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Kubo relations and radiative corrections for lepton number washout

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Abstract. The rates for lepton number washout in extensions of the Standard Model containing right-handed neutrinos are key ingredients in scenarios for baryogenesis through leptogenesis. We relate these rates to real-time correlation functions at finite temperature, without making use of any particle approximations. The relations are valid to quadratic order in neutrino Yukawa couplings and to all orders in Standard Model couplings. They take into account all spectator processes, and apply both in the symmetric and in the Higgs phase of the electroweak theory. We use the relations to compute washout rates at next-to-leading order in g, where g denotes a Standard Model gauge or Yukawa coupling, both in the non-relativistic and in the relativistic regime. Even in the non-relativistic regime the parametrically dominant radiative corrections are only suppressed by a single power of g. In the non-relativistic regime radiative corrections increase the washout rate by a few percent at high temperatures, but they are of order unity around the weak scale and in the relativistic regime.

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1 Introduction and motivation

With the Standard Model (SM) of electroweak interactions being validated up to an energy scale of several hundred GeV by the LHC experiments, and the knowledge that the SM combined with the conventional Big Bang scenario can explain neither the baryon asymmetry of the Universe, nor the existence of dark matter, it is appealing to search for an answer to these cosmological questions from other than electroweak interactions. The physics associated with right-handed (RH) neutrinos, which can also account for the observed left-handed neutrino mass differences and mixing angles, is rich enough to conceivably solve both problems [1, 2]. In this paper we concentrate on the baryon asymmetry aspect, and refer to the scenario as leptogenesis.

The original description of leptogenesis uses Boltzmann equations for the phase space density of the participating particles as a starting point. These are then integrated over momenta assuming kinetic equilibrium. The processes considered are the decays ($1 \to 2$) and inverse decays ($2 \to 1$) of the heavy RH neutrinos. In addition, so-called spectator processes, which proceed much faster than the expansion of the Universe, determine which quantities are in thermal equilibrium. Frequently also additional scattering processes ($2 \leftrightarrow 2$) and inverse decays have been included.

The $2 \leftrightarrow 2$ processes are a part of the radiative corrections. To assess the theoretical uncertainty of the analysis it would be desirable to compute the complete next-to-leading order (NLO) radiative corrections. It is not clear how to do this consistently within a description based on Boltzmann equations. This has motivated several authors to search for a first principles description of leptogenesis, without already putting in the set of assumptions and approximations which are implicit to the Boltzmann equations. Among the strategies followed are Kadanoff-Baym and similar equations for Green's functions (cf. refs. [3–5] for recent work and references). Although the starting point is exact, it may be difficult to perform systematic NLO calculations in these settings.

In this paper we propose a different route towards a first principles understanding of leptogenesis. We first formulate a rather general non-equilibrium problem. We argue that it can be described by a simple set of ordinary differential equations. The coefficients in these effective equations are shown to be related to real-time correlation functions at finite temperature. We then focus on one of the coefficients, the dissipation of lepton minus baryon number n_{L-B},[1] which in the absence of expansion is described by

$$\frac{\mathrm{d}n_{L-B}}{\mathrm{d}t} = -\gamma_{L-B}\, n_{L-B} + O(n_{L-B}^2)\,. \tag{1.1}$$

We calculate the dissipation coefficient[2] γ_{L-B} up to NLO in the SM couplings. It is shown, in particular, that the dominant NLO corrections are only suppressed by a single power of the gauge or top Yukawa couplings, and have a substantial relative influence even at temperatures much below the mass of the lightest right-handed neutrino.

In section 2 we describe the physical picture behind baryogenesis through leptogenesis. Section 3 contains a general analysis of the dissipation or washout rates of almost conserved charges, and relations of these rates to real-time correlation functions at finite temperature. We show that these rates factorize into a real-time spectral function and the inverse of a susceptibility matrix. In section 4 we specialize to the washout of lepton minus baryon number in extensions of the SM with right-handed neutrinos. We obtain a master formula, and leading-order (LO) and NLO results for its two ingredients, the spectral function and the susceptibility matrix. In section 5 we study the effect of the radiative corrections to the washout rate on the lepton asymmetry for one particular set of parameters. We summarize and conclude in section 6.

Notation. 4-vectors are denoted by lower-case italics and 3-vectors by boldface, and the metric is such that $p^2 = (p^0)^2 - \mathbf{p}^2$. For spatial integrals we use the notation $\int_{\mathbf{x}} \equiv \int \mathrm{d}^3x$, space-time integrals are denoted by $\int_x \equiv \int \mathrm{d}^4x$. Spatial momentum integrals are written as $\int_{\mathbf{k}} \equiv \int \mathrm{d}^3k/(2\pi)^3$.

2 Physical picture

We start by outlining the physical picture for how our computation fits in a generic leptogenesis framework (a recent example can be found in ref. [6]). A key observation is that in leptogenesis the system is almost in thermal equilibrium.[3] Most physical quantities rapidly fluctuate thermally around their equilibrium values. The corresponding reactions are often referred to as "spectator processes" [7, 8]. Other quantities relax to equilibrium on time scales much larger than the Hubble time, so that they can be considered conserved. A few have relaxation times of the order of the Hubble time. Only these have to be taken into account as dynamical degrees of freedom, and we will refer to them as "slow". What they are

[1] Or closely related quantities, depending on the temperature under consideration.

[2] Or, again depending on the temperature, the dissipation matrix.

[3] With the exception of leptogenesis during reheating, which is a process far from equilibrium.

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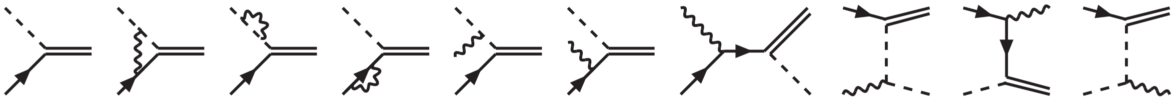

Figure 1. Examples of processes, up to $O(g^2)$, through which a net lepton number can be "washed out". Arrowed, dashed, and wiggly lines correspond to leptons, scalars, and gauge fields, whereas RH neutrinos are denoted by a double line. Additional reactions (not shown) involve the generation of antileptons or processes mediated by SM Yukawa or Higgs couplings.

depends on the Hubble rate and thus on the temperature. A non-equilibrium state is then characterized by deviations of the slowly relaxing quantities from their equilibrium values which are much larger than a typical thermal fluctuation.

One of the slowly relaxing quantities is $L-B$. It is violated by the Yukawa interactions between Standard Model leptons and right-handed (gauge singlet or "sterile") neutrinos N_I, with Majorana masses M_I. Therefore, if at some time in the evolution of the Universe $L-B$ is non-zero, these interactions tend to reduce it (examples of processes are shown in figure 1). There can also be a source term for the lepton number if the number density of right-handed neutrinos deviates from thermal equilibrium and if their interactions violate CP. This would lead to baryogenesis through leptogenesis [1].

If the slow variables have only small deviations from equilibrium, we can linearize their equations of motion. For example, assuming that the only slow degrees of freedom are the lepton minus baryon number density n_{L-B} and the right-handed neutrino phase space distribution f_N (momentum, spin and flavour indices are suppressed and it is assumed that the entanglement of the right-handed neutrinos does not play a role), the evolution equations are of the form

$$\begin{aligned} \mathcal{D}_t\, f_N &= -\gamma^{(2)}_N \left[f_N - f_{\rm eq}\right] - \gamma^{(4)}_{N,L-B}\, n_{L-B} + \dots\,, \\ \mathcal{D}_t\, n_{L-B} &= -\gamma^{(2)}_{L-B}\, n_{L-B} - \gamma^{(4)}_{L-B,N} \left[f_N - f_{\rm eq}\right] + \dots\,, \end{aligned} \tag{2.1}$$

where $\mathcal{D}_t$ is the appropriate time derivative in an expanding background, $f_{\rm eq}$ is the equilibrium distribution, and terms of higher orders in deviations from equilibrium and in time derivatives have been neglected.[4] Within this effective description, the coefficients γ are independent of the values of the slow variables n_{L-B} and $f_N - f_{\rm eq}$ and of the Hubble rate appearing in $\mathcal{D}_t$. In particular, they can be determined arbitrarily close to equilibrium and with vanishing Hubble rate, to any order in Standard Model couplings. This is the philosophy that underlies Kubo relations [9].

