)\left(A+1\right)\left(42\,A^{22} - 1975\,A^{20} + 11805\,A^{18} - 24834\,A^{16} + 31718\,A^{14}\right.$$
$$\left. - 32535\,A^{12} + 10767\,A^{10} - 2928\,A^8 + 23940\,A^6 + 37971\,A^4 - 109701\,A^2 + 56160\right)$$

$$\sigma_{[2,1]}^{10_{161}}(3) = \frac{8A^{36}}{3}\left(A-1\right)\left(A+1\right)\left(279\,A^{22} - 5185\,A^{20} + 32544\,A^{18} - 100305\,A^{16} + 182258\,A^{14}\right.$$
$$\left. - 198924\,A^{12} + 121074\,A^{10} - 45282\,A^8 + 23637\,A^6 + 8658\,A^4 - 40437\,A^2 + 22113\right)$$

$$\sigma_{[4]}^{10_{161}}(3) = \frac{4A^{36}}{3}\left(A-1\right)\left(A+1\right)\left(660\,A^{22} - 10753\,A^{20} + 68289\,A^{18} - 230418\,A^{16} + 475094\,A^{14}\right.$$
$$\left. - 627231\,A^{12} + 545397\,A^{10} - 331230\,A^8 + 147924\,A^6 - 33405\,A^4 - 16317\,A^2 + 12420\right)$$
$$\ldots \tag{6.12}$$

Now one can find the values of $\varphi_R(\Delta)$ in the table in [120, 121], and, with the help of formula (6.11), compare the results with the corresponding genus expansion of $H_{[3,1]}^{10_{161}}$. They coincide, which provides yet another nontrivial check of our results for $H_{[3,1]}$.

To understand if the newly-calculated $[3,1]$-colored HOMFLY provides new special polynomials σ_Δ, as one could naturally expect, a better understanding of non-linear relations between different $\varphi_R(\Delta)$ is needed.

6.7 Differential expansion

The naive genus (or loop) expansion in powers of $\hbar$ does not respect the polynomial property of HOMFLY. This is cured in a far less trivial "differential expansion" [63, 83, 84, 98], which also reflects the hidden "differential structure" [48] lying in the base of the Khovanov approach [28–46] to knot polynomials, very different from the $\mathcal{R}$-matrix one exploited in the present paper. In variance with naive genus expansion, the differential expansion contains only a *finite* number of terms up to the $r+s$ power of Z's, where $Z_{I|J}^{(k)} = \{Aq^{I+k}\}\{A/q^{J-k}\} = Z_{I+k|J-k} \sim \hbar^2$. The expansion in the variable h defined as $q = e^\hbar = (1+h)$, $A = e^{N\hbar} = (1+h)^N$ can also be made finite for each natural N. What breaks the polynomiality in h is the analytical continuation from integer values of N. While reasonably understood for the symmetric representations, the differential expansion remains a complete mystery beyond them, simply because almost nothing has been known so far about the generic colored HOMFLY polynomial. Our new results for $R = [3,1]$ allow one to make a new small step as compared to [74, 97, 100], where only the information about $R = [2,1]$ could be used.

From [74] we know the Z-linear term of the differential expansion for the hook diagrams:

$$H_{[r,1^{s-1}]} = 1 + \left(Z_{2r-1|2s-1} + \sum_{i=1}^{r-1} Z_{2r-s-2i|s} + \sum_{j=1}^{s-1} Z_{r|2s-r-2j}\right)\cdot G_{[1]}(A,q) + O(Z^2) \tag{6.13}$$

with the knot-dependent polynomial $G_1(A,q)$, which is one and the same for all representations, including the fundamental one with $r = s = 1$. Therefore, for representation $[3,1]$

with $r = 3$ and $s = 2$ one expects:

$$H_{[3,1]} = 1 + \Big(\underbrace{\{Aq^5\}\{A/q^3\} + \{Aq^3\}\{Aq\} + \{Aq^2\}\{A/q^2\} + \{A\}\{A/q^2\}}_{Z_{5|3} + Z_{1|1}^{(2)} + Z_{3|1}^{(-1)} + Z_{1|1}^{-1} \quad \vdots \quad [4]} \cdot G_{[1]}(A,q) \Big)$$

$$+ \{Aq^3\} \cdot O(\hbar^3) \tag{6.14}$$

We write down explicitly that the $O(Z^2)$ terms in this case are always proportional to $\{Aq^3\}$, because the transposed $H_{[2,1,1]}$ should coincide with $H_{[1]} = 1 + G_1\{Aq\}\{A/q\}$ for sl_3 group, thus, the difference of *reduced* polynomials $H_{[3,1]} - H_{[1]} = 0$ at $A = q^{-3}$. For many knots (those with the defect zero, see [98]) they are also proportional to $\{A\}$, but this is not the case, say, for the defect-two 8_{19}.

