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GKZ-hypergeometric systems for Feynman integrals

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Abstract

Basing on the systems of linear partial differential equations derived from Mellin-Barnes representations and Miller's transformation, we obtain GKZ-hypergeometric systems of one-loop self energy, one-loop triangle, two-loop vacuum, and two-loop sunset diagrams, respectively. The codimension of derived GKZ-hypergeometric system equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. Taking GKZ-hypergeometric systems of one-loop self energy, massless one-loop triangle, and two-loop vacuum diagrams as examples, we present in detail how to perform triangulation and how to construct canonical series solutions in the corresponding convergent regions. The series solutions constructed for these hypergeometric systems recover the well known results in literature.

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1. Introduction

A central target for particle physics now is to test the standard model (SM) and to search for new physics (NP) beyond the SM [1–3] after the discovery of the Higgs boson [4,5]. In order to predict the electroweak observables precisely with dimensional regularization [6,7], one should evaluate the Feynman integrals exactly in the time-space dimension $D = 4 - 2\varepsilon$ at first. Nevertheless each method presented in Ref. [8] has its blemishes since it can only be applied to the Feynman diagrams with special topologies and kinematic invariants.

It was proposed long ago to consider Feynman integrals as the generalized hypergeometric functions [9]. Certainly Feynman integrals satisfy indeed the systems of holonomic linear partial differential equations (PDEs) [10] whose singularities are determined by the Landau singularities. Recently the author of Ref. [11] shows that the D-module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) D-module [12–16].

Some Feynman integrals are already expressed as the hypergeometric series in the corresponding parameter space. In Ref. [17] the massless C_0 function is presented as the linear combination of the fourth kind of Appell function F_4 whose arguments are the dimensionless ratios among the external momentum squared, and is simplified further as the linear combination of the Gauss function ${}_2F_1$ through the quadratic transformation [18] in the literature [19]. With some special assumptions on the virtual masses, the analytic expressions of the scalar integral C_0 are given by the multiple hypergeometric functions in Ref. [20] through Mellin-Barnes representations. Taking the massless C_0 function as an example, the author of Ref. [21] presents an algorithm to evaluate the scalar integrals of one-loop vertex-type Feynman diagrams. Certainly, some analytic results of the C_0 function can also be extracted from the expressions for the scalar integrals of one-loop massive N-point Feynman diagrams [23,22]. Feynman parametrization and Mellin-Barnes contour integrals can be applied to evaluate Feynman integrals of ladder diagrams with three or four external lines [24]. In addition, the literature [25] also provides a geometrical interpretation of the analytic expressions of the scalar integrals from one-loop N-point Feynman diagrams. Using the recurrence relations respecting the time-space dimension, the authors of Refs. [26,27] formulate one-loop two-point function B_0 as the linear combination of the Gauss function ${}_2F_1$, one-loop three-point function C_0 with arbitrary external momenta and virtual masses as the linear combination of the Appell function F_1, and one-loop four-point function D_0 with arbitrary external momenta and virtual masses as the linear combination of the Lauricella-Saran function F_s with three arguments, respectively. The expression for the scalar integral C_0 is convenient for analytic continuation and numerical evaluation because continuation of the Appell functions has been analyzed thoroughly. Nevertheless, how to perform continuation of the Lauricella-Saran function F_s outside its convergent domain is still a challenge. Expressing the relevant Feynman integral as a linear combination of generalized hypergeometric functions in dimension regularization, the authors of Ref. [28] analyze Laurent expansion of these hypergeometric functions around $D = 4$. The differential-reduction algorithm to evaluate those hypergeometric functions can be found in Refs. [29–33]. A hypergeometric system of linear PDEs is given through Mellin-Barnes representation [34], where the system of linear PDEs is satisfied by the corresponding Feynman integral in the whole parameter space. Some irreducible master integrals for sunset and bubble Feynman diagrams with generic values of masses and external momenta are explicitly evaluated via their Mellin-Barnes representations in Ref. [35].

Taking some special assumptions on the virtual masses and external momenta, the author of Ref. [36] presents some GKZ-hypergeometric systems of Feynman integrals with codimension $= 0$ or codimension $= 1$ through Lee-Pomeransky parametric representations [37]. Using the triangulations of the Newton polytope of Lee-Pomeransky polynomial, the author of Ref. [38] presents GKZ-hypergeometric system of the sunset diagram of codimension = 6. He also constructs canonical series solutions which contain three redundant variables besides three independent dimensionless ratios among the external momentum squared p^2 and three virtual mass squared m_i^2 $(i = 1, 2, 3)$ under the assumption $|p^2 - m_1^2 - m_2^2 - m_3^2| \gg m_i^2$. Actually it is a common defect of GKZ-hypergeometric systems originating from Lee-Pomeransky polynomial of the corresponding Feynman diagrams that codimension is far larger than the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. To construct canonical series solutions with suitable independent variables, one should compute the restricted D-module of GKZ-hypergeometric system originating from Lee-Pomeransky representations on corresponding hyperplane in the parameter space [39–41].

Some holonomic systems of linear PDEs are given through Mellin-Barnes representations of the concerned Feynman integrals in Refs. [34,42,43]. Following the work of W. Miller [44,45], one derives GKZ-hypergeometric system of Feynman integrals, whose codimension equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. Using those holonomic systems given in Refs. [34,42,43], we present here relevant GKZ-hypergeometric systems for Feynman integrals of one-loop self-energy, two-loop vacuum, two-loop sunset, and one-loop triangle diagrams. Taking Feynman integrals of one-loop self-energy, two-loop vacuum, and massless one-loop triangle diagrams as examples, we illuminate how to construct canonical series solutions from those relevant GKZ-hypergeometric systems [46], and how to derive the convergent regions of those series with Horn's study [47]. To shorten the length of text, we don't state those mathematical concepts and theorems that have been used in our analyses here, because they can be found in some well-known mathematical textbooks [46,48–55]. Basing on Mellin-Barnes representations of one-loop Feynman diagrams or those multiloop diagrams with two vertices, we can derive GKZ-hypergeometric systems through Miller's transformation, whose codimension of GKZ-hypergeometric system equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. Using toric geometry and mirror symmetry, one also derives the PDEs satisfied by Feynman integrals of the multiloop sunset diagrams [56]. Nevertheless for generic multiloop Feynman diagrams such as that presented in Refs. [57,58], the corresponding codimension of GKZ-hypergeometric system derived is far larger than the number of independent dimensionless ratios, whether using Mellin-Barnes or Lee-Pomeransky representations. In order to construct canonical series solutions properly, the corresponding GKZ-hypergeometric system is restricted to the hyperplane in parameter space.

The generally strategy for analyzing Feynman integral includes three steps here. First we obtain the holonomic system of linear PDEs satisfied by corresponding Feynman integral through its Mellin-Barnes representation, next find GKZ-hypergeometric system via Miller's transformation, and finally construct canonical series solutions. The integration constants, i.e. the combination coefficients, are determined by the corresponding Feynman integral with some special kinematic parameters. To make the analytic continuation of those canonical series solutions from their convergent regions to the whole parameter space, one can perform some linear fractional transformations among the complex variables.

Our presentation is organized as following. Through Miller's transformation, we derive GKZ-hypergeometric systems of Feynman integrals of one-loop self energy, massless one-loop triangle, and two-loop vacuum diagrams by using the holonomic systems of linear PDEs in Refs. [34, 42,43] in section 2. Then we present in detail how to perform triangulation and how to construct canonical series solutions from those GKZ-hypergeometric systems in section 3. Actually some well-known results are recovered with the approach presented here. In section 4, we present GKZ-hypergeometric systems for the sunset diagram with three differential masses, C_0 function with one nonzero virtual mass, and C_0 function with three differential virtual masses, respectively. The conclusions are summarized in section 5.

2. GKZ-hypergeometric systems of one-loop self energy, massless one-loop triangle, and two-loop vacuum diagrams

Adopting the notation of Refs. [42,43], we write the scalar integrals of one-loop self energy, massless one-loop triangle, and two-loop vacuum diagrams respectively as

$$\begin{aligned}
B_0(p^2,\ m_1^2,\ m_2^2) &= \frac{i}{(4\pi)^2}\Big(\frac{4\pi\mu^2}{-p^2}\Big)^{2-D/2} f_B\left(\begin{matrix} a_B, & b_B \\ c_B, & c'_B\end{matrix}\,\middle|\, x_B,\ y_B\right) ,\\
C_0(p_1^2,\ p_2^2,\ p_3^2) &= \frac{i}{(4\pi)^2 p_3^2}\Big(\frac{4\pi\mu^2}{-p_3^2}\Big)^{2-D/2} f_C\left(\begin{matrix} a_C, & b_C \\ c_C, & c'_C\end{matrix}\,\middle|\, x_C,\ y_C\right) ,\\
V_2(m_1^2,\ m_2^2,\ m_3^2) &= \frac{m_3^2}{(4\pi)^4}\Big(\frac{4\pi\mu^2}{m_3^2}\Big)^{4-D} f_V\left(\begin{matrix} a_V, & b_V \\ c_V, & c'_V\end{matrix}\,\middle|\, x_V,\ y_V\right) ,
\end{aligned}\tag{1}$$

where μ denotes the renormalization energy scale, and $x_B = m_1^2/p^2$, $y_B = m_2^2/p^2$, $x_C = p_1^2/p_3^2$, $y_C = p_2^2/p_3^2$, $x_V = m_1^2/m_3^2$, $y_V = m_2^2/m_3^2$, respectively. Here the dimensionless functions f_i $(i = B,\ C,\ V)$ comply with the fourth Appell's system of linear PDEs

$$\begin{aligned}
&\Big\{\hat\vartheta_{x_i}(\hat\vartheta_{x_i}+c_i-1) - x_i(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+a_i)(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+b_i)\Big\} f_i\left(\begin{matrix} a_i, & b_i \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) = 0\,,\\
&\Big\{\hat\vartheta_{y_i}(\hat\vartheta_{y_i}+c'_i-1) - y_i(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+a_i)(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+b_i)\Big\} f_i\left(\begin{matrix} a_i, & b_i \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) = 0\,,
\end{aligned}\tag{2}$$

with the Euler operator $\hat\vartheta_x = x\partial_x$. Correspondingly those parameters in Eq. (2) are

$$\begin{aligned}
&a_{B,V} = 2 - \frac{D}{2}\,,\ \ b_{B,V} = 3 - D\,,\\
&c_{B,V} = c'_{B,V} = 2 - \frac{D}{2}\,,\\
&a_C = 1\,,\ \ b_C = 3 - \frac{D}{2}\,,\\
&c_C = c'_C = 3 - \frac{D}{2}\,.
\end{aligned}\tag{3}$$

Generally those holonomic systems presented above originate from Mellin-Barnes representations of the corresponding Feynman integrals [34,42,43]. Using the systems of linear PDEs in Eq. (2), one derives the following relations between f_i $(i = B,\ C,\ V)$ and their contiguous functions

$$\begin{aligned}
(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+a_i) f_i\left(\begin{matrix} a_i, & b_i \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) &= a_i f_i\left(\begin{matrix} a_i+1, & b_i \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) ,\\
(\hat\vartheta_{x_i}+\hat\vartheta_{y_i}+b_i) f_i\left(\begin{matrix} a_i, & b_i \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) &= b_i f_i\left(\begin{matrix} a_i, & b_i+1 \\ c_i, & c'_i\end{matrix}\,\middle|\, x_i,\ y_i\right) ,
\end{aligned}$$

$$
\begin{aligned}
&(\hat{\vartheta}_{x_i}+c_i-1)f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| x_i,\ y_i\right)=(c_i-1)f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i-1, & c'_i \end{array}\middle| x_i,\ y_i\right),\\
&(\hat{\vartheta}_{y_i}+c'_i-1)f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| x_i,\ y_i\right)=(c'_i-1)f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i-1 \end{array}\middle| x_i,\ y_i\right),\\
&\partial_{x_i}f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| x_i,\ y_i\right)=\frac{a_ib_i}{c_i}f_i\left(\begin{array}{cc} a_i+1, & b_i+1 \\ c_i+1, & c'_i \end{array}\middle| x_i,\ y_i\right),\\
&\partial_{y_i}f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| x_i,\ y_i\right)=\frac{a_ib_i}{c'_i}f_i\left(\begin{array}{cc} a_i+1, & b_i+1 \\ c_i, & c'_i+1 \end{array}\middle| x_i,\ y_i\right).
\end{aligned}
\tag{4}
$$