The coefficients $\gamma^{(2)}$ in eq. (2.1) are of $O(h^2)$, where h denotes a generic neutrino Yukawa coupling. The terms containing $\gamma^{(4)}$ violate CP, and must contain additional Yukawa couplings with makes them $O(h^4)$. If the terms omitted from eq. (2.1) are suppressed by increasingly high powers of h, and h is very small, we can make use of a perturbative expansion in h. At leading order in h^2, $\gamma^{(2)}_N$ can be determined by setting $f_N = 0$ because the contributions from the omitted higher-order terms are of $O(h^4 f^2_{\rm eq})$. Then $\gamma^{(2)}_N f_{\rm eq}$ agrees with the right-handed neutrino production rate of which a lot is known: it has been computed up to NLO in SM couplings in the non-relativistic [10–12] and relativistic [13, 14] regimes, and to LO in the ultrarelativistic regime [15, 16], which necessitates a resummation of the loop expansion.

[4]In situations where the deviation from thermal equilibrium is sizeable, e.g. in the weak washout regime, non-linear terms could also play a role. One would expect the dominant non-linear contribution to $\mathcal{D}_t\, n_{L-B}$ to be of order $h^2[f_N - f_{\rm eq}]\, n_{L-B}$.

With a similar philosophy, the CP-violating source term $\gamma^{(4)}_{L-B,N} f_{\rm eq}$ has been expressed in terms of a Green's function which could in principle be evaluated at NLO [17].

In the present paper, we are concerned with the coefficient $\gamma^{(2)}_{L-B}$, which may be called the lepton minus baryon number washout or dissipation rate. Within the range of validity of eq. (2.1), it can be computed in a system with $f_N = f_{\rm eq}$, $\mathcal{D}_t = \partial_t$, and assuming n_{L-B} to be close to its equilibrium value. If $f_N \neq f_{\rm eq}$, the rate $\mathcal{D}_t\, n_{L-B}$ changes, but the change originates from the other terms in eq. (2.1) rather than from a change of $\gamma^{(2)}_{L-B}$.

3 General analysis

3.1 Conserved and approximately conserved charges

As a first step one has to identify the spectator processes. These processes conserve some charges, which by Noether's theorem are associated with the symmetries that the corresponding interactions respect. Some of these symmetries will be broken by the slow interactions, and the corresponding charges will no longer be conserved. If some symmetries remain unbroken, there are linearly independent charges $X_{\bar a}$ which are still conserved.[5] In addition, there are charges X_a which together with the $X_{\bar a}$ form a linearly independent set, such that the original charges can be written as linear combinations of the X_a and $X_{\bar a}$. By definition the X_a are no longer conserved. Their values can now depend on time and one finds equations of motion for them of the type in eq. (2.1). The choice of the non-conserved charges X_a is not unique, because after adding a conserved charge they are still non-conserved. However, we consider a state in which the strictly conserved charges vanish. Then this ambiguity is irrelevant.

For computing the washout rates the expansion of the Universe can be ignored, as well as the interactions which are much slower than the expansion. We assume that the thermal expectation values of the X_a vanish, $\langle X_a \rangle_{\rm eq} = 0$. Now we consider a non-equilibrium system in which the X_a start with some non-zero values, which we assume to be much larger than their typical thermal fluctuation. The non-equilibrium state is completely specified by the values of X_a. Thus, their time derivatives only depend on their values and on the temperature. For sufficiently small values one can expand ${\rm d}X_a/{\rm d}t$ in powers of X_a, and keep only the linear term,

$$\frac{{\rm d}X_a}{{\rm d}t} = -\gamma_{ab} X_b \,. \tag{3.1}$$

It turns out that the coefficients γ_{ab} can be factorized into two parts (cf. eq. (3.10)): to a real-time "spectral function" $\rho_{ac}(\omega)$, and to the inverse of a static "susceptibility matrix", denoted by Ξ^{-1}_{cb}. The latter contains similar information as the flavour matrix A introduced in ref. [18], or the coefficients c_ℓ, c_φ that appear widely in leptogenesis literature (cf. ref. [19]). We define these two parts in turn.

3.2 Kubo relations for washout rates

To determine the coefficients γ_{ab} we proceed similarly to the general method[6] described in ref. [21]. We work to leading order in the neutrino Yukawa interaction (see below), but, if not stated otherwise, to all orders in other interactions.

[5] We choose this set to be complete in the sense that there are no additional charges which are linearly independent of the $X_{\bar a}$ and which are conserved. In the actual analysis, it may be possible to choose the $X_{\bar a}$ such that not all of them need to be included explicitly, cf. the discussion below eq. (3.18).

[6] For a recent application in relativistic field theory see ref. [20].

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Even in thermal equilibrium the values of the X_a are not constant in time, but instead they fluctuate around their equilibrium values which we assume to be zero. For *small frequencies* these fluctuations can be described by an effective classical theory with an equation of motion similar to eq. (3.1). The only difference compared with eq. (3.1) is that on the right-hand side there is an additional gaussian noise term. This equation of motion can be solved to write the fluctuation of X_a at time t in terms of its value at time zero plus the contribution from the noise. This can be used to compute the real-time correlation function

$$\mathcal{C}_{ab}(t) \equiv \langle X_a(t) X_b(0) \rangle \ , \tag{3.2}$$

where $\langle \cdots \rangle$ denotes the thermal average over an ensemble in which the strictly conserved charges are zero and do not fluctuate. The effective description is classical, so the ordering inside the average does not play a role. Since the noise is gaussian it drops out when taking the noise average, and one obtains

$$\mathcal{C}_{ab}(t) = \left(e^{-\gamma t}\right)_{ac} \langle X_c X_b \rangle \ , \qquad t > 0 \ . \tag{3.3}$$

Taking the one-sided Fourier (or Laplace) transform we obtain

$$\begin{aligned} \mathcal{C}^{+}_{ab}(\omega) &\equiv \int_0^\infty \mathrm{d}t \, e^{i\omega t}\, \mathcal{C}_{ab}(t) \\ &= -\left(i\omega - \gamma\right)^{-1}_{ac} \langle X_c X_b \rangle \ , \end{aligned} \tag{3.4}$$

where ω has a positive imaginary part.

The effective description of the correlation function in eq. (3.2) is valid when $\omega \ll \omega_{\rm UV}$, where $\omega_{\rm UV}$ is a characteristic frequency of the fluctuations of other, "faster" relaxation processes, the spectator processes. Now let ω be real and $\omega \gg \gamma$, where γ denotes the absolute value of the largest eigenvalue of γ_{ab}. Then one can expand eq. (3.4) and finds

$$\mathrm{Re}\, \mathcal{C}^{+}_{ab}(\omega + i0^+) = \frac{1}{\omega^2}\, \gamma_{ac} \langle X_c X_b \rangle + O\left(\omega^{-4}\right) \ . \tag{3.5}$$

On the other hand, the correlation function in eq. (3.2) can also be computed in the microscopic quantum field theory. By matching the results in the regime $\gamma \ll \omega \ll \omega_{\rm UV}$ one can then obtain the coefficients γ_{ab}, together with a consistency check on the functional form of the ω-dependence. In quantum field theory we define the correlation function as

$$C_{ab}(t) \equiv \left\langle \, \frac{1}{2}\{X_a(t), X_b(0)\} \, \right\rangle , \tag{3.6}$$

where the angular brackets indicate an average over a thermal ensemble in which the values of all strictly conserved charges are zero. Making use of text-book relations between frequency-space correlators involving anticommutators and commutators,[7] the one-sided Fourier transform can now be expressed as

$$C^{+}_{ab}(\omega) = \int \frac{\mathrm{d}\omega'}{2\pi} \, \frac{i}{\omega - \omega'} \left[\frac{1}{2} + f_{\rm B}(\omega') \right] \rho_{ab}(\omega') \ , \tag{3.7}$$

[7] For two-sided Fourier transforms, $C_{ab}(\omega) = [1/2 + f_{\rm B}(\omega)]\rho_{ab}(\omega)$, with ρ_{ab} from eq. (3.8).

