PTEP

Prog. Theor. Exp. Phys. **2022** 123A02(8 pages)
DOI: 10.1093/ptep/ptac160

IYBSSD2022
International Year
of Basic Sciences
for Sustainable Development

Noether's 1st theorem with local symmetries

Sinya Aoki*

Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-Ku, Kyoto 606-8502, Japan

*E-mail: saoki@yukawa.kyoto-u.ac.jp

Received October 8, 2022; Revised November 22, 2022; Accepted November 23, 2022; Published November 28, 2022

Noether's 2nd theorem applied to a total system states that a global symmetry which is a part of local symmetries does not provide a physically meaningful conserved charge but it instead leads to off-shell constraints as a form of conserved currents. In this paper, we propose a general method to derive a matter-conserved current associated with a special global symmetry in the presence of local symmetries. While currents derived from local symmetries of a matter sector with a covariant background gauge field are not conserved in general, we show that the current associated with a special type of a global symmetry, called a hidden matter symmetry, is on-shell conserved. We apply this derivation to a $U(1)$ gauge theory, general relativity and non-abelian gauge theory. In general relativity, the associated conserved charge agrees with the one recently proposed from a different point of view.

Subject Index A02, B00, B06, E00

1. Introduction

While Noether's 1st theorem tells us how to define a conserved current if a theory is invariant under a global continuous transformation, her 2nd theorem concludes that, if a symmetry is local, the conservation of a current is merely an identity or a constraint rather than a consequence of the dynamics of a system [1]. For example, a trivially conserved current appear as $K^\mu = \partial_\nu f^{\mu\nu}$ for an arbitrary anti-symmetric function $f^{\nu\mu} = -f^{\mu\nu}$. Noether's 2nd theorem is also applied to a global symmetry which is a part of some local symmetries [1–3]. This fact prevents us from defining a physically meaningful conserved energy in general relativity [1]. Indeed, existing definitions of energy in general relativity correspond to conserved charges of Noether's 2nd theorem, the conservation of which is physically trivial [3]. Furthermore, it brings us the question of what is a conserved electric charge in electrodynamics, which has the local $U(1)$ gauge invariance.

In this paper, we propose a general method to derive a matter-conserved current associated with a special global symmetry which is a part of local symmetries. In Sect. 2, while a total system suffers from Noether's 2nd theorem, a matter sector, which alone is still invariant under the local transformation in the presence of a covariant background gauge field, is shown to have a conserved current of Noether's 1st theorem for a special type of a global symmetry, which we call a hidden matter symmetry. We apply the general method to three examples—a $U(1)$ gauge theory, general relativity and a non-abelian gauge theory—in Sect. 3. We show that an electric charge in the $U(1)$ gauge theory is conserved due to a constant $U(1)$ transformation of the matter sector. In the case of general relativity, on the other hand, the vector for the global transformation giving a conserved current explicitly depends on the background metric like a Killing vector. We show that our definition of a conserved charge in general relativity is nothing but that which was recently proposed by a collaboration including the present author without using symmetric argument [4,5]. The background field dependence of the global transformation plays an important role in the non-abelian gauge theory as well. In Sect. 4, our conclusions and discussions are given. In particular, we stress that the difference between $U(1)$ gauge theory and others is that local gauge transformations are field-independent only in the $U(1)$ gauge theory, which is related to the absence of non-linear terms in the Lagrangian for gauge fields.

2. Noether's 1st Theorem in a theory with local symmetries

We consider a theory with local symmetries, whose Lagrangian density with a parameter λ is given by

$$L_\lambda = L_G(g, g_{,\alpha}, g_{,\alpha\beta}) + \lambda \sum_n L_n(g, h_n, h_{n,\alpha}), \tag{1}$$

where $g = \{g_j\}$ is the gauge field, $h_n = \{h_n^j\}$ represents the matter field coupled to g without its derivatives, and $g_{,\alpha} := \partial_\alpha g$, etc. Here n labels a sector of matter fields such that h_n and h_m have no direct coupling between them if $n \neq m$. We write $E^n \overset{h_n}{\approx} 0$ or $E^G \overset{g}{\approx} 0$ as an equation of motion (EOM) for h_n or g, respectively, where $\overset{\Phi}{\approx}$ means that Φ is on-shell, while $\approx$ means that all fields are on-shell. Explicitly we have

$$E^n := \frac{\partial L_n}{\partial h_n} - \partial_\alpha \left(\frac{\partial L_n}{\partial h_{n,\alpha}} \right) \overset{h_n}{\approx} 0, \tag{2}$$

$$E^G := \frac{\partial L}{\partial g} - \partial_\alpha \left(\frac{\partial L_G}{\partial g_{,\alpha}} \right) + \partial_\alpha \partial_\beta \left(\frac{\partial L_G}{\partial g_{,\alpha\beta}} \right) \overset{g}{\approx} 0. \tag{3}$$