Following the work of W. Miller [44,45], we define the auxiliary functions Φ_i $(i=B,\ C,\ V)$ through the functions f_i and additional variables s_i, t_i, u_i and v_i:

$$
\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=s_i^{a_i}t_i^{b_i}u_i^{c_i-1}v_i^{c'_i-1}f_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| x_i,\ y_i\right).
\tag{5}
$$

Miller's transformation on the functions f_i is to replace the multiplication by the parameters a_i, b_i, c_i, and c'_i in Eq. (4) by Euler operators $\hat{\vartheta}_{s_i}$, $\hat{\vartheta}_{t_i}$, $\hat{\vartheta}_{u_i}$, and $\hat{\vartheta}_{v_i}$

$$
\begin{aligned}
&\hat{\vartheta}_{s_i}\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=a_i\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\vartheta}_{t_i}\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=b_i\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\vartheta}_{u_i}\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=(c_i-1)\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\vartheta}_{v_i}\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=(c'_i-1)\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),
\end{aligned}
\tag{6}
$$

which leads naturally to the notion of GKZ-hypergeometric systems. In addition, the contiguous relations of Eq. (4) are rewritten as

$$
\begin{aligned}
&\hat{\mathcal{O}}^i_1\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=a_i\Phi_i\left(\begin{array}{cc} a_i+1, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\mathcal{O}}^i_2\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=b_i\Phi_i\left(\begin{array}{cc} a_i, & b_i+1 \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\mathcal{O}}^i_3\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=(c_i-1)\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i-1, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\mathcal{O}}^i_4\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=(c'_i-1)\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i-1 \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\mathcal{O}}^i_5\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=\frac{a_ib_i}{c_i}\Phi_i\left(\begin{array}{cc} a_i+1, & b_i+1 \\ c_i+1, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),\\
&\hat{\mathcal{O}}^i_6\Phi_i\left(\begin{array}{cc} a_i, & b_i \\ c_i, & c'_i \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right)=\frac{a_ib_i}{c'_i}\Phi_i\left(\begin{array}{cc} a_i+1, & b_i+1 \\ c_i, & c'_i+1 \end{array}\middle| \begin{array}{c} x_i,\ y_i,\ s_i,\ t_i \\ u_i,\ v_i \end{array}\right),
\end{aligned}
\tag{7}
$$

where the operators $\hat{\mathcal{O}}^i_n$, $(n=1,\cdots,6)$ are

$$\begin{aligned}
&\hat{\mathcal{O}}_1^i = s_i(\hat{\vartheta}_{x_i} + \hat{\vartheta}_{y_i} + \hat{\vartheta}_{s_i})\,,\\
&\hat{\mathcal{O}}_2^i = t_i(\hat{\vartheta}_{x_i} + \hat{\vartheta}_{y_i} + \hat{\vartheta}_{t_i})\,,\\
&\hat{\mathcal{O}}_3^i = \frac{1}{u_i}(\hat{\vartheta}_{x_i} + \hat{\vartheta}_{u_i})\,,\\
&\hat{\mathcal{O}}_4^i = \frac{1}{v_i}(\hat{\vartheta}_{y_i} + \hat{\vartheta}_{v_i})\,,\\
&\hat{\mathcal{O}}_5^i = s_i t_i u_i \partial_{x_i}\,,\\
&\hat{\mathcal{O}}_6^i = s_i t_i v_i \partial_{y_i}\,.
\end{aligned} \tag{8}$$

Those operators of Eq. (8) together with $\hat{\vartheta}_{s_i}$, $\hat{\vartheta}_{t_i}$, $\hat{\vartheta}_{u_i}$, $\hat{\vartheta}_{v_i}$ define the Lie algebra of the hypergeometric systems [44,45]. Under the variable transformation

$$\begin{aligned}
&z_{i,1} = \frac{x_i}{s_i t_i u_i}\,,\ \ z_{i,2} = \frac{y_i}{s_i t_i v_i}\,,\\
&z_{i,3} = \frac{1}{s_i}\,,\ \ z_{i,4} = \frac{1}{t_i}\,,\\
&z_{i,5} = u_i\,,\ \ z_{i,6} = v_i\,,
\end{aligned} \tag{9}$$

the equations in Eq. (6) are changed as

$$\left(\mathbf{A}\cdot\vec{\vartheta}_i\right)\Phi_i\left(\begin{array}{cc|c} a_i, & b_i & x_i,\ y_i,\ s_i,\ t_i \\ c_i, & c_i' & u_i,\ v_i \end{array}\right) = \mathbf{B}\Phi_i\left(\begin{array}{cc|c} a_i, & b_i & x_i,\ y_i,\ s_i,\ t_i \\ c_i, & c_i' & u_i,\ v_i \end{array}\right), \tag{10}$$

where

$$\begin{aligned}
&\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix},\\
&\vec{\vartheta}_i^{\,T} = (\vartheta_{z_{i,1}},\ \vartheta_{z_{i,2}},\ \vartheta_{z_{i,3}},\ \vartheta_{z_{i,4}},\ \vartheta_{z_{i,5}},\ \vartheta_{z_{i,6}})\,,\\
&\mathbf{B}^{T} = (-a_i,\ -b_i,\ c_i - 1,\ c_i' - 1)\,.
\end{aligned} \tag{11}$$

Correspondingly the universal Gröbner basis of the toric ideal associated with **A** is

$$\mathcal{U}_{\mathbf{A}} = \{\partial_{z_{i,1}}\partial_{z_{i,5}} - \partial_{z_{i,2}}\partial_{z_{i,6}},\ \partial_{z_{i,1}}\partial_{z_{i,5}} - \partial_{z_{i,3}}\partial_{z_{i,4}},\ \partial_{z_{i,2}}\partial_{z_{i,6}} - \partial_{z_{i,3}}\partial_{z_{i,4}}\}\,. \tag{12}$$

The operators $\mathbf{A}\cdot\vec{\vartheta}_i - \mathbf{B}$ and that from the set $\mathcal{U}_{\mathbf{A}}$ compose the generators of a left ideal [51] in the Weyl algebra $D = \mathbf{C}\langle z_{i,1},\ \cdots,\ z_{i,6},\ \partial_{z_{i,1}},\ \cdots,\ \partial_{z_{i,6}}\rangle$ where **C** denotes the field of complex numbers [55]. Defining the isomorphism between the commutative polynomial ring and the Weyl algebra [46]

$$\Psi:\ \mathbf{C}[z_{i,1},\ \cdots,\ z_{i,6},\ \xi_{i,1},\ \cdots,\ \xi_{i,6}] \to D,\ z_i^{\alpha}\xi_i^{\beta} \mapsto z_i^{\alpha}\partial_{z_i}^{\beta}\,, \tag{13}$$

one obtains the state polytope [53] of the preimage of the universal Gröbner basis in Eq. (12) as

$$\begin{aligned}
&\xi_{i,5} + \xi_{i,6} \geq 1,\ \xi_{i,5} \geq 0,\ \xi_{i,6} \geq 0,\\
&-\xi_{i,5} \geq -2,\ -\xi_{i,6} \geq -2,\ -\xi_{i,5} - \xi_{i,6} \geq -3
\end{aligned} \tag{14}$$

on the hyperplane

$$\begin{aligned}&\xi_{i,3}-\xi_{i,4}=0,\quad \xi_{i,2}-\xi_{i,6}=0,\\&\xi_{i,1}-\xi_{i,5}=0,\quad \xi_{i,4}+\xi_{i,5}+\xi_{i,6}=3\,.\end{aligned}\tag{15}$$

In Eq. (13) we take multi-index notation for abbreviation, i.e.

$$z_i^{\alpha}=\prod_{k=1}^{6}z_{i,k}^{\alpha_k},\quad \xi_i^{\beta}=\prod_{k=1}^{6}\xi_{i,k}^{\beta_k}\,,\tag{16}$$

where $\alpha,\ \beta\in N^6$, and $N=\{0,1,2,\cdots\}$ denotes the set of non-negative integers. The normal fan of the state polytope in Eq. (14) is the Gröbner fan of the corresponding left ideal. Because codimension $=2$ for all GKZ-hypergeometric systems, the Gröbner fan equals the hypergeometric fan. These two fans are indispensable in the construction of canonical series solutions of corresponding GKZ-hypergeometric systems.

3. Triangulation and construction of series solutions

3.1. Triangulation

For an integer $d\times n$-matrix P $(d<n)$ of rank d which satisfies the homogeneity assumption

$$(\underbrace{1,\ 1,\ \cdots,\ 1}_{n})=\sum_{l=1}^{d}q_l P_l\,,\tag{17}$$

the Gale transform of P [59] is the $n\times(n-d)$ integer-matrix Q which satisfies

$$P\cdot Q=0\,,\tag{18}$$

where P_l $(l=1,\cdots,d)$ is the l-th row vector of the integer matrix P, and the combination coefficient q_l is a rational number. The following matrix G_A is a Gale transform of A in Eq. (11):

$$G_A^T=\begin{pmatrix}1&0&-1&-1&1&0\\0&1&-1&-1&0&1\end{pmatrix},\tag{19}$$

whose column vectors compose the secondary fan Σ_A of GKZ-hypergeometric system in Eq. (10). Actually the state polytope in Eq. (14) of universal Gröbner basis indicates that the hypergeometric fan $\mathcal{H}_A$ and the Gröbner fan $\mathcal{G}_A$ all equal the secondary fan of GKZ-hypergeometric system:

$$\begin{aligned}\mathcal{H}_A=\mathcal{G}_A=\Sigma_A=&\,\mathrm{Cone}(\{\mathbf{e}_1,\mathbf{e}_2\})\bigcup\mathrm{Cone}(\{\mathbf{e}_1,-\mathbf{e}_1-\mathbf{e}_2\})\\&\bigcup\mathrm{Cone}(\{\mathbf{e}_2,-\mathbf{e}_1-\mathbf{e}_2\})\,,\end{aligned}\tag{20}$$

with $\mathbf{e}_1=(1,\ 0)^T$, $\mathbf{e}_2=(0,\ 1)^T$. The cones are defined as [52]

$$\begin{aligned}&\mathrm{Cone}(\{\mathbf{e}_1,\mathbf{e}_2\})=\{\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2\,\big|\,\lambda_1,\ \lambda_2\in\mathbf{R}_+\}\,,\\&\mathrm{Cone}(\{\mathbf{e}_1,-\mathbf{e}_1-\mathbf{e}_2\})=\{\lambda_1\mathbf{e}_1+\lambda_{12}(-\mathbf{e}_1-\mathbf{e}_2)\,\big|\,\lambda_1,\ \lambda_{12}\in\mathbf{R}_+\}\,,\\&\mathrm{Cone}(\{\mathbf{e}_2,-\mathbf{e}_1-\mathbf{e}_2\})=\{\lambda_2\mathbf{e}_2+\lambda_{12}(-\mathbf{e}_1-\mathbf{e}_2)\,\big|\,\lambda_2,\ \lambda_{12}\in\mathbf{R}_+\}\,,\end{aligned}\tag{21}$$

where $\mathbf{R}_+$ denotes the set of non-negative real numbers.