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where $f_{\rm B}$ is the Bose distribution. Here the spectral function for bosonic operators X_a and X_b is defined as

$$\rho_{ab}(\omega) \equiv \int \mathrm{d}t\, e^{i\omega t} \big\langle\, [X_a(t), X_b(0)]\, \big\rangle \ . \tag{3.8}$$

If time reversal T is a symmetry, this spectral function is real (cf. appendix B of ref. [22]). If one neglects the CP violation in the Standard Model and in the Yukawa interactions of the RH neutrinos (the latter is of $O(h^4)$), CP is a symmetry, and so is T. In this approximation ρ_{ab} is real. Then for real ω,

$$\mathrm{Re}\, C^{+}_{ab}(\omega + i0^+) = \frac{1}{2}\left[\frac{1}{2} + f_{\rm B}(\omega)\right]\rho_{ab}(\omega) \ . \tag{3.9}$$

Let us now compare eqs. (3.5) and (3.9). We are interested in $\omega \ll \omega_{\rm UV} \lesssim T$, in which case we can approximate the expression in square brackets by T/ω. Matching the two expressions gives

$$\gamma_{ab} = \frac{1}{2V}\, \omega \rho_{ac}(\omega) \left(\Xi^{-1}\right)_{cb} \ , \qquad \text{for } \gamma \ll \omega \ll \omega_{\rm UV} \ . \tag{3.10}$$

Here V is the spatial volume, and Ξ is a matrix of susceptibilities,

$$\Xi_{ab} \equiv \frac{1}{TV} \langle X_a X_b \rangle \ . \tag{3.11}$$

As will be seen later on, Ξ_{ab} is finite for $V \to \infty$, and the same is true for γ_{ab}. A consistency check is provided by whether ρ_{ac} indeed has a $1/\omega$-tail at $\omega \ll \omega_{\rm UV}$.

The restriction on ω in eq. (3.10) limits the accuracy at which the γ_{ab} can be defined. The γ_{ab} are of order $h^2\Lambda$, where h is a RH neutrino Yukawa coupling to be defined below, and $\Lambda = \max\{\pi T, M_I\}$.[8] A rough estimate gives a relative accuracy $\gamma/\omega_{\rm UV} \sim h^2$ modulo coupling constants which characterize spectator processes. It is therefore probably not meaningful to calculate the γ_{ab} beyond leading order in h^2. Of course, radiative corrections due to SM interactions can be computed and can be important.

3.3 The susceptibility matrix

In order to make use of eq. (3.10) we need to determine the susceptibility matrix Ξ defined in eq. (3.11), which turns out to require a little care (cf. ref. [23]). It is perhaps simplest to think of the problem in an ensemble in which the values of the strictly conserved charges $X_{\bar a}$ are zero and do not fluctuate.[9] We write the Hamilton operator as

$$H = H_0 + H_{\rm int} \tag{3.12}$$

where H_0 describes all free particles as well as their X_a-conserving interactions and $H_{\rm int}$ is the interaction which violates X_a-conservation. The susceptibilities are equal-time correlation functions, for which $H_{\rm int}$ is a small perturbation. Since we only consider the leading order in h, the susceptibilities can be computed using H_0, which commutes with the X_a. In this

[8] Cf. eq. (4.17) for the way that mass and thermal scales can be compared with each other.

[9] This is equivalent to a "grand canonical" ensemble in which the charges $X_{\bar a}$ do fluctuate but their expectation values are zero, cf. eq. (3.15).

approximation the X_a are conserved, and the ordering of the operators in eq. (3.11) does not matter.

To compute the susceptibilities start with a grand canonical partition function

$$\exp\left(-\Omega/T\right) = \mathrm{tr}\,\exp\left[(\mu_A X_A - H_0)/T\right] \;, \tag{3.13}$$

with chemical potentials μ_A for all charges $X_A \in \{X_a, X_{\bar a}\}$. Some of the $X_{\bar a}$ may be gauge charges. In this case the role of $\mu_{\bar a}$ is played by the zero-momentum mode of the time component of the gauge field [23]. The thermodynamic potential $\widetilde\Omega$ corresponding to fixed $X_{\bar a} = -\partial\Omega/\partial\mu_{\bar a}$ is given by the Legendre transform $\widetilde\Omega = \Omega + \mu_{\bar a}X_{\bar a}$. We are interested in $X_{\bar a} = 0$, so we have $\widetilde\Omega = \Omega$ with the $\mu_{\bar a}$ determined by

$$\frac{\partial\Omega}{\partial\mu_{\bar a}} = 0\;. \tag{3.14}$$

The required susceptibilities are

$$\Xi_{ab} = -\frac{1}{V}\,\frac{\partial^2\widetilde\Omega}{\partial\mu_a\partial\mu_b}\bigg|_{\mu=0} = -\frac{1}{V}\frac{\partial^2}{\partial\mu_a\partial\mu_b}\left\{\Omega|_{\partial\Omega/\partial\mu_{\bar a}=0}\right\}_{\mu=0}\;. \tag{3.15}$$

In order to evaluate eq. (3.15) it is sufficient to expand Ω to second order,

$$\Omega = \Omega_0 - \frac{V}{2}\mu_A\chi_{AB}\mu_B + O(\mu^4)\;, \tag{3.16}$$

with the grand canonical susceptibilities

$$\chi_{AB} \equiv \frac{1}{TV}\langle X_A X_B\rangle_{\text{grand canonical}} = -\frac{1}{V}\,\frac{\partial^2\Omega}{\partial\mu_A\partial\mu_B}\bigg|_{\mu=0}\;. \tag{3.17}$$

Then eq. (3.14) reads

$$\chi_{\bar a\,\bar b}\,\mu_{\bar b} = -\chi_{\bar a\,b}\mu_b\;. \tag{3.18}$$

Here we see which strictly conserved charges need to be included: the ones which are correlated with strictly conserved charges which are correlated with one of the X_a. The remaining ones need not be taken into consideration. This is the case, e.g., for non-abelian gauge charges in the symmetric phase. Solving eq. (3.18) gives $\mu_{\bar a} = -\left(\xi^{-1}\right)_{\bar a\bar b}\chi_{\bar b c}\,\mu_c$ where the matrix ξ is defined by

$$\xi_{\bar a\bar b} \equiv \chi_{\bar a\bar b}\;. \tag{3.19}$$

Inserting this into eq. (3.15) we find

$$\Xi_{ab} = \chi_{ab} - \chi_{a\bar a}\left(\xi^{-1}\right)_{\bar a\bar b}\chi_{\bar b b}\;. \tag{3.20}$$

4 Lepton number washout rate

4.1 Master formula

So far the discussion was general and did not make any use of the specifics of the interaction which breaks the X_a-symmetry. In the basic leptogenesis scenario the SM is extended by adding right-handed neutrino fields N_I with Majorana masses M_I (we employ a basis in which the Majorana mass matrix is diagonal). In the simplest realization they interact with the SM particles via a Yukawa coupling to ordinary, left-handed lepton doublets $\ell_i \equiv \ell_{\mathrm{L}i}$ and the Higgs doublet φ as follows:

$$\mathcal{L}_{\text{int}} = -\overline{N}\, h\, \widetilde{\varphi}^{\dagger}\, \ell + \text{h.c.} \ . \tag{4.1}$$

Here $\widetilde{\varphi} \equiv i\sigma^2\varphi^*$ with the Pauli matrix σ^2 is the isospin conjugate of φ, and the Yukawa couplings are written as a matrix in flavour space, $h = (h_{Ij})$.

In the simplest case it is only the interaction in eq. (4.1) which is responsible for inducing slowly evolving processes. In more complicated cases some SM Yukawa interactions proceed at a similar rate, and these interactions have to be taken into account as well [24]. Here we restrict ourselves to the first situation. What the relevant conserved and quasi-conserved charges $X_{\bar{a}}$ and X_a are, depends on the expansion rate of the Universe and thus on the temperature. One has to take into account all interactions which are much faster than the expansion. The quasi-conserved charges may contain many different fields, for instance both lepton and quark fields. For this reason it is convenient to compute the spectral function of the time derivatives $\dot{X}_a$ of the charges rather than of the charges directly, and then use

$$\omega^2 \rho_{ab}(\omega) = \int \mathrm{d}t\, e^{i\omega t} \left\langle \left[\dot{X}_a(t), \dot{X}_b(0) \right] \right\rangle \ . \tag{4.2}$$

The operators $\dot{X}_a$ only contain the fields which interact via $\mathcal{L}_{\text{int}}$. Furthermore, $\dot{X}_a$ contains h explicitly. Since we compute only to leading order in h, the thermal average on the right-hand side of eq. (4.2) can be taken in an ensemble with $h = 0$. Therefore we can re-express eq. (3.10) as

$$\gamma_{ab} = \frac{1}{2V} \lim_{\omega\to 0} \frac{1}{\omega} \int \mathrm{d}t\, e^{i\omega t} \left\langle \left[\dot{X}_a(t), \dot{X}_c(0) \right] \right\rangle_0 (\Xi^{-1})_{cb} \ , \tag{4.3}$$

where the subscript 0 indicates that $h = 0$ in the average, so that h only appears in the operators. Equation (4.3) has some similarity with Kubo formulas for transport coefficients [9, 22], in particular for flavour diffusion (cf. ref. [25]).