We assume that L_G and L_n are scalar density under infinitesimal local transformations generated by ξ^a as $\delta_\xi x^\alpha = X_a^\alpha(x)\xi^a(x)$:

$$\begin{aligned} \delta_\xi g(x) &= G_a(g)\xi^a(x) + G_a^\beta(g)\xi^a_{,\beta}(x), \\ \delta_\xi h_n(x) &= H_{n:a}(h_n, g)\xi^a(x) + H_{n:a}^\beta(h_n, g)\xi^a_{,\beta}(x), \end{aligned} \tag{4}$$

where a summation over a or β is implicitly assumed. It is useful to define $\bar\delta_\xi \Phi := \delta_\xi \Phi - \Phi_{,\beta}\delta_\xi x^\beta$ with $\Phi = g, h_n$, since it commutes with derivatives as $\bar\delta_\xi \Phi_{,\alpha} = \partial_\alpha \bar\delta_\xi \Phi$ [6].

2.1 *A total system and Noether's 2nd theorem*

A local symmetry of an action integral for a total Lagrangian L_λ leads to

$$\int d^d x \left[E^G \bar\delta_\xi g + \lambda \sum_n E^n \bar\delta_\xi h_n + \partial_\alpha N_\lambda^\alpha[\xi] \right] = 0, \tag{5}$$

where

$$N_\lambda^\alpha[\xi] := N_G^\alpha[\xi] + \lambda \sum_n N_n^\alpha[\xi], \tag{6}$$

$$N_G^\alpha[\xi] := \frac{\partial L_G}{\partial g_{,\alpha}} \bar\delta_\xi g + \frac{\partial L_G}{\partial g_{,\alpha\beta}} \partial_\beta \bar\delta_\xi g - \partial_\beta \left(\frac{\partial L_G}{\partial g_{,\alpha\beta}} \right) \bar\delta_\xi g + L_G \delta_\xi x^\alpha, \tag{7}$$

$$N_n^\alpha[\xi] := \frac{\partial L_n}{\partial h_{n,\alpha}} \bar{\delta}_\xi h_n + L_n \delta_\xi x^\alpha. \tag{8}$$

Thus the current $N_\lambda^\alpha[\xi]$ is on-shell conserved as

$$\partial_\alpha N_\lambda^\alpha[\xi] = -E^G \bar{\delta}_\xi g - \lambda \sum_n E^n \bar{\delta}_\xi h_n \approx 0 \tag{9}$$

for an arbitrary ξ including a constant ξ. Therefore Noether's 1st theorem for a constant ξ seems to hold at first sight. The fact that $\partial_\alpha N_\lambda^\alpha[\xi] \approx 0$ for an arbitrary ξ, however, implies stronger relations which spoils the usefulness of the on-shell conserved current $N_\lambda^\alpha[\xi]$, as shown below.

Using Eq. (4), we rewrite Eq. (5) as

$$\int d^d x \left\{ O_{\lambda:a} \xi^a + \partial_\alpha K_\lambda^\alpha[\xi] \right\} = 0, \tag{10}$$

where $O_{\lambda:\, a}$ is a ξ-independent function of fields g, h_n and their derivatives, and

$$K_\lambda^\alpha[\xi] := K_G^\alpha[\xi] + \lambda \sum_n K_n^\alpha[\xi] = N_\lambda^\alpha[\xi] + \left(E^G G_a^\alpha + \lambda \sum_n E^n H_{n:a}^\alpha \right) \xi^a, \tag{11}$$

$$K_G^\alpha[\xi] := N_G^\alpha[\xi] + \left(E^G - \lambda \sum_n \frac{\partial L_n}{\partial g} \right) G_a^\alpha \xi^a, \tag{12}$$

$$K_n^\alpha[\xi] := N_n^\alpha[\xi] + \left(\frac{\partial L_n}{\partial g} G_a^\alpha + E^n H_{n:a}^\alpha \right) \xi^a. \tag{13}$$