For a generic weight vector $\omega\in$ Cone($\{\mathbf{e}_1,\mathbf{e}_2\}$), the corresponding triangulation [53] $\triangle_\omega=\{\sigma_1^a,\ \sigma_2^a,\ \sigma_3^a,\ \sigma_4^a\}$ is unimodular, and supports the toric ideal

$$I=\langle\partial_{z_{i,1}}\partial_{z_{i,5}}-\partial_{z_{i,3}}\partial_{z_{i,4}},\partial_{z_{i,2}}\partial_{z_{i,6}}-\partial_{z_{i,3}}\partial_{z_{i,4}}\rangle\;, \tag{22}$$

which corresponds to the initial monomial ideal $in_\omega(I)=\langle\partial_{z_{i,1}}\partial_{z_{i,5}},\partial_{z_{i,2}}\partial_{z_{i,6}}\rangle$ with $i=B,\ V,\ C$. Each facet $\sigma_j^a,\ (j=1,\ 2,\ 3,\ 4)$ of the simplicial complex $\triangle_\omega$ is the index set of an invertible 4×4-submatrix $A_{\sigma_j^a}$ of A:

$$\begin{aligned}&\sigma_1^a=\{1,\ 2,\ 3,\ 4\}\;,\ \ \sigma_2^a=\{1,\ 3,\ 4,\ 6\}\;,\\&\sigma_3^a=\{2,\ 3,\ 4,\ 5\}\;,\ \ \sigma_4^a=\{3,\ 4,\ 5,\ 6\}\;.\end{aligned} \tag{23}$$

Certainly four standard pairs [46] $(1,\ \sigma_j^a),\ (j=1,\ 2,\ 3,\ 4)$ produce the following exponent vectors of initial monomials of series solutions

$$\begin{aligned}&p_{\sigma_1^a}=(1-c_i,\ 1-c_i',\ c_i+c_i'-a_i-2,\ c_i+c_i'-b_i-2,\ 0,\ 0)\;,\\&p_{\sigma_2^a}=(c_i-b_i-1,\ 0,\ b_i-a_i-c_i+1,\ 1-c_i,\ 0,\ c_i'-1)\;,\\&p_{\sigma_3^a}=(0,\ c_i'-1,\ 1-a_i-c_i',\ 1-b_i-c_i',\ c_i-1,\ 0)\;,\\&p_{\sigma_4^a}=(0,\ 0,\ -a_i,\ -b_i,\ c_i-1,\ c_i'-1)\;.\end{aligned} \tag{24}$$

For a generic weight vector $\omega\in$ Cone($\{\mathbf{e}_2,-\mathbf{e}_1-\mathbf{e}_2\}$) (Fig. 1), the corresponding triangulation $\triangle_\omega=\{\sigma_1^b,\ \sigma_2^b,\ \sigma_3^b,\ \sigma_4^b\}$ is also unimodular, and supports similarly the toric ideal

$$I=\langle\partial_{z_{i,2}}\partial_{z_{i,6}}-\partial_{z_{i,1}}\partial_{z_{i,5}},\partial_{z_{i,3}}\partial_{z_{i,4}}-\partial_{z_{i,1}}\partial_{z_{i,5}}\rangle\;, \tag{25}$$

which corresponds to the initial monomial ideal $in_\omega(I)=\langle\partial_{z_{i,2}}\partial_{z_{i,6}},\partial_{z_{i,3}}\partial_{z_{i,4}}\rangle$. Each facet $\sigma_j^b,\ (j=1,\ 2,\ 3,\ 4)$ of the simplicial complex $\triangle_\omega$ is the index set of an invertible 4×4-submatrix $A_{\sigma_i^b}$ of A:

$$\begin{aligned}&\sigma_1^b=\{1,\ 2,\ 3,\ 5\}\;,\ \ \sigma_2^b=\{1,\ 2,\ 4,\ 5\}\;,\\&\sigma_3^b=\{1,\ 3,\ 5,\ 6\}\;,\ \ \sigma_4^b=\{1,\ 4,\ 5,\ 6\}\;.\end{aligned} \tag{26}$$

Correspondingly four standard pairs $(1,\ \sigma_j^b),\ (j=1,\ 2,\ 3,\ 4)$ induce the following exponent vectors of initial monomials for series solutions

$$\begin{aligned}&p_{\sigma_1^b}=(2-c_i-c_i',\ c_i+c_i'-b_i-2,\ b_i-a_i',\ 0,\ 1-c_i',\ 0)\;,\\&p_{\sigma_2^b}=(c_i'-a_i-1,\ 1-c_i',\ 0,\ a_i-b_i,\ c_i+c_i'-a_i-2,\ 0)\;,\\&p_{\sigma_3^b}=(-b_i,\ 0,\ b_i-a_i,\ 0,\ c_i-b_i-1,\ c_i'-1)\;,\\&p_{\sigma_4^b}=(-a_i,\ 0,\ 0,\ a_i-b_i,\ c_i-a_i-1,\ c_i'-1)\;.\end{aligned} \tag{27}$$

Finally for a generic weight vector $\omega\in$ Cone($\{\mathbf{e}_1,-\mathbf{e}_1-\mathbf{e}_2\}$), the corresponding triangulation $\triangle_\omega=\{\sigma_1^c,\ \sigma_2^c,\ \sigma_3^c,\ \sigma_4^c\}$ is unimodular, and supports the toric ideal

$$I=\langle\partial_{z_{i,1}}\partial_{z_{i,5}}-\partial_{z_{i,2}}\partial_{z_{i,6}},\partial_{z_{i,3}}\partial_{z_{i,4}}-\partial_{z_{i,2}}\partial_{z_{i,6}}\rangle\;, \tag{28}$$

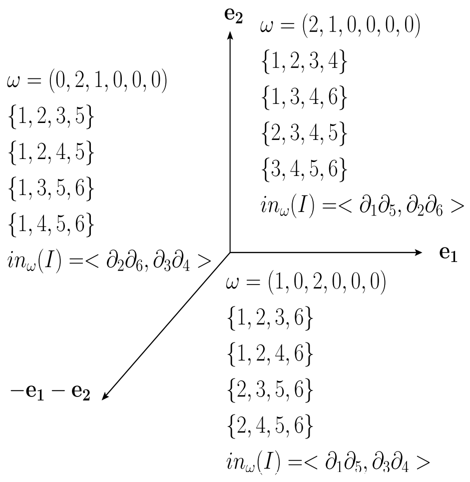

Fig. 1. The secondary fan of GKZ-hypergeometric system in Eq. (10), ω in each cone is a representative weight vector.

which corresponds to the initial monomial ideal $in_\omega(I)=\langle\partial_{z_{i,1}}\partial_{z_{i,5}},\partial_{z_{i,3}}\partial_{z_{i,4}}\rangle$. Each facet σ_j^c, $(j=1,\ 2,\ 3,\ 4)$ of the simplicial complex $\triangle_\omega$ is the index set of an invertible $4\times$ 4-submatrix $A_{\sigma_j^c}$ of matrix A:

$$\begin{aligned}&\sigma_1^c=\{1,\ 2,\ 3,\ 6\}\,,\ \ \sigma_2^c=\{1,\ 2,\ 4,\ 6\}\,,\\&\sigma_3^c=\{2,\ 3,\ 5,\ 6\}\,,\ \ \sigma_4^c=\{2,\ 4,\ 5,\ 6\}\,.\end{aligned}\tag{29}$$

Correspondingly four standard pairs $(1,\ \sigma_j^c)$, $(j=1,\ 2,\ 3,\ 4)$ give the following exponent vectors of initial monomials for series solutions

$$\begin{aligned}&p_{\sigma_1^c}=(1-c_i,\ c_i-b_i-1,\ a_i-b_i,\ 0,\ 0,\ c_i+c_i'-b_i-2)\,,\\&p_{\sigma_2^c}=(1-c_i,\ c_i-a_i-1,\ 0,\ a_i-b_i,\ 0,\ c_i+c_i'-a_i-2)\,,\\&p_{\sigma_3^c}=(0,\ -b_i,\ b_i-a_i,\ 0,\ c_i-1,\ c_i'-b_i-1)\,,\\&p_{\sigma_4^c}=(0,\ -a_i,\ 0,\ a_i-b_i,\ c_i-1,\ c_i'-a_i-1)\,.\end{aligned}\tag{30}$$

3.2. Construction of canonical series solutions

The integer kernel of the matrix A is defined as

$$\begin{aligned}\ker_{\mathbf{Z}}(A)&=\{u\in\mathbf{Z}^6:\ \ A\cdot u=0\}\\&=\{N(1,\ 0,\ -1,\ -1,\ 1,\ 0),\ N(-1,\ 0,\ 1,\ 1,\ -1,\ 0),\\&\quad N(0,\ 1,\ -1,\ -1,\ 0,\ 1),\ N(0,\ -1,\ 1,\ 1,\ 0,\ -1),\\&\quad N(-1,\ 1,\ 0,\ 0,\ -1,\ 1),\ N(1,\ -1,\ 0,\ 0,\ 1,\ -1)\}\,.\end{aligned}\tag{31}$$

The vector $u\in\ker_{\mathbf{Z}}(A)$ can be decomposed into positive and negative part, $u=u_+-u_-$, where u_+ and u_- are non-negative vectors with disjoint supports. In order to construct canonical series solutions of GKZ-hypergeometric system, we define the negative support of any vector $v = (v_1, \cdots, v_n) \in \mathbf{R}^n$ as

$$nsupp(v) = \{i \in \{1,\ 2,\ \cdots,\ n\}\ :\ v_i \text{ is a negative integer}\}\ . \tag{32}$$

Furthermore we introduce the following subset of $\ker_{\mathbf{Z}}(A)$

$$N_p = \{u \in \ker_{\mathbf{Z}}(A)\ :\ nsupp(p) = nsupp(p+u)\}\ . \tag{33}$$

With an exponent vector p of the initial monomial, the corresponding canonical series solution of the hypergeometric system Eq. (10) is well-defined:

$$\phi_p = \sum_{u\in N_p} \frac{[p]_{u_-}}{[p+u]_{u_+}} z_i^{p+u}\ , \tag{34}$$

where the abbreviations

$$\begin{aligned}
z_i^p &= \prod_{j=1}^{6} z_{i,j}^{p_j}\ ,\\
[p]_{u_-} &= \prod_{k:u_k<0} \prod_{j=1}^{-u_k} (p_k - j + 1)\ ,\\
[p+u]_{u_+} &= \prod_{k:u_k>0} \prod_{j=1}^{u_k} (p_k + j)\ .
\end{aligned} \tag{35}$$

For one-loop self energy and two-loop vacuum, $p_{\sigma_1^a} = (D/2-1,\ D/2-1,\ -D/2,\ -1, 0,\ 0)$, where $p_{\sigma_1^a,4} = -1$ is a negative integer. To construct canonical series solution properly, we perturb the vector $\mathbf{B}$ in the direction of $\mathbf{B}' = (0,\ 1,\ 0,\ 0)^T$. Choosing the perturbed parameter vector $\mathbf{B} + \epsilon\mathbf{B}'$, we modify the exponent vector as $p'_{\sigma_1^a} = (D/2-1,\ D/2-1,\ -D/2,\ -1+\epsilon, 0,\ 0)$. Correspondingly the set

$$\begin{aligned}
N_{p'_{\sigma_1^a}} = &\{N(1,\ 0,\ -1,\ -1,\ 1,\ 0)\ , N(0,\ 1,\ -1,\ -1,\ 0,\ 1)\ ,\\
&N(1,\ 0,\ -1,\ -1,\ 1,\ 0) + N(0,\ 1,\ -1,\ -1,\ 0,\ 1)\}\ .
\end{aligned} \tag{36}$$

