Nucl. Phys. B 998 (2024) 116400

Contents lists available at ScienceDirect

Nuclear Physics, Section B

journal homepage: www.elsevier.com/locate/nuclphysb

High Energy Physics – Theory

(2,0) theory on $S^5 \times S^1$ and quantum M2 branes

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ARTICLE INFO

Editor: Stephan Stieberger

ABSTRACT

The superconformal index Z of the 6d (2,0) theory on $S^5 \times S^1$ (which is related to the localization partition function of 5d SYM on S^5) should be captured at large N by the quantum M2 brane theory in the dual M-theory background. Generalizing the type IIA string theory limit of this relation discussed in arXiv:2111.15493 and arXiv:2304.12340, we consider semiclassically quantized M2 branes in a half-supersymmetric 11d background which is a twisted product of thermal AdS_7 and S^4. We show that the leading non-perturbative term at large N is reproduced precisely by the 1-loop partition function of an "instanton" M2 brane wrapped on $S^1 \times S^2$ with $S^2 \subset S^4$. Similarly, the (2,0) theory analog of the BPS Wilson loop expectation value is reproduced by the partition function of a "defect" M2 brane wrapped on thermal $AdS_3 \subset AdS_7$. We comment on a curious analogy of these results with similar computations in arXiv:2303.15207 and arXiv:2307.14112 of the partition function of quantum M2 branes in $AdS_4 \times S^7/\mathbb{Z}_k$ which reproduced the corresponding localization expressions in the ABJM 3d gauge theory.

Contents

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https://doi.org/10.1016/j.nuclphysb.2023.116400
Received 4 October 2023; Received in revised form 10 November 2023; Accepted 12 November 2023

Available online 17 November 2023

1. Introduction

The 6d (2,0) superconformal field theory should be describing the low-energy dynamics of N coincident M5 branes. It is expected to be dual [1] to 11d M-theory theory on the $\mathrm{AdS}_7\times S^4$ background, which is a limit of the M5 brane solution of 11d supergravity [2][2]

$$ds^2_{11}=a^2\left(ds^2_{\mathrm{AdS}_7}+\tfrac14 ds^2_{S^4}\right),\qquad F_4=dC_3=\pi^2a^3\,\mathrm{vol}_{S^4}\ ,\qquad a^3=8\pi N\,\ell_P^3\ . \tag{1.1}$$

Due to the lack of an intrinsic definition of the (2,0) theory and having only N as a free parameter, it is not clear how to define non-trivial observables (computable, e.g., by localization) that can be used to test this AdS/CFT duality.[3]

To introduce an extra parameter one may consider some "orbifolding" of (1.1) (by analogy, e.g., with the ABJM theory [10] of multiple M2 branes on $\mathbb{R}^8/\mathbb{Z}_k$ dual to M-theory on $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$). One option is to consider the (2,0) theory on $S^5\times S^1_\beta$ where β is the length of the circle. The dual M-theory background may then have the AdS_7 part with the corresponding $S^5\times S^1_\beta$ boundary, i.e. $ds^2_{\mathrm{AdS}_7}=dx^2+\sinh^2x\,dS_5+\cosh^2x\,dy^2$ where $y\equiv y+\beta$ and dS_5 is the metric of a unit-radius 5-sphere.[4]

Dimensionally reducing on the y-circle, i.e. considering the limit of $\beta\to 0$, the M5 brane solution will reduce to the D4 brane solution of type IIA 10d supergravity, while the (2,0) theory on $S^5\times S^1_\beta$ is expected to be related to the maximally supersymmetric 5d SYM theory on S^5. The 5d SYM theory does not have a first-principles definition being nonrenormalizable, i.e. the (2,0) theory should be thought of as its UV completion (cf. [12]). Yet this relation may be useful at a heuristic level as one may attempt to define free energy of the SYM theory on S^5 by analogy with 4d SYM theory where it can be computed from localization.

It turns out that the requirement of preservation of 16 real supersymmetries demands introducing an extra R-symmetry twist in the (2,0) theory on $S^5\times S^1_\beta$, or a twist in the S^4 part of the background (1.1). This was understood in [13] when constructing the type IIA solution which corresponds to a D4 brane world volume wrapped on S^5. The 11d uplift of this solution is related by an analytic continuation to the following 11d background [13–15]

$$ds^2_{11}=a^2\Big(\big[dx^2+\sinh^2x\,dS_5+\cosh^2x\,dy^2\big]+\tfrac14\big[du^2+\cos^2u\,dS_2+\sin^2u\,(dz+idy)^2\big]\Big), \tag{1.2}$$

$$C_3=-\tfrac18 a^3\cos^2u\,\mathrm{vol}_{S^2}\wedge(dz+idy)\ . \tag{1.3}$$

Here the S^4 part $du^2+\cos^2u\,dS_2+\sin^2u\,dz^2$ got the 2π periodic angle z shifted by iy where $y\in(0,\beta)$ is the circular 11d coordinate.[5] This background is related to (1.1) by a periodic identification and a coordinate shift so is an obvious solution of the 11d supergravity.[6] We will denote the first 7d part of (1.2) as $\mathrm{AdS}_{7,\beta}$ and the 4d part as $\tilde S^4$ and somewhat loosely refer to (1.2) as a "direct product" $\mathrm{AdS}_{7,\beta}\times\tilde S^4$.

Our aim in this paper is to consider the quantum M2 brane in the (N,β) dependent background (1.2), (1.3) and compute its partition function in the semiclassical (large tension $\mathrm{T}_2=a^3T_2=\frac{2}{\pi}N\gg 1$) expansion near particular classical solutions similarly to how that was done in the $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ case in [18,19]. This will represent an M-theory generalization of the type IIA string theory semiclassical computations done in the limit $\beta\to0,\ N\to\infty$ with fixed $N\beta$ in [15,20].

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[2] Here $ds^2_{\mathrm{AdS}_7}$ and $ds^2_{S^4}$ are the metrics of the unit-radius AdS_7 and S^4. We shall often use the notation $dS_n\equiv ds^2_{S^n}$. ℓ_P is the 11d Planck constant related to the gravitational constant in the (Euclidean) 11d supergravity action $S_{11}=-\frac{1}{2\kappa^2_{11}}\int d^{11}x\sqrt{G}(R-\frac{1}{2\cdot4!}F^2_{MNKL}+\dots)$ as $2\kappa^2_{11}=(2\pi)^8\ell_P^9$ and to the M2 brane tension as $T_2=\frac{1}{(2\pi)^2\ell_P^3}$. Also, vol_{S^4} is the normalized volume 4-form of S^4, i.e. $\int_{S^4}\mathrm{vol}_{S^4}=1$ with $\mathrm{vol}(S^4)=\frac{8\pi^2}{3}$.

[3] Almost all of the available information comes from the 11d supergravity effective action and supersymmetry considerations that may be used, e.g., to determine the M-theory predictions for the a- and c- conformal anomaly coefficients of the (2,0) theory (see, e.g., [3–7]) and thus, in particular, the expression for its free energy on S^6 (that should have the same structure as the free energy of the $\mathcal{N}=4$ SYM on S^4): $F\sim \mathrm{a}(N)\log\Lambda+\mathrm{const}$. One may also find a defect conformal anomaly by using M2 brane probe in $\mathrm{AdS}_7\times S^4$ background as discussed in [8,9] and refs. there.

[4] In general, introducing a thermal circle one would need to consider also black hole like geometry with the corresponding asymptotics [11]. This will not be the case here as we will be interested in the background corresponding to a superconformal index with an extra R-symmetry twisting and periodic fermions.

[5] This complex background becomes real after $y\to it$ with the time-like direction t here playing the role of the 11d circle.

[6] Note that near $x=0,\ u=0$ and relevant part of the metric becomes $dy^2+du^2+u^2(dz+idy)^2$ so that it may be thought of as a special case of a (complex or time-like) Melvin twist discussed in [16] and, in particular, in 11d context in [17]).

We will provide a check of the AdS_7/CFT_6 correspondence in this setting by establishing matching of quantum M2 brane results with the large N expansion of the supersymmetric partition function of the (2,0) theory on $S^5\times S^1_\beta$ with R-symmetry twist (and periodic fermions), identified with the corresponding superconformal index computed in [21,22].[7]

As the non-abelian (2,0) theory does not have an explicit Lagrangian formulation, its supersymmetric partition function on $S^5\times S^1$ that should be equal to the index cannot be computed directly, but it may be interpreted as a partition function of the 5d SYM theory (assuming the latter has a well-defined UV completion). Then the superconformal index may be interpreted as a (properly defined) localization result for the partition function of 5d SYM on S^5 with $g^2_{\rm YM}$ proportional to β up to a length scale factor. By analogy with the 4d SYM theory, this suggests also to consider the localization expression for the BPS Wilson loop expectation value (cf. [24]) which may be then compared with an M2 brane semiclassical computation as in [18].

Denoting by $Z_N(\beta)$ the index of the $(2,0)$ theory on $S^5\times S^1_\beta$ one finds for the large N, fixed β expansion of the corresponding free energy $F_N(\beta)=-\log Z_N(\beta)$ [21,22]

$$F_N(\beta)=F_N^{\rm pert}(\beta)+F_N^{\rm np}(\beta)\,,\qquad F_N^{\rm pert}(\beta)=-(\tfrac16 N^3-\tfrac18 N)\beta+\sum_{n=1}^\infty c_n\,e^{-n\beta}\,, \tag{1.4}$$

$$F_N^{\rm np}(\beta)=\frac{1}{4\sinh^2\frac{\beta}{2}}\,e^{-N\beta}+\mathcal{O}(e^{-2N\beta})\,. \tag{1.5}$$

For the natural analog of the Wilson loop one finds for large N [21,22]

$$\langle W\rangle=\frac{1}{2\sinh\frac{\beta}{2}}e^{N\beta}+\mathcal{O}(N^0)\,. \tag{1.6}$$

Below we will reproduce the expressions (1.5) and (1.6) on the M-theory side, by performing semiclassical M2-brane computations in the background (1.2), (1.3). In the case of the non-perturbative contribution to free energy in (1.5), the classical M2 brane solution will be wrapped on $S^1_\beta\times S^2$ with $S^1_\beta\subset {\rm AdS}_{7,\beta}$ and $S^2\subset\tilde S^4$. In the case of the Wilson loop (1.6), the dual M2 brane solution will be wrapped on ${\rm AdS}_{3,\beta}\subset{\rm AdS}_{7,\beta}$, where ${\rm AdS}_{3,\beta}$ is the "thermal" AdS_3 background.

In both cases, the exponents in (1.5) and (1.6) will come from the classical M2 brane action while the β dependent prefactors will be precisely reproduced by the one-loop M2 brane fluctuation determinants as in [18,19]. Our results will generalize to the finite β case the analogous computations in the type IIA string-theory limit in [15,20].

The plan of the paper is as follows. In section 2 we review the localization results for the $(2,0)$ theory superconformal index and the analog of the supersymmetric Wilson loop, leading to (1.5), (1.6). In section 3 we discuss the general structure of the M2 brane semiclassical partition function. Section 4 presents the details of the calculation of this partition function in the case of the $S^1_\beta\times S^1$ M2 brane instanton background reproducing (1.5). Section 5 addresses similar computation in the case of the M2 brane wrapped on ${\rm AdS}_{3,\beta}$ reproducing the Wilson loop expectation value in (1.6). Section 6 contains a summary and concluding remarks. Appendices contain some technical details used in the main part of the paper.

2. Localization expressions for the free energy and Wilson loop

The superconformal index of $U(N)$ (2,0) theory on $S^5\times S^1_\beta$ was found [21,22] to be given by a matrix model which is the same as for the supersymmetric 3d pure Chern-Simons theory solved in [25]. The result may be represented as a product of two factors[8]

$$Z_N(q)\equiv e^{-F_N(\beta)}=Z_N^{(0)}(q)\,Z_N^{\rm inst}(q)\,,\qquad q\equiv e^{-\beta}\,, \tag{2.1}$$

$$Z_N^{(0)}(q)=\Big(\frac{\beta}{2\pi}\Big)^{N/2}e^{\frac{N(N^2-1)}{6}\beta}\prod_{n=1}^{N-1}(1-e^{-n\beta})^{N-n}\,,\qquad Z_N^{\rm inst}(q)=\Big[\eta\Big(\frac{2\pi i}{\beta}\Big)\Big]^{-N}\,. \tag{2.2}$$

We shall refer to Z_N as partition function. $F_N(\beta)=-\log Z_N(q)$ may be interpreted as a "supersymmetric" free energy.[9]

To study the expansion of the partition function Z_N at large N and fixed β, it is convenient to apply a modular transformation to the η-function factor $Z_N^{\rm inst}(q)$ in Z_N. This gives

$$Z_N(q)=q^{\epsilon_0(N)}\,\widehat{Z}_N(q)\,,\qquad \epsilon_0(N)=-\tfrac16 N(N^2-1)-\tfrac{1}{24}N=-\tfrac16 N^3+\tfrac18 N\,, \tag{2.3}$$

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[7] Ref. [21] started with the abelian 6d $(2,0)$ theory (i.e. tensor multiplet) with 32 supersymmetries and by introducing a Scherk-Schwarz-like R-symmetry twist obtained a theory on $S^5\times S^1$ with 16 supersymmetries and a subgroup $SO(2)\times SO(3)$ of the original $SO(5)$ R-symmetry. The $SO(2)\subset SO(5)$ twist was necessary to have constant spinors on S^5. Upon dimensional reduction, the R-symmetry twist leads to extra mass terms in the 5d SYM action. The construction was then extended to the non-abelian case via 5d SYM connection, and using supersymmetric localization provided the expression for the perturbative partition function in the form of a matrix model [21], which was supplemented by all instanton corrections in [22]. The $SO(2)$ twist corresponds to the introduction of a chemical potential coupled to the R-charge and the corresponding localization matrix model computes the (unrefined) superconformal index of the $(2,0)$ theory (see also [23]).

[8] Here the Dedekind function is $\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^\infty(1-q^n)$ where $q=e^{2\pi i\tau}$. Its modular transformation is $\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau)$.

[9] In the interpretation of $F_N(\beta)$ as a free energy of 5d SYM theory on S^5 one may set $\beta=\frac{g^2_{\rm YM}}{2\pi R}$ where R is an effective length scale.

$$\hat{Z}_N(q) = \prod_{n=1}^{N}\prod_{m=0}^{\infty} \frac{1}{1-q^{n+m}} \ , \tag{2.4}$$

where $\epsilon_0(N)$ is the "supersymmetric Casimir energy" [26–28].

The partition function (2.4) has an expansion in powers of q with integer N-dependent coefficients. The coefficients take finite values for large N: the $N \to \infty$ limit $\hat{Z}_\infty(q)$ of $\hat{Z}_N$ is the MacMahon function

$$\hat{Z}_N(q) \overset{N\to\infty}{\to} \hat{Z}_\infty(q) \ , \qquad \hat{Z}_\infty(q) = \prod_{n=1}^{\infty}(1-q^n)^{-n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \cdots \ . \tag{2.5}$$

The expression $\hat{Z}_\infty$ may be interpreted as the unrefined superconformal index counting BPS states of 11d supergravity on $\text{AdS}_7 \times S^4$, i.e. given by the sum over Kaluza-Klein states of the S^4 compactification [21].