Noether's 2nd theorem tells us that $O_{\lambda:\, a} = 0$ and $\partial_\alpha K_\lambda^\alpha[\xi] = 0$ without using an EOM [1,6]. Thus $K_\lambda^\alpha[\xi]$ is nothing but an explicit example of $K^\alpha = \partial_\beta f^{\alpha\beta}$ with $f^{\beta\alpha} = -f^{\alpha\beta}$, the conservation of which is trivial. Eq. (11) says that the conserved current $N_\lambda^\alpha[\xi]$ is equal to the trivial one up to EOMs as

$$N_\lambda^\alpha[\xi] = K_\lambda^\alpha[\xi] - \left(E^G G_a^\alpha + \lambda \sum_n E^n H_{n:a}^\alpha \right) \xi^a \approx K_\lambda^\alpha[\xi], \tag{14}$$

which is the conclusion from an application of Noether's 2nd theorem to a global symmetry as a part of local symmetries [1].

2.2 *Noether's 1st theorem for a matter sector*

In order to escape from Noether's 2nd theorem for the total system, we consider the matter action, the invariance of which under the local transformation gives

$$\int d^d x \left[\frac{\partial L_n}{\partial g} \bar{\delta}_\xi g + E^n \bar{\delta}_\xi h_n + \partial_\alpha N_n^\alpha[\xi] \right] = 0, \tag{15}$$

where

$$N_n^\alpha[\xi] := \frac{\partial L_n}{\partial h_{n,\alpha}} \bar{\delta}_\xi h_n + L_n \delta_\xi x^\alpha. \tag{16}$$

Thus the current is *not* conserved for a general ξ as

$$\partial_\alpha N_n^\alpha[\xi] = -\frac{\partial L_n}{\partial g} \bar{\delta}_\xi g - E^n \bar{\delta}_\xi h_n \overset{h_n}{\approx} -\frac{\partial L_n}{\partial g} \bar{\delta}_\xi g, \tag{17}$$

even if the EOM of the matter is employed.

If a vector ζ_n however satisfies

$$\frac{\partial L_n}{\partial g} \bar{\delta}_{\zeta_n} g := \sum_j \frac{\partial L_n}{\partial g_j} \bar{\delta}_{\zeta_n} g_j \overset{h_n}{\approx} 0 \tag{18}$$

for a fixed (background) g, which may or may not satisfy its EOM, $N_n^\alpha[\zeta_n]$ is on-shell conserved as

$$\partial_\alpha N_n^\alpha[\zeta_n] \overset{h_n}{\approx} 0. \tag{19}$$

Thus $N_n^\alpha[\zeta_n]$ is a physically meaningful conserved current of Noether's 1st theorem for a global symmetry generated by a particular $\zeta_n(x)$, which we call a hidden matter symmetry. Since Eq. (18) usually becomes one simple linear partial differential equation, it is easy to find a non-trivial solution in general, as discussed in Ref. [5] for the case of general relativity. In addition, an explicit solution has been known for time-dependent but spherically symmetric spacetimes in general relativity [7]. Since the EOM for g is not required here, the conservation holds for an off-shell g as well as an on-shell g, as long as g is kept fixed in L_n. In this paper, the special vector is denoted as $\zeta_n(x)$ in order to avoid complicated expressions, instead of using the more precise expression $\zeta_n[g, h_n](x)$, which makes its dependence on g and h_n manifest.

Furthermore, since $K_n^\alpha[\xi]$ is trivially conserved as a consequence of Noether's 2nd theorem for a matter action, we redefine the current $N_n^\alpha[\xi]$ as

$$J_n^\alpha[\xi] := N_n^\alpha[\xi] - K_n^\alpha[\xi] = -\left(\frac{\partial L_n}{\partial g} G_a^\alpha + E^n H_{n:a}^\alpha\right)\xi^a, \tag{20}$$

which satisfies $\partial_\alpha J_n^\alpha[\xi] = \partial_\alpha N_n^\alpha[\xi]$.