Using Eq. (34), one derives

$$\begin{aligned}
\phi_{p'_{\sigma_1^a}} = &\frac{(z_{i,1}z_{i,2})^{D/2-1}}{z_{i,3}^{D/2} z_{i,4}^{1-\epsilon}} \Big\{ 1 + \sum_{n_1=1}^{\infty} \prod_{j=1}^{n_1} \frac{(1-\frac{D}{2}-j)(\epsilon-j)}{j(\frac{D}{2}-1+j)} \Big(\frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}\Big)^{n_1}\\
&+ \sum_{n_2=1}^{\infty} \prod_{j=1}^{n_2} \frac{(1-\frac{D}{2}-j)(\epsilon-j)}{j(\frac{D}{2}-1+j)} \Big(\frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}}\Big)^{n_2}\\
&+ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\prod_{j=1}^{n_1+n_2} (1-\frac{D}{2}-j)(\epsilon-j)}{\prod_{j_1=1}^{n_1} j_1(\frac{D}{2}-1+j_1) \prod_{j_2=1}^{n_2} j_2(\frac{D}{2}-1+j_2)} \Big(\frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}\Big)^{n_1} \Big(\frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}}\Big)^{n_2} \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(z_{i,1}z_{i,2})^{D/2-1}}{z_{i,3}^{D/2} z_{i,4}^{1-\epsilon}} \Big\{ 1 + \sum_{n_1=1}^{\infty} \frac{(\frac{D}{2})_{n_1}(1-\epsilon)_{n_1}}{n_1!(\frac{D}{2})_{n_1}} \Big(\frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}\Big)^{n_1} \\
&+ \sum_{n_2=1}^{\infty} \frac{(\frac{D}{2})_{n_2}(1-\epsilon)_{n_2}}{n_2!(\frac{D}{2})_{n_2}} \Big(\frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}}\Big)^{n_2} \\
&+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \frac{(\frac{D}{2})_{n_1+n_2}(1-\epsilon)_{n_1+n_2}}{n_1!n_2!(\frac{D}{2})_{n_1}(\frac{D}{2})_{n_2}} \Big(\frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}\Big)^{n_1} \Big(\frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}}\Big)^{n_2} \Big\}. \qquad (37)
\end{aligned}
$$

Then

$$
\begin{aligned}
\phi_{p_{\sigma_1^a}} &= \lim_{\epsilon\to 0} \phi_{p'_{\sigma_1^a}} \\
&= \frac{(z_{i,1}z_{i,2})^{D/2-1}}{z_{i,3}^{D/2} z_{i,4}} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2} \\ \frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}, \frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}} \right), \qquad (38)
\end{aligned}
$$

where F_4 denotes the fourth Appell function [49]. Similarly we have

$$
\begin{aligned}
\phi_{p_{\sigma_2^a}} &= z_{i,1}^{D/2-1} z_{i,3}^{-1} z_{i,4}^{D/2-2} z_{i,6}^{1-D/2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}, \frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}} \right), \\
\phi_{p_{\sigma_3^a}} &= z_{i,2}^{D/2-1} z_{i,3}^{-1} z_{i,4}^{D/2-2} z_{i,5}^{1-D/2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}, \frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}} \right), \\
\phi_{p_{\sigma_4^a}} &= z_{i,3}^{D/2-2} z_{i,4}^{D-3} z_{i,5}^{1-D/2} z_{i,6}^{1-D/2} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,3}z_{i,4}}, \frac{z_{i,2}z_{i,6}}{z_{i,3}z_{i,4}} \right), \\
\phi_{p_{\sigma_1^b}} &= z_{i,1}^{D/2-2} z_{i,2}^{D/2-1} z_{i,3}^{1-D/2} z_{i,5}^{-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,3}z_{i,4}}{z_{i,1}z_{i,5}}, \frac{z_{i,2}z_{i,6}}{z_{i,1}z_{i,5}} \right), \\
\phi_{p_{\sigma_2^b}} &= z_{i,1}^{-1} z_{i,2}^{D/2-1} z_{i,4}^{D/2-1} z_{i,5}^{-D/2} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2} \\ \frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,3}z_{i,4}}{z_{i,1}z_{i,5}}, \frac{z_{i,2}z_{i,6}}{z_{i,1}z_{i,5}} \right), \\
\phi_{p_{\sigma_3^b}} &= z_{i,1}^{D-3} z_{i,3}^{1-D/2} z_{i,5}^{D/2-2} z_{i,6}^{1-D/2} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,3}z_{i,4}}{z_{i,1}z_{i,5}}, \frac{z_{i,2}z_{i,6}}{z_{i,1}z_{i,5}} \right), \\
\phi_{p_{\sigma_4^b}} &= z_{i,1}^{D/2-2} z_{i,4}^{D/2-1} z_{i,5}^{-1} z_{i,6}^{1-D/2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,3}z_{i,4}}{z_{i,1}z_{i,5}}, \frac{z_{i,2}z_{i,6}}{z_{i,1}z_{i,5}} \right), \\
\phi_{p_{\sigma_1^c}} &= z_{i,1}^{D/2-1} z_{i,2}^{D/2-2} z_{i,3}^{1-D/2} z_{i,6}^{-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,2}z_{i,6}}, \frac{z_{i,3}z_{i,4}}{z_{i,2}z_{i,6}} \right), \\
\phi_{p_{\sigma_2^c}} &= z_{i,1}^{D/2-1} z_{i,2}^{-1} z_{i,4}^{D/2-1} z_{i,6}^{-D/2} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2} \\ \frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,2}z_{i,6}}, \frac{z_{i,3}z_{i,4}}{z_{i,2}z_{i,6}} \right), \\
\phi_{p_{\sigma_3^c}} &= z_{i,2}^{D-3} z_{i,3}^{1-D/2} z_{i,5}^{1-D/2} z_{i,6}^{D/2-2} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,2}z_{i,6}}, \frac{z_{i,3}z_{i,4}}{z_{i,2}z_{i,6}} \right), \\
\phi_{p_{\sigma_4^c}} &= z_{i,2}^{D/2-2} z_{i,4}^{D/2-1} z_{i,5}^{1-D/2} z_{i,6}^{-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{z_{i,1}z_{i,5}}{z_{i,2}z_{i,6}}, \frac{z_{i,3}z_{i,4}}{z_{i,2}z_{i,6}} \right). \qquad (39)
\end{aligned}
$$

For the fourth Appell function

$$F_4\left(\begin{array}{cc}a, & b\\ c, & c'\end{array}\middle| x,\ y\right)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(b)_{n_1+n_2}}{n_1!n_2!(c)_{n_1}(c')_{n_1}}x^{n_1}y^{n_2}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}x^{n_1}y^{n_2}\ , \tag{40}$$

the adjacent ratios of the coefficients are

$$\Phi_1'(n_1,n_2)=\frac{A_{1+n_1,n_2}}{A_{n_1,n_2}}=\frac{(1+n_1+n_2)(3-D/2+n_1+n_2)}{(1+n_1)(3-D/2+n_1)}\ ,$$
$$\Phi_2'(n_1,n_2)=\frac{A_{n_1,1+n_2}}{A_{n_1,n_2}}=\frac{(1+n_1+n_2)(3-D/2+n_1+n_2)}{(1+n_2)(3-D/2+n_2)}\ . \tag{41}$$

To investigate the absolutely and uniformly convergent region of the double series in Eq. (40), one takes

$$r_x=t_x^2=\lim_{\lambda\to\infty}\frac{1}{\Phi_1'(\lambda n_1,\lambda n_2)}=\frac{n_1^2}{(n_1+n_2)^2}=\frac{1}{(1+t)^2}\ ,$$
$$r_y=t_y^2=\lim_{\lambda\to\infty}\frac{1}{\Phi_2'(\lambda n_1,\lambda n_2)}=\frac{n_2^2}{(n_1+n_2)^2}=\frac{t^2}{(1+t)^2}\ , \tag{42}$$

with $t=n_2/n_1,\ r_x=|x|,\ r_y=|y|$, respectively. The generator of the principal ideal $\langle t_x^2(1+t)^2-1,\ t_y^2(1+t)^2-t^2\rangle\bigcap\mathbf{C}[t_x,t_y]$ is

$$g(t_x,\ t_y)=1-2(t_x^2+t_y^2)+(t_x^2-t_y^2)^2$$
$$=(-1+t_x-t_y)(1+t_x-t_y)(-1+t_x+t_y)(1+t_x+t_y)\ , \tag{43}$$

where $\mathbf{C}[t_x,t_y]$ denotes the polynomial ring of $t_x,\ t_y$ on the field $\mathbf{C}$. Since $t_x,\ t_y\geq 0$ the equation $g(t_x,\ t_y)=0$ gives the Cartesian curve of the double power series in Eq. (40) as

$$\sqrt{|x|}+\sqrt{|y|}=1\ . \tag{44}$$

For convenience we denote the region surrounded by the coordinate axes and the Cartesian curve in the positive quadrant of the plane Or_xr_y by C, and denote the rectangle by D in the positive quadrant of the plane Or_xr_y bounded by the coordinate axes and the straight lines parallel to the coordinate axes $r_x=1$, and $r_y=1$. According to Horn's study of convergence of the hypergeometric series [47], one finds the well-known conclusion [49] that the double power series in Eq. (40) absolutely and uniformly converges in the intersection of the regions C and D in the plane Or_xr_y, i.e. $\sqrt{|x|}+\sqrt{|y|}<1$.

Finally we set $z_{i,1}=x_i,\ z_{i,2}=y_i,\ z_{i,k}=1$ with $i=B,\ V$ and $k=3,\cdots,6$, and formulate the Feynman integrals as the linear combinations of canonical series solutions in the corresponding parameter space.

- For $|x_i|\leq 1,\ |y_i|\leq 1$, the Feynman integral is

$$S_{a,i}(x_i,y_i)=A_{a,i}(x_iy_i)^{D/2-1}F_4\left(\begin{array}{cc}1, & \frac{D}{2}\\ \frac{D}{2}, & \frac{D}{2}\end{array}\middle| x_i,\ y_i\right)$$

$$
\begin{aligned}
&+ B_{a,i} x_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| x_i,\ y_i \right) \\
&+ C_{a,i} y_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| x_i,\ y_i \right) \\
&+ D_{a,i} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| x_i,\ y_i \right) , \qquad (45)
\end{aligned}
$$

which is convergent in the region $\sqrt{|x_i|} + \sqrt{|y_i|} < 1$.

- For $|x_i| \geq 1,\ |y_i| \leq 1$, the Feynman integral is

$$
\begin{aligned}
S_{b,i}(x_i, y_i) = &A_{b,i} x_i^{D/2-2} y_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{1}{x_i},\ \frac{y_i}{x_i} \right) \\
&+ B_{b,i} x_i^{-1} y_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2} \\ \frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{1}{x_i},\ \frac{y_i}{x_i} \right) \\
&+ C_{b,i} x_i^{D-3} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{1}{x_i},\ \frac{y_i}{x_i} \right) \\
&+ D_{b,i} x_i^{D/2-2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{1}{x_i},\ \frac{y_i}{x_i} \right) , \qquad (46)
\end{aligned}
$$

which is convergent in the region $1 + \sqrt{|y_i|} < \sqrt{|x_i|}$.

- For $|x_i| \leq 1,\ |y_i| \geq 1$, the Feynman integral is

$$
\begin{aligned}
S_{c,i}(x_i, y_i) = &A_{c,i} x_i^{D/2-1} y_i^{D/2-2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ \frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{x_i}{y_i},\ \frac{1}{y_i} \right) \\
&+ B_{c,i} x_i^{D/2-1} y_i^{-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2} \\ \frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{x_i}{y_i},\ \frac{1}{y_i} \right) \\
&+ C_{c,i} y_i^{D-3} F_4 \left(\begin{array}{cc} 3-D, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & 2-\frac{D}{2} \end{array} \middle| \frac{x_i}{y_i},\ \frac{1}{y_i} \right) \\
&+ D_{c,i} y_i^{D/2-2} F_4 \left(\begin{array}{cc} 1, & 2-\frac{D}{2} \\ 2-\frac{D}{2}, & \frac{D}{2} \end{array} \middle| \frac{x_i}{y_i},\ \frac{1}{y_i} \right) , \qquad (47)
\end{aligned}
$$

which is convergent in the region $1 + \sqrt{|x_i|} < \sqrt{|y_i|}$.