As already mentioned, the choice of the broken charges is not unique: adding some linear combination of the conserved ones we again obtain a charge which is not conserved. It is possible to choose the symmetries so that they only act on SM particles. Let the left-handed leptons transform as

$$\ell_i \to \left(e^{i\alpha_a T_a^{\ell}} \right)_{ij} \ell_j \ , \tag{4.4}$$

with Hermitian matrices T_a^{ℓ}. The interaction Lagrangian in eq. (4.1) is not invariant under this transformation. Following the usual steps to derive Noether's theorem one finds

$$X_a = \int_{\mathbf{x}} \left[\sum_i \overline{\ell}_i \gamma^0 T_a^{\ell} \ell_i + (\text{contributions from other fields}) \right] , \tag{4.5}$$

and

$$\dot{X}_a = i \int_{\mathbf{x}} \left[\overline{N}\, h\, \widetilde{\varphi}^{\dagger}\, T_a^{\ell}\, \ell - \overline{\ell}\, T_a^{\ell}\, \widetilde{\varphi}\, h^{\dagger} N \right] . \tag{4.6}$$

Given that the thermal average in eq. (4.3) is performed with $h = 0$, we can integrate out the RH neutrinos treating them as free fields. In a basis where the Majorana mass matrix is diagonal, a straightforward calculation[10] yields for eq. (4.3)

$$\gamma_{ab} = -\frac{1}{2}\sum_I \int_{\mathbf{k}} \frac{f'_{\rm F}(E_I)}{2E_I} \times h_{Ii}\,{\rm tr}\left[\not{k}\left(T^\ell_a\big[\,\widetilde{\rho}(k)+\widetilde{\rho}(-k)\,\big]T^\ell_c + T^\ell_c\big[\,\widetilde{\rho}(k)+\widetilde{\rho}(-k)\,\big]T^\ell_a \right)_{ij}\right] h^*_{Ij}\left(\Xi^{-1}\right)_{cb}\,, \qquad (4.7)$$

where the trace refers to the spinor indices, and $k^0 = E_I \equiv (\mathbf{k}^2 + M_I^2)^{1/2}$. The prime is a derivative with respect to energy, and $f_{\rm F}$ is the Fermi-Dirac distribution. We have introduced the spectral function for the composite operator of the SM fields that appears in eq. (4.6),

$$\widetilde{\rho}_{ij\alpha\beta}(k) \equiv \int_x e^{ik\cdot x}\Big\langle \Big\{ (\widetilde{\varphi}^\dagger \ell_{i\alpha})(x), (\bar{\ell}_{j\beta}\,\widetilde{\varphi})(0) \Big\} \Big\rangle_0\,, \qquad (4.8)$$

where α, β are Dirac spinor indices. Note that for fermionic operators the spectral function is defined with anticommutators.

Equations (4.7), (4.8), together with the expression for the Ξ matrix in eq. (3.15) for charges like those in eq. (4.5) constitute the main formal results of this paper. We stress again that these expressions are valid to any order in SM couplings.

If charged lepton Yukawa interactions can be neglected, $\widetilde{\rho}$ is invariant under $U(3)_\ell$ and thus $\widetilde{\rho}_{ij} \propto \delta_{ij}$. Writing $\widetilde{\rho}_{ij\alpha\beta}(k) = \delta_{ij}\,\widetilde{\rho}_{\alpha\beta}(k)$, we can then re-express eq. (4.7) as

$$\gamma_{ab} = \frac{1}{2}\sum_I h_{Ii}\left\{T^\ell_a, T^\ell_c\right\}_{ij} h^*_{Ij}\left(\Xi^{-1}\right)_{cb}\,\mathcal{W}(M_I)\,, \qquad (4.9)$$

$$\mathcal{W}(M_I) \equiv -\int_{\mathbf{k}} \frac{f'_{\rm F}(E_I)}{2E_I}\,(\not{k})_{\beta\alpha}\big[\widetilde{\rho}_{\alpha\beta}(k) + \widetilde{\rho}_{\alpha\beta}(-k)\big]\,. \qquad (4.10)$$

4.2 Spectral function

We proceed to discussing how the real-time part of the washout rate, i.e. the weighted integral over $\widetilde{\rho}$ in eqs. (4.7) and (4.10), can be evaluated in practice.

4.2.1 Leading order in the non-relativistic regime

Consider first the LO contributions to $\widetilde{\rho}$ in the symmetric phase in a regime $M_I \gg \pi T$. In this case one can neglect the thermal masses of the SM particles[11] which can then be treated as massless and free. For timelike k ($k^2 > 0$) but with k^0 of either sign one obtains

$$\widetilde{\rho}_{\alpha\beta}(k) = 2\int_{\mathbf{p}} \frac{1}{2p^0}\,(P_{\rm L}\,\not{p})_{\alpha\beta}\int_{\mathbf{q}}\frac{1}{2q^0}\big[1 - f_{\rm F}(p^0) + f_{\rm B}(q^0)\,\big] \times (2\pi)^4\Big[\delta^4(p+q-k) + \delta^4(p+q+k)\Big]\,, \qquad (4.11)$$

[10] For instance, making use of the imaginary-time formalism, (4.3) contains a 2-point correlator of the right-handed neutrino fields and of the composite operators to which they couple according to (4.6). The former is of the familiar form $(\not{k}+M_I)/(k^2-M_I^2)$, with $k^0 \to ik_n$; the mass in the numerator is projected out by the Dirac trace. The latter can be expressed in a spectral representation as $\Sigma(p_n,\mathbf{p}) = \int {\rm d}p^0/(2\pi)\,\widetilde{\rho}(p^0,\mathbf{p})/(p^0 - ip_n)$, where $\widetilde{\rho}$ is from (4.8). Matsubara sums can be carried out, and generate Fermi-Dirac distributions. A subsequent analytic continuation yields a retarded real-time correlator, and its cut yields the spectral function needed in (4.3). This contains structures like $f_{\rm F}(E_I+\omega) - f_{\rm F}(E_I)$, which after taking the limit $\lim_{\omega\to 0}(\ldots)/\omega$ leave over $f'_{\rm F}(E_I)$.

[11] Thermal masses have to be taken into account for $M_I \lesssim \sqrt{g}T$ [14].

where $p^0 \equiv |\mathbf{p}|$, $q^0 \equiv |\mathbf{q}|$, and the left chiral projector is defined as $P_\mathrm{L} \equiv (1-\gamma_5)/2$. For eq. (4.10) we get

$$\begin{aligned}\mathcal{W}(M_I) = & -4 \int_\mathbf{k} \frac{1}{2E_I} \int_\mathbf{p} \frac{1}{2p^0} \int_\mathbf{q} \frac{1}{2q^0} \\ & \times 2p\cdot k\, f_\mathrm{F}'(E_I) \left[1 - f_\mathrm{F}(p^0) + f_\mathrm{B}(q^0)\right] (2\pi)^4 \delta^4(p+q-k) \,. \end{aligned} \tag{4.12}$$

When $M_I \gg \pi T$, the Bose and Fermi distributions in the square brackets are exponentially suppressed with $\exp(-M_I/T)$ and can be neglected. Omitting terms of order $\exp(-M_I/T)$ also in $f_\mathrm{F}'(E_I)$, it is straightforward to perform the integrals, yielding

$$\gamma_{ab} = \frac{1}{16\pi^3} \sum_I M_I^3\, K_1(M_I/T)\, h_{Ii} \left\{T^\ell_a, T^\ell_c\right\}_{ij} h^*_{Ij} \left(\Xi^{-1}\right)_{cb} \,, \qquad \pi T \ll M_I \,. \tag{4.13}$$

Here K_1 is a modified Bessel function of the second kind. Once a LO susceptibility from section 4.3.1 is inserted, this reduces to a standard result, cf. eq. (4.26).