Under Eq. (18), the new current $J_n^\alpha[\zeta_n]$ is also conserved as

$$\partial_\alpha J_n^\alpha[\zeta_n] = -\frac{\partial L_n}{\partial g}\bar\delta_{\zeta_n} g - E^n \bar\delta_{\zeta_n} h_n \overset{h_n}{\approx} 0, \tag{21}$$

where the conserved current $J_n^\alpha[\zeta_n]$ is given by

$$J_n^\alpha[\zeta_n] = -\left(\frac{\partial L_n}{\partial g} G_a^\alpha + E^n H_{n:a}^\alpha\right)\zeta_n^a \overset{h_n}{\approx} -\frac{\partial L_n}{\partial g} G_a^\alpha \zeta_n^a. \tag{22}$$

While $N_n^\alpha[\zeta_n]$ is related to the variation of L_n with respect to the matter field h_n, the on-shell $J_n^\alpha[\zeta_n]$ is determined by couplings between g and h_n in L_n. Since $N_n^\alpha[\zeta_n]$ and $J_n^\alpha[\zeta_n]$ are equivalent up to the trivially conserved current $K_n^\alpha[\xi]$, we focus our attention on $J_n^\alpha[\zeta_n]$ in the remainder of this paper. Thus, Eqs. (22) and (21), together with Eq. (18), are the main formulae used in this paper.

3. Examples

We apply the method in the previous section to three examples, in order to construct a matter-conserved charge for the hidden matter symmetry in each case.

3.1 *U(1) gauge theory*

For a $U(1)$ gauge theory, we denote a gauge field A_μ, and a matter field ψ_{nj}, which transform under a $U(1)$ gauge symmetry by $\theta(x)$ as

$$\delta_\theta A_\mu = \frac{1}{e}\theta_{,\mu},\ \delta_\theta \psi_{nj} = i q_{nj}\theta\psi_{nj}, \tag{23}$$

where e is a charge unit and q_{nj} is a $U(1)$ representation of ψ_{nj}.

Eq. (20) leads to

$$J_n^\alpha[\theta] = -\frac{1}{e}\frac{\partial L_n}{\partial A_\alpha}\theta = N_n^\alpha[\theta] = \sum_j i q_{nj}\frac{\partial L_n}{\psi_{nj,\alpha}}\theta, \tag{24}$$

which satisfies

$$\partial_\alpha J_n^\alpha[\theta] = -\frac{1}{e}\frac{\partial L_n}{\partial A_\alpha}\partial_\alpha\theta - i\sum_j E^{nj} q_{nj}\psi_{nj}\theta, \tag{25}$$

where E^{nj} is an EOM for ψ_{nj}. Therefore, for a constant $\theta(x) = \theta_0$, the generator of the hidden matter symmetry of this theory, $J_n^\alpha[\theta_0]$ becomes the on-shell conserved current from Noether's 1st theorem for a global $U(1)$ symmetry generated by the constant θ_0, and is equal to (a part of) the current in the EOM for A_μ as

$$\frac{\partial L_G}{\partial A_\mu} - \partial_\alpha\left(\frac{\partial L_G}{\partial A_{\mu,\alpha}}\right) = \lambda e\sum_n J_n^\mu[\theta_0 = 1]. \tag{26}$$

The fact that the global transformation generated by θ_0 is independent of the gauge field A_μ is the reason why a conserved electric charge can be easily defined in electrodynamics at the classical level, and this property of the charge conservation can be faithfully carried over to QED. Note that the electric charge is separately conserved in each n even for the $U(1)$ gauge field A_μ to be dynamical, so that there is no charge exchange between different sectors labeled by n.

A distinction between matter fields and gauge fields becomes non-trivial if a $U(1)$ gauge symmetry is contained non-trivially as part of a larger gauge symmetry. In the case of the electromagnetic $U(1)_{\rm em}$ symmetry of the $U(1)_{\rm Y}\otimes SU(2)$ electroweak gauge symmetry, for example, the current $J_n^\alpha[\theta_0]$ defined from the matter action L_n without including W gauge bosons is *not* conserved, since $\delta_{\theta_0} W_\mu^\pm \neq 0$. Thus we should include the action for W gauge bosons in our definition of the matter action to ensure its invariance under the constant $U(1)_{\rm em}$ transformation.