In order to determine those integration constants, i.e. the combination coefficients $A_{\sigma,i}$, $B_{\sigma,i}$, $C_{\sigma,i}$, $D_{\sigma,i}$ with $\sigma = a,\ b,\ c$ and $i = B,\ V$, we utilize expressions of the Feynman integrals at some special points of the parameter space. For the Feynman integral of one-loop self energy diagram $B_0(p^2,\ m_1^2,\ m_2^2)$, we employ the following expressions

$$
\begin{aligned}
&B_0(\Lambda^2,\ 0,\ 0) = \frac{i\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{(4\pi)^2\Gamma(D-2)} \Big(\frac{4\pi\mu^2}{-\Lambda^2}\Big)^{2-D/2} , \\
&B_0(0,\ \Lambda^2,\ 0) = B_0(0,\ 0,\ \Lambda^2) = -\frac{i\Gamma(1-\frac{D}{2})}{(4\pi)^2} \Big(\frac{4\pi\mu^2}{\Lambda^2}\Big)^{2-D/2} , \\
&B_0(\Lambda^2,\ \Lambda^2,\ 0) = B_0(\Lambda^2,\ 0,\ \Lambda^2) = \frac{i\Gamma(2-\frac{D}{2})}{(4\pi)^2(D-3)} \Big(\frac{4\pi\mu^2}{\Lambda^2}\Big)^{2-D/2} ,
\end{aligned}
$$

$$B_0(0,\ \Lambda^2,\ \Lambda^2)=\frac{i\Gamma(2-\frac{D}{2})}{(4\pi)^2}\Big(\frac{4\pi\mu^2}{\Lambda^2}\Big)^{2-D/2}\;. \tag{48}$$

Using above expressions, one derives the combination coefficients as

$$\begin{aligned}
&A_{a,B}=0,\quad B_{a,B}=C_{a,B}=(-)^{D/2-1}\Gamma(1-\frac{D}{2})\;,\\
&D_{a,B}=\Gamma^2(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})\;,\\
&A_{b,B}=A_{c,B}=C_{b,B}=C_{c,B}=0\;,\\
&B_{b,B}=B_{c,B}=-D_{b,B}=-D_{c,B}=\Gamma(1-\frac{D}{2})\;.
\end{aligned} \tag{49}$$

In a similar way, the combination coefficients involved in the Feynman integral of the two-loop vacuum diagram are written as

$$\begin{aligned}
&A_{a,V}=-B_{a,V}=-C_{a,V}=\Gamma(3-\frac{D}{2})\Gamma^2(1-\frac{D}{2})\;,\\
&D_{a,V}=-2\Gamma^2(3-\frac{D}{2})\Gamma(\frac{D}{2}-1)\Gamma(2-D)\;,\\
&A_{b,V}=-B_{b,V}=D_{b,V}=A_{a,V}\,,\ C_{b,V}=D_{a,V}\;,\\
&A_{c,V}=-B_{c,V}=D_{c,V}=A_{a,V}\,,\ C_{c,V}=D_{a,V}\;.
\end{aligned} \tag{50}$$

For the triangulation $\triangle_\omega=\{\sigma_1^a,\ \sigma_2^a,\ \sigma_3^a,\ \sigma_4^a\}$ of the massless one-loop triangle diagram, the canonical series solutions are constructed as

$$\begin{aligned}
&\phi_{p_{\sigma_1^a}}=z_{C,1}^{D/2-2}z_{C,2}^{D/2-2}z_{C,3}^{3-D}z_{C,4}^{1-D/2}F_4\left(\begin{array}{cc}D-3, & \frac{D}{2}-1\\ \frac{D}{2}-1, & \frac{D}{2}-1\end{array}\middle|\,\frac{z_{C,1}z_{C,5}}{z_{C,3}z_{C,4}},\ \frac{z_{C,2}z_{C,6}}{z_{C,3}z_{C,4}}\right)\;,\\
&\phi_{p_{\sigma_2^a}}=z_{C,1}^{D/2-2}z_{C,3}^{1-D/2}z_{C,4}^{-1}z_{C,6}^{2-D/2}F_4\left(\begin{array}{cc}1, & \frac{D}{2}-1\\ \frac{D}{2}-1, & 3-\frac{D}{2}\end{array}\middle|\,\frac{z_{C,1}z_{C,5}}{z_{C,3}z_{C,4}},\ \frac{z_{C,2}z_{C,6}}{z_{C,3}z_{C,4}}\right)\;,\\
&\phi_{p_{\sigma_3^a}}=z_{C,2}^{D/2-2}z_{C,3}^{1-D/2}z_{C,4}^{-1}z_{C,5}^{2-D/2}F_4\left(\begin{array}{cc}1, & \frac{D}{2}-1\\ 3-\frac{D}{2}, & \frac{D}{2}-1\end{array}\middle|\,\frac{z_{C,1}z_{C,5}}{z_{C,3}z_{C,4}},\ \frac{z_{C,2}z_{C,6}}{z_{C,3}z_{C,4}}\right)\;,\\
&\phi_{p_{\sigma_4^a}}=z_{C,3}^{-1}z_{C,4}^{D/2-3}z_{C,5}^{2-D/2}z_{C,6}^{2-D/2}F_4\left(\begin{array}{cc}1, & 3-\frac{D}{2}\\ 3-\frac{D}{2}, & 3-\frac{D}{2}\end{array}\middle|\,\frac{z_{C,1}z_{C,5}}{z_{C,3}z_{C,4}},\ \frac{z_{C,2}z_{C,6}}{z_{C,3}z_{C,4}}\right)\;.
\end{aligned} \tag{51}$$

Similarly the canonical series solutions corresponding to the triangulation $\triangle_\omega=\{\sigma_1^b,\ \sigma_2^b,\ \sigma_3^b,\ \sigma_4^b\}$ are written as

$$\begin{aligned}
&\phi_{p_{\sigma_1^b}}=z_{C,1}^{-1}z_{C,2}^{D/2-2}z_{C,3}^{2-D/2}z_{C,5}^{1-D/2}F_4\left(\begin{array}{cc}1, & \frac{D}{2}-1\\ 3-\frac{D}{2}, & \frac{D}{2}-1\end{array}\middle|\,\frac{z_{C,3}z_{C,4}}{z_{C,1}z_{C,5}},\ \frac{z_{C,2}z_{C,6}}{z_{C,1}z_{C,5}}\right)\;,\\
&\phi_{p_{\sigma_2^b}}=z_{C,1}^{1-D/2}z_{C,2}^{D/2-2}z_{C,3}^{D/2-2}z_{C,4}^{3-D}F_4\left(\begin{array}{cc}D-3, & \frac{D}{2}-1\\ \frac{D}{2}-1, & \frac{D}{2}-1\end{array}\middle|\,\frac{z_{C,3}z_{C,4}}{z_{C,1}z_{C,5}},\ \frac{z_{C,2}z_{C,6}}{z_{C,1}z_{C,5}}\right)\;,\\
&\phi_{p_{\sigma_3^b}}=z_{C,1}^{D/2-3}z_{C,3}^{2-D/2}z_{C,5}^{-1}z_{C,6}^{2-D/2}F_4\left(\begin{array}{cc}1, & 3-\frac{D}{2}\\ 3-\frac{D}{2}, & 3-\frac{D}{2}\end{array}\middle|\,\frac{z_{C,3}z_{C,4}}{z_{C,1}z_{C,5}},\ \frac{z_{C,2}z_{C,6}}{z_{C,1}z_{C,5}}\right)\;,\\
&\phi_{p_{\sigma_4^b}}=z_{C,1}^{-1}z_{C,4}^{D/2-2}z_{C,5}^{1-D/2}z_{C,6}^{2-D/2}F_4\left(\begin{array}{cc}1, & \frac{D}{2}-1\\ \frac{D}{2}-1, & 3-\frac{D}{2}\end{array}\middle|\,\frac{z_{C,3}z_{C,4}}{z_{C,1}z_{C,5}},\ \frac{z_{C,2}z_{C,6}}{z_{C,1}z_{C,5}}\right)\;.
\end{aligned} \tag{52}$$

For the triangulation $\triangle_\omega=\{\sigma_1^c,\ \sigma_2^c,\ \sigma_3^c,\ \sigma_4^c\}$, the canonical series solutions are

$$\phi_{p_{\sigma_1^c}} = z_{C,1}^{D/2-2} z_{C,2}^{-1} z_{C,3}^{2-D/2} z_{C,6}^{1-D/2} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & 3-\frac{D}{2} \end{array} \middle| \frac{z_{C,1}z_{C,5}}{z_{C,2}z_{C,6}}, \frac{z_{C,3}z_{C,4}}{z_{C,2}z_{C,6}} \right),$$
$$\phi_{p_{\sigma_2^c}} = z_{C,1}^{D/2-2} z_{C,2}^{1-D/2} z_{C,4}^{D/2-2} z_{C,6}^{3-D} F_4 \left(\begin{array}{cc} D-3, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & \frac{D}{2}-1 \end{array} \middle| \frac{z_{C,1}z_{C,5}}{z_{C,2}z_{C,6}}, \frac{z_{C,3}z_{C,4}}{z_{C,2}z_{C,6}} \right),$$
$$\phi_{p_{\sigma_3^c}} = z_{C,2}^{D/2-3} z_{C,3}^{2-D/2} z_{C,5}^{2-D/2} z_{C,6}^{-1} F_4 \left(\begin{array}{cc} 1, & 3-\frac{D}{2} \\ 3-\frac{D}{2}, & 3-\frac{D}{2} \end{array} \middle| \frac{z_{C,1}z_{C,5}}{z_{C,2}z_{C,6}}, \frac{z_{C,3}z_{C,4}}{z_{C,2}z_{C,6}} \right),$$
$$\phi_{p_{\sigma_4^c}} = z_{C,2}^{-1} z_{C,4}^{D/2-2} z_{C,5}^{2-D/2} z_{C,6}^{1-D/2} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ 3-\frac{D}{2}, & \frac{D}{2}-1 \end{array} \middle| \frac{z_{C,1}z_{C,5}}{z_{C,2}z_{C,6}}, \frac{z_{C,3}z_{C,4}}{z_{C,2}z_{C,6}} \right). \quad (53)$$

Similarly setting $z_{C,1} = x_C$, $z_{C,2} = y_C$, and $z_{C,k} = 1$ with $k = 3, \cdots, 6$, one formulates the Feynman integral as the linear combinations of canonical series solutions in the corresponding parameter space.