There is no yet commonly accepted choice of the next terms of the differential expansion for non-symmetric representations. We now suggest an improved (as compared to [100]) differential expansion for $[2, 1]$:

$$H_{[2,1]} = 1 + \Big(\{Aq^3\}\{A/q^3\} + \{Aq^2\}\{A\} + \{A\}\{A/q^2\} \Big) \cdot G_{[1]}(A,q) + \tag{6.15}$$

$$+ [3]\{Aq^2\}\{A/q^2\} \frac{\{Aq^3\} \cdot G_{[2]}(A,q) + \{A/q^3\} \cdot G_{[2]}(A,q^{-1})}{[2]}$$

$$+ \{Aq^2\}\{A/q^2\} \Big(\{Aq^3\}\{A/q^3\} \cdot G_{[3]}(A,1) + \{q\}^2 \cdot G_{[2,1]}(A,q) \Big)$$

In this case, $G_{[21]}$ is of the order $\hbar^2$ like $G_{[3]}$ (we remind that the symmetric representation coefficients are always $G_{[r]} \sim \hbar^{r-1}$, but they are explicitly divisible by factors $\{Aq^i\}$ only for small *defects*, see [98]). We remind that $G_{[21]}$ drops out of the differential expansion for the special polynomial (i.e. at $q = 1$). Also

$$G_{[2]}(A, q^{-1}) = G_{[2]}(A, q^{-1}) = G_{[1,1]}(A, q) \qquad \text{and} \qquad G_{[2,1]}(A, q^{-1}) = G_{[2,1]}(A, q) \tag{6.16}$$

what makes the expression explicitly symmetric under the transposition of Young diagram.

Searching for a counterpart of (6.15) for $R = [31]$, we use the three properties:

$$H_{[3,1]} - H_{[2]} \sim \{A/q^2\} \implies H_{[3,1]} \overset{A=q^2}{=} H_{[2]}$$

$$H_{[3,1]} - H_{[1]} \sim \{Aq^3\} \implies H_{[2,1,1]} \overset{A=q^3}{=} H_{[1]}$$

$$H_{[3,1]} - H_{[2]} \sim \{Aq^3\} \implies H_{[2,1,1]} \overset{A=q^3}{=} H_{[1,1]} \tag{6.17}$$

which implies that

$$H_{[3,1]} = 1 + \Big(\{Aq^5\}\{A/q^3\} + \{Aq^3\}\{Aq\} + \{Aq^2\}\{A/q^2\} + \{A\}\{A/q^2\} \Big) \cdot G_{[1]}(A,q) +$$

$$+ \{Aq^3\}\{Aq^2\}\{A/q\} \cdot G_2(A,q) + \{Aq^3\}\{A/q^2\} \cdot \Big(\text{something} \Big) \tag{6.18}$$

"Something" should complement this expression so that it contains six terms of the order of G_2, four terms of the order of G_3 and one term of the order of G_4, with possible corrections by G_{21} and G_{31}. Eq. (6.18) is true for all our examples, but this only checks

their consistency with representation theory. Less trivial (but more speculative) conjectures about the non-symmetric differential expansion will be presented elsewhere.

Since we devoted a piece of this paper to implications of the evolution method, it deserves mentioning that description of the knot polynomials in terms of the differential expansion parameters G_R is badly consistent with the evolution considered in section 5.2. This is clear already from the fact that unity, the first term in the differential expansion is *not* an eigenvalue of the $\mathcal{R}$-matrix in the $R^{\otimes 2}$ channel (it *is* in the channel $R \otimes \bar{R}$, and this explains a nice consistency of the evolution and differential expansion for the peculiar family of *twist* knots [63, 83, 92]). In general the interplay between the two hidden structures, the evolution and differential expansion coefficients leads to a very interesting and important puzzle in knot theory.

7 Conclusion

In this paper we report the results of tedious calculation of the Racah (mixing) $\mathcal{U}_Q$ matrices for $[3,1]^{\otimes 3} \longrightarrow Q$ and their application to study of the 3-strand knots and links. These results confirm previous conjectures and slightly extend our understanding of complicated issues like evolution method [56, 83] and differential expansion [48, 84]. This advance was made possible by application of the highest weight method, which we developed in [100]. The size of this paper does not reflect the actual complexity and amount of calculations, we just commented briefly on various problems encountered and solved on the way, and provided simple illustrations for the results at different steps of calculations. For the full set of newly-derived *mixing matrices* (additionally converted to the block form) see [105]: they are too big for a paper, but are nicely handled by the eigenvalue hypothesis [72, 73]. Also at [105] there are the [3, 1]-*colored HOMFLY polynomials* for the 3-strand knots up to 10 crossings and the *evolution coefficients* for the entire infinite next-to-twist *family* $(n, -1 \,|\, 1, 1) = \{(n+3)_2, (2-n)_3\}$. In this paper, we also updated *the list of properties* in section 6, including *the improved differential expansion* for asymmetric representations.