4.2.2 Next-to-leading order

In general the spectral function in eq. (4.8) contains two independent Dirac structures at finite temperature [26]. However the Dirac trace in eq. (4.7) is exactly the same as appears in the right-handed neutrino production rate, projecting out a particular linear combination of the Dirac structures. Assuming as before that $\widetilde{\rho}_{ij\alpha\beta}(k) = \delta_{ij}\, \widetilde{\rho}_{\alpha\beta}(k)$, the production rate of flavour I reads

$$\gamma_I^+ = \sum_i |h_{Ii}|^2\, \mathcal{P}(M_I) \,, \tag{4.14}$$

$$\mathcal{P}(M_I) \equiv \int_\mathbf{k} \frac{f_\mathrm{F}(E_I)}{2E_I} (\not{K})_{\beta\alpha} \left[\widetilde{\rho}_{\alpha\beta}(k) + \widetilde{\rho}_{\alpha\beta}(-k)\right] . \tag{4.15}$$

Comparing with eq. (4.10), it is seen that $\mathcal{W}$ differs from $\mathcal{P}$ only through a weight, $-f_\mathrm{F}'(E_I)$ versus $f_\mathrm{F}(E_I)$.[12] Therefore, $\mathcal{W}$ can be extracted from known results for $\mathcal{P}$.

The extraction of $\mathcal{W}$ is particularly simple in the non-relativistic regime $\pi T \ll M_I$, where $f_\mathrm{F}'(E_I) = -f_\mathrm{F}(E_I)/T$. Displaying contributions involving the $\mathrm{U}_Y(1)$, $\mathrm{SU}_\mathrm{L}(2)$ and $\mathrm{SU}(3)$ gauge couplings g_1, g_2 and g_3, the Higgs self-coupling λ, with the tree-level value $\lambda = g_2^2 m_H^2/(8m_W^2)$, as well as the top Yukawa coupling h_t, the Dirac trace reads [11]

$$(\not{K})_{\beta\alpha} \left[\widetilde{\rho}_{\alpha\beta}(k) + \widetilde{\rho}_{\alpha\beta}(-k)\right] = \frac{M_I^2}{2\pi} \left[1 + c_1 + \frac{c_2\, \mathbf{k}^2}{M_I^2} + O\left(\frac{\mathbf{k}^4}{M_I^4}\right)\right] , \tag{4.16}$$

where

$$\begin{aligned} c_1 = & -\frac{\lambda T^2}{M_I^2}\left(1 - \frac{3m_\varphi}{\pi T}\right) - |h_t|^2 \left[\frac{3}{(4\pi)^2}\left(\ln\frac{\bar\mu^2}{M_I^2} + \frac{7}{2}\right) + \frac{7\pi^2 T^4}{60 M_I^4}\right] \\ & + (g_1^2 + 3g_2^2)\left[\frac{3}{4(4\pi)^2}\left(\ln\frac{\bar\mu^2}{M_I^2} + \frac{29}{6}\right) - \frac{\pi^2 T^4}{80 M_I^4}\right] + O\left(g^4, \frac{g^2 T^6}{M_I^6}\right) , \end{aligned} \tag{4.17}$$

$$c_2 = -|h_t|^2 \frac{7\pi^2 T^4}{45 M_I^4} - (g_1^2 + 3g_2^2)\frac{\pi^2 T^4}{60 M_I^4} + O\left(\frac{g^4 T^4}{M_I^4}, \frac{g^2 T^6}{M_I^6}\right) . \tag{4.18}$$

[12] An intuitive reason for the difference is that in the production rate the combination $\sim f_\mathrm{F}(E_I + \mu) + f_\mathrm{F}(E_I - \mu)$ appears whereas in the dissipation rate it is the difference $\sim f_\mathrm{F}(E_I + \mu) - f_\mathrm{F}(E_I - \mu)$ that plays a role. Here μ is a chemical potential induced by the Yukawa interaction.

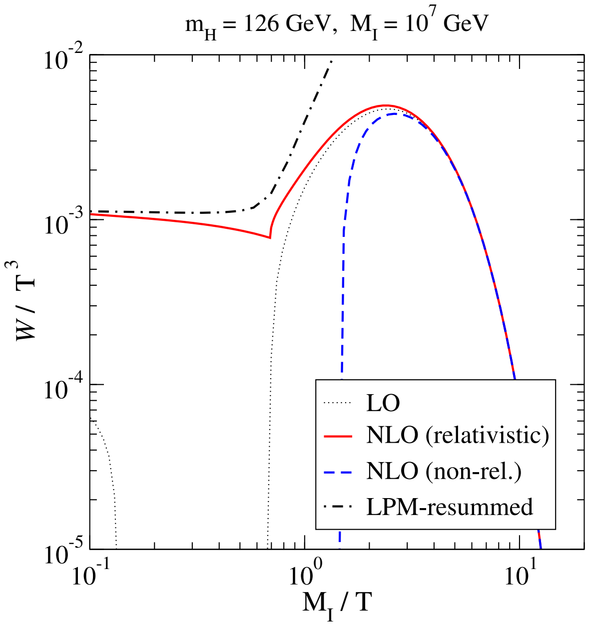

Figure 2. The washout rate expressed through $\mathcal{W}$ as defined in eq. (4.10). Shown are the LO result (dotted line); the relativistic NLO result (solid line, from ref. [14]); the NLO result in the non-relativistic approximation (dashed line, from eq. (4.20)); and the LPM-resummed result valid in the ultrarelativistic regime $M_I \lesssim gT$ (dash-dotted line, from ref. [16]).

Here $\bar\mu$ is the $\overline{\text{MS}}$ renormalization scale related to the neutrino Yukawa couplings appearing in eq. (4.9), and m_φ is the thermal Higgs mass parameter,

$$m_\varphi^2 \equiv m_0^2 + \frac{T^2}{16}\Big(g_1^2 + 3g_2^2 + 4|h_t|^2 + 8\lambda\Big) , \tag{4.19}$$

where $m_0^2 < 0$ is the vacuum value, and we assume $m_\varphi^2 > 0$. Integrating over the momenta in eq. (4.10) yields corrections to eq. (4.13):

$$\begin{aligned}\gamma_{ab} = \frac{1}{16\pi^3}\sum_I M_I^3 &\left[(1+c_1)\, K_1\left(\frac{M_I}{T}\right) + \frac{3c_2 T}{M_I} K_2\left(\frac{M_I}{T}\right) + O\left(\frac{T^{17/2}e^{-M_I/T}}{M_I^{17/2}}\right)\right] \\ &\times h_{Ii}\left\{T_a^\ell, T_c^\ell\right\}_{ij} h_{Ij}^* \left(\Xi^{-1}\right)_{cb} , \qquad \pi T \ll M_I .\end{aligned} \tag{4.20}$$

If the temperature is increased to $T \gtrsim M_I/4$, which may be relevant e.g. for setting the initial conditions for leptogenesis, we leave the non-relativistic regime [14]. The NLO production rate in the relativistic regime ($\pi T \sim M_I$) and the LO production rate in the ultrarelativistic regime ($gT \gtrsim M_I$) are also known but only in numerical form [14, 16].[13] The results of refs. [14, 16] for $\mathcal{P}(M_I)/T^4$ are plotted in figure 5 of ref. [14]. The results of refs. [14, 16] for $\mathcal{W}(M_I)/T^3$, obtained by changing the weight from $f_F(E_I)$ to $-f_F'(E_I)$, are shown in figure 2. It is seen how the perturbative expansion breaks down and resummations are necessary for $T \gtrsim M_I$, and how the subsequent washout rate is strongly enhanced compared with a naive tree-level analysis ("LO" in the plot). It is also clear that the non-relativistic

[13] For $gT \gtrsim M_I$, multiple gauge interactions need to be resummed to obtain the correct LO result; the resummation can be expressed as a solution of an inhomogeneous differential equation [15, 16].

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expansion of eq. (4.20) breaks down for $T \gtrsim M_I/4$. (To be more precise, the non-relativistic expansion shows convergence only for $T \lesssim M_I/15$, but the smallness of *any* loop corrections allows it to be used in practice up to somewhat higher temperatures [14].)

4.3 Susceptibility matrix

The susceptibilities as given by eq. (3.11) or eq. (3.15) measure the correlations in fluctuations of the slowly varying charges X_a when the other charges $X_{\bar{a}}$ are constrained to vanish. The susceptibility matrix influences the lepton number washout rate as indicated in eq. (3.10), (4.7) and (4.9). Here we show how the susceptibilities can be computed in practice, first at leading order and then including corrections of $O(g)$ and $O(g^2)$. We consider the Standard Model with left-handed quarks ($q \equiv q_{\rm L}$) and leptons ($\ell \equiv \ell_{\rm L}$); as well as right-handed up-type quarks ($u \equiv u_{\rm R}$), down-type quarks ($d \equiv d_{\rm R}$), and charged leptons ($e \equiv e_{\rm R}$).

