Review Article

Quasi-Normal Modes: The "Electrons" of Black Holes as "Gravitational Atoms"? Implications for the Black Hole Information Puzzle

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Some recent important results on black hole (BH) quantum physics concerning the *BH effective state* and the natural correspondence between Hawking radiation and BH quasi-normal modes (QNMs) are reviewed, clarified, and refined. Such a correspondence permits one to naturally interpret QNMs as quantum levels in a semiclassical model. This is a model of BH somewhat similar to the historical semiclassical model of the structure of a hydrogen atom introduced by Bohr in 1913. In a certain sense, QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies, "triggered" by emissions (Hawking radiation) and absorption of particles, represent the energy "shells" of the "gravitational hydrogen atom." Important consequences on the BH information puzzle are discussed. In fact, it is shown that the time evolution of this "Bohr-like BH model" obeys *a time dependent Schrödinger equation* which permits the final BH state to be *a pure* quantum state instead of a mixed one. Thus, information comes out in BH evaporation in agreement with the assumption by 't Hooft that Schröedinger equations can be used universally for all dynamics in the universe. We also show that, in addition, our approach solves the entanglement problem connected with the information paradox.

1. Introduction

An intriguing and largely used framework to obtain Hawking radiation [1] is the tunnelling mechanism; see [2–8], for example. Considering an object classically stable, if it becomes unstable in a quantum mechanical approach, it is natural to suspect that a tunnelling process works. The famous Hawking's mechanism of particles creation by BHs [1] is described in a modern way as tunnelling of particles near the BH horizon [2–8]. Let us assume that a virtual particle pair is created just inside the horizon. Then, the virtual particle having positive energy can tunnel out and materializes outside the BH as a real particle. The same happens when one considers a virtual particle pair created

just outside the BH horizon. In that case, the particle having negative energy can tunnel inwards. The result of both of the situations is that the BH absorbs the particle having negative energy. Thus, the BH mass decreases and the particle having positive energy propagates towards infinity. Subsequent emissions of quanta appear, in turn, as Hawking radiation. The remarkable result of Parikh and Wilczek [2, 3] has shown a probability of emission which is compatible with a nonstrictly thermal spectrum, different from the original result of Hawking [1] and a recent result of Banerjee and Majhi [7], who found a perfect black body spectrum through a reformulation of the tunnelling mechanism. In [8] we have recently improved the tunnelling approach in [2, 3], showing that the obtained probability of emission

is really associated with the two following distributions [8, 9]:

$$\langle n \rangle_{\text{boson}} = \frac{1}{\exp\left[4\pi\left(2M - \omega\right)\omega\right] - 1},$$

$$\langle n \rangle_{\text{fermion}} = \frac{1}{\exp\left[4\pi\left(2M - \omega\right)\omega\right] + 1},$$

(1)

for bosons and fermions, respectively, which are *nonstrictly* thermal. The derivation of the two distributions (1) will be sketched in Section 2 of this paper.

The nonprecise black body spectrum has important implications for the BH information puzzle. In fact, arguments that information is lost during BH evaporation partially rely on the assumption of strictly black body spectrum [3, 10]. The nonstrictly thermal behavior in [2, 3] implies that emissions of subsequent Hawking quanta are countable [8, 9, 11–21] and, in turn, generates a natural correspondence between Hawking radiation and BH QNMs [9, 11–15, 21], permitting natural interpretation of QNMs as quantum levels [9, 11–15, 21]. The system composed of Hawking radiation and BH QNMs is somewhat similar to the semiclassical Bohr model of the structure of a hydrogen atom [22–24]. In the BH model in [9, 11–15, 21], during a quantum jump, a discrete amount of energy is indeed radiated and, for large values of the principal quantum number n, the analysis becomes independent of the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another and the absolute values of the QNMs frequencies represent the energy "shells" [9, 13]. In Bohr model [22–24] electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$, where h is the Planck constant and f is the transition frequency. In the BH model in [9, 11–15, 21], QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to equations which are in full agreement with previous literature of BH thermodynamics, like [25–27]. More, it will be shown that the BH model in [9, 11–15, 21] is also in agreement with the famous result of Bekenstein on the *area quantization* [28]. Bekenstein [29] was also, to our knowledge, the first who viewed BHs as similar to Bohr atoms, although our model in [9, 11–15, 21] is more detailed than Bekenstein's original intuition.