This list of 5 boldfaced items describes the main results reported in the present paper.

The next steps in study could be:

●   Finding the inclusive Racah matrices $R^{\otimes 3} \longrightarrow all$ for higher representations R. This can look like impossible dream, but there is certain evidence that general formulas can exist. Still, this requires an essential progress in the highest weight method, we devote a separate publication to these perspectives.

●   Finding the Racah matrices $\mathcal{S}_R$ for $R \otimes \bar{R} \otimes \bar{R} \longrightarrow R$ and $\bar{\mathcal{S}}_R$ for $R \otimes R \otimes \bar{R} \longrightarrow R$. These are much simpler, because only one outgoing representation $Q = R$ is requested. They are more complicated for the highest weight method, because $\bar{R}$ depends on N (of sl_N) and so do the highest weights. As explained in [75–77, 97, 102], the knowledge of these matrices allows one to handle all arborescent (double fat) knots [9, 122, 123], and merging that knowledge with the results of the present paper extends this to *the fingered 3-strand knots* [1, 100, 102] which include at least the entire Rolfsen table up to 10 intersections and, after applying the power *family* method from these papers, many more.

The Racah matrices $\mathcal{S}$ and $\bar{\mathcal{S}}$ for the symmetric representations are well known [93–95, 106], for $R = [2,1]$ they were found in [91].

• In fact, a simple trick allows one to extract $\mathcal{S}$ from comparison of two different expressions, 3-strand and Pretzel ones for a peculiar two-parametric evolution family $(m, -1|n, -1) = \text{Pr}(m, n, \bar{2})$. The second matrix $\bar{\mathcal{S}}$ can then be built from $\mathcal{S}$ by equation (63) from [97]. The result, requiring a generalization of our analysis in section5, is presented in [124].

• A serious shortcut to the Racah matrix calculus can be provided by the eigenvalue hypothesis of [72, 73], see also [101], expressing these matrices through the much simpler eigenvalues of quantum $\mathcal{R}$-matrices (for which there is a general formula through the eigenvalues of the Casimir/cut-and-join-operator [56, 120, 121]). Even if the very hypothesis is true, it suffers from two uncertainties: the sign choice (well defined are the squares of matrix elements) and the problem of degenerate $\mathcal{R}$-matrix eigenvalues. One of the results of our studies in this paper is a confirmation of a natural assumption: that this degeneracy just implies a decomposition of mixing matrices into simpler (smaller-size) blocks, which are provided by the same eigenvalue hypothesis. Further work in this direction looks quite promising, and, perhaps, can provide general formulas for the Racah matrices much sooner than any alternative approach.

Acknowledgments

References

[1] A. Mironov and A. Morozov, *Towards effective topological field theory for knots*, *Nucl. Phys.* **B 899** (2015) 395 [`arXiv:1506.00339`] [INSPIRE].

[2] J.W. Alexander, *Topological invariants of knots and links*, *Trans. Amer. Math. Soc.* **30** (1928) 275.

[3] V.F.R. Jones, *Index for subfactors*, *Invent. Math.* **72** (1983) 1.

[4] V.F.R. Jones, *A polynomial invariant for knots via von Neumann algebras*, *Bull. Amer. Math. Soc.* **12** (1985) 103.

[5] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, *Annals Math.* **126** (1987) 335 [INSPIRE].

[6] L. Kauffman, *State models and the Jones polynomial*, *Topology* **26** (1987) 395.

[7] P. Freyd et al., *A new polynomial invariant of knots and links*, *Bull. Amer. Math. Soc.* **12** (1985) 239.

[8] J.H. Przytycki and K.P. Traczyk, *Invariants of links of Conway type*, *Kobe J. Math.* **4** (1987) 115.

[9] J.H. Conway, *Algebraic properties*, in *Computational problems in abstract algebra, Proc. Conf. Oxford U.K.* 1967, J. Leech ed., Pergamon Press, Oxford U.K. and New York U.S.A. (1970), pg. 329.

[10] S.-S. Chern and J. Simons, *Characteristic forms and geometric invariants*, *Annals Math.* **99** (1974) 48 [INSPIRE].

[11] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [INSPIRE].

[12] N. Yu. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, *Commun. Math. Phys.* **127** (1990) 1 [INSPIRE].

[13] E. Guadagnini, M. Martellini and M. Mintchev, *Chern-Simons field theory and quantum groups*, in *Proc. Clausthal*, (1989), pg. 307 [INSPIRE].