4.3.1 Leading order

The susceptibilities defined by eq. (3.11) are, at LO, determined by free field theory. It turns out that at NLO it is helpful to first evaluate the grand canonical potential Ω, and then extract the answer directly from the second relation in eq. (3.15). However, we start by showing how at LO the results can also be obtained from the 2-point correlators in eq. (3.17) through eq. (3.20).

For eq. (3.17) one has to compute fluctuations of charges, which for free fields are in one-to-one correspondence with particle number fluctuations. Start with the fluctuation $\langle Q^2 \rangle$ where Q is the difference of particle and antiparticle number of a single fermion species. By a single fermion species we mean either one chiral fermion or a single spin state of a Dirac fermion. For only one fermionic degree of freedom, we have the particle number fluctuation $\langle (\Delta N)^2 \rangle = VT^3/12$. The fluctuations of particles and antiparticles are uncorrelated which implies $\langle Q^2 \rangle = VT^3/6$.

The fluctuations of the charges

$$Q_a = \int_{\mathbf{x}} \overline{\psi} \gamma^0 T_a \psi = \int_{\mathbf{x}} \psi^\dagger T_a \psi \tag{4.21}$$

of a set of left-chiral fermion fields ψ_i, $\psi_i = P_{\rm L} \psi_i$, are given by

$$\langle Q_a Q_b \rangle = V \int_{\mathbf{x}} \Big\langle \Big(\psi^\dagger T_a \psi \Big)(x) \Big(\psi^\dagger T_b \psi \Big)(0) \Big\rangle \ . \tag{4.22}$$

The free propagators are flavour diagonal which directly implies $\langle Q_a Q_b \rangle = VT\chi_{ab}$ with the susceptibilities

$$\chi_{ab} = \mathrm{tr}\,(T_a T_b) \frac{T^2}{6} \ . \tag{4.23}$$

We also need the fluctuation of the weak hypercharge Y_φ of the Higgs field. For a single scalar particle species (including the antiparticle) of unit charge the charge fluctuation is $\langle Q^2 \rangle = VT^3/3$. Therefore, counting both isospin states,

$$\langle Y_\varphi^2 \rangle = VT \times \frac{2}{3} y_\varphi^2 T^2 \ , \tag{4.24}$$

with the Higgs hypercharge $y_\varphi = 1/2$.

Consider now the simplest realistic case, with only one right-handed neutrino N_1 at a temperature $T \gg 10^{13}$ GeV. Then only top Yukawa and gauge interactions are in equilibrium. All other SM Yukawa interactions, as well as strong and weak sphalerons can be neglected. It is then sufficient to consider only the left-handed leptons ℓ, the Higgs, the 3rd family quark doublet q_3 and the right-handed top t. Without $H_{\rm int}$ we have a $\mathrm{U}(3)_\ell$ symmetry as well as two U(1) symmetries, generated by the baryon number B_{q_3t} carried by q_3 and t, and by the hypercharge $Y_{q_3t\varphi}$ carried by q_3, t, and φ (we denote $X_{AB\cdots} \equiv X_A + X_B + \cdots$).[14] Using a $\mathrm{U}(3)_\ell$ transformation we choose the fields such that N_1 only couples to the ℓ of one family, which we denote by ℓ_{N_1}. Without $H_{\rm int}$ the corresponding lepton number L_{N_1} is conserved. It is broken by $H_{\rm int}$, together with $Y_{q_3t\varphi}$, leaving $Y_{q_3t\varphi\ell_{N_1}}$ unbroken. Thus the only X_a can be chosen as L_{N_1}, and the set of $X_{\bar a}$ consists of $Y_{q_3t\varphi\ell_{N_1}}$ and B_{q_3t}.[15] If we arrange the charges in this order we obtain at LO

$$\chi = \begin{pmatrix} 1/3 & -1/6 & 0 \\ -1/6 & 1/2 & 1/6 \\ 0 & 1/6 & 1/6 \end{pmatrix} T^2 \, . \tag{4.25}$$

Then from eq. (3.20) Ξ is just a number, $\Xi = T^2/4$. In this scenario $h_{Ii}\{T_a, T_c\}_{ij} h^*_{Ij} = 2|h_{11}|^2$. Undoing the $\mathrm{U}(3)_\ell$ rotation, eq. (4.13) subsequently gives the LO result [6] (it corresponds to $c_\ell = 1$, $c_\varphi = 2/3$ in the notation of ref. [19])

$$\gamma_{L_{N_1}} = \sum_i \frac{|h_{1i}|^2}{2\pi^3} \frac{M_1^3}{T^2} K_1\left(\frac{M_1}{T}\right) \, . \tag{4.26}$$

4.3.2 Next-to-leading order

We now include Standard Model interactions. The up-type, down-type, and charged lepton Yukawa couplings are denoted by h_{uij}, h_{dij}, h_{eij}, respectively, where $i,j \in \{1,2,3\}$ label families.

In order to determine the susceptibilities, it is convenient to first compute the pressure [27], $P(T,\mu)$, as a function of the temperature and the chemical potentials associated with all charges X_A. The pressure determines the grand canonical potential through

$$\Omega = -P(T,\mu) V \, . \tag{4.27}$$

One should include only interactions which are in thermal equilibrium (see the general discussion in sections 2 and 3). We assume this is the case for the gauge interactions, and, depending on the Hubble rate and thus on the temperature, some Standard Model Yukawa interactions.

If one collects all fermion fields in one big spinor ψ, the fermionic contribution to the conserved charges can be written as

$$X_A = \int_{\mathbf{x}} \overline{\psi} \gamma^0 T_A \psi \, , \tag{4.28}$$

with hermitian matrices T_A. Including the chemical potentials corresponds to adding a term $\overline{\psi}\gamma^0 \mu \psi$ with $\mu \equiv \mu_A T_A$ to the Lagrangian. For simplicity we carry out the computation in

[14] In principle one could have chosen the total hypercharge Y as one of the $X_{\bar a}$. However, this would be rather inconvenient because there are a lot of conserved charges which are correlated with Y and which would then all have to be included in the set of X_A. Note also that when employing (4.23) one has to keep in mind that the T_a may contain unit matrices in colour or weak isospin space which would contribute factors of $N_{\rm colour}$ or $N_{\rm weak\,isospin}$ to the trace (cf. section 4.3.2).

[15] Without $H_{\rm int}$ the charges corresponding to the off-diagonal generators which mix ℓ_{N_1} with the other families are conserved as well, and they are also broken by $H_{\rm int}$. Here we consider only the dissipation of L_{N_1}; the evolution of diagonal and off-diagonal charges decouples at leading order.

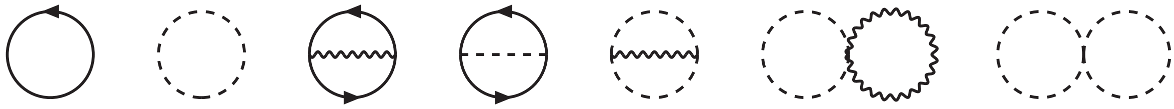

Figure 3. 1- and 2-loop graphs contributing to the pressure $P(T,\mu)$, from which the lepton number susceptibilities can be extracted according to eq. (4.35). Solid, dashed, and wiggly lines correspond to fermions, scalars, and gauge fields, respectively.

the symmetric phase of the electroweak theory. In this situation the T_A commute with weak isospin rotations in addition to colour rotations. They are thus block diagonal and can be written as $T_A = T_A^q \otimes \mathbb{1}_{\text{colour}} \otimes \mathbb{1}_{\text{weak isospin}} + T_A^u \otimes \mathbb{1}_{\text{colour}} + \cdots$. Correspondingly, the chemical potential matrix takes the form $\mu = \mu_q \otimes \mathbb{1}_{\text{colour}} \otimes \mathbb{1}_{\text{weak isospin}} + \mu_u \otimes \mathbb{1}_{\text{colour}} + \cdots$, where μ_q, $\mu_u, \ldots$ are matrices in family space. Then the fermion propagators are matrices in family space as well.