- For $|x_C| \le 1,\ |y_C| \le 1$, the Feynman integral is

$$\begin{aligned} S_{a,C}(x_C, y_C) = &A_{a,C}(x_C y_C)^{D/2-2} F_4 \left(\begin{array}{cc} D-3, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & \frac{D}{2}-1 \end{array} \middle| x_C, y_C \right) \\ &+ B_{a,C} x_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & 3-\frac{D}{2} \end{array} \middle| x_C, y_C \right) \\ &+ C_{a,C} y_i^{D/2-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ 3-\frac{D}{2}, & \frac{D}{2}-1 \end{array} \middle| x_C, y_C \right) \\ &+ D_{a,C} F_4 \left(\begin{array}{cc} 1, & 3-\frac{D}{2} \\ 3-\frac{D}{2}, & 3-\frac{D}{2} \end{array} \middle| x_C, y_C \right), \end{aligned} \quad (54)$$

  which is convergent in the region $\sqrt{|x_C|} + \sqrt{|y_C|} < 1$.
- For $|x_C| \ge 1,\ |y_C| \le 1$, the Feynman integral is

$$\begin{aligned} S_{b,C}(x_C, y_C) = &A_{b,C} x_C^{-1} y_C^{D/2-2} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ 3-\frac{D}{2}, & \frac{D}{2}-1 \end{array} \middle| \frac{1}{x_C}, \frac{y_C}{x_C} \right) \\ &+ B_{b,C} x_C^{1-D/2} y_C^{D/2-2} F_4 \left(\begin{array}{cc} D-3, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & \frac{D}{2}-1 \end{array} \middle| \frac{1}{x_C}, \frac{y_C}{x_C} \right) \\ &+ C_{b,C} x_C^{D/2-3} F_4 \left(\begin{array}{cc} 1, & 3-\frac{D}{2} \\ 3-\frac{D}{2}, & 3-\frac{D}{2} \end{array} \middle| \frac{1}{x_C}, \frac{y_C}{x_C} \right) \\ &+ D_{b,C} x_C^{-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & 3-\frac{D}{2} \end{array} \middle| \frac{1}{x_C}, \frac{y_C}{x_C} \right) \end{aligned} \quad (55)$$

  which is convergent in the region $1 + \sqrt{|y_C|} < \sqrt{|x_C|}$.
- For $|x_C| \le 1,\ |y_C| \ge 1$, the Feynman integral is

$$\begin{aligned} S_{c,C}(x_C, y_C) = &A_{c,C} x_C^{D/2-2} y_C^{-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & 3-\frac{D}{2} \end{array} \middle| \frac{x_C}{y_C}, \frac{1}{y_C} \right) \\ &+ B_{c,C} x_C^{D/2-2} y_C^{1-D/2} F_4 \left(\begin{array}{cc} D-3, & \frac{D}{2}-1 \\ \frac{D}{2}-1, & \frac{D}{2}-1 \end{array} \middle| \frac{x_C}{y_C}, \frac{1}{y_C} \right) \\ &+ C_{c,C} y_C^{D/2-3} F_4 \left(\begin{array}{cc} 1, & 3-\frac{D}{2} \\ 3-\frac{D}{2}, & 3-\frac{D}{2} \end{array} \middle| \frac{x_C}{y_C}, \frac{1}{y_C} \right) \end{aligned}$$

$$+ D_{c,C}\, y_C^{-1} F_4 \left(\begin{array}{cc} 1, & \frac{D}{2}-1 \\ 3-\frac{D}{2}, & \frac{D}{2}-1 \end{array} \middle| \frac{x_C}{y_C},\ \frac{1}{y_C} \right) , \tag{56}$$

which is convergent in the region $1+\sqrt{|x_C|} < \sqrt{|y_C|}$.

Using the expressions of Feynman integral of the massless triangle diagram at some special kinematic points $(\Lambda^2,\ \Lambda^2,\ 0)$, $(\Lambda^2,\ 0,\ 0)$ etc, one obtains those integration constants as

$$\begin{aligned}
&A_{a,C} = -\Gamma^2(2-\frac{D}{2})\Gamma(\frac{D}{2}-1) , \\
&B_{a,C} = C_{a,C} = -D_{a,C} = \frac{\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{(2-\frac{D}{2})\Gamma(D-3)} , \\
&B_{b,C} = B_{c,C} = A_{a,C} ,\ \ D_{b,C} = A_{c,C} = B_{a,C} , \\
&A_{b,C} = D_{c,C} = C_{a,C} ,\ \ C_{b,C} = C_{c,C} = D_{a,C} .
\end{aligned} \tag{57}$$

Actually Feynman integrals presented here can be written in terms of Gauss function [19] by the well-known reduction of Appell function of the fourth kind [18], then the analytic continuation of those Feynman integrals is made to the whole parameter space through the transformations of Gauss functions.

4. GKZ-hypergeometric systems of other Feynman integrals

4.1. Sunset diagram with three differential masses

In order to make the notation less cluttered, we adopt the multi-index convention [55], and write Feynman integral of the two-loop sunset diagram as

$$\Sigma_{\ominus}(p^2) = -\frac{p^2\Gamma^2(3-\frac{D}{2})}{(4\pi)^4}\left(\frac{4\pi\mu^2}{-p^2}\right)^{4-D} T_{123}^{p}(\mathbf{a},\ \mathbf{b};\ \mathbf{x}) , \tag{58}$$

with the multi-index notations $\mathbf{a}=(a_1, a_2)$, $\mathbf{b}=(b_1, b_2, b_3)$, and $\mathbf{x}=(x_1, x_2, x_3)$. Where $a_1 = 3-D$, $a_2 = 4-3D/2$, $b_1 = b_2 = b_3 = 2-D/2$, and $x_1 = m_1^2/p^2$, $x_2 = m_2^2/p^2$, $x_3 = m_3^2/p^2$. Certainly the dimensionless function T_{123}^{p} complies with the third Lauricella's system of linear PDEs [42,43]

$$\left\{\hat{\vartheta}_{x_k}(\hat{\vartheta}_{x_k}+b_k-1) - x_k(\sum_{i=1}^{3}\hat{\vartheta}_{x_i}+a_1)(\sum_{i=1}^{3}\hat{\vartheta}_{x_i}+a_2)\right\}T_{123}^{p} = 0 , (k=1,\ 2,\ 3) . \tag{59}$$

Using the system of linear PDEs, one derives the following relations between T_{123}^{p} and its contiguous functions as

$$\begin{aligned}
&(\sum_{i=1}^{3}\hat{\vartheta}_{x_i}+a_j)T_{123}^{p}(\mathbf{a},\ \mathbf{b};\ \mathbf{x}) = a_j T_{123}^{p}(\mathbf{a}+\mathbf{n}_{2,j},\ \mathbf{b};\ \mathbf{x}),\ (j=1,\ 2) , \\
&(\hat{\vartheta}_{x_k}+b_k-1)T_{123}^{p}(\mathbf{a},\ \mathbf{b};\ \mathbf{x}) = (b_k-1)T_{123}^{p}(\mathbf{a},\ \mathbf{b}-\mathbf{n}_{3,k};\ \mathbf{x}) , \\
&\partial_{x_k} T_{123}^{p}(\mathbf{a},\ \mathbf{b};\ \mathbf{x}) = \frac{a_1 a_2}{b_k} T_{123}^{p}(\mathbf{a}+\mathbf{n}_2,\ \mathbf{b}+\mathbf{n}_{3,k};\ \mathbf{x}) , (k=1,\ 2,\ 3) .
\end{aligned} \tag{60}$$

Where $\mathbf{n}_{2,j} \in \mathbf{R}^2$, $(j=1,2)$ denotes the row vector whose entry is zero except that the j-th entry is 1, and the row vector $\mathbf{n}_2=(1,1)$. In addition $\mathbf{n}_{3,k} \in \mathbf{R}^3$, $(k=1,2,3)$ denotes the row vector whose entries are zero except that the j-th entry is 1 and $\mathbf{n}_3 = (1, 1, 1)$. Following the work of W. Miller [44,45], we define the auxiliary function Φ as

$$\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\ \mathbf{v}^{\mathbf{b}-\mathbf{n}_3}\ T_{123}^{p}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x})\ . \tag{61}$$

Where the row vectors $\mathbf{u} = (u_1, u_2) \in \mathbf{R}^2$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{R}^3$, and the multi-index notations $\mathbf{u}^{\mathbf{a}} = u_1^{a_1} u_2^{a_2}$, $\mathbf{v}^{\mathbf{b}-\mathbf{n}_3} = \prod_{i=1}^{3} v_i^{b_i-1}$. Miller's transformation [44,45] on the function T_{123}^{p} is to replace the multiplication by the parameter a_j, b_k in Eq. (60) by Euler operators $\hat{\vartheta}_{u_j}$, $\hat{\vartheta}_{v_k}$:

$$\begin{aligned}
&\hat{\vartheta}_{u_j}\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}),\ (j = 1, 2)\ ,\\
&\hat{\vartheta}_{v_k}\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1)\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}),\ (k = 1, 2, 3)\ ,
\end{aligned} \tag{62}$$

which induces the notion of GKZ-hypergeometric system naturally. In addition, the contiguous relations of the function defined in Eq. (61) are given as

$$\begin{aligned}
&\hat{\mathcal{O}}_1\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi(\mathbf{a}+\mathbf{n}_{2,j},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}),\ (j = 1, 2)\ ,\\
&\hat{\mathcal{O}}_{2+k}\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1)\Phi(\mathbf{a},\ \mathbf{b}-\mathbf{n}_{3,k};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\\
&\hat{\mathcal{O}}_{5+k}\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \frac{a_1 a_2}{b_k}\Phi(\mathbf{a}+\mathbf{n}_2,\ \mathbf{b}+\mathbf{n}_{3,k};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}),\ (k = 1, 2, 3)\ ,
\end{aligned} \tag{63}$$

where the operators $\hat{\mathcal{O}}_n^i\,(n = 1, \cdots, 8)$ are

$$\begin{aligned}
&\hat{\mathcal{O}}_1 = u_1(\sum_{i=1}^{3}\hat{\vartheta}_{x_i} + \hat{\vartheta}_{u_1})\ ,\\
&\hat{\mathcal{O}}_2 = u_2(\sum_{i=1}^{3}\hat{\vartheta}_{x_i} + \hat{\vartheta}_{u_2})\ ,\\
&\hat{\mathcal{O}}_{2+k} = \frac{1}{v_k}(\hat{\vartheta}_{x_k} + \hat{\vartheta}_{v_k})\ ,\\
&\hat{\mathcal{O}}_{5+k} = u_1 u_2 v_k \partial_{x_k}\ ,(k = 1,\ 2,\ 3)\ .
\end{aligned} \tag{64}$$

Those operators together with $\hat{\vartheta}_{u_j}$, $\hat{\vartheta}_{v_k}$ define the Lie algebra of the hypergeometric system [44, 45] in Eq. (59). Through the transformation of indeterminates

$$\begin{aligned}
&z_1 = \frac{1}{u_1}\ ,\ \ z_2 = \frac{1}{u_2}\ ,\\
&z_{2+k} = v_i\ , z_{5+k} = \frac{x_k}{u_1 u_2 v_k}\ ,\ \ (k = 1,\ 2,\ 3)\ ,
\end{aligned} \tag{65}$$

the equations in Eq. (62) are changed as

$$\Big(\mathbf{A}_{\ominus}\cdot\vec{\vartheta}\Big)\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{B}_{\ominus}\Phi(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ , \tag{66}$$

where

$$\mathbf{A}_\ominus = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$
$$\vec{\vartheta}^{\,T} = (\vartheta_{z_1}, \vartheta_{z_2}, \vartheta_{z_3}, \vartheta_{z_4}, \vartheta_{z_5}, \vartheta_{z_6}, \vartheta_{z_7}, \vartheta_{z_8}),$$
$$\mathbf{B}_\ominus^T = (-a_1, -a_2, b_1 - 1, b_2 - 1, b_3 - 1). \tag{67}$$

Correspondingly the universal Gröbner basis of the toric ideal associated with $\mathbf{A}_\ominus$ is

$$\mathcal{U}_{\mathbf{A}_\ominus} = \{\partial_{z_1}\partial_{z_2} - \partial_{z_3}\partial_{z_6}, \partial_{z_1}\partial_{z_2} - \partial_{z_4}\partial_{z_7}, \partial_{z_1}\partial_{z_2} - \partial_{z_5}\partial_{z_8},$$
$$\partial_{z_3}\partial_{z_6} - \partial_{z_4}\partial_{z_7}, \partial_{z_3}\partial_{z_6} - \partial_{z_5}\partial_{z_8}, \partial_{z_4}\partial_{z_7} - \partial_{z_5}\partial_{z_8}\}. \tag{68}$$