It is important to recall that Bohr model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. In the same way, one expects the Bohr-like BH model to be an approximated model with respect to the definitive, but at the present time unknown, BH model arising from a definitive theory of quantum gravity.

The time evolution of the Bohr-like BH model obeys a *time dependent Schrödinger equation* which permits writing the physical state and the correspondent *wave function* in terms of an *unitary* evolution matrix instead of a density matrix [9]. The BH final state results in turn in being a *pure*

quantum state instead of mixed one [9]. The fundamental consequence is that information comes out in BH evaporation in terms of pure states in an unitary time dependent evolution [9]. Thus, the BH evolution is in full agreement with the assumption by 't Hooft that Schrödinger equations can be used universally for all dynamics in the universe and this endorses the conclusion that BH evaporation is information preserving [9]. In addition, we will see that the present approach permits also solving the entanglement problem connected with the BH information puzzle [9].

2. Deviation from Standard Distributions for Bosons and Fermions

A problem on the tunnelling approach for Hawking radiation was that, in [4–6] and in other papers in the literature, the analysis has been finalized almost only to obtain the Hawking temperature through a comparison of the probability of emission of an outgoing particle with the Boltzmann factor. In the interesting work [7], the problem was apparently solved. In fact, analysing the tunnelling mechanism in a slightly different way, the authors of [7] found a perfect black body spectrum for Hawking radiation. On the other and, the result in [7] is in contrast with the result in [2, 3] that, indeed, has shown a probability of emission which is compatible with a nonstrictly black body spectrum having associated two nonstrictly thermal distributions. Introducing a BH effective state [8, 9, 11–15, 21], we solved such a contradiction in [8], considering a modification of the analysis in [7]. The final result, which we review in this section, is a nonstrictly black body spectrum with associated two nonstrictly thermal distributions in BH evaporation.

For the sake of simplicity, in all this paper we work with Planck units; that is, $G = c = k_B = \hbar = 1/4\pi\epsilon_0 = 1$.

Considering a Schwarzschild BH, the Schwarzschild line element is [8, 30]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right)$$

(2)

(historical notes to this notion can be found in [31]). The Schwarzschild radius (event horizon) is given by $r_H = 2M$ [7, 8, 30] and $1/4M$ is the BH surface gravity. As Hawking radiation will be discussed like tunnelling, the radial trajectory is the only relevant [2, 3, 7, 8]. Following [7], the (normalized) physical states of the system for bosons and fermions are [7]

$$|\Psi\rangle_{\text{boson}} = \left(1 - \exp\left(-8\pi M\omega\right)\right)^{1/2}$$
$$\cdot \sum_n \exp\left(-4\pi n M\omega\right)\left|n_{\text{out}}^{(L)}\right\rangle \otimes \left|n_{\text{out}}^{(R)}\right\rangle$$

$$|\Psi\rangle_{\text{fermion}} = \left(1 + \exp\left(-8\pi M\omega\right)\right)^{-1/2}$$
$$\cdot \sum_n \exp\left(-4\pi n M\omega\right)\left|n_{\text{out}}^{(L)}\right\rangle \otimes \left|n_{\text{out}}^{(R)}\right\rangle.$$

(3)

In the following, the analysis will be focused only on bosons. For fermions, the treatment is indeed the same [7]. The density matrix operator of the system is given by [7]

$$\widehat{\rho}_{\text{boson}} \equiv |\Psi\rangle_{\text{boson}} \langle\Psi|_{\text{boson}}$$

$$= (1 - \exp(-8\pi M\omega))$$

$$\cdot \sum_{n,m} \exp\left[-4\pi(n+m)M\omega\right] \left|n_{\text{out}}^{(L)}\right\rangle \tag{4}$$

$$\otimes \left|n_{\text{out}}^{(R)}\right\rangle \left\langle m_{\text{out}}^{(R)}\right| \otimes \left\langle m_{\text{out}}^{(L)}\right| .$$

One traces out the ingoing modes obtaining the density matrix for the outgoing (right) modes as [7]

$$\widehat{\rho}_{\text{boson}}^{(R)} = (1 - \exp(-8\pi M\omega))$$

$$\cdot \sum_{n} \exp(-8\pi n M\omega) \left|n_{\text{out}}^{(R)}\right\rangle \left\langle n_{\text{out}}^{(R)}\right| . \tag{5}$$

Then, the average number of particles detected at infinity is [7]

$$\langle n\rangle_{\text{boson}} = \text{tr}\left[\widehat{n}\widehat{\rho}_{\text{boson}}^{(R)}\right] = \frac{1}{\exp(8\pi M\omega) - 1}. \tag{6}$$