[14] E. Guadagnini, M. Martellini and M. Mintchev, *Chern-Simons holonomies and the appearance of quantum groups*, *Phys. Lett.* **B 235** (1990) 275 [INSPIRE].

[15] V.G. Turaev and O.Y. Viro, *State sum invariants of 3 manifolds and quantum 6j symbols*, *Topology* **31** (1992) 865 [INSPIRE].

[16] A. Morozov and A. Smirnov, *Chern-Simons theory in the temporal gauge and knot invariants through the universal quantum R-matrix*, *Nucl. Phys.* **B 835** (2010) 284 [arXiv:1001.2003] [INSPIRE].

[17] A. Smirnov, *Notes on Chern-Simons theory in the temporal gauge*, in *Proc. of International School of Subnuclear Phys.*, Erice Italy (2009) [*Subnucl. Ser.* **47** (2011) 489] [arXiv:0910.5011] [INSPIRE].

[18] R.K. Kaul and T.R. Govindarajan, *Three-dimensional Chern-Simons theory as a theory of knots and links*, *Nucl. Phys.* **B 380** (1992) 293 [hep-th/9111063] [INSPIRE].

[19] R.K. Kaul and T.R. Govindarajan, *Three-dimensional Chern-Simons theory as a theory of knots and links. 2. Multicolored links*, *Nucl. Phys.* **B 393** (1993) 392 [INSPIRE].

[20] P. Rama Devi, T.R. Govindarajan and R.K. Kaul, *Three-dimensional Chern-Simons theory as a theory of knots and links. 3. Compact semisimple group*, *Nucl. Phys.* **B 402** (1993) 548 [hep-th/9212110] [INSPIRE].

[21] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, *Knot invariants from rational conformal field theories*, *Nucl. Phys.* **B 422** (1994) 291 [hep-th/9312215] [INSPIRE].

[22] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, *Representations of composite braids and invariants for mutant knots and links in Chern-Simons field theories*, *Mod. Phys. Lett.* **A 10** (1995) 1635 [hep-th/9412084] [INSPIRE].

[23] P. Ramadevi and T. Sarkar, *On link invariants and topological string amplitudes*, *Nucl. Phys.* **B 600** (2001) 487 [hep-th/0009188] [INSPIRE].

[24] Zodinmawia and P. Ramadevi, SU(N) *quantum Racah coefficients & non-torus links*, *Nucl. Phys.* **B 870** (2013) 205 [arXiv:1107.3918] [INSPIRE].

[25] Zodinmawia and P. Ramadevi, *Reformulated invariants for non-torus knots and links*, arXiv:1209.1346 [INSPIRE].

[26] L. Rozansky, *On p-adic propreties of the Witten-Reshetikhin-Turaev invariant*, `math/9806075`.

[27] A. Yu. Morozov, *Are there p-adic knot invariants?*, [*Theor. Math. Phys.* **187** (2016) 447](https://doi.org/) [*Teor. Mat. Fiz.* **187** (2016) 3] [`arXiv:1509.04928`] [ɪɴSPIRE].

[28] M. Khovanov, *A categorification of the Jones polynomial*, [*Duke Math. J.* **101** (2000) 359](https://doi.org/) [`math/9908171`] [ɪɴSPIRE].

[29] M. Khovanov, *Patterns in knot cohomology I*, [*Exper. Math.* **12** (2003) 365](https://doi.org/) [`math/0201306`].

[30] M. Khovanov, *Categorifications of the colored Jones polynomial*, [*J. Knot Theor.* **14** (2005) 111](https://doi.org/) [`math/0302060`].

[31] M. Khovanov, *sl(3) link homology*, [*Algebr. Geom. Topol.* **4** (2004) 1045](https://doi.org/) [`math/0304375`].

[32] M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, [*Int. J. Math.* **18** (2007) 869](https://doi.org/) [`math/0510265`].

[33] M. Khovanov, *Link homology and categorification*, `math/0605339`.

[34] M. Khovanov, *Categorifications from planar diagrammatics*, `arXiv:1008.5084`.

[35] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, [*Algebr. Geom. Topol.* **2** (2002) 337](https://doi.org/) [`math/0201043`].

[36] D. Bar-Natan, *Khovanov's homology for tangles and cobordisms*, [*Geom. Topol.* **9** (2005) 1443](https://doi.org/) [`math/0410495`].

[37] D. Bar-Natan, *Fast Khovanov homology computations*, [*J. Knot Theor.* **16** (2007) 243](https://doi.org/) [`math/0606318`].

[38] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, *Fund. Math.* **199** (2008) 191 [`math/0401268`].