Finite N corrections to the partition function can be read off from (2.4) after writing it in the following equivalent form

$$\hat{Z}_N(q) = \hat{Z}_\infty(q) \prod_{n=0}^{\infty}\prod_{m=0}^{\infty}(1-q^{N+n+m+1}) \ . \tag{2.6}$$

Expanding $\log \hat{Z}_N$ in powers of q^N, summing over n, m, and exponentiating back gives

$$\begin{aligned} \hat{Z}_N(q) &= \hat{Z}_\infty(q)\Big[1 - \frac{q}{(1-q)^2}\, q^N + \frac{2q^3}{(1-q^2)^2(1-q)^2}\, q^{2N} + \cdots\Big] \\ &= \hat{Z}_\infty(q)\Big[1 - \frac{1}{4\sinh^2\frac{\beta}{2}} e^{-N\beta} + \frac{1}{32\sinh^4\frac{\beta}{2}\cosh^2\frac{\beta}{2}}\, e^{-2N\beta} + \cdots\Big] \ . \end{aligned} \tag{2.7}$$

Combining (2.3) and (2.7), we can write the large N, fixed β expansion of the free energy F_N in (2.1) as a sum of a perturbative and non-perturbative parts

$$F_N(\beta) = F_N^{\text{pert}}(\beta) + F_N^{\text{np}}(\beta) \ , \tag{2.8}$$

$$F_N^{\text{pert}}(\beta) = \epsilon_0(N)\,\beta + \hat{F}(\beta) \ , \qquad \hat{F}(\beta) \equiv -\log \hat{Z}_\infty(q) = \sum_{n=1}^{\infty} c_n\, e^{-n\beta} \ , \tag{2.9}$$

$$F_N^{\text{np}}(\beta) = \frac{1}{4\sinh^2\frac{\beta}{2}}\, e^{-N\beta} + \mathcal{O}(e^{-2N\beta}) \ , \tag{2.10}$$

where $\epsilon_0(N)$ is given in (2.3) and c_n in (2.9) following from (2.5) are $c_1 = -1,\ c_2 = -\frac{5}{2},\ c_3 = -\frac{10}{3}, \ldots$.

The leading $N^3\beta$ term in the perturbative part of (2.9) where $\epsilon_0 = -\frac{1}{24}(4N^3 - 3N)$ as in (2.3) should originate from the 11d supergravity action $\int R + \ldots$ evaluated on the corresponding dual background $\text{AdS}_{7,\beta} \times \tilde{S}^4$ in (1.2), (1.3).[10] The first subleading $N\beta$ term in F_N^{pert} should originate from the R^4 invariant in the 11d effective action, by analogy with the case of the 11d effective action evaluated on the standard $\text{AdS}_7 \times S^4$ background, reproducing [4] the order N term in the coefficient $\text{a} = 4N^3 - \frac{9}{4}N - \frac{7}{4}$ of the conformal anomaly of the (2,0) theory on S^6.[11] Let us note that in general the supersymmetric Casimir energy of a 6d (2,0) supersymmetric theory on $S^5 \times S^1$ should be related to the conformal c-anomaly coefficient as [27] $\epsilon_0 = -\frac{1}{24}\text{c}$. For the $SU(N)$ (2,0) theory one has $\text{c} = 4N^3 - 3N - 1$ [3,6] which is thus consistent with (2.3) (the -1 term is absent in the $U(N)$ case).

The term $\hat{F}(\beta)$ in (2.9) should be reproduced by the 1-loop 11d supergravity partition function on $\text{AdS}_{7,\beta} \times \tilde{S}^4$ (with periodic boundary conditions on fermions). The supergravity index $\hat{Z}_\infty(q)$ was found in [21] from the BPS KK spectrum of S^4 compactification of 11d supergravity [31], adding also an R-charge shift to the Hamiltonian (conjugate to the Euclidean "time" y) when defining the index. This R-charge shift corresponds effectively to computing a supersymmetric partition function on $\text{AdS}_{7,\beta} \times \tilde{S}^4$ with the $z \to z + iy$ shift in a S^4 angle as in (1.2). There is again an analogy with how the constant N^0 term in the a-coefficient of 6d conformal anomaly is found from the 1-loop 11d supergravity effective action on $\text{AdS}_7 \times S^4$ with S^6 boundary [5].

[10] The computation of the N^3 term in the free energy from the supergravity action in thermal $\text{AdS}_7 \times S^4$ has a priori no reason to match the coefficient in the index asymptotics, see a discussion in Appendix A. Reproducing the coefficient of this leading N^3 contribution attempted in [29,30] requires adding finite "counterterms" to the low-dimensional effective supergravity action that were claimed to be needed to preserve supersymmetry. Let us also note that, in view of the relation between the supersymmetric Casimir energy and the c-coefficient of the conformal anomaly [27], one may expect that to match the former on the supergravity side one may need a more subtle procedure than just directly evaluating the supergravity action on the $\text{AdS}_7 \times S^4$ background: to capture the c-anomaly one needs to perturb the AdS_7 boundary metric to have a non-zero 6d Weyl tensor [3].

[11] Note that while in the case of AdS_7 with S^6 boundary the value of 11d effective action is proportional to $\text{vol}(\text{AdS}_7) = \frac{\pi^3}{3}\log\varepsilon$ (where $\varepsilon \to 0$ is an IR cutoff) and thus computes the a-anomaly coefficient, in the case of the $S^5 \times S^1_\beta$ boundary we have $\text{vol}(\text{AdS}_{7,\beta}) = -\frac{5\pi^4}{48}\beta$ (see Appendix A) and thus the local $\int(R + R^4)$ part of the 11d effective action evaluated on $\text{AdS}_{7,\beta} \times S^4$ is finite and linear in β.

The large N expansion (2.7) of the superconformal index of the (2,0) theory was interpreted in [32,33] as representing the 11d supergravity index $\hat{Z}_\infty(q)$ corrected by the contributions of other BPS states corresponding to wrapped M2 branes (that here play the role of "giant gravitons", cf. [34]).

Below we will prove that the leading $-[4\sinh^2\frac{\beta}{2}]^{-1}e^{-N\beta}$ term in (2.7) or in (2.10) originates precisely from the partition function of M2 brane wrapped on $S^1_\beta\times S^2$, in full analogy with how that happened [19] for the instanton M2 brane in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ background in the ABJM case.

By analogy with the familiar $\mathcal{N}=4$ SYM case [35], it is possible to insert into the matrix model integral found in [21,22] a counterpart of the Wilson loop operator $W(X)=\mathrm{Tr}\, e^X$ (where X is the matrix which is the integration variable). One may interpret $\langle W\rangle$ as the expectation value of a suitable [21,22] supersymmetric Wilson loop in the SYM theory on S^5 (cf. [24])[12] or rather of a corresponding 2-defect operator in the (2,0) theory on $S^5\times S^1_\beta$ that wraps S^1 of S^5 as well S^1_β. The resulting matrix model expectation value is [21,22] (using the original Wilson loop computation in $U(N)$ Chern-Simons matrix model [36])

$$\langle W\rangle = e^{\frac{N\beta}{2}}\,\frac{\sinh\frac{N\beta}{2}}{\sinh\frac{\beta}{2}} = \frac{1}{2\sinh\frac{\beta}{2}}e^{N\beta} - \frac{1}{2\sinh\frac{\beta}{2}}\ . \tag{2.11}$$

On the M-theory side this expression is expected to be reproduced by the M2 brane semiclassical contributions of the two saddle points: of $\mathrm{AdS}_{3,\beta}$ corresponding to M2 ending on S^1 of the S^5 boundary of $\mathrm{AdS}_{7,\beta}$ part of (1.2) (having non-zero classical action) and of a degenerate M2 brane wrapping only S^1_β (with zero action). As we will show below, the fluctuation determinants near the first saddle point reproduce precisely the prefactor $[2\sinh\frac{\beta}{2}]^{-1}$ in (2.11), which is again in full analogy with a similar computation in the $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ case in [18].

3. Semiclassical expansion of M2 brane path integral

Our aim will be to consider a semiclassical expansion of the Euclidean M2 brane path integral near particular classical solutions in the "twisted" version (1.2), (1.3) of the $\mathrm{AdS}_7\times S^4$ background. While the M2 brane action [37] is highly non-linear, when expanded near a classical solution with a non-degenerate induced 3d metric it can be straightforwardly quantized in a static gauge. Then the leading 1-loop result for its partition function is well defined (has no UV logarithmic divergences) [38–40,9,18,19].

The bosonic part of the M2 brane action may be written as

$$S = S_{\rm V} + S_{\rm WZ},\qquad S_{\rm V} = T_2\int d^3\xi\,\sqrt{g},\qquad g_{ab}=\partial_a X^M\partial_b X^N\,G_{MN}(X), \tag{3.1}$$

$$S_{\rm WZ} = -i\,T_2\int d^3\xi\,\frac{1}{3!}\varepsilon^{abc}C_{MNK}(X)\,\partial_a X^M\partial_b X^N\partial_c X^K,\qquad T_2=\frac{1}{(2\pi)^2\ell_P^3}\ . \tag{3.2}$$

Here $S_{\rm V}$ is the induced volume (or Dirac-Nambu-Goto) term, while $S_{\rm WZ}$ represents the coupling to the C_3 potential of 11d supergravity. The explicit form of the fermionic part of the M2 brane action is also known, in particular, for the cases of the maximally supersymmetric $\mathrm{AdS}_4\times S^7$ or $\mathrm{AdS}_7\times S^4$ backgrounds [41,42]. It can also be found for the $\mathrm{AdS}_{7,\beta}\times\tilde{S}^4$ background (1.2), (1.3) related to $\mathrm{AdS}_7\times S^4$ by an "orbifolding" and coordinate redefinition. The 1-loop computation discussed below will require only the knowledge of the quadratic fermionic term in the M2 brane action expanded near a bosonic background $X^M(\xi)$ [41,39,43,17,44]

$$S_F = iT_2\int d^3\xi\Big[\sqrt{g}\,g^{ab}\partial_a X^M\,\bar\theta\,\Gamma_M\,\hat D_b\theta - \frac{1}{2}\varepsilon^{abc}\partial_a X^M\partial_b X^N\,\bar\theta\,\Gamma_{MN}\,\hat D_c\theta + \ldots\Big]\ , \tag{3.3}$$

$$g_{ab}=\partial_a X^M\partial_b X^N G_{MN}(X),\qquad G_{MN}=E^A_M E^A_N,\qquad \Gamma_M = E^A_M(X)\Gamma_A\ , \tag{3.4}$$

$$\hat D_a = \partial_a X^M\,\hat D_M,\qquad \hat D_M = \mathrm{D}_M - \tfrac{1}{288}(\Gamma^{PNKL}{}_M + 8\Gamma^{PNK}\delta^L_M)F_{PNKL}\ , \tag{3.5}$$

where $\hat D_M$ is the generalized 11d spinor covariant derivative [45] and $\mathrm{D}_M=\partial_M+\frac{1}{4}\Gamma_{AB}\omega^{AB}_M$.[13]

The action (3.1) computed on the twisted $\mathrm{AdS}_7\times S^4$ background (1.2), (1.3) depends on the effective dimensionless M2 brane tension

$$\mathrm{T}_2 = a^3 T_2 = \frac{2}{\pi}N\ . \tag{3.6}$$

Thus the semiclassical large tension expansion of the M2 brane partition function should correspond to the large N expansion on the dual field theory side.

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[12] As discussed in [22], representing S^5 as a Hopf fibration over $\mathbb{CP}^2$ suggests the following field theory analog of this operator: $W=\mathrm{Tr}\Big[\mathcal{P}\exp\oint ds\,\big(iA_m\dot x^m+\phi|\dot x|\big)\Big]$, where $x^m(s)$ wraps the Hopf fiber.

[13] In the static gauge $X^a=\xi^a,\ X^I=0\ (I=1,\ldots,8)$ the natural κ-symmetry gauge is like in flat space [41,39]: $(1+\Gamma)\theta=0,\ \ \Gamma=\frac{1}{6\sqrt{g}}\varepsilon^{abc}\partial_a X^M\partial_b X^N\partial_c X^K\Gamma_{MNK}$ or alternatively $(1+\Gamma^{1\ldots 8})\theta=0$.

In general, for an M2 solution with a non-vanishing classical action $S_{\rm cl} = {\rm T}_2 \bar S_{\rm cl}$ (where $\bar S_{\rm cl}$ represents the total value of the sum of the volume and the WZ term in (3.1), (3.2)) the M2 brane partition function Z expanded near this background will contain a factor $e^{-S_{\rm cl}} = e^{-{\rm T}_2 \bar S_{\rm cl}} = e^{-pN}$ where p may depend on the parameter β of the background (1.2), (1.3).

Given an M2 brane classical solution $X^M = X^M(\xi)$ $(M = 1, \dots, 11)$ we may choose the static gauge identifying three of the X^M coordinates with the M2 world-volume coordinates ξ^a $(a = 1, 2, 3)$ and also fix a κ-symmetry gauge for the fermions. Then the remaining 8 bosonic and 8 fermionic fluctuations will produce a β-dependent 1-loop prefactor in the M2 brane partition function Z

$$ {\rm Z} = \int [dX\, d\theta]\, e^{-S[X,\theta]} = \mathcal{Z}_1\, e^{-{\rm T}_2 \bar S_{\rm cl}} \big[1 + \mathcal{O}({\rm T}_2^{-1})\big] , \qquad S_{\rm cl} = {\rm T}_2\, \bar S_{\rm cl} , \tag{3.7} $$

$$ \mathcal{Z}_1 = e^{-\Gamma_1} , \qquad \Gamma_1 = \frac{1}{2} \sum_k \nu_k \log\det \Delta_k , \tag{3.8} $$

where Δ_k are 2nd-derivative fluctuation operators and $\nu_k = \pm 1$ for the bosons and fermions.

Below we will consider the M2 branes wrapped on S^1_β. The leading non-perturbative $e^{-N\beta}$ term in the free energy (2.10) will be reproduced by the solution that wraps also the S^2 in the $\tilde S^4$ part of the metric in (1.2). We will also consider the solution that wraps an ${\rm AdS}_{3,\beta}$ part of ${\rm AdS}_{7,\beta}$ in (1.2) (ending on the big circle of S^5) that will reproduce the leading $e^{N\beta}$ term in the Wilson loop expectation value in (2.11).