The dependence of $P(T,\mu)$ on the chemical potentials is needed only up to quadratic order (cf. eq. (3.17)). The results of section 4.3.1 correspond to (unresummed) 1-loop contributions to the pressure. The Standard Model interactions enter at two loops. All 1- and 2-loop Feynman diagrams are displayed in figure 3. Since gauge interactions are flavour blind, the chemical potentials can be diagonalized. In the Lagrangian

$$\mathcal{L}_{\text{SM-Yukawa}} = -\overline{u}\, h_u\, \widetilde{\varphi}^\dagger\, q - \overline{d}\, h_d\, \varphi^\dagger\, q - \overline{e}\, h_e\, \varphi^\dagger\, \ell + \text{ h.c.} \tag{4.29}$$

we include only Yukawa interactions which are in equilibrium. The terms included have to be invariant under the symmetry transformations generated by the X_A. This implies relations between the T_A and the in-equilibrium Yukawa couplings,

$$\begin{aligned} -T_A^u h_u + h_u T_A^\varphi + h_u T_A^q &= 0\,, \\ -T_A^d h_d - h_d T_A^\varphi + h_d T_A^q &= 0\,, \\ -T_A^e h_e - h_e T_A^\varphi + h_e T_A^\ell &= 0\,, \end{aligned} \tag{4.30}$$

where T_A^φ is simply a number. Multiplying by μ_A one finds relations between the chemical potentials,

$$\begin{aligned} -\mu_u h_u + h_u(\mu_q + \mu_\varphi) &= 0\,, \\ -\mu_d h_d + h_d(\mu_q - \mu_\varphi) &= 0\,, \\ -\mu_e h_e + h_e(\mu_\ell - \mu_\varphi) &= 0\,. \end{aligned} \tag{4.31}$$

Yukawa couplings mediating reactions not in equilibrium have to be omitted.

By making use of eqs. (4.31) and their hermitian conjugates, as well as substitutions of sum-integration variables, the 2-loop computation can be reduced to products of the following 1-loop sum-integrals:

$$\begin{aligned} T\sum_{p_n}\int_{\mathbf{p}} \frac{1}{(p_n - i\mu)^2 + \mathbf{p}^2 + m_\varphi^2} &= \int_{\mathbf{p}} \frac{1 + f_{\text{B}}(E_\varphi - \mu) + f_{\text{B}}(E_\varphi + \mu)}{2E_\varphi} \\ &= \frac{T^2}{12}\left(1 - \frac{3m_\varphi}{\pi T}\right) + \frac{\mu^2}{8\pi^2}\left(\frac{\pi T}{m_\varphi} - 1\right) + \ldots\,, \end{aligned} \tag{4.32}$$

$$\begin{aligned} T\sum_{\{p_n\}}\int_{\mathbf{p}} \frac{1}{(p_n - i\mu)^2 + \mathbf{p}^2} &= \int_{\mathbf{p}} \frac{1 - f_{\text{F}}(|\mathbf{p}| - \mu) - f_{\text{F}}(|\mathbf{p}| + \mu)}{2|\mathbf{p}|} \\ &= -\frac{T^2}{24} - \frac{\mu^2}{8\pi^2}\,. \end{aligned} \tag{4.33}$$

Here p_n denotes bosonic and $\{p_n\}$ fermionic Matsubara frequencies. The parameter m_φ^2 is the thermal Higgs mass given by eq. (4.19), $E_\varphi \equiv (\mathbf{p}^2 + m_\varphi^2)^{1/2}$, and we assume $\mu^2 \ll m_\varphi^2 \ll (\pi T)^2$. The bosonic result in eq. (4.32) is an expansion with higher orders omitted, whereas the fermionic result in eq. (4.33) is exact (in dimensional regularization). It is important to keep in mind that the divergent $1/m_\varphi$ terms, appearing through (4.32), need to be "daisy resummed" (or "thermal mass resummed") in order to obtain a consistent weak-coupling expansion [28].

A straightforward computation making use of eqs. (4.32), (4.33) and implementing the appropriate resummation yields[16]

$$\begin{aligned}
\frac{12[P(T,\mu)-P(T,0)]}{T^2} &= 6\left[1-\frac{3}{8\pi^2}\left(\frac{g_1^2}{36}+\frac{3g_2^2}{4}+\frac{4g_3^2}{3}\right)\right]\mathrm{tr}(\mu_q^2)+3\left[1-\frac{3}{8\pi^2}\left(\frac{4g_1^2}{9}+\frac{4g_3^2}{3}\right)\right]\mathrm{tr}(\mu_u^2)\\
&+3\left[1-\frac{3}{8\pi^2}\left(\frac{g_1^2}{9}+\frac{4g_3^2}{3}\right)\right]\mathrm{tr}(\mu_d^2)\\
&+2\left[1-\frac{3}{8\pi^2}\left(\frac{g_1^2}{4}+\frac{3g_2^2}{4}\right)\right]\mathrm{tr}(\mu_\ell^2)+\left[1-\frac{3}{8\pi^2}g_1^2\right]\mathrm{tr}(\mu_e^2)\\
&+4\left[1-\frac{3m_\varphi}{2\pi T}+\frac{3}{4\pi^2}\left(2\lambda+\frac{g_1^2+3g_2^2}{8}\right)\right]\mu_\varphi^2\\
&+3\left[\frac{1}{4\pi^2}\,\mathrm{tr}(h_u h_u^\dagger)\mu_\varphi^2-\frac{3}{8\pi^2}\,\mathrm{tr}\left(h_u^\dagger h_u\mu_q^2+h_u h_u^\dagger\mu_u^2\right)\right]\\
&+3\left[\frac{1}{4\pi^2}\,\mathrm{tr}(h_d h_d^\dagger)\mu_\varphi^2-\frac{3}{8\pi^2}\,\mathrm{tr}\left(h_d^\dagger h_d\mu_q^2+h_d h_d^\dagger\mu_d^2\right)\right]\\
&+\left[\frac{1}{4\pi^2}\,\mathrm{tr}(h_e h_e^\dagger)\mu_\varphi^2-\frac{3}{8\pi^2}\,\mathrm{tr}\left(h_e^\dagger h_e\mu_\ell^2+h_e h_e^\dagger\mu_e^2\right)\right]+O(\mu^4)\,. \qquad (4.34)
\end{aligned}$$

The leading correction is the term proportional to $m_\varphi/(\pi T) \sim g/\pi$. It is the leading contribution of soft ($\mathbf{k} \sim m_\varphi$) Higgs bosons, from the thermal mass resummed [28] 1-loop diagram. Thus, whereas NLO corrections to the spectral function are of $O(g^2)$ in the relativistic and non-relativistic regimes, corrections to susceptibilities already start at $O(g)$. Numerically, $m_\varphi \lesssim 0.6T$ everywhere in the symmetric phase, so the correction is less than 30%. The terms of $O(g^2)$ in eq. (4.34) contribute to the susceptibilities at next-to-next-to-leading order (NNLO). We expect that additional contributions of the same order appear at two loops when also the gauge boson thermal masses are resummed, but we have not calculated these terms.

According to eq. (3.15), the desired matrix is obtained from

$$\Xi_{ab} = \frac{\partial^2}{\partial\mu_a\partial\mu_b}\Big\{ P(T,\mu)|_{\partial P/\partial\mu_{\bar a}=0}\Big\}\,. \qquad (4.35)$$

Relevant for us is the inverse Ξ^{-1}, cf. eqs. (4.7), (4.9). Let us give a few examples:

(i) Very high temperatures ($T \gtrsim 10^{13}$ GeV). This case was already discussed at leading order in section 4.3.1. Here only the top Yukawa interaction, the gauge interactions, and the Higgs self-coupling have to be included ($h_{uij} \to h_t\,\delta_{i3}\delta_{j3}$, $h_{dij} \to 0$, $h_{eij} \to 0$). Denoting by μ_Y, μ_B, and μ_L the chemical potentials of $Y_{q_3 t\varphi\ell_{N_1}}$, $B_{q_3 t}$, and L_{N_1}, the chemical potentials in eq. (4.34) are

$$\mu_{q_3} = \frac{\mu_Y}{6}+\frac{\mu_B}{3}, \qquad \mu_{u_3} = \frac{2\mu_Y}{3}+\frac{\mu_B}{3}, \qquad \mu_{\ell_1} = -\frac{\mu_Y}{2}+\mu_L, \qquad \mu_\varphi = \frac{\mu_Y}{2}; \quad (4.36)$$

[16] Higher orders could be worked out with the same techniques as employed in ref. [29].

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all other chemical potentials vanish. From eqs. (4.34), (4.35) we obtain for the inverse of the susceptibility

$$\Xi^{-1} = \frac{4}{T^2}\left\{1 + \frac{1}{16\pi^2}\left[\frac{4(\pi T + m_\varphi)m_\varphi}{T^2} + \frac{37g_1^2}{36} + \frac{11g_2^2}{4} + \frac{2g_3^2}{3} - \frac{|h_t|^2}{12} - 4\lambda\right]\right\}. \quad (4.37)$$

The leading term here agrees with that obtained from eq. (4.25). The correction represents, numerically, an increase of the washout rate of about 4%.