The operators $\mathbf{A}_\ominus \cdot \vec{\vartheta} - \mathbf{B}_\ominus$ and that from the set $\mathcal{U}_{\mathbf{A}_\ominus}$ compose the generators of a left ideal in the Weyl algebra $D = \mathbf{C}\langle z_1, \cdots, z_8, \partial_{z_1}, \cdots, \partial_{z_8}\rangle$. Defining an isomorphism between the commutative polynomial ring and the Weyl algebra [46]

$$\Psi: \ \mathbf{C}[z_1, \cdots, z_8, \xi_1, \cdots, \xi_8] \to D, \ z_i^\alpha \xi_i^\beta \mapsto z_i^\alpha \partial_{z_i}^\beta, \tag{69}$$

one obtains the state polytope [53] of the preimage of the universal Gröbner basis in Eq. (68)

$$\xi_6 + \xi_7 + \xi_8 \geq 3, \ \xi_6 + \xi_7 \geq 1, \ \xi_6 + \xi_8 \geq 1,$$
$$\xi_7 + \xi_8 \geq 1, \ \xi_6 \geq 0, \ \xi_7 \geq 0, \ \xi_8 \geq 0, \ -\xi_8 \geq -3,$$
$$-\xi_7 \geq -3, \ -\xi_6 \geq -3, \ -\xi_7 - \xi_8 \geq -5, \ -\xi_6 - \xi_8 \geq -5,$$
$$-\xi_6 - \xi_7 \geq -5, \ -\xi_6 - \xi_7 - \xi_8 \geq -6, \tag{70}$$

on the hyperplane

$$\xi_1 = \xi_2, \quad \xi_3 = \xi_6, \quad \xi_4 = \xi_7,$$
$$\xi_5 = \xi_8, \quad \xi_2 + \xi_6 + \xi_7 + \xi_8 = 6. \tag{71}$$

The normal fan of the state polytope in Eq. (70) is the Gröbner fan of the left ideal generated by the operators in Eq. (66) and Eq. (68). Because codimension = 3 for GKZ-hypergeometric system here, the Gröbner fan refines the secondary fan which is composed by the column vectors of a Gale transform of the matrix $A_\ominus$ in Eq. (67). With these fans, one constructs canonical basis series solutions in C-type Lauricella functions with three variables [43,60,61]. In order to make analytic continuation of Lauricella functions from their convergent regions to the whole parameter space, we should perform some linear fractional transformations among the complex variables $z_1, \cdots, z_8$. We will release our calculation results further elsewhere.

4.2. C_0 function with one nonzero mass

In this case, the scalar integral

$$C_0^a(p_1^2, p_2^2, p_3^2, m^2) = \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2 - m^2)(q + p_1)^2 (q - p_2)^2}$$
$$= \frac{i(-)^{D/2}(p_3^2)^{D/2-3}}{(4\pi)^{D/2}} F_{a,p_3}(\mathbf{a}, \mathbf{b}; \mathbf{x}) \tag{72}$$

where the row vectors $\mathbf{a}=(a_1,a_2,a_3)\in\mathbf{R}^3$, $\mathbf{b}=(b_1,b_2)\in\mathbf{R}^2$, and $\mathbf{x}=(\xi_{33},x_{13},x_{23})\in\mathbf{R}^3$, respectively. Additionally the parameters $a_1=4-D$, $a_2=3-D/2$, $a_3=1$, $b_1=b_2=3-D/2$, $p_3^2=(p_1+p_2)^2$, and the dimensionless ratios $\xi_{33}=-m^2/p_3^2$, $x_{ij}=p_i^2/p_j^2$, $(i,\ j=1,2,3)$. The dimensionless function F_{a,p_3} satisfies the holonomic hypergeometric system of linear PDEs

$$\begin{aligned}&\Big\{(a_1+\hat{\vartheta}_{\xi_{33}})(a_2+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}})(a_3+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}})\\&-\frac{1}{\xi_{33}}\hat{\vartheta}_{\xi_{33}}\prod_{i=1}^{2}(b_i+\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{x_{i3}})\Big\}F_{a,p_3}=0\,,\\&\Big\{(a_2+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}})(a_3+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}})\\&-\frac{1}{x_{j3}}\hat{\vartheta}_{x_{j3}}(b_j+\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{x_{j3}})\Big\}F_{a,p_3}=0\,,\ \ (j=1,\ 2)\,.\end{aligned}\tag{73}$$

Defining Miller's transformation on the function F_{a,p_3}

$$\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{n}_2}F_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x})\tag{74}$$

with $\mathbf{u}=(u_1,u_2,u_3)\in\mathbf{R}^3$, $\mathbf{v}=(v_1,v_2)\in\mathbf{R}^2$, one replaces the multiplication of the parameter a_k, b_j by Euler operators $\hat{\vartheta}_{u_k}$, $\hat{\vartheta}_{v_j}$:

$$\begin{aligned}&\hat{\vartheta}_{u_k}\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=a_k\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\&\hat{\vartheta}_{v_j}\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=(b_j-1)\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\end{aligned}\tag{75}$$

where $k=1,\ 2,\ 3$ and $j=1,\ 2$, respectively. In addition, the contiguous relations of the auxiliary function are given as

$$\begin{aligned}&\hat{\mathcal{O}}_k\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=a_k\Phi_{a,p_3}(\mathbf{a}+\mathbf{n}_{3,k},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\&\hat{\mathcal{O}}_{3+j}\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=(b_j-1)\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b}-\mathbf{n}_{2,j};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\&\hat{\mathcal{O}}_6\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\frac{a_1a_2a_3}{b_1b_2}\Phi_{a,p_3}(\mathbf{a}+\mathbf{n}_3,\ \mathbf{b}+\mathbf{n}_2;\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\&\hat{\mathcal{O}}_{6+j}\Phi_{a,p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\frac{a_2a_3}{b_j}\Phi_{a,p_3}(\mathbf{a}+\mathbf{n}_{3,2}+\mathbf{n}_{3,3},\ \mathbf{b}+\mathbf{n}_{2,j};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\end{aligned}\tag{76}$$

where the operators $\hat{\mathcal{O}}_n^i\,(n=1,\cdots,8)$ are defined as

$$\begin{aligned}&\hat{\mathcal{O}}_1=u_1(\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{u_1})\,,\\&\hat{\mathcal{O}}_2=u_2(\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}+\hat{\vartheta}_{u_2})\,,\\&\hat{\mathcal{O}}_3=u_3(\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}+\hat{\vartheta}_{u_3})\,,\end{aligned}$$

$$\hat{\mathcal{O}}_{3+j} = \frac{1}{v_j}(\hat{\vartheta}_{\xi_{33}} + \hat{\vartheta}_{x_{j3}} + \hat{\vartheta}_{v_j}) \;(j=1,\,2)\,,$$
$$\hat{\mathcal{O}}_{6} = u_1 u_2 u_3 v_1 v_2 \partial_{\xi_{33}}\,,$$
$$\hat{\mathcal{O}}_{6+j} = u_2 u_3 v_j \partial_{x_{j3}}\,,(j=1,\,2)\,. \tag{77}$$

Those operators above together with $\hat{\vartheta}_{u_k}$, $\hat{\vartheta}_{v_j}$ define the Lie algebra of the hypergeometric system [44,45] in Eq. (73). Through the transformation of indeterminates

$$z_k^a = \frac{1}{u_k}\,,\; z_{3+j}^a = v_j\,,$$
$$z_6^a = \frac{\xi_{33}}{u_1 u_2 u_3 v_1 v_2}\,, z_{6+j}^a = \frac{x_{j3}}{u_2 u_3 v_j}\,,\;(k=1,\,2,\,3,\;\; j=1,\,2)\,, \tag{78}$$

the equations in Eq. (75) are rewritten as

$$\Big(\mathbf{A}_a \cdot \vec{\vartheta}_a\Big)\Phi_{a,p_3}(\mathbf{a},\,\mathbf{b};\;\mathbf{x},\,\mathbf{u},\,\mathbf{v}) = \mathbf{B_a}\Phi_{a,p_3}(\mathbf{a},\,\mathbf{b};\;\mathbf{x},\,\mathbf{u},\,\mathbf{v})\,, \tag{79}$$

where

$$\mathbf{A}_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$
$$\vec{\vartheta}_a^{\,T} = (\vartheta_{z_1^a},\,\vartheta_{z_2^a},\,\vartheta_{z_3^a},\,\vartheta_{z_4^a},\,\vartheta_{z_5^a},\,\vartheta_{z_6^a},\,\vartheta_{z_7^a},\,\vartheta_{z_8^a})\,,$$
$$\mathbf{B}_a^{\,T} = (-a_1,\,-a_2,\,-a_3,\,b_1,\,b_2)\,. \tag{80}$$

Correspondingly the universal Gröbner basis of the toric ideal associated with $\mathbf{A}_a$ is

$$\mathcal{U}_{\mathbf{A}_a} = \{\partial_{z_1^a}\partial_{z_7^a} - \partial_{z_5^a}\partial_{z_6^a},\,\partial_{z_1^a}\partial_{z_8^a} - \partial_{z_4^a}\partial_{z_6^a},\,\partial_{z_2^a}\partial_{z_3^a} - \partial_{z_4^a}\partial_{z_7^a},\,\partial_{z_2^a}\partial_{z_3^a} - \partial_{z_5^a}\partial_{z_8^a},$$
$$\partial_{z_4^a}\partial_{z_7^a} - \partial_{z_5^a}\partial_{z_8^a},\,\partial_{z_1^a}\partial_{z_2^a}\partial_{z_3^a} - \partial_{z_4^a}\partial_{z_5^a}\partial_{z_6^a},\,\partial_{z_1^a}\partial_{z_7^a}\partial_{z_8^a} - \partial_{z_2^a}\partial_{z_3^a}\partial_{z_6^a}\}\,. \tag{81}$$

Certainly one can calculate the state polytope [53] corresponding to the universal Gröbner basis $\mathcal{U}_{\mathbf{A}_a}$ whose normal fan coincides with the Gröbner fan, then construct canonical series solutions in the convergent regions which are presented in Ref. [43]. In order to perform the analytic continuation of canonical series solutions from the convergent regions to the whole parameter space, one utilizes some linear fractional transformations among the complex variables $z_1^a, \cdots, z_8^a$, then chooses $u_k = 1,\; v_j = 1\;(k=1,2,3,\; j=1,2)$ finally.

4.3. C_0 function with three differential masses

The massive C_0 function is generally written as

$$C_0(p_1^2,\,p_2^2,\,p_3^2,\,m_1^2,\,m_2^2,\,m_3^2)$$
$$= \int \frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2-m_3^2)((q+p_1)^2-m_2^2)((q-p_2)^2-m_1^2)}$$
$$= \frac{i(p_3^2)^{D/2-3}}{(4\pi)^{D/2}}F_{p_3}(\mathbf{a},\,\mathbf{b};\;\mathbf{x}) \tag{82}$$

with the row vectors $\mathbf{a}=(a_1,a_2,a_3)\in\mathbf{R}^3$, $\mathbf{b}=(b_1,b_2)\in\mathbf{R}^2$, and $\mathbf{x}=(\xi_{13},\xi_{23},\xi_{33},x_{13},x_{23})\in\mathbf{R}^5$, respectively. Here the parameters $a_k,\ b_j\ (k=1,2,3,\ j=1,2)$ are taken the same values as in Eq. (72) and the dimensionless ratios $\xi_{ij}=-m_i^2/p_j^2$, $x_{ij}=p_i^2/p_j^2$. Furthermore the dimensionless function F_{p_3} complies with the hypergeometric system of linear PDEs

$$\begin{aligned}
&\Big\{\Big[3-\frac{D}{2}+\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}\Big]\Big[1+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}\Big]\Big[4-D+\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}}\Big]\\
&-\frac{1}{\xi_{33}}\hat{\vartheta}_{\xi_{33}}\prod_{i=1}^{3}\Big[2-\frac{D}{2}+\hat{\vartheta}_{\xi_{i3}}+\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{x_{(3-i)3}}\Big]\Big\}F_{p_3}=0\,,\\
&\Big\{\Big[3-\frac{D}{2}+\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}\Big]\Big[4-D+\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}}\Big]\\
&+\frac{1}{\xi_{j3}}\hat{\vartheta}_{\xi_{j3}}\Big[2-\frac{D}{2}+\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{\xi_{j3}}+\hat{\vartheta}_{x_{(3-j)3}}\Big]\Big\}F_{p_3}=0\,,\\
&\Big\{\Big[3-\frac{D}{2}+\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}\Big]\Big[1+\hat{\vartheta}_{\xi_{33}}+\sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}}\Big]\\
&-\frac{1}{x_{j3}}\hat{\vartheta}_{x_{j3}}\Big[2-\frac{D}{2}+\hat{\vartheta}_{\xi_{33}}+\hat{\vartheta}_{\xi_{(3-j)3}}+\hat{\vartheta}_{x_{j3}}\Big]\Big\}F_{p_3}=0,\ (j=1,\ 2)\,.
\end{aligned}\tag{83}$$