4. $S^1_\beta \times S^2$ M2 solution: matching non-perturbative free energy

Let us consider the classical M2 brane solution that is wrapped on S^1_β in ${\rm AdS}_{7,\beta}$ and S^2 in the $\tilde S^4$ part of the metric (1.2). It is an analog of the instanton M2 brane in ${\rm AdS}_4 \times S^7/\mathbb{Z}_k$ discussed in [19]. Explicitly, we may choose the coordinate y in (1.2) to be ξ^3 (assuming now that $\xi^3 \in (0, \beta)$) and the coordinates of the unit-radius S^2 to be ξ^1 and ξ^2, with the rest of the coordinates in (1.2) being trivial, i.e. $x = 0$, $u = 0$, etc.[14]

The corresponding value of the classical M2 brane action in (3.1) (cf. (3.6)) is then[15]

$$ S_{\rm V,cl} = T_2\, a^3 {\rm r}^2\, {\rm vol}(S^1_\beta \times S^2) = \frac{1}{4} {\rm T}_2\, \beta\, 4\pi = 2N\beta , \qquad {\rm r} \equiv \frac{1}{2} , \tag{4.1} $$

$$ S_{\rm WZ,cl} = -i\, T_2 \int C_3 = -\frac{1}{8} T_2 a^3 \int dy \wedge {\rm vol}_{S^2} = -\frac{1}{8} {\rm T}_2\, \beta\, 4\pi = -N\beta , \tag{4.2} $$

$$ S_{\rm cl} = S_{\rm V,cl} + S_{\rm WZ,cl} = N\beta . \tag{4.3} $$

Thus $e^{-S_{\rm cl}}$ matches the exponential factor in the leading term in the non-perturbative part of free energy (2.10).[16]

4.1. *Quadratic fluctuation Lagrangian*

To discuss fluctuations near this classical solution we will choose a natural static gauge, i.e. set the fluctuations of y and S^2 coordinates to be zero. Let us first discuss fluctuations in the ${\rm AdS}_{7,\beta}$ directions of (1.2) parametrizing its metric as

$$ ds^2_{{\rm AdS}_{7,\beta}} = \frac{(1+\frac{1}{4}\chi^2)^2}{(1-\frac{1}{4}\chi^2)^2}\, dy^2 + \frac{d\chi^p d\chi^p}{(1-\frac{1}{4}\chi^2)^2}, \qquad \chi^p = (\chi_1, \dots, \chi_6), \qquad y \equiv y + \beta. \tag{4.4} $$

In the static gauge $y = \xi_3$ the 6 fluctuations χ_p are thus functions of $\xi^3 \equiv \xi^3 + \beta$ and the unit 2-sphere coordinates. As C_3 in (1.3) does not involve ${\rm AdS}_{7,\beta}$ coordinates, we need to consider the quadratic fluctuation term of the S_V part of the M2 brane action (3.1) only. Let us introduce the notation g_{ij} $(i, j = 1, 2)$ for the unit-radius S^2 metric so that the 3d induced metric may be written as

$$ ds^2 = g_{ab} d\xi^a d\xi^b = g_{ij}(\xi) d\xi^i d\xi^j + d\xi^3 d\xi^3 , \qquad g_{ij}(\xi) d\xi^i d\xi^j = d\xi_1^2 + \sin^2\xi_1\, d\xi_2^2 . \tag{4.5} $$

Then expanding to quadratic order in χ_p we get

$$ S_{\rm V} = {\rm T}_2 {\rm r}^2 \int d^3\xi \sqrt{g}\, \big(1 + \mathcal{L}_{2,\rm V} + \cdots\big), \tag{4.6} $$

$$ \mathcal{L}_{2,\rm V}(\chi) = \frac{1}{2{\rm r}^2} \Big[g^{ij}\, \partial_i \chi^p \partial_j \chi^p + {\rm r}^2 \chi^p \chi^p + {\rm r}^2 (\partial_3 \chi^p)^2 \Big] . \tag{4.7} $$

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[14] Keeping a general constant value of the coordinate u and computing the classical action one can check that $u = 0$ is an extremum. Note also that the shift of z by iy in (1.2) is irrelevant at the classical level at the $u = 0$ point.

[15] Here we introduced for convenience the notation r for the relative factor $\frac{1}{2}$ between the radii of ${\rm AdS}_7$ and S^4 metrics in (1.1) and (1.2).

[16] A similar computation in the type IIA string limit (i.e. $\beta \to 0$ with $\xi = \beta N =$ fixed) was done in [20]. Wrapping M2 on 2-sphere n times we get a "multi-instanton" contribution $S_{\rm cl} = nN\beta$ and thus may match the subleading $e^{-nN\beta}$ terms in the free energy $F^{\rm np}$ in (2.10). Note that if we consider an "anti-instanton" solution with reversed orientation of the $S^2 \to S^2$ map the contribution of the C_3 term (3.2) in the action will then have the opposite sign and thus we will get $S_{\rm cl} = 2N\beta + N\beta = 3N\beta$. This "anti-instanton" solution should not be supersymmetric and thus presumably should not be contributing to the free energy (we thank the authors of [20] for this suggestion).

The overall factor $\frac{1}{\mathrm{r}^2}$ here can be rescaled away by redefining χ^p. Expanding χ_p in Fourier modes in the periodic ξ^3 coordinate we get an equivalent 2d theory on S^2 for a tower of 6 scalar fields χ^p_n with masses ($\partial_3 \to i\frac{2\pi}{\beta}n$)

$$M^2_{\chi,n} = \mathrm{r}^2(1+n^2_\beta) = \frac{1}{4}+\frac{1}{4}n^2_\beta\,, \qquad n_\beta \equiv \frac{2\pi n}{\beta}\,, \qquad n=0,\pm1,\pm2,\dots\,. \tag{4.8}$$

The remaining 2 fluctuations in $\tilde{S}^4$ directions of (1.2) correspond to u and z coordinates which represent a 2-sphere subspace $du^2+\sin^2 u\, dz^2$. Using the Cartesian parametrization for this 2-sphere[17]

$$du^2+\sin^2 u\, dz^2 = \frac{dA^2+dB^2}{[1+\frac{1}{4}(A^2+B^2)]^2}\,, \tag{4.9}$$

we may use A and B as the two fluctuation fields. Rescaling them by $(\mathrm{T}_2)^{1/2}$ (cf. (4.6)) we then get the following counterpart of (4.7) coming from the volume part of the M2 brane action in (3.1)

$$\begin{aligned}\mathscr{L}_{2,\mathrm{V}}(A,B) =& \frac{1}{2}\Big[g^{ij}(\partial_i A\partial_j A+\partial_i B\partial_j B)\\ &+\mathrm{r}^2(\partial_3 A)^2+\mathrm{r}^2(\partial_3 B)^2-(\mathrm{r}^2+2)(A^2+B^2)+4i\,\mathrm{r}^2 A\partial_3 B\Big]\,.\end{aligned} \tag{4.10}$$

Here the mixing term $A\partial_3 B$ is due to the presence of $dy=d\xi^3$ in the $(dz+i\,dy)^2$ term in (1.2) (cf. [17]).

For the contribution of the WZ term in (3.2) with C_3 in (1.3) one finds using that $\partial_3 y=1$ (cf. (4.2))

$$S_{\mathrm{WZ}} = -iT_2\int C_3 = \frac{i}{8}\mathrm{T}_2\int \cos^3 u\,(\partial_3 z+i)\,d\xi_3\wedge \mathrm{vol}_{S^2} = -\frac{1}{8}\mathrm{T}_2\int d^3\xi\,\sqrt{g}\,\cos^3 u\,(1-i\partial_3 z). \tag{4.11}$$

Expanding to quadratic order in the fluctuations A, B we get the following addition to (4.10)

$$\mathscr{L}_{2,\mathrm{WZ}}(A,B) = \frac{3}{16\mathrm{r}^2}(A^2+B^2)-\frac{3}{8\mathrm{r}^2}i\,A\partial_3 B\,. \tag{4.12}$$

Summing up (4.10) and (4.12) gives (setting $\mathrm{r}=\frac{1}{2}$ and ignoring a total derivative)

$$\mathscr{L}_2(A,B) = \frac{1}{2}g^{ij}(\partial_i A\partial_j A+\partial_i B\partial_j B)+\mathscr{L}_{2,M}(A,B)\,, \tag{4.13}$$

$$\mathscr{L}_{2,M}(A,B) = -\frac{3}{8}(A^2+B^2)+\frac{1}{8}\left[(\partial_3 A)^2+(\partial_3 B)^2\right]-i\,A\partial_3 B\,. \tag{4.14}$$

Setting $\phi=\frac{A+iB}{\sqrt{2}}$, $\bar\phi=\frac{A-iB}{\sqrt{2}}$ we get

$$\mathscr{L}_{2,M}(\phi) = \frac{1}{2}\begin{pmatrix}\phi & \bar\phi\end{pmatrix}\begin{pmatrix}0 & -\frac{3}{4}+\partial_3-\frac{1}{4}\partial_3^2\\ -\frac{3}{4}-\partial_3-\frac{1}{4}\partial_3^2 & 0\end{pmatrix}\begin{pmatrix}\phi\\ \bar\phi\end{pmatrix}\,. \tag{4.15}$$

Expanding $\phi(\xi)$ in Fourier modes in ξ^3 we get an effective 2d Lagrangian for a tower of complex scalars on S^2 (cf. (4.8))

$$\mathscr{L}_2(\phi) = \sum_{n=-\infty}^{\infty}\left(g^{ij}\partial_i\phi_n\partial_j\bar\phi_n+M^2_{\phi,n}\phi_n\bar\phi_n\right)\,, \tag{4.16}$$

$$M^2_{\phi,n} = -\frac{3}{4}+in_\beta+\frac{1}{4}n^2_\beta = 1+\frac{1}{4}(n_\beta+2i)^2\,. \tag{4.17}$$

In the limit $\beta\to 0$ the current problem should reduce to the type IIA string computation considered in [20]: the string spectrum should be the $n=0$ level of the M2 brane spectrum. Indeed, the $n=0$ values of the masses of the 6 fluctuations in (4.8) and 2 fluctuations in (4.17) agree with the bosonic string fluctuation masses in Table 1 of [20].

The fermionic part of the M2 brane action directly corresponds (upon double dimensional reduction as in [46]) to the fermionic part of type IIA superstring action. In the superstring limit one finds [20] that the quadratic part of the GS action is equivalent to 8 fermions in S^2 geometry with the square of the Dirac operator containing the mass term with $M^2=-\frac{1}{4}$. Explicitly, the 2d Dirac operator is given by (cf. [20,19]): $\mathcal{D}=i\sigma^k \mathrm{D}_k+M\sigma_3$ where σ_a are the three Pauli matrices with the σ_3 term originating from the terms with Γ_{11} factors in the membrane action (3.3). Its square is $\Delta_{\frac{1}{2}}=-\mathrm{D}^2+\frac{1}{4}R^{(2)}+M^2$, where $R^{(2)}=2$ is the curvature of the 2-sphere. In the type IIA string limit [20] one gets $M=-\frac{1}{2}i$.

Starting directly with the M2 brane action (3.3), in the present case with $y=x^{11}=\xi^3$ there are two different $M\sigma_3$ contributions to the fermionic $\mathcal{D}$ operator. One is coming from the non-zero y-component of F_4 field strength corresponding to C_3 in (1.3) that

[17] Explicitly, $u=\arccos\frac{[1-\frac{1}{4}(A^2+B^2)]^2}{[1+\frac{1}{4}(A^2+B^2)]^2}$, $z=\arctan\frac{B}{A}$.

gets contracted with Γ^y or Γ^{11} in (3.5). This corresponds upon double dimensional reduction to a similar term in the type IIA string action leading precisely to the above $-\frac{1}{2}i$ contribution to M.

In addition, there is a contribution of the $\Gamma_{11}\partial_3$ term in the covariant derivative in (3.3), (3.5) (cf. also Eq. (4.29) in [19]).[18] This gives an extra $-\frac{1}{2}n_\beta$ contribution to the fermion mass M so that in total $M=-\frac{1}{2}n_\beta-\frac{1}{2}i$.

As a result, we get 8 towers of 2d fermions on S^2 with

$$M^2_{\theta,n}=\frac{1}{4}(n_\beta+i)^2=-\frac{1}{4}+\frac{i}{2}n_\beta+\frac{1}{4}n^2_\beta\ . \tag{4.18}$$

Combined with the 6+2 towers of bosons in (4.8) and (4.17) this represents the complete M2 brane fluctuation spectrum.

4.2. One-loop M2 brane partition function

The expressions for the determinants of the standard bosonic and fermionic massive field operators on S^2 in the 1-loop contribution in (3.8) are well known. In general, using spectral zeta-function regularization one has $\log\det\Delta=-\zeta_\Delta(0)\log\Lambda^2-\zeta'_\Delta(0)$. Like in [18,19] the total coefficient $\zeta_\Delta(0)$ of the log UV divergence vanishes if we use the Riemann zeta-function regularization of the sum over the modes (that removes power divergences)

$$\zeta_{\rm tot}(0)=\sum_{n\in\mathbb{Z}}2=2+4\zeta_R(0)=0\ . \tag{4.19}$$

Here the coefficient 2 is related to the value of the Euler number of S^2 (cf. [47]). The finite $-\zeta'_\Delta(0)$ parts of $\log\det\Delta$ for the bosonic ($\Delta_0=-D^2+M^2$) and fermionic ($\Delta_{\frac{1}{2}}=-{\rm D}^2+\frac{1}{2}+M^2$) fields on S^2 are given by (we follow the notation in [20,19])

$$\log\det\Delta_0=s_{\frac{1}{2}}(\tfrac{1}{4}-M^2)\ ,\qquad \log\det\Delta_{\frac{1}{2}}=s_0(-M^2)\ , \tag{4.20}$$

$$s_p(\mu)\equiv-4\zeta'(-1,p)+\int_0^\mu dx\,\Big[\psi(p+\sqrt{x})+\psi(p-\sqrt{x})\Big]\ , \tag{4.21}$$

where $\zeta'(x,a)$ is the derivative of the Hurwitz ζ-function over x and ψ is the logarithmic derivative of the Γ-function.

As a result, combining together the contributions of the above (6+2) bosonic and 8 fermionic determinants and summing over n we find

$$\Gamma_1=\sum_{n\in\mathbb{Z}}U\Big(\frac{2\pi n}{\beta}\Big),\qquad U(v)=3s_{\frac{1}{2}}\big(-\tfrac{v^2}{4}\big)+s_{\frac{1}{2}}\big((1-\tfrac{iv}{2})^2\big)-4s_0\big((\tfrac{1}{2}-\tfrac{iv}{2})^2\big)\ . \tag{4.22}$$

Using (4.21) we observe that[19]

$$U(0)=i\pi-2\log 2\ ,\qquad U(v)+U(-v)=-4\log 2+2\log(1+v^2),\qquad v>0\ . \tag{4.23}$$

Thus, like in the case of the instanton M2 brane solution in $AdS_4\times S^7/\mathbb{Z}_k$ in [19], all non-trivial ψ-function dependent terms from (4.21) cancel out in the sum of the bosonic and fermionic contributions[20] and we end up with

$$\begin{aligned}\Gamma_1&=i\pi-2\log 2+\sum_{n=1}^{\infty}\Big[-4\log 2+2\log\Big(1+\frac{4\pi^2n^2}{\beta^2}\Big)\Big]\\&=i\pi-2\log 2\big(1+2\zeta_R(0)\big)+2\log\big(2\sinh\frac{\beta}{2}\big)=\log\big(-4\sinh^2\frac{\beta}{2}\big).\end{aligned} \tag{4.24}$$

As a result, the 1-loop factor in the M2-brane partition function (3.7) on this M2 instanton background is given by

$$\mathcal{Z}_1=e^{-\Gamma_1}=-\frac{1}{4\sinh^2\frac{\beta}{2}}\ . \tag{4.25}$$

Taking into account that, as discussed in [19], the field-theory free energy should be matched by *minus* the M2 brane partition function, we thus reproduce the prefactor in the leading non-perturbative term in the free energy in (2.10). This generalizes to finite β case the matching found in the string theory limit in [20].