3.2 *General relativity*

Under a general coordinate transformation by $\delta_\xi x^\alpha = \xi^\alpha$, the gravity (metric) and matter field, $g_{\mu\nu}$ and ϕ_n, transform as

$$\delta_\xi g_{\mu\nu} = \bar\delta_\xi g_{\mu\nu} + g_{\mu\nu,\alpha}\xi^\alpha,\ \ \bar\delta_\xi g_{\mu\nu} = -(\nabla_\mu\xi_\nu + \nabla_\mu\xi_\mu),$$
$$\delta_\xi\phi_n = H_n{}^\alpha{}_\beta\,\xi^\beta_{,\alpha}. \tag{27}$$

A (k, l) tensor $T^A{}_B := T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ transforms as

$$\delta_\xi T^A{}_B = -\mathcal{L}_\xi T^A{}_B + T^A{}_{B,\alpha}\xi^\alpha, \tag{28}$$

where $\mathcal{L}_\xi$ is the Lie derivative with the vector ξ.

For a matter sector n, Eqs. (20) and (21) read

$$J_n^\alpha[\xi] = (2\mathcal{T}_n{}^\alpha{}_\beta - E^n H_n{}^\alpha{}_\beta)\xi^\beta \overset{\phi_n}{\approx} 2\mathcal{T}_n^{\alpha\beta}\xi_\beta,$$
$$\partial_\alpha J_n^\alpha[\xi] = \mathcal{T}_n^{\alpha\beta}\bar\delta_\xi g_{\alpha\beta} - E^n\bar\delta_\xi\phi_n \overset{\phi_n}{\approx} \mathcal{T}_n^{\alpha\beta}\bar\delta_\xi g_{\alpha\beta}, \tag{29}$$

where E^n is the EOM for the matter ϕ_n, and

$$\mathcal{T}_n^{\alpha\beta} = \sqrt{-g}\,T_n^{\alpha\beta} := \frac{\partial L_n}{\partial g_{\alpha\beta}} \tag{30}$$

is the symmetric energy momentum tensor (density), which appears in the EOM for $g_{\mu\nu}$ as

$$E^{g_{\mu\nu}} := \frac{\partial L_G}{\partial g_{\mu\nu}} - \partial_\alpha\left(\frac{\partial L_G}{\partial g_{\mu\nu,\alpha}}\right) + \partial_\alpha\partial_\beta\left(\frac{\partial L_G}{\partial g_{\mu\nu,\alpha\beta}}\right) + \lambda\sum_n \mathcal{T}_n^{\mu\nu} \overset{g_{\mu\nu}}{\approx} 0. \tag{31}$$

On the other hand, $N_n^\alpha[\xi]$ consists of the canonical energy momentum tensor, and is related to $J_n^\alpha[\xi]$ made of the symmetric $\mathcal{T}_n^{\alpha\beta}$, by subtracting the trivially conserved current $K_n^\alpha[\xi]$.

If the vector ζ_n satisfies

$$\mathcal{T}_n^{\mu\nu}[\nabla_\mu(\zeta_n)_\nu + \nabla_\nu(\zeta_n)_\mu] \overset{\phi_n}{\approx} 0, \tag{32}$$

$J_n^\alpha[\zeta^n] \overset{\phi_n}{\approx} 2\mathcal{T}_n^{\alpha\beta}(\zeta_n)_\beta$ is an on-shell conserved current, and

$$Q_n[\zeta^n] = 2\int_\Sigma d\Sigma_\alpha\, T_n^{\alpha\beta}(\zeta_n)_\beta \tag{33}$$

gives a conserved charge of Noether's 1st theorem in general relativity for the global hidden matter symmetry generated by $\zeta_n(x)$, together with a symmetric energy momentum tensor $T_n^{\alpha\beta}$. The integration is performed over a $d-1$ dimensional space-like hyper-surface Σ with the hyper-surface element $d\Sigma_\alpha$, and appropriate boundary conditions are assumed for the conservation of $Q_n[\zeta_n]$. The condition (32) for the vector ζ_n and the corresponding conserved charge (33) in general relativity are identical to those previously proposed from a completely different argument [4,5]. Note that $Q_n[\zeta_n]$ is invariant under general coordinate transformations.