Defining Miller's transformation on the function F_{p_3}

$$\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_2}F_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x})\tag{84}$$

with $\mathbf{u}=(u_1,u_2,u_3)\in\mathbf{R}^3$, $\mathbf{v}=(v_1,v_2)\in\mathbf{R}^2$, we similarly replace the multiplication of the parameters $a_k,\ b_j$ by Euler operators $\hat{\vartheta}_{u_k},\ \hat{\vartheta}_{v_j}$:

$$\begin{aligned}
&\hat{\vartheta}_{u_k}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=a_k\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\
&\hat{\vartheta}_{v_j}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=(b_j-1)\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})
\end{aligned}\tag{85}$$

where $k=1,\ 2,\ 3$ and $j=1,\ 2$ respectively. Similarly the contiguous relations of the function defined in Eq. (84) are

$$\begin{aligned}
&\hat{\mathcal{O}}_k\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=a_k\Phi_{p_3}(\mathbf{a}+\mathbf{n}_{3,k},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\
&\hat{\mathcal{O}}_{3+j}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=(b_j-1)\Phi_{p_3}(\mathbf{a},\ \mathbf{b}-\mathbf{n}_{2,j};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\
&\hat{\mathcal{O}}_{5+j}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\frac{a_2a_3}{b_j}\Phi_{p_3}(\mathbf{a}+\mathbf{n}_{3,1}+\mathbf{n}_{3,2},\ \mathbf{b}+\mathbf{n}_{2,j};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\
&\hat{\mathcal{O}}_8\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\frac{a_1a_2a_3}{b_1b_2}\Phi_{p_3}(\mathbf{a}+\mathbf{n}_3,\ \mathbf{b}+\mathbf{n}_2;\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,\\
&\hat{\mathcal{O}}_{8+j}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})=\frac{a_2a_3}{b_j}\Phi_{p_3}(\mathbf{a}+\mathbf{n}_{3,2}+\mathbf{n}_{3,3},\ \mathbf{b}+\mathbf{n}_{2,j};\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\,,
\end{aligned}\tag{86}$$

where the operators $\hat{\mathcal{O}}_n\,(n=1,\cdots,10)$ are defined as

$$
\begin{aligned}
&\hat{\mathcal{O}}_1 = u_1(\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}} + \hat{\vartheta}_{u_1}) ,\\
&\hat{\mathcal{O}}_2 = u_2(\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}} + \sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}} + \hat{\vartheta}_{u_2}) ,\\
&\hat{\mathcal{O}}_3 = u_3(\hat{\vartheta}_{\xi_{33}} + \sum_{i=1}^{2}\hat{\vartheta}_{x_{i3}} + \hat{\vartheta}_{u_3}) ,\\
&\hat{\mathcal{O}}_{3+j} = \frac{1}{v_j}(\sum_{i=1}^{3}\hat{\vartheta}_{\xi_{i3}} - \hat{\vartheta}_{\xi_{j3}} + \hat{\vartheta}_{x_{j3}} + \hat{\vartheta}_{v_j})\ (j=1,\ 2) ,\\
&\hat{\mathcal{O}}_6 = u_1 u_2 v_1 \partial_{\xi_{23}} ,\\
&\hat{\mathcal{O}}_7 = u_1 u_2 v_2 \partial_{\xi_{13}} ,\\
&\hat{\mathcal{O}}_8 = u_1 u_2 u_3 v_1 v_2 \partial_{\xi_{33}} ,\\
&\hat{\mathcal{O}}_{8+j} = u_2 u_3 v_j \partial_{x_{j3}} , (j=1,\ 2) .
\end{aligned}
\tag{87}
$$

Those operators together with $\hat{\vartheta}_{u_i}$, $\hat{\vartheta}_{v_j}$ define the Lie algebra of the hypergeometric system [44, 45] in Eq. (83). Through the transformation of indeterminates

$$
\begin{aligned}
&z_i = \frac{1}{u_i} ,\ z_{3+j} = v_j ,\ z_6 = \frac{\xi_{13}}{u_1 u_2 v_2} ,\ z_7 = \frac{\xi_{23}}{u_1 u_2 v_1} ,\\
&z_8 = \frac{\xi_{33}}{u_1 u_2 u_3 v_1 v_2} , z_{8+j} = \frac{x_{j3}}{u_2 u_3 v_j} ,\ (i=1,\ 2,\ 3,\ \ j=1,\ 2) ,
\end{aligned}
\tag{88}
$$

the equations in Eq. (85) are

$$
\left(\mathbf{A}\cdot\vec{\vartheta}\right)\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{B}\Phi_{p_3}(\mathbf{a},\ \mathbf{b};\ \ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) ,
\tag{89}
$$

where

$$
\begin{aligned}
&\mathbf{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1
\end{pmatrix} ,\\
&\vec{\vartheta}^{\,T} = (\vartheta_{z_1},\ \vartheta_{z_2},\ \vartheta_{z_3},\ \vartheta_{z_4},\ \vartheta_{z_5},\ \vartheta_{z_6},\ \vartheta_{z_7},\ \vartheta_{z_8},\ \vartheta_{z_9},\ \vartheta_{z_{10}}) ,\\
&\mathbf{B}^{T} = (-a_1,\ -a_2,\ -a_3,\ b_1,\ b_2) .
\end{aligned}
\tag{90}
$$

Correspondingly the universal Gröbner basis of the toric ideal associated with **A** is

$$
\begin{aligned}
\mathcal{U}_{\mathbf{A}} = \{&\partial_{z_1}\partial_{z_2} - \partial_{z_4}\partial_{z_7},\ \partial_{z_1}\partial_{z_2} - \partial_{z_5}\partial_{z_6},\ \partial_{z_1}\partial_{z_9} - \partial_{z_3}\partial_{z_7},\ \partial_{z_1}\partial_{z_9} - \partial_{z_5}\partial_{z_8},\\
&\partial_{z_1}\partial_{z_{10}} - \partial_{z_3}\partial_{z_6},\ \partial_{z_1}\partial_{z_{10}} - \partial_{z_4}\partial_{z_8},\ \partial_{z_2}\partial_{z_3} - \partial_{z_4}\partial_{z_9},\ \partial_{z_2}\partial_{z_3} - \partial_{z_5}\partial_{z_{10}},\\
&\partial_{z_2}\partial_{z_8} - \partial_{z_6}\partial_{z_9},\ \partial_{z_2}\partial_{z_8} - \partial_{z_7}\partial_{z_{10}},\ \partial_{z_3}\partial_{z_6} - \partial_{z_4}\partial_{z_8},\ \partial_{z_3}\partial_{z_7} - \partial_{z_5}\partial_{z_8},\\
&\partial_{z_4}\partial_{z_7} - \partial_{z_5}\partial_{z_6},\ \partial_{z_4}\partial_{z_9} - \partial_{z_5}\partial_{z_{10}},\ \partial_{z_6}\partial_{z_9} - \partial_{z_7}\partial_{z_{10}},\ \partial_{z_1}\partial_{z_2}\partial_{z_3} - \partial_{z_4}\partial_{z_5}\partial_{z_8},
\end{aligned}
$$

$$\begin{aligned}&\partial_{z_1}\partial_{z_2}\partial_{z_8}-\partial_{z_3}\partial_{z_6}\partial_{z_7},\ \partial_{z_1}\partial_{z_2}\partial_{z_9}-\partial_{z_5}\partial_{z_7}\partial_{z_{10}},\ \partial_{z_1}\partial_{z_2}\partial_{z_{10}}-\partial_{z_4}\partial_{z_6}\partial_{z_9},\\&\partial_{z_1}\partial_{z_4}\partial_{z_9}-\partial_{z_3}\partial_{z_5}\partial_{z_6},\ \partial_{z_1}\partial_{z_5}\partial_{z_{10}}-\partial_{z_3}\partial_{z_4}\partial_{z_7},\ \partial_{z_1}\partial_{z_6}\partial_{z_9}-\partial_{z_4}\partial_{z_7}\partial_{z_8},\\&\partial_{z_1}\partial_{z_7}\partial_{z_{10}}-\partial_{z_5}\partial_{z_6}\partial_{z_8},\ \partial_{z_1}\partial_{z_9}\partial_{z_{10}}-\partial_{z_2}\partial_{z_3}\partial_{z_8},\ \partial_{z_2}\partial_{z_3}\partial_{z_6}-\partial_{z_4}\partial_{z_7}\partial_{z_{10}},\\&\partial_{z_2}\partial_{z_3}\partial_{z_7}-\partial_{z_5}\partial_{z_6}\partial_{z_9},\ \partial_{z_2}\partial_{z_4}\partial_{z_8}-\partial_{z_5}\partial_{z_6}\partial_{z_{10}},\ \partial_{z_2}\partial_{z_5}\partial_{z_8}-\partial_{z_4}\partial_{z_7}\partial_{z_9},\\&\partial_{z_3}\partial_{z_6}\partial_{z_9}-\partial_{z_5}\partial_{z_8}\partial_{z_{10}},\ \partial_{z_3}\partial_{z_7}\partial_{z_{10}}-\partial_{z_4}\partial_{z_8}\partial_{z_9}\}\,. \end{aligned} \tag{91}$$

The normal fan of corresponding state polytope of the universal Gröbner basis is the Gröbner fan, canonical series solutions are obtained similarly in the convergent regions. To perform the analytic continuation of canonical series solutions from the convergent regions to the whole parameter space, one can use the linear fractional transformations among the complex variables $z_1, \cdots, z_{10}$, then sets $u_k = 1,\ v_j = 1$ $(k = 1, 2, 3,\ j = 1, 2)$ finally.

5. Summary

Using the system of linear PDEs satisfied by the corresponding Feynman integral in Refs. [42, 43], we present GKZ-hypergeometric systems of one-loop self energy, one-loop triangle, two-loop vacuum, and the two-loop sunset diagrams, respectively. In those GKZ-hypergeometric systems the codimension equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared.

Actually one can derive GKZ-hypergeometric systems from Mellin-Barnes representations for the one-loop Feynman diagrams and those multiloop diagrams with two vertices, whose codimension equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. Nevertheless for the generic multiloop Feynman diagrams, the corresponding codimension of GKZ-hypergeometric system is far larger than the number of independent dimensionless ratios, whether using Mellin-Barnes or Lee-Pomeransky representations. In order to construct canonical series solutions properly, the corresponding GKZ-hypergeometric system should be restricted to the hyperplane in parameter space.

Taking GKZ-hypergeometric systems of one-loop self energy, one-loop massless triangle, and two-loop vacuum diagrams as examples, we present in detail how to perform triangulation and how to construct canonical series solutions in the corresponding convergent regions. The analytic continuation of those series solutions is performed through some well known reduction of Appell function of the fourth kind. In order to make analytic continuation of those series solutions of GKZ-hypergeometric systems of the massive sunset and massive one-loop triangle diagrams etc., one can perform the linear fractional transformations among the complex variables.

One of the techniques not involved here is how to project GKZ-hypergeometric system to the restricting hyperplane. Another calculation not contained here is how to make analytic continuation of those canonical series solutions from their convergent regions to the whole parameter space through linear fractional transformation among the complex variables. Algorithm for the first problem has been presented in literature already, and the second problem is attributed to a problem of integer programming [62] in principle. We will release our results relating to those topics in near future elsewhere.

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