[39] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology II*, [*Geom. Topol.* **12** (2008) 1387](https://doi.org/) [`math/0505056`].

[40] M. Khovanov and L. Rozansky, *Virtual crossings, convolutions and a categorification of the SO(2N) Kauffman polynomial*, `math/0701333`.

[41] N. Carqueville and D. Murfet, *Computing Khovanov-Rozansky homology and defect fusion*, [*Algebr. Geom. Topol.* **14** (2014) 489](https://doi.org/) [`arXiv:1108.1081`] [ɪɴSPIRE].

[42] V. Dolotin and A. Morozov, *Introduction to Khovanov homologies. I. Unreduced Jones superpolynomial*, [*JHEP* **01** (2013) 065](https://doi.org/) [`arXiv:1208.4994`] [ɪɴSPIRE].

[43] V. Dolotin and A. Morozov, *Introduction to Khovanov homologies. II. Reduced Jones superpolynomials*, [*J. Phys. Conf. Ser.* **411** (2013) 012013](https://doi.org/) [`arXiv:1209.5109`] [ɪɴSPIRE].

[44] V. Dolotin and A. Morozov, *Introduction to Khovanov homologies. III. A new and simple tensor-algebra construction of Khovanov-Rozansky invariants*, [*Nucl. Phys.* **B 878** (2014) 12](https://doi.org/) [`arXiv:1308.5759`] [ɪɴSPIRE].

[45] A. Anokhina and A. Morozov, *Towards R-matrix construction of Khovanov-Rozansky polynomials. I. Primary T-deformation of HOMFLY*, [*JHEP* **07** (2014) 063](https://doi.org/) [`arXiv:1403.8087`] [ɪɴSPIRE].

[46] S. Nawata and A. Oblomkov, *Lectures on knot homology*, `arXiv:1510.01795` [ɪɴSPIRE].

[47] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B 577** (2000) 419 [hep-th/9912123] [INSPIRE].

[48] N.M. Dunfield, S. Gukov and J. Rasmussen, *The superpolynomial for knot homologies*, *Exper. Math.* **15** (2006) 129 [math/0505662] [INSPIRE].

[49] M. Aganagic and S. Shakirov, *Knot homology and refined Chern-Simons index*, *Commun. Math. Phys.* **333** (2015) 187 [arXiv:1105.5117] [INSPIRE].

[50] M. Aganagic and S. Shakirov, *Refined Chern-Simons theory and knot homology*, *Proc. Symp. Pure Math.* **85** (2012) 3 [arXiv:1202.2489] [INSPIRE].

[51] M. Aganagic and S. Shakirov, *Refined Chern-Simons theory and topological string*, arXiv:1210.2733 [INSPIRE].

[52] A. Mironov, A. Morozov, S. Shakirov and A. Sleptsov, *Interplay between MacDonald and Hall-Littlewood expansions of extended torus superpolynomials*, *JHEP* **05** (2012) 070 [arXiv:1201.3339] [INSPIRE].

[53] E. Gorsky and A. Negut, *Refined knot invariants and Hilbert schemes*, *J. Math. Pure. Appl.* **104** (2015) 403 [arXiv:1304.3328] [INSPIRE].

[54] I. Cherednik and I. Danilenko, *DAHA and iterated torus knots*, *Algebr. Geom. Topol.* **16** (2016) 843 [arXiv:1408.4348].

[55] S. Arthamonov and S. Shakirov, *Refined Chern-Simons theory in genus two*, arXiv:1504.02620 [INSPIRE].

[56] P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov, *Superpolynomials for toric knots from evolution induced by cut-and-join operators*, *JHEP* **03** (2013) 021 [arXiv:1106.4305] [INSPIRE].

[57] I. Cherednik, *Jones polynomials of torus knots via DAHA*, arXiv:1111.6195 [INSPIRE].

[58] S. Gukov and M. Stošić, *Homological algebra of knots and BPS states*, *Proc. Symp. Pure Math.* **85** (2012) 125 [arXiv:1112.0030] [INSPIRE].

[59] A. Mironov, A. Morozov and An. Morozov, *Character expansion for HOMFLY polynomials. I. Integrability and difference equations*, in *Strings, Gauge Fields, and the Geometry Behind: The Legacy of Maximilian Kreuzer*, A. Rebhan, L. Katzarkov, J. Knapp, R. Rashkov and E. Scheidegger eds., World Scientific Publishing Co. Pte. Ltd., Singapore (2013), pg. 101 [arXiv:1112.5754].

[60] A. Mironov, A. Morozov and A. Morozov, *Character expansion for HOMFLY polynomials. II. Fundamental representation. Up to five strands in braid*, *JHEP* **03** (2012) 034 [arXiv:1112.2654] [INSPIRE].