If the metric $g_{\mu\nu}$ allows a Killing vector ξ_K, Eq. (32) is satisfied by ξ_K, and thus $Q_n[\xi_K]$ is a conserved charge associated with the isometry of $g_{\mu\nu}$[3,4]. In this case, ξ_K depends only on the metric $g_{\mu\nu}$ and is independent of matter fields ϕ_n for all n, so that the symmetry generated by $\xi_K[g_{\mu\nu}]$ is a standard type of global symmetry for all matter sectors in the curved spacetime determined by $g_{\mu\nu}$. In particular, if the Killing vector ξ_{K_0} is stationary, $\mathcal{E}_n := Q_n[\xi_{K_0}]$ defines ``energy'' for the sector n in general relativity [3,4], which is separately conserved in each sector. Since the $\lambda \to 0$ limit implies a solution to Eq. (31) with zero cosmological constant to be a Minkowski metric, $\mathcal{E}_n$ with an appropriately normalized ξ_{K_0} becomes the standard definition of energy in the flat spacetime as

$$\lim_{\lambda\to 0} \mathcal{E}_n = \int_{x^0=\text{fixed}} (d^{d-1}x)_0\, T^{00}. \tag{34}$$

Even if the metric $g_{\mu\nu}$ has no Killing vector, we can always find the vector ζ_n which satisfies Eq. (32) [5], so that Eq. (33) defines an associated conserved charge, which is separately conserved in each sector even if the metric $g_{\mu\nu}$ is dynamical. If ζ_n is time-like, $Q_n[\zeta_n]$ defines energy or its generalization, which may be identified as "entropy" in some cases in general relativity[5]. Note that $\zeta_n \neq \zeta_m$ in general if $n \neq m$.

3.3 *Non-abelian gauge theory*

For a non-abelian gauge theory, the local gauge transformation by $\theta = \theta^a t_a$ generates

$$\delta_\theta A_\mu = i[\theta, A_\mu] + \frac{1}{g_s}\partial_\mu\theta,\ \ A_\mu := A_\mu^a t_a$$
$$\delta_\theta \psi_{nj} = i\theta^a t_a^{R_{nj}} \psi_{nj}, \tag{35}$$

where g_s is the coupling constant, t_a is the generator in the fundamental representation of the gauge group with an implicit summation over a, while $t_a^{R_{nj}}$ is the generator in an irreducible representation R_{nj} for matter j in sector n, so that $t_a^{R_{nj}}$ and ψ_{nj} implicitly have gauge indices such that $(t_a^{R_{nj}})^A{}_B$ and $(\psi_{nj})^A$, respectively, where A, B run from 1 to $\dim(R_{nj})$.

For a matter sector n, Eqs. (20) and (21) imply

$$J_n^\alpha[\theta] := -\frac{1}{g_s}\text{tr}(T_n^\alpha\theta),\ \ (T_n^\alpha)^B{}_A := \frac{\partial L_n}{\partial(A_\alpha)^A{}_B}, \tag{36}$$

$$\partial_\alpha J_n^\alpha[\theta] = -\text{tr}(T_n^\alpha \delta_\theta A_\alpha) - \sum_j (E^{nj}\delta_\theta\psi_{nj}), \tag{37}$$

where E^{nj} is an EOM for ψ_{nj}, and T_n^α appears in an EOM for A_μ as $\lambda \sum_n T_n^\alpha$.

If $\omega_n = \omega_n^a t_a$ (more precisely denoted as $w_n[A_\mu, \psi_{nj}]$) satisfies

$$\mathrm{tr}(T_n^\alpha \delta_{\omega_n} A_\alpha) = \mathrm{tr}\, T_n^\alpha \left\{ i[\omega_n, A_\alpha] + \frac{1}{g_s}\partial_\alpha \omega_n \right\} = 0, \tag{38}$$