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[18] To find the quadratic fermionic term in the M2 brane action what matters is the form of the classical bosonic $X^M(\xi)$ background that gives the induced 3-bein contracted with Γ_M. In the present case this is coming from the y-dependent terms in the metric (1.2). On the classical solution $u=0,\ x=0,\ y=\xi^1$, the only term that is relevant originates simply from the dy^2 term in (1.2).

[19] Recall that $\zeta'_R(-1)=\frac{1}{12}-\log{\rm A}$ and $\zeta'(-1,\frac{1}{2})=-\frac{1}{24}-\frac{1}{24}\log 2+\frac{1}{2}\log{\rm A}$ where A is Glaisher's constant.

[20] Note that these cancellations would not happen if we were to ignore the $z\to z+iy$ twist in the metric (1.2) which appears to be consistence with its need for preservation of supersymmetry.

5. $AdS_{3,\beta}$ M2 solution: matching Wilson loop expectation value

In analogy with the discussion in the $AdS_4 \times S^7/\mathbb{Z}_k$ case in [18], the leading term in the circular BPS Wilson loop expectation value in (2.11) is to be reproduced by the M2 brane partition function expanded near the solution that ends on a circle of the S^5 part of the boundary and is also wrapped on the 11d y-circle of $AdS_{7,\beta}$ while being point-like in $\tilde{S}^4$ part of in (1.2). This should generalize to the M-theory (finite β) level the related computation done in the type IIA string theory limit in [15].

Denoting by $\varphi \equiv \varphi + 2\pi$ the circular coordinate of S^5, the relevant $AdS_{3,\beta} \subset AdS_{7,\beta}$ part of the metric (1.2) and thus the induced metric for the classical M2 solution $x = \xi^1, \ \varphi = \xi^2, \ y = \xi^3 \equiv \xi^3 + \beta$ will be that of "thermal" $AdS_{3,\beta}$

$$ds^2_{AdS_{3,\beta}} = dx^2 + \sinh^2 x\, d\varphi^2 + \cosh^2 x\, dy^2 \ \to \ g_{ab}(\xi) d\xi^a d\xi^b = d\xi_1^2 + \sinh^2\xi_1\, d\xi_2^2 + \cosh^2\xi_1\, d\xi_3^2 \,. \tag{5.1}$$

The corresponding classical M2 brane action gets only the volume contribution (3.1), i.e.

$$S_{\rm V,cl} = T_2\, {\rm vol}(AdS_{3,\beta}) = -N\beta \,. \tag{5.2}$$

The computation of the regularized volume of $AdS_{2n+1,\beta}$ with boundary $S^{2n-1} \times S^1_\beta$ is reviewed in Appendix A. Explicitly,

$${\rm vol}(AdS_{3,\beta}) = \int_0^\beta dy \int_0^{2\pi} d\varphi \int_0^{x_0} dx\, \sinh x \cosh x = \beta\pi \sinh^2 x_0 = \frac{1}{4}\pi\beta\Big(\frac{1}{\epsilon^2} - 2 + \epsilon^2\Big) \to -\frac{1}{2}\pi\beta \,, \tag{5.3}$$

where we set $x_0 = -\log\epsilon$ as IR cutoff ($\epsilon \to 0$) and dropped power divergence. Using (3.6) we thus get the value in (5.2) which indeed matches the exponent of the first term in (2.11) (see also [48]). The second term in (2.11) may be expected to come from an M2 brane solution with vanishing 3-volume but this remains to be clarified.

5.1. Quadratic fluctuation Lagrangian

Choosing the static gauge in which the fluctuations of x, φ and y are set to zero one can check (see below) that since the classical solution is trivial in the $\tilde{S}^4$ directions, the only contribution to the quadratic fluctuation action comes from the volume part (3.1) of the M2 brane action.

The part of the quadratic fluctuation Lagrangian depending only on the $AdS_{7,\beta}$ coordinates in (1.2) is represented by the four S^5 directions that have trivial classical values. Parametrizing the S^5 metric as[21]

$$dS_5 = \frac{(1-\frac{1}{4}w^2)^2}{(1+\frac{1}{4}w^2)^2}\, d\varphi^2 + \frac{dw_r dw_r}{(1+\frac{1}{4}w^2)^2}, \qquad r = 1,\dots,4 \,, \tag{5.4}$$

and expanding in powers of w_r we get from (3.1) (we rescale away the overall factor of tension)

$$S_{2,\rm V}(w) = \frac{1}{2}\int d^3\xi \sqrt{g}\, g^{ab} \sinh^2\xi_1 \big(\partial_a w_r \partial_b w_r - \delta_{a2}\delta_{b2} w^2\big) \,. \tag{5.5}$$

Setting

$$w_r = \frac{1}{\sinh\xi_1}\tilde{w}_r \,, \tag{5.6}$$

and integrating by parts we get

$$S_{2,\rm V}(\tilde{w}) = \int d^3\xi\, \sqrt{g}\, \mathcal{L}_{2,\rm V}(\tilde{w}) \,, \qquad \mathcal{L}_{2,\rm V}(\tilde{w}) = \frac{1}{2}\big(g^{ab}\partial_a\tilde{w}_r\, \partial_b\tilde{w}_r + 3\tilde{w}_r\tilde{w}_r\big) \,. \tag{5.7}$$

To find the contribution of the other 4 bosonic fluctuations corresponding to $\tilde{S}^4$ directions in (1.2) we note that the leading part of the $\tilde{S}^4$ metric expanded near $u = 0$ is $\frac{1}{4}[du^2 + dS_2 + u^2(dz + i\, d\xi_3)^2]$. Using Cartesian coordinates (A, B) to parametrize the (u, z) plane and v_k $(k = 1, 2)$ for S^2, i.e.

$$A = u\cos z \,, \quad B = u \sin z \,, \qquad dS_2 = \frac{dv_k dv_k}{(1+\frac{1}{4}v^2)^2} \,, \tag{5.8}$$

we get the quadratic fluctuation Lagrangian (rescaling all 4 fields by factor of r $= \frac{1}{2}$; here $i, j = 1, 2$)[22]

$$\mathcal{L}_{2,\rm V}(v_k, A, B) = \frac{1}{2} g^{ab}\big(\partial_a v_k \partial_b v_k + \partial_a A \partial_b A + \partial_a B \partial_b B\big) - \frac{1}{2}\frac{1}{\cosh^2\xi_1}(A^2 + B^2) + \frac{2i}{\cosh^2\xi_1}\, A\partial_3 B$$

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[21] Same result for quadratic fluctuations is found if we use the Hopf fibration parametrization of the S^5 metric, i.e. $dS_5 = (d\varphi' + A)^2 + ds^2_{\mathbb{CP}^2}$ where A depends on $\mathbb{CP}^2$ coordinates.

[22] Explicitly, we use that $\int d^3\xi \sinh\xi_1 \cosh\xi_1 \frac{1}{\cosh^2\xi_1}[(\partial_3 A)^2 + (\partial_3 B)^2 - A^2 - B^2 + 2i(A\partial_3 B - B\partial_3 A)] = \int d^3\xi\, \sqrt{g}\, g^{33}\big[(\partial_3 A - iB)^2 + (\partial_3 B + iA)^2\big]$.

$$= \frac{1}{2}\Big(g^{ab}\partial_a v_k \partial_b v_k + g^{ij}(\partial_i A \partial_j A + \partial_i B \partial_j B) + g^{33}[(\partial_3 A - iB)^2 + (\partial_3 B + iA)^2]\Big). \tag{5.9}$$

Note that the (A, B) mixing term may be formally diagonalized by a ξ^3-dependent "rotation"

$$A = \cos\psi\, X + \sin\psi\, Y, \qquad B = -\sin\psi\, X + \cos\psi\, Y, \qquad \psi = i\xi^3\ , \tag{5.10}$$

$$(\partial_3 A - iB)^2 + (\partial_3 B + iA)^2 = (\partial_3 X)^2 + (\partial_3 Y)^2. \tag{5.11}$$

Since ξ^3 is periodic, this redefinition is only formal as it shifts the value of the S^1_β mode number (cf. (4.8)) as $n_\beta \to n_\beta + i$ and this should be taken into account.

Indeed, here we have a coupling of the complex scalar $A + iB$ to a constant 3d gauge potential with the component $\mathcal{A}_3 = -i$ in the S^1_β direction which can not be gauged away.[23] Its origin is related to the presence of the twist $z \to z + iy$ in the metric (1.2). This shift is similar to what we found in the $\tilde{S}^4$ part of the fluctuation Lagrangian (4.17) in the S^2 instanton case where $n_\beta + 2i$ rather than $n_\beta + i$ was due to the contribution of the WZ term.[24]

Finally, let us note that the C_3 coupling term (3.2) evaluated on the background (1.3) gives

$$S_{\rm WZ} = -iT_2 \int C_3 = \frac{i}{8}{\rm T}_2 \int \cos^3 u\,(dz + idy) \wedge \frac{dv_1 \wedge dv_2}{(1 + \frac{1}{4}(v_1^2 + v_2^2))^2}\ . \tag{5.12}$$

Since $y = \xi^3$, expanding to quadratic order in the fields projected on the world-volume this reduces to a total derivative term $\epsilon^{ij}\epsilon^{kl}\partial_i v_k \partial_j v_l$ and thus does not indeed contribute to the leading order.

As for the fermionic fluctuation Lagrangian, it can be found by a generalization of its string theory ($\beta \to 0$) limit discussed in [15]. We should get 8 fermions in ${\rm AdS}_{3,\beta}$ with $\mathcal{D} = i\sigma^k {\rm D}_k + M\sigma_3$ where $M = \frac{3}{2}$. The ∂_3 derivative term in ${\rm D}_k$ produces (upon Fourier expansion in ξ^3) a mode number n_β contribution as in (4.8), (4.18). Also, as in the case of the (A, B) fields in (5.9), here the covariant derivative contains (in addition to the standard ${\rm AdS}_{3,\beta}$ spin connection) a constant $U(1)$ potential term, reflecting again the presence of the twist in the metric (1.2), i.e. we have (cf. [17])

$${\rm D}_3 = \partial_3 - i\mathcal{A}_3 + \ldots, \qquad \mathcal{A}_3 = -\tfrac{1}{2} i\ . \tag{5.13}$$

5.2. One-loop M2 brane partition function

The fluctuation Lagrangian represents a collection of massive bosons and fermions propagating in ${\rm AdS}_{3,\beta}$, i.e. in "thermal" ${\rm AdS}_3$ with $S^1 \times S^1_\beta$ boundary. The expressions for the corresponding determinants are well-known from the literature (see, e.g., [49–51]).

For a scalar field with mass M one finds [50][25]

$$\Gamma^{(\Delta)}(\beta) \equiv \frac{1}{2}\log\det(-D^2 + M^2) = E_c(\Delta)\,\beta - \sum_{n=1}^{\infty} \frac{e^{-\beta n\Delta}}{n(1 - e^{-\beta n})^2}\ , \tag{5.14}$$

$$\Delta = 1 + \sqrt{1 + M^2}\ . \tag{5.15}$$

Here Δ is the conformal dimension of the "dual boundary field" and E_c is the Casimir energy

$$E_c(\Delta) = \frac{1}{2\Gamma(z)} \int_0^\infty d\beta\ \beta^{z-1}\ \frac{e^{-\beta\Delta}}{(1 - e^{-\beta})^2}\Big|_{z\to -1} = \frac{1}{24}(\Delta - 1)(1 - 4\Delta + 2\Delta^2)\ . \tag{5.16}$$

For $\beta \to \infty$ we have $\Gamma^{(\Delta)}(\beta) = E_c(\Delta)\,\beta + \mathcal{O}(e^{-\beta\Delta})$, while for $\beta \to 0$ one finds (see Appendix B)

$$\Gamma^{(\Delta)}(\beta) = -\frac{\zeta(3)}{\beta^2} + \frac{\pi^2(\Delta - 1)}{6\beta} - \mathscr{C}(\Delta) + \frac{1}{12}(5 - 12\Delta + 6\Delta^2)\log\beta + \frac{(1 - 20\Delta + 50\Delta^2 - 40\Delta^3 + 10\Delta^4)}{2880}\beta^2 + \mathcal{O}(\beta^4)\ , \tag{5.17}$$

$$\mathscr{C}(\Delta) = (\Delta - 1)\Big[\frac{1}{2}\log(2\pi) - \log\Gamma(\Delta)\Big] + \zeta'(-1, \Delta)\ . \tag{5.18}$$

We still need to address the following subtlety: the scalars A, B in (5.9) are not just massless scalars in ${\rm AdS}_{3,\beta}$ but are coupled also to a flat but topologically non-trivial $U(1)$ gauge potential in ξ^3 direction that leads to a shift $n'_\beta = n_\beta + i$ of the S^1_β mode number. To

[23] Equivalently, this is the $SO(2)$ gauge field coupled to $\Phi_k = (A, B)$ via $D_3\Phi_k = \partial_3\Phi_k + \varepsilon_{kl}\mathcal{A}_3\Phi_l$, cf. [17].

[24] Again, the origin of this shift can be traced to the structure of the metric in (1.2): in view of the definition of A, B in (5.8), redefining $z \to z + i\xi_3$ translates into the rotation (5.10).

[25] This expression was found in [50] (for the Casimir contribution see [52]) by applying the method of images to the heat kernel for the thermal quotient of ${\rm AdS}_3$. It is rederived in an alternative way in Appendix C below by using the explicit expansion in modes along the two boundary circles $S^1 \times S^1_\beta$, *cf.* (C.17).

account for the effect of such coupling on the scalar determinant we may use the path integral representation for the log det or heat kernel of the fluctuation operator in (5.9) defined on the complex scalar $A + iB$ in which the coupling to a background 3d gauge field $\mathcal{A}_a$ appears as a phase factor $\exp[i\int d\tau\, \mathcal{A}\cdot\dot{x}]$. For constant

$$\mathcal{A}_3 = -i\kappa \tag{5.19}$$

this gives a factor of $e^{m\kappa\beta}$ where m is the number of times the worldline $x(\tau)$ wraps around the thermal circle (in (5.9) we have $\kappa = 1$). This implies the following modification of (5.14)

$$\Gamma^{(\Delta,\kappa)}(\beta) = \frac{1}{2}\Big[\Gamma^{(\Delta+\kappa)}(\beta) + \Gamma^{(\Delta-\kappa)}(\beta)\Big] = E_c(\Delta,\kappa)\beta - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-\beta n\Delta}(e^{-\beta\kappa}+e^{\beta\kappa})}{(1-e^{-\beta n})^2}\ , \tag{5.20}$$

$$E_c(\Delta,\kappa) = \frac{1}{2}[E_c(\Delta+\kappa)+E_c(\Delta-\kappa)] = \frac{1}{24}(\Delta-1)(1-4\Delta+2\Delta^2+6\kappa^2)\ . \tag{5.21}$$

This is derived directly using the $S^1\times S^1_\beta$ mode expansion in Appendix C, see (C.25).