[61] A. Anokhina, A. Mironov, A. Morozov and A. Morozov, *Racah coefficients and extended HOMFLY polynomials for all 5-, 6- and 7-strand braids*, *Nucl. Phys.* **B 868** (2013) 271 [arXiv:1207.0279] [INSPIRE].

[62] A. Anokhina, *On R-matrix approaches to knot invariants*, arXiv:1412.8444 [INSPIRE].

[63] H. Itoyama, A. Mironov, A. Morozov and A. Morozov, *HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations*, *JHEP* **07** (2012) 131 [arXiv:1203.5978] [INSPIRE].

[64] H. Itoyama, A. Mironov, A. Morozov and A. Morozov, *Character expansion for HOMFLY polynomials. III. All 3-strand braids in the first symmetric representation*, *Int. J. Mod. Phys.* **A 27** (2012) 1250099 [arXiv:1204.4785] [INSPIRE].

[65] S. Nawata, P. Ramadevi, Zodinmawia and X. Sun, *Super-A-polynomials for twist knots*, *JHEP* **11** (2012) 157 [arXiv:1209.1409] [ɪɴSPIRE].

[66] H. Fuji, S. Gukov, M. Stošić and P. Sulkowski, *3d analogs of Argyres-Douglas theories and knot homologies*, *JHEP* **01** (2013) 175 [arXiv:1209.1416] [ɪɴSPIRE].

[67] K. Liu and P. Peng, *Proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, *J. Diff. Geom.* **85** (2010) 479 [arXiv:0704.1526] [ɪɴSPIRE].

[68] S. Zhu, *Colored HOMFLY polynomials via skein theory*, *JHEP* **10** (2013) 229 [arXiv:1206.5886] [ɪɴSPIRE].

[69] A. Morozov, *The first-order deviation of superpolynomial in an arbitrary representation from the special polynomial*, *JETP Lett.* **97** (2013) 171 [arXiv:1211.4596] [ɪɴSPIRE].

[70] A. Morozov, *Special colored superpolynomials and their representation-dependence*, *JHEP* **12** (2012) 116 [arXiv:1208.3544] [ɪɴSPIRE].

[71] A. Mironov and A. Morozov, *Equations on knot polynomials and 3d/5d duality*, *AIP Conf. Proc.* **1483** (2012) 189 [arXiv:1208.2282] [ɪɴSPIRE].

[72] H. Itoyama, A. Mironov, A. Morozov and A. Morozov, *Eigenvalue hypothesis for Racah matrices and HOMFLY polynomials for 3-strand knots in any symmetric and antisymmetric representations*, *Int. J. Mod. Phys.* **A 28** (2013) 1340009 [arXiv:1209.6304] [ɪɴSPIRE].

[73] I. Tuba and H. Wenzl, *Representations of the braid group B_3 and of* $SL(2,Z)$, *Pacif. J. Math.* **197** (2001) 491 [math/9912013].

[74] A. Anokhina, A. Mironov, A. Morozov and A. Morozov, *Knot polynomials in the first non-symmetric representation*, *Nucl. Phys.* **B 882** (2014) 171 [arXiv:1211.6375] [ɪɴSPIRE].

[75] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, *Chirality of knots 9_{42} and 10_{71} and Chern-Simons theory*, *Mod. Phys. Lett.* **A 9** (1994) 3205 [hep-th/9401095] [ɪɴSPIRE].

[76] S. Nawata, P. Ramadevi and Zodinmawia, *Colored HOMFLY polynomials from Chern-Simons theory*, *J. Knot Theor.* **22** (2013) 1350078 [arXiv:1302.5144] [ɪɴSPIRE].

[77] Zodinmawia, *Knot polynomials from* $SU(N)$ *Chern-Simons theory, superpolynomials and super-A-polynomials*, Ph.D. thesis, IIT, Mumbai India (2014).

[78] A. Mironov, A. Morozov and A. Sleptsov, *Genus expansion of HOMFLY polynomials*, *Theor. Math. Phys.* **177** (2013) 1435 [*Teor. Mat. Fiz.* **177** (2013) 179] [arXiv:1303.1015] [ɪɴSPIRE].

[79] A. Mironov, A. Morozov and A. Sleptsov, *On genus expansion of knot polynomials and hidden structure of Hurwitz τ-functions*, *Eur. Phys. J.* **C 73** (2013) 2492 [arXiv:1304.7499] [ɪɴSPIRE].

[80] A. Sleptsov, *Hidden structures of knot invariants*, *Int. J. Mod. Phys.* **A 29** (2014) 1430063 [ɪɴSPIRE].