$J_n^\alpha[\omega_n]$ becomes the conserved current for a hidden matter symmetry as

$$\partial_\alpha J_n^\alpha[\omega_n] \overset{\psi_n}{\approx} 0. \tag{39}$$

We then define the conserved non-abelian charge of Noether's 1st theorem as

$$Q_n[\omega_n] = \int_{x^0=\mathrm{fixed}} (d^{d-1}x)_0\, J_n^0[\omega_n], \tag{40}$$

which is invariant under the gauge transformation Ω by

$$A'_\mu = \Omega\, A_\mu\, \Omega^\dagger - \frac{1}{ig_s}\Omega\, \partial_\mu \Omega^\dagger, \quad \omega'_n = \Omega\, \omega_n\, \Omega^\dagger, \tag{41}$$

since T_n^α and $\delta_{\omega_n} A_\alpha$ are covariant as

$$(T_n^\alpha)' = \Omega\, T_n^\alpha\, \Omega^\dagger, \quad \delta_{\omega'_n} A'_\alpha = i[\omega'_n, A'_\alpha] + \frac{1}{g_s}\partial_\alpha \omega'_n = \Omega(\delta_{\omega_n} A_\alpha)\Omega^\dagger, \tag{42}$$

so that $J_n^\alpha[\omega_n]$ and the condition Eq. (38) are gauge invariant.

Note again that the non-abelian gauge charge is separately conserved in each sector n even when the non-abelian gauge field A_μ^a is dynamical, so that there is no non-abelian charge exchange among different sectors.

4. Conclusions and discussions

In this letter, we have proposed a general method to define the conserved current/charge for the global hidden matter symmetry in the presence of local symmetries, which is then applied to the $U(1)$ gauge theory, general relativity, and the non-abelian gauge theory. We have shown that the conserved electric charge in the electrodynamics can be interpreted as this type of conserved charge for the constant $U(1)$ transformation. It also leads to a new type of conserved charges in general relativity and the non-abelian gauge theory. Indeed, the symmetric argument in our proposal theoretically explains the existence of a new conserved charge in general relativity recently proposed by different considerations [4,5].

In the standard view, the electric field does not carry an electric charge because of the absence of non-linear terms, while the gravitational field or the gluon carries energy or color, respectively, due to non-linear interactions. In our approach, however, the absence or presence of non-linear terms in the theory makes gauge transformations field-independent or -dependent, respectively, so that the global hidden matter transformation, which leads to the conservation, is constant and field-independent for the $U(1)$ theory but field-dependent for others. Regardless of this difference, we stress that the conserved charge in our method is carried only by matter fields for all cases. Of course, it would be better if we could define a physically meaningful conserved charge such as energy or non-abelian gauge charge including both matter and gauge contributions for a total system. Unfortunately, we have failed to find a global symmetry so far for such charges in the presence of a local symmetry.

We finally point out that our method can be extend to more general cases that the matter sector contains derivatives of gauge fields. For example, in the case that $L_n = L_n(g, g_{,\alpha}, h_n,$

$h_{n,\alpha}$), Eqs. (22), (21) and (18) still hold under the replacement of

$$\frac{\partial L_n}{\partial g} \to \frac{\partial L_n}{\partial g} - \partial_\beta \left(\frac{\partial L_n}{\partial g_{,\beta}} \right). \tag{43}$$

Acknowledgement

This work is supported in part by the Grant-in-Aid of the Japanese Ministry of Education, Sciences and Technology, Sports and Culture (MEXT) for Scientific Research (Nos. JP22H00129). The author thanks Prof. Yoshimasa Hidaka and Mr. Takumi Hayashi for useful discussions.

Funding

Open Access funding: SCOAP^3.

References

[1] E. Noether, Gott. Nachr. **1918**, 235 (1918), [arXiv:physics/0503066 [physics]] [Search inSPIRE].
[2] A. Deriglazov, Classical Mechanics: Hamiltonian and Lagrangian Formalism. Springer, Cham. p.329. (2017).
[3] S. Aoki and T. Onogi, Int. J. Mod. Phys. A **37**, 2250129 (2022) [arXiv:2201.09557 [hep-th]] [Search inSPIRE].
[4] S. Aoki, T. Onogi, and S. Yokoyama, Int. J. Mod. Phys. A **36**. 2150098 (2021) [arXiv:2005.13233 [gr-qc]] [Search inSPIRE].
[5] S. Aoki, T. Onogi, and S. Yokoyama, Int. J. Mod. Phys. A **36**, 2150201 (2021) [arXiv:2010.07660 [gr-qc]] [Search inSPIRE].
[6] R. Utiyama, Prog. Theor. Phys. **72**, 83 (1984).
[7] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).