The determinant of the squared massive Dirac operator in $AdS_{3,\beta}$, i.e. $\Delta_{\frac{1}{2}} = -D^2 + \frac{1}{4}R^{(3)} + M^2$, where $R^{(3)} = -6$, is given by the same expression as in (5.14) but instead of the relation between Δ and M in the scalar case in (5.15) here one has (see, e.g., [53])

$$\Delta = 1 + |M|\ . \tag{5.22}$$

Eq. (5.22) is the $d = 3$ case of the standard AdS_d/CFT_{d-1} relation for the fermions $\Delta = \frac{d-1}{2} + |M|$ (see, e.g., [54]).[26] The generalization to the case of the presence of a constant gauge potential $\mathcal{A}_3 = -i\kappa$ is straightforward as this coupling is via the D_3 term in the covariant derivative and thus the same as in the scalar case. It is given again by (5.20).

We are now ready to compute the total contribution to the 1-loop effective action (3.8) in the present case. According to the discussion in the previous subsection we have 4 scalars with $M^2 = 3$ in (5.7), 2 massless scalars v_k in (5.9), two scalars (A, B) in (5.9), (5.11) with $M^2 = 0$ coupled to a constant potential (5.19) with $\kappa = 1$ and 8 fermions with $M = \frac{3}{2}$ coupled to (5.19) with $\kappa = \frac{1}{2}$ (see (5.13)).[27]

Thus we get from (5.14), (5.20)

$$\begin{aligned}\Gamma_1 &= 4\,\Gamma^{(3,0)}(\beta) + 2\,\Gamma^{(2,0)}(\beta) + 2\,\Gamma^{(2,1)}(\beta) - 8\,\Gamma^{(\frac{5}{2},\frac{1}{2})}(\beta)\\ &= \frac{\beta}{2} - \sum_{n=1}^{\infty}\frac{e^{-\beta n}}{n} = \frac{\beta}{2} + \log(1-e^{-\beta}) = \log\Big(2\sinh\frac{\beta}{2}\Big)\ .\end{aligned} \tag{5.23}$$

Like in other similar cases of supersymmetric M2 brane 1-loop effective actions we observe remarkable cancellations of all "complicated" contributions that happen in the sum over all fields.[28]

The final result for the "defect" M2 brane 1-loop partition function is very simple

$$\mathcal{Z}_1 = \frac{1}{2\sinh\frac{\beta}{2}}\ , \tag{5.24}$$

and thus matches the prefactor in the leading term in the Wilson loop expectation value in (2.11).

6. Summary and concluding remarks

Let us summarize what we have found above. We considered the semiclassical expansion of the M2 brane partition function Z (3.7) in the 11d background $AdS_{7,\beta}\times\tilde{S}^4$ (1.2), (1.3) which is an S^1_β-compactified and "twisted" version of the maximally supersymmetric $AdS_7\times S^4$ limit (1.1) of the multiple M5 brane solution of 11d supergravity. The main dimensionless parameters are β (the ratio of the length of 11-circle to the scale a of AdS_7 in (1.1), (1.2)) and the effective M2 brane tension T_2 (or N)

$$\mathrm{T}_2 = a^3 T_2 = \frac{2}{\pi}N, \qquad T_2 = \frac{1}{(2\pi)^2\ell_P^3}, \qquad a = 2(\pi N)^{1/3}\ell_P\ . \tag{6.1}$$

[26] In general, for a spin s field in AdS_3 with the operator $-D_s^2 + \mu^2$ one has $\Delta = 1 + \sqrt{\mu^2 + s + 1}$. Thus for $s = \frac{1}{2}$ we get $\Delta = 1 + \sqrt{\mu^2 + \frac{3}{2}}$. Since here $\mu^2 = \frac{1}{4}R^{(3)} + M^2 = -\frac{3}{2} + M^2$ we get $\Delta = 1 + |M|$.

[27] Note that the corresponding values of Δ with multiplicities 4, 4 and 8 are 3, 2 and $\frac{5}{2}$. This hints at an effective 3d supersymmetry, but its realization for the above system of 8+8 scalars and fermions on $AdS_{3,\beta}$ should be non-trivial as it appears to require the presence of the flat connection in scalar and fermion covariant derivatives originating from the twist in $\tilde{S}^4$.

[28] One may draw an analogy of these cancellations with what happens in the case of supersymmetric partition functions on $S^1\times S^d$ that are equivalent to superconformal indices and thus effectively receive contributions only from BPS states. Indeed, the prefactor $\frac{1}{4\sinh^2\frac{\beta}{2}} = \frac{q}{(1-q)^2}$ of the M2 brane instanton contribution $e^{-N\beta}$ in (1.5), (2.7) that we reproduced as the M2 brane partition function in (4.25) may be also interpreted [32] as the superconformal index of $k = 1$ abelian ABJM theory [55] or as a supersymmetric partition function of a single $\mathcal{N} = 8$ 3d scalar supermultiplet in $S^1_\beta\times S^2$ background with extra twist on S^2 required for supersymmetry (i.e. corresponding to the presence of rotation generator in the definition of the 3d superconformal index). Similar relation may somehow apply also to the WL computation in this section.

Our first example was the “instanton” M2 brane solution that is wrapped on S^1_β of $\mathrm{AdS}_{7,\beta}$ and S^2 of $\tilde S^4$. We found that in this case (see (4.1)-(4.3), (4.25))

$$S^1_\beta\times S^2\ :\qquad \mathrm{Z}=-\frac{1}{\big(2\sinh\frac{\beta}{2}\big)^2}\,e^{-\mathrm{T}_2\bar S_{\rm cl}}\big[1+\mathcal{O}(\mathrm{T}_2^{-1})\big]\ ,\qquad \bar S_{\rm cl}=(1-\tfrac12)\pi\beta=\tfrac12\pi\beta\ . \tag{6.2}$$

We also studied the “defect” M2 brane solution wrapped on the “thermal” $\mathrm{AdS}_{3,\beta}$ part of $\mathrm{AdS}_{7,\beta}$ that corresponds to “open” M2 brane ending on the $S^1\times S^1_\beta$ at the boundary of $\mathrm{AdS}_{7,\beta}$ (thus representing a Wilson-surface like “defect” in the (2,0) theory that generalizes the circular BPS Wilson loop in gauge theory).[29] In this case (see (5.2), (5.24))

$$\mathrm{AdS}_{3,\beta}\ :\qquad \mathrm{Z}=\frac{1}{2\sinh\frac{\beta}{2}}\,e^{-\mathrm{T}_2\bar S_{\rm cl}}\big[1+\mathcal{O}(\mathrm{T}_2^{-1})\big]\ ,\qquad \bar S_{\rm cl}=-\tfrac12\pi\beta\ . \tag{6.3}$$

It is useful to compare these results with what was found in [18,19] for similar M2 brane solutions in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ M-theory background dual to $U_k(N)\times U_{-k}(N)$ 3d Chern-Simons-matter ABJM theory [10]. This 11d background is the supersymmetric $\mathbb{Z}_k$ orbifold of the $\mathrm{AdS}_4\times S^7$ which is a limit of the multiple M2 brane solution of 11d supergravity (cf. (1.1), (6.1)):

$$ds^2_{11}=R^2\big(\tfrac14 ds^2_{\mathrm{AdS}_4}+ds^2_{S^7/\mathbb{Z}_k}\big), \tag{6.4}$$

$$ds^2_{S^7/\mathbb{Z}_k}=ds^2_{\mathbb{CP}^3}+(d\mathrm{y}+A)^2\ ,\qquad \mathrm{y}\equiv\mathrm{y}+\mathrm{b}\ ,\qquad \mathrm{b}\equiv\frac{2\pi}{k}\ , \tag{6.5}$$

$$F_4=-\tfrac38 iR^3\,\mathrm{vol}_{\mathrm{AdS}_4}\ ,\qquad R=\big(32\pi^2Nk\big)^{1/6}\ell_P\ ,\qquad \mathrm{T}_2=R^3T_2=\frac{\sqrt{2k}}{\pi}\sqrt{N}\ . \tag{6.6}$$

We are assuming the Euclidean signature and A depends on the 6 coordinates of $\mathbb{CP}^3$. Here the dimensionless parameters are k and N, or b and the effective tension T_2.

The M2 brane “instanton” solution considered in [19] is the 11d uplift of the IIA string $\mathbb{CP}^1$ instanton of [56]: it is wrapped on the 11d circle y of dimensionless length $\mathrm{b}=\frac{2\pi}{k}$ and on $\mathbb{CP}^1\subset\mathbb{CP}^3$, so that it has the $S^3/\mathbb{Z}_k$ world-volume metric. In this case one finds for the M2 brane partition function [19][30]

$$S^3/\mathbb{Z}_k\ :\qquad \mathrm{Z}=\frac{1}{\big(2\sin\mathrm{b}\big)^2}\,e^{-\mathrm{T}_2\bar S_{\rm cl}}\big[1+\mathcal{O}(\mathrm{T}_2^{-1})\big]\ ,\qquad \bar S_{\rm cl}=\mathrm{vol}(S^3/\mathbb{Z}_k)=\pi\mathrm{b}=\tfrac{2\pi^2}{k}\ . \tag{6.7}$$

This corresponds to the leading $e^{-2\pi\sqrt{\frac{2N}{k}}}$ term in the large N non-perturbative part of the localization result [57] for the free energy of the ABJM theory on S^3.

Another M2 brane solution in (6.4) considered in [58,18] has world-volume of $\mathrm{AdS}_2\times S^1/\mathbb{Z}_k$ where $S^1/\mathbb{Z}_k$ corresponds to the y-circle in (6.4) and $\mathrm{AdS}_2\subset\mathrm{AdS}_4$ has the S^1 boundary. It may be interpreted as a dual of the circular BPS Wilson loop in the ABJM theory. In this case [18]

$$\mathrm{AdS}_2\times S^1/\mathbb{Z}_k\ :\quad \mathrm{Z}=\frac{1}{2\sin\mathrm{b}}\,e^{-\mathrm{T}_2\bar S_{\rm cl}}\big[1+\mathcal{O}(\mathrm{T}_2^{-1})\big],\quad \bar S_{\rm cl}=\tfrac14\mathrm{vol}(\mathrm{AdS}_2)\mathrm{b}=-\tfrac12\pi\mathrm{b}=-\tfrac{\pi^2}{k}\ . \tag{6.8}$$

This matches the leading large N term $[2\sin\frac{2\pi}{k}]^{-1}e^{\pi\sqrt{\frac{2N}{k}}}$ in the localization result [59] for the $\frac12$-BPS Wilson loop in the ABJM theory, in the limit of large N with k fixed.

Comparing (6.2), (6.3) with (6.7), (6.8) we observe close similarities. This suggests some relation by analytic continuation of both the backgrounds and the M2 brane solutions. Indeed, the maximally supersymmetric $\mathrm{AdS}_7\times S^4$ and $\mathrm{AdS}_4\times S^7$ backgrounds are related by a formal analytic continuation (like the one between AdS_n and S^n, i.e. $dx^2+\sinh^2x\,dS_{n-1}\to-(dr^2+\sin^2r\,dS_{n-1})$, $r=ix$) and the same will apply to the M2 brane actions in these backgrounds.

The compactification $y\equiv y+\beta$ of the circle in $\mathrm{AdS}_{7,\beta}$ part of (1.2) suggests an analogy with the discrete orbifolding $\mathrm{y}\equiv\mathrm{y}+\mathrm{b}$ in $S^7/\mathbb{Z}_k$ part of (6.4) and thus a similar role of β and b, which is indeed evident from the comparison of (6.2), (6.3) with (6.7), (6.8). Such analytic continuation suggests that the “instanton” $S^1_\beta\times S^2$ M2 solution in $\mathrm{AdS}_{7,\beta}\times\tilde S^4$ may be related to the “Wilson loop” $\mathrm{AdS}_2\times S^1/\mathbb{Z}_k$ solution in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$, and vice versa, the “defect” $\mathrm{AdS}_{3,\beta}$ solution in $\mathrm{AdS}_{7,\beta}\times\tilde S^4$ may be related to the “instanton” $S^3/\mathbb{Z}_k$ solution in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$.[31]

[29] The similar AdS_3 “defect” M2 brane solution considered in [9] has S^2 boundary instead of $S^1\times S^1_\beta$ and thus has logarithmically divergent classical action related to the defect conformal anomaly.

[30] Here we ignore the overall factor 4 that accounts for contribution of the anti-instanton saddle and also for the effect of resolution of the 0-mode degeneracy (see [20,19]).

[31] The factor of 2 mismatch in powers of sinh/sin prefactors in the corresponding M2 brane partition functions may be related to the fact that the analytic continuation maps a world-volume with S^1 times a 2-sphere topology to S^1 times a disk (AdS_2) one.

Still, some details do not match: the $\mathbb{Z}_k$ orbifold of S^7 in the Hopf fibration parametrization is not equivalent to an analytic continuation of an orbifold of AdS_7 with $S^5 \times S^1$ boundary.[32] Also, there is no analog of the $z \to z + iy$ twist in $AdS_{7,\beta} \times \tilde{S}^4$ in (1.2) on the $AdS_4 \times S^7/\mathbb{Z}_k$ side. Thus the reason for the close similarity between the expressions in (6.2), (6.3) and (6.7), (6.8) calls for further insight.

CRediT authorship contribution statement

Authors contributed equally.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are grateful to J. Figueroa-O'Farrill, V.G.M. Puletti, J. Russo and K. Zarembo for useful discussions. We also thank P. Bomans, F. Gautason, V.G.M. Puletti and J. van Muiden for helpful comments on the draft. MB was supported by the INFN grants GSS and GAST. SG is supported in part by the US NSF under Grant No. PHY-2209997. AAT is supported by the STFC grant ST/T000791/1 and ST/X000575/1.

Appendix A. Renormalized volume of AdS_{2n+1} with boundary $S^{2n-1} \times S^1$

As is well known, the regularized volume of global AdS_{2n+1} with S^{2n} boundary has logarithmic IR divergence, $\mathrm{vol}(AdS_{2n+1}) = -\frac{2(-1)^n\pi^n}{n!}\log\varepsilon$ (where $\varepsilon \to 0$); in particular, $\mathrm{vol}(AdS_7) = \frac{\pi^3}{3}\log\varepsilon$ (see, e.g., [61]). At the same time, in the case of $S^{2n-1}\times S^1$ boundary the volume contains only power divergences and thus is finite after one drops them. This is analogous to the case of AdS_{2n} with S^{2n-1} boundary where $\mathrm{vol}(AdS_{2n}) = \frac{(-1)^n(2\pi)^n}{(2n-1)!!}$.