[81] A. Anokhina, A. Mironov, A. Morozov and A. Morozov, *Colored HOMFLY polynomials as multiple sums over paths or standard Young tableaux*, *Adv. High Energy Phys.* **2013** (2013) 931830 [arXiv:1304.1486] [ɪɴSPIRE].

[82] E. Gorsky, S. Gukov and M. Stošić, *Quadruply-graded colored homology of knots*, arXiv:1304.3481 [ɪɴSPIRE].

[83] A. Mironov, A. Morozov and A. Morozov, *Evolution method and "differential hierarchy" of colored knot polynomials*, *AIP Conf. Proc.* **1562** (2013) 123 [arXiv:1306.3197] [INSPIRE].

[84] S.B. Arthamonov, A. Mironov and A. Morozov, *Differential hierarchy and additional grading of knot polynomials*, *Theor. Math. Phys.* **179** (2014) 509 [*Teor. Mat. Fiz.* **179** (2014) 147] [arXiv:1306.5682] [INSPIRE].

[85] A. Anokhina and A. Morozov, *Cabling procedure for the colored HOMFLY polynomials*, *Theor. Math. Phys.* **178** (2014) 1 [*Teor. Mat. Fiz.* **178** (2014) 3] [arXiv:1307.2216] [INSPIRE].

[86] S. Arthamonov, A. Mironov, A. Morozov and A. Morozov, *Link polynomial calculus and the AENV conjecture*, *JHEP* **04** (2014) 156 [arXiv:1309.7984] [INSPIRE].

[87] S. Nawata, P. Ramadevi and Zodinmawia, *Colored Kauffman homology and super-A-polynomials*, *JHEP* **01** (2014) 126 [arXiv:1310.2240] [INSPIRE].

[88] S. Gukov, S. Nawata, I. Saberi, M. Stošić and P. Sulkowski, *Sequencing BPS spectra*, *JHEP* **03** (2016) 004 [arXiv:1512.07883] [INSPIRE].

[89] A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov, *On genus expansion of superpolynomials*, *Nucl. Phys.* **B 889** (2014) 757 [arXiv:1310.7622] [INSPIRE].

[90] E. Witten, *Two lectures on the Jones polynomial and Khovanov homology*, arXiv:1401.6996 [INSPIRE].

[91] J. Gu and H. Jockers, *A note on colored HOMFLY polynomials for hyperbolic knots from WZW models*, *Commun. Math. Phys.* **338** (2015) 393 [arXiv:1407.5643] [INSPIRE].

[92] A. Mironov, A. Morozov and A. Morozov, *On colored HOMFLY polynomials for twist knots*, *Mod. Phys. Lett.* **A 29** (2014) 1450183 [arXiv:1408.3076] [INSPIRE].

[93] D. Galakhov, D. Melnikov, A. Mironov, A. Morozov and A. Sleptsov, *Colored knot polynomials for arbitrary pretzel knots and links*, *Phys. Lett.* **B 743** (2015) 71 [arXiv:1412.2616] [INSPIRE].

[94] D. Galakhov, D. Melnikov, A. Mironov and A. Morozov, *Knot invariants from Virasoro related representation and pretzel knots*, *Nucl. Phys.* **B 899** (2015) 194 [arXiv:1502.02621] [INSPIRE].

[95] A. Mironov, A. Morozov and A. Sleptsov, *Colored HOMFLY polynomials for the pretzel knots and links*, *JHEP* **07** (2015) 069 [arXiv:1412.8432] [INSPIRE].

[96] S. Nawata, P. Ramadevi and V.K. Singh, *Colored HOMFLY polynomials can distinguish mutant knots*, arXiv:1504.00364 [INSPIRE].

[97] A. Mironov, A. Morozov, A. Morozov, P. Ramadevi and V.K. Singh, *Colored HOMFLY polynomials of knots presented as double fat diagrams*, *JHEP* **07** (2015) 109 [arXiv:1504.00371] [INSPIRE].

[98] Ya. Kononov and A. Morozov, *On the defect and stability of differential expansion*, *JETP Lett.* **101** (2015) 831 [*Pisma Zh. Eksp. Teor. Fiz.* **101** (2015) 931] [arXiv:1504.07146] [INSPIRE].

[99] Ya. Kononov and A. Morozov, *Factorization of colored knot polynomials at roots of unity*, *Phys. Lett.* **B 747** (2015) 500 [arXiv:1505.06170] [INSPIRE].

[100] A. Mironov, A. Morozov, A. Morozov and A. Sleptsov, *Colored knot polynomials: HOMFLY in representation* [2, 1], *Int. J. Mod. Phys.* **A 30** (2015) 1550169 [`arXiv:1508.02870`] [inSPIRE].

[101] A. Mironov and A. Morozov, *Universal Racah matrices and adjoint knot polynomials: arborescent knots*, *Phys. Lett.* **B 755** (2016) 47 [`arXiv:1511.09077`] [inSPIRE].