(ii) As a second example, we consider the same temperature as above ($T \gtrsim 10^{13}$ GeV), but allow for three right-handed neutrinos, and consider the evolution of three lepton densities. For a basis in which the matrix h is diagonal we have $(T_a^\ell)_{ij} = \delta_{ai}\delta_{aj}$ and

$$\mu_{q_3} = \frac{\mu_Y}{6} + \frac{\mu_B}{3}, \qquad \mu_{u_3} = \frac{2\mu_Y}{3} + \frac{\mu_B}{3}, \qquad \mu_{\ell_i} = -\frac{\mu_Y}{2} + \mu_{Li} \qquad \mu_\varphi = \frac{\mu_Y}{2}, \quad (4.38)$$

with all other chemical potentials set to zero. The same exercise now leads to

$$\begin{aligned}\Xi^{-1} &= \frac{3}{T^2}\left\{1 + \frac{3(g_1^2 + 3g_2^2)}{32\pi^2}\right\}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \\ &+ \frac{1}{T^2}\left\{1 + \frac{1}{16\pi^2}\left[\frac{16(\pi T + m_\varphi)m_\varphi}{T^2}\right.\right. \\ &\qquad\left.\left. - \frac{7g_1^2}{18} - \frac{5g_2^2}{2} + \frac{8g_3^2}{3} - \frac{|h_t|^2}{3} - 16\lambda\right]\right\}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}. \quad (4.39)\end{aligned}$$

The diagonal components of this matrix agree with eq. (4.37). The dissipation matrix of eq. (4.9) becomes $\gamma_{ab} = |h_{aa}|^2\mathcal{W}(M_a)\,\Xi^{-1}_{ab}$. This is non-symmetric and non-diagonal; the non-diagonal components determine how the lepton numbers L_b, $b \neq a$, influence the evolution of L_a (cf. eq. (3.1)).

(iii) The final example is a "low" temperature ($10^2\,\text{GeV} \lesssim T \lesssim 10^5\,\text{GeV}$), such that all Standard Model interactions are in equilibrium. Among them are strong sphalerons, but they have no particular effect since the chirality flipping processes are also mediated by the quark Yukawa interactions. The electroweak sphalerons violate lepton and baryon numbers. The SM interactions conserve the charges $X_i = L_i - B/3$ and the hypercharge Y. Unless some of the neutrino Yukawa couplings vanish, H_{int} breaks all X_i-symmetries, leaving only Y conserved. Then,

$$\begin{aligned}\mu_{q_i} &= \frac{\mu_Y}{6} + \frac{\mu}{3}, & \mu_{u_i} &= \frac{2\mu_Y}{3} + \frac{\mu}{3}, & \mu_{d_i} &= -\frac{\mu_Y}{3} + \frac{\mu}{3}, \\ \mu_{\ell_i} &= -\frac{\mu_Y}{2} - \mu_{X_i}, & \mu_{e_i} &= -\mu_Y - \mu_{X_i}, & \mu_\varphi &= \frac{\mu_Y}{2}, \end{aligned} \quad (4.40)$$

where $\mu \equiv \frac{1}{3}\sum_i \mu_{X_i}$, and $i = 1,2,3$. The extremization in eq. (4.35) takes place with respect to μ_Y, which then leads to

$$\begin{aligned}\Xi^{-1} &= \frac{2}{T^2}\left\{1+\frac{3(g_1^2+g_2^2)}{(4\pi)^2}\right\}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\\&\quad+\frac{40}{237T^2}\left\{1+\frac{27}{790\pi^2}\left[\frac{16\pi m_\varphi}{T}+\frac{312m_\varphi^2}{79T^2}\right.\right.\\&\qquad\left.\left.-\frac{3749g_1^2}{288}-\frac{1813g_2^2}{288}+\frac{121g_3^2}{3}-\frac{11|h_t|^2}{24}-16\lambda\right]\right\}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}. \qquad (4.41)\end{aligned}$$

Numerically, the correction appearing in the second structure of eq. (4.41) is 23%, and it is dominated by the term proportional to g_3^2. Close to the electroweak crossover, where $m_\varphi \sim g^2T/\pi$, the perturbative expansion associated with the Matsubara zero modes breaks down, and non-perturbative methods are needed for determining Ξ.

For $T \lesssim 130\,\text{GeV}$, the sphaleron processes violating $B+L$ are so slow that B is effectively conserved [30]. Then Ξ is a different 3×3 matrix from the above. In a narrow temperature range around $T\sim 130\,\text{GeV}$, both B and L_i need to be treated as separate slow variables, and Ξ is a 4×4 matrix. For practical purposes it may be sufficient to solve separate 3-variable non-equilibrium problems in the regimes $T \gtrsim 130\,\text{GeV}$ and $T \lesssim 130\,\text{GeV}$ and just match the solutions at $T\sim 130\,\text{GeV}$ by requiring continuity.

5 Lepton asymmetry

To get an idea on the numerical effect of radiative corrections we have computed the lepton asymmetry in a scenario with $M_1 = 10^{14}\,\text{GeV}$, and $M_I \gg M_1$ for $I\neq 1$. This corresponds to the example in section 4.3.1 and example (i) in section 4.3.2. For the washout factor

$$K \equiv \left.\frac{\Gamma_0}{H}\right|_{T=M_1}, \qquad (5.1)$$

where

$$\Gamma_0 = \frac{M_1}{8\pi}\sum_i |h_{1i}|^2 \qquad (5.2)$$

is the tree-level decay rate of N_1, we have used $K=7$. We have started the evolution with zero initial asymmetry and thermal N_1-number density at $T=M_1$. We have used the non-relativistic approximation [6], and solved the evolution equations until $T=M_1/10$, below which the asymmetry hardly changes any more.

We find that the effect of the $O(g)$ corrections to Ξ on the asymmetry is about 3%. The order $O(g^2)$ corrections to Ξ and to $\widetilde{\rho}$ are 1.3% and 1%, and the total effect of the corrections on the final asymmetry is $\sim 5\%$.

If, in contrast, a scenario like in ref. [2] is considered, in which temperatures around the weak scale play a role and the dynamics takes place in the ultrarelativistic regime ($M_I \ll T$), then it is clear from figure 2 and from the discussion below eq. (4.41) that effects of order 100% are to be expected. We have not carried out numerics for this scenario, however.

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6 Summary and conclusions

In this paper we have obtained a relation, eq. (4.7), between the lepton number washout rate relevant for leptogenesis, and finite temperature equilibrium correlation functions. The washout rate factorizes into a real-time spectral function, and an inverse susceptibility matrix which is determined by equilibrium thermodynamics (cf. eq. (3.15)). This relation does not make use of any particle approximation, and is valid to all orders in Standard Model couplings and at any temperature. The main approximation made is that we have worked to order h^2 in neutrino Yukawa couplings, which should be a good approximation in many popular leptogenesis scenarios.

We have computed explicitly the spectral function and the susceptibility matrix to next-to-leading order (NLO) in Standard Model couplings for temperatures above the electroweak crossover temperature but below the mass M_1 of the lightest right-handed neutrino, i.e., when the right-handed neutrinos are non-relativistic ($150\,\text{GeV} \lesssim \pi T \ll M_1$). This is particularly relevant for leptogenesis in the strong washout regime.

We find that even in the non-relativistic regime there are corrections only suppressed by $O(g)$. They originate from Higgs effects on the susceptibility matrix, cf. eqs. (4.37), (4.39), (4.41). In contrast, NLO corrections to the spectral function are of $O(g^2)$ in this regime, cf. eq. (4.17). Numerically, the $O(g)$ corrections are a few percent, except for temperatures close to the electroweak crossover, where they can be substantially larger.

In the relativistic regime $M_1 \lesssim \pi T$, the susceptibilities remain unmodified since they are insensitive to M_1. In contrast, the spectral functions become increasingly sensitive to infrared corrections, and extensive resummations are needed for obtaining even the complete *leading-order* results for $M_1 \lesssim gT$. We have shown that fortunately, the results can be inferred, after minor modifications, from existing computations of the right-handed neutrino production rate [14, 16]. Numerical results are shown in figure 2. The lepton number washout rate of the relativistic regime plays a role at the initial stage of the classic leptogenesis process, erasing some of the lepton asymmetry that is being generated when right-handed neutrinos are produced from the Standard Model plasma, and would also be relevant for scenarios in which the right-handed neutrino masses are at or below the weak scale ([2] and references therein).

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