To find the volume of AdS_{2n+1} with $S^{2n-1} \times S^1$ boundary[33]

$$ds^2 = dx^2 + \sinh^2 x\, dS_{2n-1} + \cosh^2 x\, dy^2\,, \qquad y \equiv y + 2\pi\,. \tag{A.1}$$

Let us introduce an IR cutoff $0 < x \le x_0$ in the volume integral

$$\mathrm{vol}(AdS_{2n+1}) = \mathrm{vol}(S^{2n-1}\times S^1)\int_0^{x_0} dx\, \cosh x\, \sinh^{2n-1} x = \mathrm{vol}(S^{2n-1}\times S^1)\,\frac{1}{2n}\sinh^{2n} x_0\,. \tag{A.2}$$

A natural cutoff is $r = \varepsilon^2 \to 0$ in Fefferman-Graham coordinates $ds^2 = \frac{1}{4r^2}dr^2 + \frac{1}{r}g_{mn}(x,r)dx^m dx^n$ which is related to x_0 as $x_0 = -\log\varepsilon$.[34] Dropping $\frac{1}{\varepsilon^k}$ power divergences in (A.2) and setting $\varepsilon \to 0$ gives (using that $\mathrm{vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$)

$$\mathrm{vol}(AdS_{2n+1}) = \mathrm{vol}(S^{2n-1}\times S^1)\,\frac{(1-\varepsilon^2)^{2n}}{2^{2n+1}\, n\,\varepsilon^{2n}} \;\to\; \mathrm{vol}(S^{2n-1}\times S^1)\frac{(-1)^n\Gamma(n+\frac{1}{2})}{2n^2\sqrt{\pi}\,\Gamma(n)} = \frac{(-1)^n\pi^{n+1}(2n)!}{2^{2n-1}\,(n!)^3}\,. \tag{A.3}$$

In particular,

$$\mathrm{vol}(AdS_3) = -\pi^2, \qquad \mathrm{vol}(AdS_5) = \frac{3\pi^3}{8}, \qquad \mathrm{vol}(AdS_7) = -\frac{5\pi^4}{48}\,. \tag{A.4}$$

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32 The S^7 metric can be parametrized as $Z_r^* Z_r = 1$ $(r = 1,2,3,4)$ with $Z_r = e^{i\mathrm{y}} W_r$ where $\mathrm{y} = \mathrm{y} + 2\pi$ and $W_r^* W_r = 1$ parametrize $\mathbb{CP}^3$ so that (see, e.g., [60]) $dS_7 = ds^2_{\mathbb{CP}^3} + (d\mathrm{y} + A)^2$, where A depends on $\mathbb{CP}^3$ coordinates. Alternatively, we may set $Z_1 = \cos r\, e^{iy}$, $Z_i = \sin r\, U_i$ $(i = 1,2,3)$, $U_i^* U_i = 1$ where U_i parametrize S^5. Then the S^7 metric is $dS_7 = dr^2 + \sin^2 r\, dS_5 + \cosh^2 r\, dy^2$. To relate this to the first Hopf fibration parametrization of the metric we need to redefine U_i by e^{iy} and identify y with y. Then the orbifold of y will act also on S^5. But orbifolding $y \equiv y + \mathrm{b}$ in the second form of the metric does not act on S^5. Thus the two orbifolds are not equivalent.

33 This space may be viewed as "thermal" AdS_{2n+1}, i.e. is obtained from Minkowski signature AdS_{2n+1} by analytic continuation and periodical identification of the Euclidean time.

34 For comparison, in the case of AdS_{2n+1} with S^{2n} boundary, i.e. $ds^2 = dx^2 + \sinh^2 x\, dS_{2n}$, we get $\int_0^{x_0} dx\, \sinh^{2n} x = \frac{(-1)^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\, x_0 + \cdots$, where dots stand for powers of $\sinh x_0$ leading to powers of $\frac{1}{\varepsilon}$ and subleading finite terms. Multiplying by $\mathrm{vol}(S^{2n})$, one gets $\mathrm{vol}(AdS_{2n+1}) = -\frac{2(-1)^n\pi^n}{n!}\log\varepsilon$.

As an application, let compute the value of the 11d supergravity action on $AdS_{7,\beta}\times S^4$ of radius a where β is the length of the S^1 circle y as in (1.2). Compactifying on S^4 (that has radius $\frac{a}{2}$) we get

$$S_{11} = -\frac{1}{2\kappa_{11}^2}\,(\frac{a}{2})^4\,\mathrm{vol}(S^4)\int d^7x\sqrt{g}\,\big(R^{(7)}-2\Lambda\big), \qquad 2\kappa_{11}^2 = (2\pi)^8\ell_P^9\,, \tag{A.5}$$

where for the AdS_7 solution one has $R^{(7)} = -\frac{42}{a^2}$, $\Lambda = -\frac{15}{a^2}$. Since $\mathrm{vol}(S^4) = \frac{8\pi^2}{3}$, we get

$$S_{11} = -\frac{1}{(2\pi)^8}(\frac{a}{\ell_P})^9\,\frac{8\pi^2}{3}(\frac{1}{2})^4(-12)\,\frac{\beta}{2\pi}\,\mathrm{vol}(AdS_7) = \frac{\pi}{(2\pi)^8}(\frac{a}{\ell_P})^9\beta\,\mathrm{vol}(AdS_7)\,. \tag{A.6}$$

Using (A.4), i.e. $\mathrm{vol}(AdS_7) = -\frac{5\pi^4}{48}$, and (1.1) implying $(\frac{a}{\ell_P})^9 = 2^9\pi^3N^3$ we end up with

$$S_{11} = -\frac{5}{24}N^3\beta\,. \tag{A.7}$$

The same result is found also for the "twisted" $AdS_{7,\beta}\times\tilde{S}^4$ background in (1.2) (the shift $z\to z+iy$ along the S^4 isometry z-direction does not change the value of the 11d volume form $\sim dy\wedge dz\wedge\ldots$). At the same time, the leading large N term in the free energy in (2.9), is $F_N^{\rm pert} = -\frac{1}{6}N^3\beta+\ldots$, so there is a 5/4 mismatch with (A.7).

This discrepancy was noted in [62,48], see also [63]. A way to resolve it at the level of 7d gauged supergravity with extra (non-invariant) counterterms was suggested in [30]. It is unclear at the moment how to reach the same conclusion directly at the level of 11d supergravity action, i.e. to see how the leading-order term can distinguish between the standard and "supersymmetric" free energy. One may contemplate adding some non-invariant boundary terms, but this issue needs further clarification.

Appendix B. $\beta\to 0$ expansion of scalar free energy in thermal $AdS_{3,\beta}$

Here we discuss several methods to compute the small β expansion of the non-Casimir part of the scalar log det in (5.14), i.e. of the function

$$f(\beta;\Delta) \equiv \sum_{n=1}^{\infty}\frac{q^{n\Delta}}{n(1-q^n)^2}, \qquad q = e^{-\beta q}\,, \qquad \Delta\geq 2, \tag{B.1}$$

that can be written equivalently as

$$f(\beta;\Delta) = -\sum_{\ell,\ell'=0}^{\infty}\log(1-q^{\ell+\ell'+\Delta}) = -\sum_{n=0}^{\infty}(n+1)\log(1-q^{n+\Delta}). \tag{B.2}$$

The first method is to expand f in (B.1) at small β and sum the terms using Riemann zeta-function regularization (i.e. multiplying by n^s, summing, and taking the finite part of the $s\to 0$ limit). This gives

$$\begin{aligned}
\text{(I):}\qquad f(\beta;\Delta) =& \frac{\zeta(3)}{\beta^2} - \frac{\pi^2(-1+\Delta)}{6\beta} + \frac{1}{12}\gamma_{\rm E}(5-12\Delta+6\Delta^2) + \frac{1}{24}(-1+\Delta)(1-4\Delta+2\Delta^2)\beta \\
&+ \frac{(-1+20\Delta-50\Delta^2+40\Delta^3-10\Delta^4)}{2880}\beta^2 \\
&+ \frac{(-5+42\Delta+63\Delta^2-420\Delta^3+525\Delta^4-252\Delta^5+42\Delta^6)}{3628800}\beta^4+\cdots.
\end{aligned} \tag{B.3}$$

The constant $\gamma_{\rm E}$ term is regularization dependent and is related to the dropped pole $\sim\frac{1}{s}$.

Another method is to expand f in (B.2) at small β, multiply by $(n+\Delta)^s$, sum over n and then take the finite part of the $s\to 0$ limit. This way we obtain

$$\begin{aligned}
\text{(II):}\qquad f(\beta;\Delta) =& \mathscr{C}(\Delta) - \frac{1}{12}(5-12\Delta+6\Delta^2)\log\beta + \frac{1}{24}(-1+\Delta)(1-4\Delta+2\Delta^2)\beta \\
&+ \frac{(-1+20\Delta-50\Delta^2+40\Delta^3-10\Delta^4)}{2880}\beta^2 \\
&+ \frac{(-5+42\Delta+63\Delta^2-420\Delta^3+525\Delta^4-252\Delta^5+42\Delta^6)}{3628800}\beta^4+\cdots\,,
\end{aligned} \tag{B.4}$$

$$\mathscr{C}(\Delta) = (\Delta-1)\Big[\frac{1}{2}\log(2\pi) - \log\Gamma(\Delta)\Big] + \zeta'(-1,\Delta). \tag{B.5}$$

Comparing to (B.3), we see that we miss the $\frac{1}{\beta^2}$ and $\frac{1}{\beta}$ terms and the $\gamma_{\rm E}$ term is replaced by the $\log\beta$ term.

A rigorous (third) method is to follow [64].[35] Starting again from (B.2) and differentiating over β gives

$$f'(\beta;\Delta) = -\sum_{n=0}^{\infty}\frac{(n+1)(n+\Delta)}{e^{(\Delta+n)\beta}-1} = -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(n+1)(n+\Delta)e^{-(\Delta+n)m\beta}. \tag{B.6}$$

Now using that $e^{-x} = \frac{1}{2\pi i}\int_C ds\, x^{-s}\,\Gamma(s)$ (where the contour C is along the imaginary axis with large enough real part of s) gives the Mellin representation

$$f'(\beta;\Delta) = \frac{1}{2\pi i}\int_C ds \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(\Delta+n)^{-s}m^{-s}\beta^{-s}\Gamma(s)(n+1)(n+\Delta) = \frac{1}{2\pi i}\int_C ds\, \beta^{-s}G(s), \tag{B.7}$$

$$G(s) = \Gamma(s)[\zeta(-2+s,\Delta)+(1-\Delta)\zeta(-1+s,\Delta)]\zeta(s). \tag{B.8}$$

Closing the contour to the left we get for the $\beta\to 0$ expansion (up to exponentially suppressed terms denoted by dots)

$$f'(\beta;\Delta) = -\sum_{n=0}^{\infty}\operatorname*{Res}_{s=3-n}(\beta^{-s}G(s)) + \dots \tag{B.9}$$

Integrating this over β gives

$$\begin{aligned}\text{(III):}\qquad f(\beta;\Delta) =& \mathscr{C}(\Delta) + \frac{\zeta(3)}{\beta^2} - \frac{\pi^2(\Delta-1)}{6\beta} - \frac{1}{12}(5-12\Delta+6\Delta^2)\log\beta\\ &+ \frac{1}{24}(-1+\Delta)(1-4\Delta+2\Delta^2)\beta + \frac{(-1+20\Delta-50\Delta^2+40\Delta^3-10\Delta^4)}{2880}\beta^2\\ &+ \frac{(-5+42\Delta+63\Delta^2-420\Delta^3+525\Delta^4-252\Delta^5+42\Delta^6)}{3628800}\beta^4+\cdots,\end{aligned} \tag{B.10}$$

where $\mathscr{C}(\Delta)$ is yet undetermined integration constant. By doing numerics, we found that (B.10) is the correct expansion with $\mathscr{C}(\Delta)$ being the same as in (B.5). The expansion (B.10) reproduces the two singular $\frac{1}{\beta^2}$ and $\frac{1}{\beta}$ terms in (B.3) and the logarithm in (B.4).

For example, this gives for $\Delta = 3$

$$f(\beta;3) = \frac{\zeta(3)}{\beta^2} - \frac{\pi^2}{3\beta} + \frac{1}{12}\Big(1-12\log\frac{\mathsf{A}}{2\pi}\Big) - \frac{23}{12}\log\beta + \frac{7\beta}{12} - \frac{121\beta^2}{2880} + \frac{251\beta^4}{725760}+\cdots. \tag{B.11}$$

Appendix C. Scalar determinant in $\mathrm{AdS}_{3,\beta}$ from expansion in modes on $S^1\times S^1_\beta$

Here we derive the expression in (5.14) by directly expanding in Fourier modes in the two $S^1\times S^1_\beta$ boundary angles.[36] Let us start with the scalar operator $\hat{K}\equiv\Delta_0 = -D^2+M^2$ in the $\mathrm{AdS}_{3,\beta}$ metric (5.1) in the explicit coordinate form (here $M^2=\Delta(\Delta-2)$ as in (5.15))

$$\hat{K} = -\frac{1}{\sinh\xi_1\cosh\xi_1}\partial_1(\sinh\xi_1\cosh\xi_1\partial_1) - \frac{1}{\sinh^2\xi_1}\partial_2^2 - \frac{1}{\cosh^2\xi_1}\partial_3^2 + \Delta(\Delta-2)\,. \tag{C.1}$$

Redefining $\xi_1\to\frac{1}{2}\rho$ and expanding in modes so that $\partial_2\to i\,m,\ \partial_3\to i\,n_{_\beta} = i\,\frac{2\pi}{\beta}n$, we get a "radial" 1d operator

$$K_{m,n} = -\frac{4}{\sinh\rho}\frac{d}{d\rho}\Big(\sinh\rho\frac{d}{d\rho}\Big) + \frac{m^2}{\sinh^2\frac{\rho}{2}} + \frac{n_{_\beta}^2}{\cosh^2\frac{\rho}{2}} + \Delta(\Delta-2), \qquad n_{_\beta} = \frac{2\pi}{\beta}n\,, \quad n,m\in\mathbb{Z}. \tag{C.2}$$

By applying the Gelfand-Yaglom theorem (see, e.g., [65]) we have

$$\log\frac{\det K_{m,n}}{\det K_{m,0}} = \lim_{\rho\to\infty}\log\frac{\psi_{m,n}(\rho)}{\psi_{m,0}(\rho)}, \tag{C.3}$$

$$K_{m,n}\,\psi_{m,n}(\rho) = 0, \qquad \psi_{m,n}(\rho)\stackrel{\rho\to 0}{\to}\rho^{|m|}+\cdots. \tag{C.4}$$

The solution of (C.4) is

$$\psi_{m,n}(\rho) = 2^{|m|}(\tanh\frac{\rho}{2})^{|m|}(\cosh\frac{\rho}{2})^{-\Delta}\,{}_2F_1\Big(\frac{\Delta+|m|-in_{_\beta}}{2},\frac{\Delta+|m|+in_{_\beta}}{2},1+|m|,\tanh^2\frac{\rho}{2}\Big), \tag{C.5}$$

and as a consequence of (C.3)

[35] Another rigorous approach is based on the temperature inversion relations as in [49].