[102] A. Mironov, A. Morozov, A. Morozov, A. Sleptsov, P. Ramadevi and V.K. Singh, *Tabulating knot polynomials for arborescent knots*, `arXiv:1601.04199` [inSPIRE].

[103] S. Garoufalidis, A.D. Lauda and T.T.Q. Lê, *The colored HOMFLYPT function is q-holonomic*, `arXiv:1604.08502` [inSPIRE].

[104] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum toroidal $\mathfrak{gl}_1$ algebra: plane partitions*, *Kyoto J. Math.* **52** (2012) 621 [`arXiv:1110.5310`].

[105] *Knotebook webpage*, http://knotebook.org.

[106] S. Nawata, P. Ramadevi and Zodinmawia, *Multiplicity-free quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$*, *Lett. Math. Phys.* **103** (2013) 1389 [`arXiv:1302.5143`] [inSPIRE].

[107] A. Mironov, R. Mkrtchyan and A. Morozov, *On universal knot polynomials*, *JHEP* **02** (2016) 078 [`arXiv:1510.05884`] [inSPIRE].

[108] P. Vogel, *The universal Lie algebra*, preprint, http://webusers.imj-prg.fr/~pierre.vogel/, (1999).

[109] M. Kontsevich, *Vassiliev's knot invariants*, *Adv. Soviet Math.* **16** (1993) 137.

[110] M. Alvarez, J.M.F. Labastida and E. Perez, *Vassiliev invariants for links from Chern-Simons perturbation theory*, *Nucl. Phys.* **B 488** (1997) 677 [`hep-th/9607030`] [inSPIRE].

[111] S. Chmutov, S. Duzhin and J. Mostovoy, *Introduction to Vassiliev knot invariants*, Cambridge University Press, Cambridge U.K. (2012) [ISBN:978-1-107-02083-2] [`arXiv:1103.5628`].

[112] M. Alvarez and J.M.F. Labastida, *Numerical knot invariants of finite type from Chern-Simons perturbation theory*, *Nucl. Phys.* **B 433** (1995) 555 [*Erratum ibid.* **B 441** (1995) 403] [`hep-th/9407076`] [inSPIRE].

[113] J.M.F. Labastida, *Chern-Simons gauge theory: ten years after*, *AIP Conf. Proc.* **484** (1999) 1 [`hep-th/9905057`] [inSPIRE].

[114] P. Dunin-Barkowski, A. Sleptsov and A. Smirnov, *Kontsevich integral for knots and Vassiliev invariants*, *Int. J. Mod. Phys.* **A 28** (2013) 1330025 [`arXiv:1112.5406`] [inSPIRE].

[115] D. Bar-Natan and S. Morrison, *The Knot atlas webpage*, http://katlas.org.

[116] A.S. Alexandrov, A. Mironov and A. Morozov, *Instantons and merons in matrix models*, *Physica* **D 235** (2007) 126 [`hep-th/0608228`] [inSPIRE].

[117] A. Alexandrov, A. Mironov and A. Morozov, *BGWM as second constituent of complex matrix model*, *JHEP* **12** (2009) 053 [`arXiv:0906.3305`] [inSPIRE].

[118] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, *Commun. Num. Theor. Phys.* **1** (2007) 347 [`math-ph/0702045`] [inSPIRE].

[119] N. Orantin, *Symplectic invariants, Virasoro constraints and Givental decomposition*, arXiv:0808.0635 [INSPIRE].

[120] A. Mironov, A. Morozov and S. Natanzon, *Complete set of cut-and-join operators in Hurwitz-Kontsevich theory*, *Theor. Math. Phys.* **166** (2011) 1 [*Teor. Mat. Fiz.* **166** (2011) 3] [arXiv:0904.4227] [INSPIRE].

[121] A. Mironov, A. Morozov and S. Natanzon, *Algebra of differential operators associated with Young diagrams*, *J. Geom. Phys.* **62** (2012) 148 [arXiv:1012.0433] [INSPIRE].

[122] A. Caudron, *Classification des noeuds et des enlacements* (in French), Publ. Math. Orsay **82-4**, University of Paris XI, Orsay France (1982).

[123] F. Bonahon and L.C. Siebenmann, *New geometric splittings of classical knots and the classification and symmetries of arborescent knots*, http://www-bcf.usc.edu/∼fbonahon/Research/Preprints/BonSieb.pdf, (2010).

[124] A. Mironov, A. Morozov, A. Morozov and A. Sleptsov, *Racah matrices and hidden integrability in evolution of knots*, *Phys. Lett.* **B 760** (2016) 45 [arXiv:1605.04881] [INSPIRE].