[36] As usual, the determinant will be defined using analytic regularization so that power divergences will be ignored (there is no logarithmic divergence in the present 3d case).

$$\log \frac{\det K_{m,n}}{\det K_{m,0}} = \log \frac{\Gamma(\frac{\Delta}{2}+\frac{|m|}{2})^2}{\Gamma(\frac{\Delta}{2}+\frac{|m|}{2}-i\frac{n_\beta}{2})\Gamma(\frac{\Delta}{2}+\frac{|m|}{2}+i\frac{n_\beta}{2})}. \tag{C.6}$$

Thus

$$\Gamma^{(\Delta)}(\beta) \equiv \frac{1}{2}\log\det\hat{K} = \frac{1}{2}\sum_{n,m\in\mathbb{Z}}\log\det K_{m,n} = -\frac{1}{2}\sum_{n,m\in\mathbb{Z}}\log\Big[\Gamma\Big(\frac{\Delta}{2}+\frac{|m|}{2}-i\frac{n_\beta}{2}\Big)\Gamma\Big(\frac{\Delta}{2}+\frac{|m|}{2}+i\frac{n_\beta}{2}\Big)\Big], \tag{C.7}$$

where we dropped n-independent term as $\sum_{n\in\mathbb{Z}} 1 = 1 + 2\zeta_R(0) = 0$. As in [66] we may use that

$$\log\big[\Gamma(x+iy)\Gamma(x-iy)\big] = 2\log\Gamma(x) - \sum_{k=0}^{\infty}\log\Big[1+\frac{y^2}{(x+k)^2}\Big]. \tag{C.8}$$

Then from (C.7) we get

$$\Gamma^{(\Delta)}(\beta) = \frac{1}{2}\sum_{n,m\in\mathbb{Z}}\sum_{k=0}^{\infty}\log\Big[1+\frac{n_\beta^2}{(\Delta+|m|+2k)^2}\Big]. \tag{C.9}$$

The set $|m|+2k$ with $m\in\mathbb{Z}$ and $k\in\mathbb{N}_0$ can be replaced by a sum over $k\in\mathbb{N}_0$ with multiplicity $k+1$. Thus,

$$\Gamma^{(\Delta)}(\beta) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}(k+1)\log\Big[1+\frac{n_\beta^2}{(\Delta+k)^2}\Big]. \tag{C.10}$$

The $n=0$ term vanishes and separating the divergent part of the sum over n we get[37]

$$\Gamma^{(\Delta)}(\beta) = \Gamma^{(\Delta)}_{\rm div}(\beta) + \Gamma^{(\Delta)}_{\rm reg}(\beta)\ , \tag{C.11}$$

$$\Gamma^{(\Delta)}_{\rm div}(\beta) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(k+1)\log\frac{n_\beta^2}{(\Delta+k)^2}, \qquad \Gamma^{(\Delta)}_{\rm reg}(\beta) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(k+1)\log\Big[1+\frac{(\Delta+k)^2}{n_\beta^2}\Big]. \tag{C.12}$$

Computing $\Gamma^{(\Delta)}_{\rm div}(\beta)$ using again the Riemann zeta-function regularization gives

$$\begin{aligned}\Gamma^{(\Delta)}_{\rm div}(\beta) &= \sum_{k=0}^{\infty}(k+1)\sum_{n=1}^{\infty}\log\frac{n_\beta^2}{(\Delta+k)^2} = \sum_{k=0}^{\infty}(k+1)\sum_{n=1}^{\infty}\Big[-2\log(\Delta+k)+2\log\frac{2\pi}{\beta}+2\log n\Big]\\ &= \sum_{k=0}^{\infty}(k+1)\Big[\log(\Delta+k)-\log\frac{2\pi}{\beta}+\log(2\pi)\Big] = \sum_{k=0}^{\infty}(k+1)\Big[\log(\Delta+k)+\log\beta\Big]\ .\end{aligned} \tag{C.13}$$

Here the sum over k may also be computed using zeta-function regularization but it is useful not to do this before combining it with $\Gamma^{(\Delta)}_{\rm reg}(\beta)$.

Since

$$\sum_{n=1}^{\infty}\log\Big(1+\frac{a^2}{n^2}\Big) = \log\frac{\sinh(\pi a)}{\pi a} = \pi a - \log(\pi a) - \log 2 + \log(1-e^{-2\pi a})\ , \tag{C.14}$$

we find that $\Gamma^{(\Delta)}_{\rm reg}(\beta)$ in (C.12) (here for $a = \frac{1}{2\pi}(\Delta+k)\beta$) may be written as

$$\Gamma^{(\Delta)}_{\rm reg}(\beta) = \sum_{k=0}^{\infty}(k+1)\Big[\frac{1}{2}(\Delta+k)\beta - \log((\Delta+k)\beta) + \log(1-e^{-(\Delta+k)\beta})\Big]. \tag{C.15}$$

Adding (C.13) and (C.15) gives

$$\Gamma^{(\Delta)}(\beta) = \tfrac{1}{2}\beta\sum_{k=0}^{\infty}(k+1)(\Delta+k) + \sum_{k=0}^{\infty}(k+1)\log\Big(1-e^{-(\Delta+k)\beta}\Big). \tag{C.16}$$

Doing the sum in the first term using Hurwitz zeta-function regularization gives finally the expression [50] equivalent (cf. (B.1), (B.2)) to the one in (5.14), (5.16)

$$\Gamma^{(\Delta)}(\beta) = \frac{1}{24}(\Delta-1)(1-4\Delta+2\Delta^2)\,\beta + \sum_{k=0}^{\infty}(k+1)\log(1-q^{\Delta+k})\ . \tag{C.17}$$

[37] Note that here the "reg" part may still contain a divergent contribution from the sum over k (see below).

Note that in the $\beta \to 0$ expansion the first "Casimir" term cancels against the linear in β term in the second term in (C.17) (see (B.10)).

C.1. Including the twist $\partial_3 \to \partial_3 - \kappa$ or $n_{_\beta} \to n_{_\beta} + i\kappa$

Let us now consider the determinant of the scalar operator including the coupling to the flat gauge potential in the ξ^3 direction (5.13), (5.19), i.e. $\partial_3 \to \partial_3 - \kappa$ or $n_{_\beta} \to n_{_\beta} + i\kappa$ where $n_{_\beta} = \frac{\beta}{2\pi}$. Repeating the above calculation with $n_{_\beta} \to n_{_\beta} + i\kappa$ we get in (C.14)

$$\sum_{n\in\mathbb{Z}} \log\Big(1+\frac{a^2}{n^2}\Big) \to \sum_{n\in\mathbb{Z}} \log\Big(1+\frac{a^2}{(n+i\frac{\beta\kappa}{2\pi})^2}\Big), \tag{C.18}$$

where the sum can be computed using

$$\sum_{n\in\mathbb{Z}} \log\Big(1+\frac{a^2}{(n+ib)^2}\Big) = \log\Big|1-\frac{\sinh^2(\pi a)}{\sinh^2(\pi b)}\Big| . \tag{C.19}$$

This leads to the following modification of the expression (C.11), (C.12) for the determinant in (C.7)

$$\Gamma^{(\Delta,\kappa)}(\beta) = \Gamma^{(\Delta,\kappa)}_{\rm div}(\beta) + \Gamma^{(\Delta,\kappa)}_{\rm reg}(\beta) , \qquad \Gamma^{(\Delta,\kappa)}_{\rm div}(\beta) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}(k+1)\log\frac{(n_{_\beta}+i\kappa)^2}{(\Delta+k)^2}, \tag{C.20}$$

$$\Gamma^{(\Delta,\kappa)}_{\rm reg}(\beta) = \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\log\Big[\frac{\sinh^2(\frac{(k+\Delta)\beta}{2})}{\sinh^2(\frac{\beta\kappa}{2})}-1\Big]. \tag{C.21}$$

This can be written in a form similar to (B.1) as follows. For the divergent part of the sum over n we get (ignoring again a sum of a constant assuming ζ_R regularization)

$$\begin{aligned}\Gamma^{(\Delta,\kappa)}_{\rm div}(\beta) &= \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Big[\log(\frac{\beta^2\kappa^2}{4\pi^2}) + \sum_{n=1}^{\infty}\log\big[(n+i\frac{\beta\kappa}{2\pi})^2\big] + \sum_{n=1}^{\infty}\log\big[(n-i\frac{\beta\kappa}{2\pi})^2\big]\Big]\\ &= \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Big[\log(\frac{\beta^2\kappa^2}{4\pi^2}) + 2\log\big(\frac{4\pi}{\beta\kappa}\sinh\frac{\beta\kappa}{2}\big)\Big] = \sum_{k=0}^{\infty}(k+1)\ \log(2\sinh\frac{\beta\kappa}{2}) .\end{aligned} \tag{C.22}$$

Using that

$$\log\Big[\frac{\sinh^2\frac{(k+\Delta)\beta}{2}}{\sinh^2\frac{\kappa\beta}{2}}-1\Big] = -2\log(2\sinh\tfrac{\beta\kappa}{2}) + \beta(k+\Delta) + \log[(1-q^{k+\Delta+\kappa})(1-q^{k+\Delta-\kappa})] , \tag{C.23}$$

for the $\Gamma^{(\Delta,\kappa)}_{\rm reg}(\beta)$ part we get[38]

$$\begin{aligned}\Gamma^{(\Delta,\kappa)}_{\rm reg}(\beta) &= \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Big[-2\log(2\sinh\tfrac{\beta\kappa}{2}) + \beta(k+\Delta) + \sum_{\pm}\log(1-q^{k+\Delta\pm\kappa})\Big]\\ &= -\sum_{k=0}^{\infty}(k+1)\ \log(2\sinh\tfrac{\beta\kappa}{2}) + \tfrac{1}{24}(\Delta-1)(1-4\Delta+2\Delta^2+6\kappa^2)\beta + \tfrac{1}{2}\sum_{\pm}\sum_{\ell,\ell'=0}^{\infty}\log(1-q^{\ell+\ell'+\Delta\pm\kappa})\\ &= -\sum_{k=0}^{\infty}(k+1)\ \log(2\sinh\frac{\beta\kappa}{2}) + \tfrac{1}{24}(\Delta-1)(1-4\Delta+2\Delta^2+6\kappa^2)\beta - \tfrac{1}{2}\sum_{\pm}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(\Delta\pm\kappa)}}{(1-q^n)^2}.\end{aligned} \tag{C.24}$$

Adding together (C.22) and (C.24) we finally get the finite expression quoted in (5.20), (5.21)

$$\Gamma^{(\Delta,\kappa)}(\beta) = \frac{1}{24}(\Delta-1)(1-4\Delta+2\Delta^2+6\kappa^2)\beta - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(\Delta+\kappa)}}{(1-q^n)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(\Delta-\kappa)}}{(1-q^n)^2} . \tag{C.25}$$

The small β expansion of $\Gamma^{(\Delta,\kappa)}(\beta)$ can be found as in (B.1), (B.10):

$$\Gamma^{(\Delta,\kappa)}(\beta) = -\frac{\zeta(3)}{\beta^2} - \frac{\pi^2(\Delta-1)}{6\beta} - \mathcal{C}(\Delta,\kappa) + \frac{1}{12}(5-12\Delta+6\Delta^2+6\kappa^2)\log\beta - \sum_{n=1}^{\infty}\mathcal{C}_{2n}(\Delta,\kappa)\,\beta^{2n}, \tag{C.26}$$

where

[38] The Casimir term is computed by splitting $\Delta = \frac{1}{2}(\Delta+\kappa) + \frac{1}{2}(\Delta-\kappa)$ and using Hurwitz zeta function regularization, i.e. introducing a factor $(k+\Delta\pm\kappa)^s$ and dropping singular terms in the limit $s \to 0$.

$$\mathcal{C}(\Delta,\kappa)=\frac{1}{2}[\mathcal{C}(\Delta+\kappa)+\mathcal{C}(\Delta-\kappa)]\ , \tag{C.27}$$

$$\mathcal{C}_2(\Delta,\kappa)=\frac{-1+20\Delta-50\Delta^2+40\Delta^3-10\Delta^4}{2880}+\frac{1}{288}(-5+12\Delta-6\Delta^2)\kappa^2-\frac{\kappa^4}{288}, \tag{C.28}$$

$$\mathcal{C}_4(\Delta,\kappa)=\frac{-5+42\Delta+63\Delta^2-420\Delta^3+525\Delta^4-252\Delta^5+42\Delta^6}{3628800}$$
$$+\frac{(1-20\Delta+50\Delta^2-40\Delta^3+10\Delta^4)\kappa^2}{57600}+\frac{(5-12\Delta+6\Delta^2)\kappa^4}{34560}+\frac{\kappa^6}{86400}\ ,\ \dots \tag{C.29}$$

C.2. Alternative derivation by Poisson resummation

An alternative way to derive the expression for the log det in (C.7) is to apply the Poisson resummation trick

$$\sum_{n\in\mathbb{Z}} f(n)=\sum_{\ell\in\mathbb{Z}}\widetilde{f}(\ell), \qquad \widetilde{f}(\ell)=\mathcal{F}[f]\equiv\int_{-\infty}^{\infty} dn\, f(n)\, e^{-2\pi i\ell n}. \tag{C.30}$$

Since

$$\mathcal{F}\Big[\log(1+a^2n^2)\Big]=-\frac{1}{|\ell|}\exp\Big(-\frac{2\pi|\ell|}{|a|}\Big), \tag{C.31}$$

this gives

$$\Gamma^{(\Delta)}(\beta)=\frac{1}{2}\sum_{\ell\in\mathbb{Z}}\sum_{k=0}^{\infty}(k+1)\frac{1}{|\ell|}e^{-|\ell|(\Delta+k)\beta}\ . \tag{C.32}$$

If we separate the $\ell=0$ term, we obtain

$$\Gamma^{(\Delta)}(\beta)=\text{“}\ell=0\text{ term”}+\sum_{k=0}^{\infty}(k+1)\log(1-e^{-(\Delta+k)\beta}). \tag{C.33}$$

This can be generalized to the case of a non-zero κ-shift using that

$$\mathcal{F}\Big[\log(1+a^2(n+ib)^2)\Big]=e^{2\pi\ell b}\ \mathcal{F}[\log(1+a^2n^2)], \tag{C.34}$$

which leads to the last term in (C.24).

References

[1] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. 38 (1999) 1113, arXiv:hep-th/9711200; O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Large N field theories, string theory and gravity, Phys. Rep. 323 (2000) 183, arXiv:hep-th/9905111.

[2] R. Gueven, Black p-brane solutions of D = 11 supergravity theory, Phys. Lett. B 276 (1992) 49; G.W. Gibbons, P.K. Townsend, Vacuum interpolation in supergravity via super p-branes, Phys. Rev. Lett. 71 (1993) 3754, arXiv:hep-th/9307049.

[3] M. Henningson, K. Skenderis, The holographic Weyl anomaly, J. High Energy Phys. 9807 (1998) 023, arXiv:hep-th/9806087.

[4] A.A. Tseytlin, R4 terms in 11 dimensions and conformal anomaly of (2, 0) theory, Nucl. Phys. B 584 (2000) 233, arXiv:hep-th/0005072.

[5] M. Beccaria, G. Macorini, A.A. Tseytlin, Supergravity one-loop corrections on AdS_7 and AdS_3, higher spins and AdS/CFT, Nucl. Phys. B 892 (2015) 211, arXiv:1412.0489.

[6] C. Beem, L. Rastelli, B.C. van Rees, $\mathcal{W}$ symmetry in six dimensions, J. High Energy Phys. 1505 (2015) 017, arXiv:1404.1079.

[7] K. Ohmori, H. Shimizu, Y. Tachikawa, K. Yonekura, Anomaly polynomial of general 6d SCFTs, PTEP 2014 (2014) 103B07, arXiv:1408.5572.

[8] N. Drukker, M. Probst, M. Trépanier, Surface operators in the 6d N = (2, 0) theory, J. Phys. A 53 (2020) 365401, arXiv:2003.12372.

[9] N. Drukker, S. Giombi, A.A. Tseytlin, X. Zhou, Defect CFT in the 6d (2, 0) theory from M2 brane dynamics in $AdS_7\times S^4$, J. High Energy Phys. 07 (2020) 101, arXiv:2004.04562.

[10] O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, $\mathcal{N}=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 10 (2008) 091, arXiv:0806.1218.

[11] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. 2 (1998) 505, arXiv:hep-th/9803131.

[12] N. Seiberg, Notes on theories with 16 supercharges, Nucl. Phys. Proc. Suppl. 67 (1998) 158, arXiv:hep-th/9705117; M.R. Douglas, On D=5 super Yang-Mills theory and (2, 0) theory, J. High Energy Phys. 02 (2011) 011, arXiv:1012.2880; N. Lambert, C. Papageorgakis, M. Schmidt-Sommerfeld, M5-branes, D4-branes and quantum 5D super-Yang-Mills, J. High Energy Phys. 01 (2011) 083, arXiv:1012.2882.

[13] N. Bobev, P. Bomans, F.F. Gautason, Spherical branes, J. High Energy Phys. 08 (2018) 029, arXiv:1805.05338.

[14] M. Mezei, S.S. Pufu, Y. Wang, Chern-Simons theory from M5-branes and calibrated M2-branes, J. High Energy Phys. 08 (2019) 165, arXiv:1812.07572.

[15] F.F. Gautason, V.G.M. Puletti, Precision holography for 5D super Yang-Mills, J. High Energy Phys. 03 (2022) 018, arXiv:2111.15493.

[16] G.W. Gibbons, Quantized flux tubes in Einstein-Maxwell theory and noncompact internal spaces, in: 22nd Winter School of Theoretical Physics: Fields and Geometry, vol. 5, 1986, pp. 597–615; J.G. Russo, A.A. Tseytlin, Magnetic flux tube models in superstring theory, Nucl. Phys. B 461 (1996) 131, arXiv:hep-th/9508068; A.A. Tseytlin, Closed superstrings in magnetic field: instabilities and supersymmetry breaking, Nucl. Phys. Proc. Suppl. 49 (1996) 338, arXiv:hep-th/9510041.

[17] J.G. Russo, A.A. Tseytlin, Green-Schwarz superstring action in a curved magnetic Ramond-Ramond background, J. High Energy Phys. 04 (1998) 014, arXiv:hep-th/9804076.

[18] S. Giombi, A.A. Tseytlin, Wilson loops at large N and the quantum M2-brane, Phys. Rev. Lett. 130 (2023) 201601, arXiv:2303.15207.

[19] M. Beccaria, S. Giombi, A.A. Tseytlin, Instanton contributions to the ABJM free energy from quantum M2 branes, arXiv:2307.14112.

[20] F.F. Gautason, V.G.M. Puletti, J. van Muiden, Quantized strings and instantons in holography, J. High Energy Phys. 08 (2023) 218, arXiv:2304.12340.
[21] H.-C. Kim, S. Kim, M5-branes from gauge theories on the 5-sphere, J. High Energy Phys. 05 (2013) 144, arXiv:1206.6339.
[22] H.-C. Kim, J. Kim, S. Kim, Instantons on the 5-sphere and M5-branes, arXiv:1211.0144.
[23] S. Kim, K. Lee, Indices for 6 dimensional superconformal field theories, J. Phys. A 50 (2017) 443017, arXiv:1608.02969.
[24] D. Young, Wilson loops in five-dimensional super-Yang-Mills, J. High Energy Phys. 02 (2012) 052, arXiv:1112.3309.
[25] M. Mariño, Les Houches Lectures on Matrix Models and Topological Strings, vol. 10, 2004, arXiv:hep-th/0410165.
[26] H.-C. Kim, S. Kim, S.-S. Kim, K. Lee, The general M5-brane superconformal index, arXiv:1307.7660;
B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, D. Martelli, The Casimir energy in curved space and its supersymmetric counterpart, J. High Energy Phys. 07 (2015) 043, arXiv:1503.05537.
[27] N. Bobev, M. Bullimore, H.-C. Kim, Supersymmetric Casimir energy and the anomaly polynomial, J. High Energy Phys. 09 (2015) 142, arXiv:1507.08553.
[28] P. Benetti Genolini, D. Cassani, D. Martelli, J. Sparks, The holographic supersymmetric Casimir energy, Phys. Rev. D 95 (2017) 021902, arXiv:1606.02724.
[29] P. Benetti Genolini, D. Cassani, D. Martelli, J. Sparks, Holographic renormalization and supersymmetry, J. High Energy Phys. 02 (2017) 132, arXiv:1612.06761.
[30] N. Bobev, P. Bomans, F.F. Gautason, J.A. Minahan, A. Nedelin, Supersymmetric Yang-Mills, spherical branes, and precision holography, J. High Energy Phys. 03 (2020) 047, arXiv:1910.08555.
[31] M. Gunaydin, P. van Nieuwenhuizen, N. Warner, General construction of the unitary representations of Anti-de Sitter superalgebras and the spectrum of the S^4 compactification of eleven-dimensional supergravity, Nucl. Phys. B 255 (1985) 63;
J. Bhattacharya, S. Bhattacharyya, S. Minwalla, S. Raju, Indices for superconformal field theories in 3, 5 and 6 dimensions, J. High Energy Phys. 02 (2008) 064, arXiv:0801.1435.
[32] R. Arai, S. Fujiwara, Y. Imamura, T. Mori, D. Yokoyama, Finite-N corrections to the M-brane indices, J. High Energy Phys. 11 (2020) 093, arXiv:2007.05213.
[33] Y. Imamura, Analytic continuation for giant gravitons, PTEP 2022 (2022) 103B02, arXiv:2205.14615.
[34] D. Gaiotto, J.H. Lee, The giant graviton expansion, arXiv:2109.02545.
[35] N. Drukker, D.J. Gross, An exact prediction of N=4 SUSYM theory for string theory, J. Math. Phys. 42 (2001) 2896, arXiv:hep-th/0010274.
[36] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (1989) 351;
A. Kapustin, B. Willett, I. Yaakov, Exact results for Wilson loops in superconformal Chern-Simons theories with matter, J. High Energy Phys. 03 (2010) 089, arXiv:0909.4559.
[37] E. Bergshoeff, E. Sezgin, P.K. Townsend, Supermembranes and eleven-dimensional supergravity, Phys. Lett. B 189 (1987) 75.
[38] M.J. Duff, T. Inami, C.N. Pope, E. Sezgin, K.S. Stelle, Semiclassical quantization of the supermembrane, Nucl. Phys. B 297 (1988) 515.
[39] E. Bergshoeff, E. Sezgin, P.K. Townsend, Properties of the eleven-dimensional super membrane theory, Ann. Phys. 185 (1988) 330.
[40] L. Mezincescu, R.I. Nepomechie, P. van Nieuwenhuizen, Do supermembranes contain massless particles?, Nucl. Phys. B 309 (1988) 317;
S. Forste, Membrany corrections to the string anti-string potential in M5-brane theory, J. High Energy Phys. 05 (1999) 002, arXiv:hep-th/9902068.
[41] E. Bergshoeff, M.J. Duff, C.N. Pope, E. Sezgin, Supersymmetric supermembrane vacua and singletons, Phys. Lett. B 199 (1987) 69.
[42] E. Bergshoeff, M.J. Duff, C.N. Pope, E. Sezgin, Compactifications of the eleven-dimensional supermembrane, Phys. Lett. B 224 (1989) 71;
M.J. Duff, C.N. Pope, E. Sezgin, A stable supermembrane vacuum with a discrete spectrum, Phys. Lett. B 225 (1989) 319;
B. de Wit, K. Peeters, J. Plefka, A. Sevrin, The M theory two-brane in $AdS_4 \times S^7$ and $AdS_7 \times S^4$, Phys. Lett. B 443 (1998) 153, arXiv:hep-th/9808052;
P. Claus, Super M-brane actions in AdS(4) × S7 and AdS(7) × S4, Phys. Rev. D 59 (1999) 066003, arXiv:hep-th/9809045;
P. Pasti, D.P. Sorokin, M. Tonin, On gauge fixed superbrane actions in AdS superbackgrounds, Phys. Lett. B 447 (1999) 251, arXiv:hep-th/9809213.
[43] A.A. Tseytlin, On dilaton dependence of type II superstring action, Class. Quantum Gravity 13 (1996) L81, arXiv:hep-th/9601109;
B. de Wit, K. Peeters, J. Plefka, Superspace geometry for supermembrane backgrounds, Nucl. Phys. B 532 (1998) 99, arXiv:hep-th/9803209.
[44] J.A. Harvey, G.W. Moore, Superpotentials and membrane instantons, arXiv:hep-th/9907026.
[45] E. Cremmer, B. Julia, The SO(8) supergravity, Nucl. Phys. B 159 (1979) 141.
[46] M.J. Duff, P.S. Howe, T. Inami, K.S. Stelle, Superstrings in D = 10 from Supermembranes in D = 11, Phys. Lett. B 191 (1987) 70.
[47] S. Giombi, A.A. Tseytlin, Strong coupling expansion of circular Wilson loops and string theories in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$, J. High Energy Phys. 10 (2020) 130, arXiv:2007.08512.
[48] J.A. Minahan, A. Nedelin, M. Zabzine, 5D super Yang-Mills theory and the correspondence to AdS_7/CFT_6, J. Phys. A 46 (2013) 355401, arXiv:1304.1016.
[49] G.W. Gibbons, M.J. Perry, C.N. Pope, Partition functions, the Bekenstein bound and temperature inversion in anti-de Sitter space and its conformal boundary, Phys. Rev. D 74 (2006) 084009, arXiv:hep-th/0606186.
[50] S. Giombi, A. Maloney, X. Yin, One-loop partition functions of 3D gravity, J. High Energy Phys. 0808 (2008) 007, arXiv:0804.1773.
[51] R. Gopakumar, R.K. Gupta, S. Lal, The heat kernel on AdS, J. High Energy Phys. 1111 (2011) 010, arXiv:1103.3627.
[52] S. Giombi, I.R. Klebanov, A.A. Tseytlin, Partition functions and Casimir energies in higher spin AdS_{d+1}/CFT_d, Phys. Rev. D 90 (2014) 024048, arXiv:1402.5396.
[53] J.R. David, M.R. Gaberdiel, R. Gopakumar, The heat kernel on AdS_3 and its applications, J. High Energy Phys. 1004 (2010) 125, arXiv:0911.5085;
S. Datta, J.R. David, Higher spin quasinormal modes and one-loop determinants in the BTZ black hole, J. High Energy Phys. 03 (2012) 079, arXiv:1112.4619;
S. Datta, J.R. David, Higher spin fermions in the BTZ black hole, J. High Energy Phys. 07 (2012) 079, arXiv:1202.5831;
A. Kakkar, S. Sarkar, Phases of theories with fermions in AdS, J. High Energy Phys. 06 (2023) 009, arXiv:2303.02711.
[54] M. Henningson, K. Sfetsos, Spinors and the AdS / CFT correspondence, Phys. Lett. B 431 (1998) 63, arXiv:hep-th/9803251.
[55] D. Gang, E. Koh, K. Lee, J. Park, ABCD of 3d $\mathcal{N} = 8$ and 4 superconformal field theories, arXiv:1108.3647.
[56] A. Cagnazzo, D. Sorokin, L. Wulff, String instanton in AdS(4) x CP3, J. High Energy Phys. 05 (2010) 009, arXiv:0911.5228.
[57] N. Drukker, M. Marino, P. Putrov, Nonperturbative aspects of ABJM theory, J. High Energy Phys. 11 (2011) 141, arXiv:1103.4844;
Y. Hatsuda, S. Moriyama, K. Okuyama, Instanton bound states in ABJM theory, J. High Energy Phys. 05 (2013) 054, arXiv:1301.5184.
[58] M. Sakaguchi, H. Shin, K. Yoshida, Semiclassical analysis of M2-brane in $AdS_4 x S^7/Z_k$, J. High Energy Phys. 12 (2010) 012, arXiv:1007.3354.
[59] A. Klemm, M. Mariño, M. Schiereck, M. Soroush, Aharony-Bergman-Jafferis–Maldacena Wilson loops in the Fermi gas approach, Z. Naturforsch. A 68 (2013) 178, arXiv:1207.0611.
[60] T. Nishioka, T. Takayanagi, On type IIA Penrose limit and $\mathcal{N} = 6$ Chern-Simons theories, J. High Energy Phys. 08 (2008) 001, arXiv:0806.3391.
[61] D.E. Diaz, H. Dorn, Partition functions and double-trace deformations in AdS/CFT, J. High Energy Phys. 0705 (2007) 046, arXiv:hep-th/0702163.
[62] J. Kallen, J.A. Minahan, A. Nedelin, M. Zabzine, N^3-behavior from 5D Yang-Mills theory, J. High Energy Phys. 10 (2012) 184, arXiv:1207.3763.
[63] J.A. Minahan, Matrix models for 5d super Yang–Mills, J. Phys. A 50 (2017) 443015, arXiv:1608.02967.
[64] J.L. Cardy, Operator content and modular properties of higher dimensional conformal field theories, Nucl. Phys. B 366 (1991) 403.
[65] G.V. Dunne, Functional determinants in quantum field theory, J. Phys. A 41 (2008) 304006, arXiv:0711.1178;
M. Kruczenski, A. Tirziu, Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling, J. High Energy Phys. 05 (2008) 064, arXiv:0803.0315;
J. Aguilera-Damia, A. Faraggi, L. Pando Zayas, V. Rathee, G.A. Silva, Functional determinants of radial operators in AdS_2, J. High Energy Phys. 06 (2018) 007, arXiv:1802.06789.
[66] F. Denef, S.A. Hartnoll, S. Sachdev, Black hole determinants and quasinormal modes, Class. Quantum Gravity 27 (2010) 125001, arXiv:0908.2657.