Trace (6) has been summed over all the eigenstates and a bit of algebra has been used to obtain the final result; see [7] for details. Equation (6) represents the well-known Bose-Einstein distribution. A similar analysis easily gives the well-known Fermi-Dirac distribution [7]

$$\langle n\rangle_{\text{fermion}} = \frac{1}{\exp(8\pi M\omega) + 1}. \tag{7}$$

Both distributions (6) and (7) represent a black body spectrum with the famous Hawking temperature [1, 7]:

$$T_H \equiv \frac{1}{8\pi M}. \tag{8}$$

Thus, we shortly reviewed the result in [7], which is very important. In fact, considering a reformulation of the tunnelling mechanism, a perfect black body spectrum is found, in full agreement with the original result by Hawking [1]. On the other hand, one immediately notes that this result is in contrast with the result in [2, 3]. The probability of emission which corresponds to the two distributions (6) and (7) is indeed [1–3]

$$\Gamma \sim \exp\left(-\frac{\omega}{T_H}\right). \tag{9}$$

But an exact calculation of the action for a tunnelling spherically symmetric particle gives the important correction [2, 3]:

$$\Gamma \sim \exp\left[-\frac{\omega}{T_H}\left(1 - \frac{\omega}{2M}\right)\right]. \tag{10}$$

This result is in contrast with the one in [7] because it releases the additional term $\omega/2M$ as a deviation from the strict

thermality [2, 3]. The key point is that in [7] the dynamical BH geometry due to the energy conservation has not been taken into due account like in [2, 3]. The energy conservation forces indeed the BH to contract during the emission of the particle [2, 3]. In this way, the horizon recedes from its original radius and becomes smaller at the end of the emission process [2, 3]. Thus, BHs do not exactly emit like perfect black bodies [2, 3].

In fact, the tunnelling is a *discrete* instead of *continuous* process [8, 11] because two different *countable* BH physical states have to be considered: the first before the emission of the particle and the latter after the emission of the particle [8, 11]. Then, the emission of the particle is interpreted like a *quantum transition* of frequency ω between the two different discrete states [8, 11]. The tunnelling mechanism works indeed considering a trajectory in imaginary or complex time joining two separated classical turning points [2, 3]. The important consequence is that the radiation spectrum is also discrete [8, 11]. Based on its importance, this issue has to be clarified in a better way. If one fixes the Hawking temperature, the statistical probability distribution (10) is a continuous function. But the Hawking temperature in (10) varies in time with a behavior which is *discrete.* The reason of this discrete character is that the forbidden region traversed by the emitting particle has a *finite* size [3]. Considering a strictly thermal approximation, the turning points have zero separation. Thus, it is not clear what joining trajectory one has to consider because there is not barrier [3]. The problem is solved if one argues that it is the forbidden finite region from $r_{\text{initial}} = 2M$ to $r_{\text{final}} = 2(M - \omega)$ that the tunnelling particle traverses which acts like barrier [3]. Then, the intriguing explanation is that it is the particle itself which generates a tunnel through the BH horizon [3].

If one wants to take into due account the dynamical geometry of the BH during the emission of the particle, a BH *effective state* can be introduced [8, 9, 11–15, 21]. In fact, introducing the *effective temperature* [8, 9, 11–15, 21]

$$T_E(\omega) \equiv \frac{2M}{2M - \omega}T_H = \frac{1}{4\pi(2M - \omega)}, \tag{11}$$

(10) can be rewritten in a Hawking-Boltzmann-like form similar to the original probability found by Hawking:

$$\Gamma \sim \exp\left[-\beta_E(\omega)\omega\right] = \exp\left(-\frac{\omega}{T_E(\omega)}\right), \tag{12}$$

with $\exp[-\beta_E(\omega)\omega]$ being the *effective Boltzmann factor* with [8, 9, 11–15, 21]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}. \tag{13}$$

Therefore, the effective temperature replaces the Hawking temperature in the equation of the probability of emission [8, 9, 11–15, 21]. Let us discuss the physical interpretation of $T_E(\omega)$ following [9]. The probability of emission of Hawking quanta found in [2, 3], that is, (10), shows that the BH does *not* emit like a perfect black body, having no strictly thermal behavior. On the other hand, the temperature in Bose-Einstein and Fermi-Dirac distributions is a perfect black body

temperature. Thus, when we have deviations from the strictly thermal behavior, that is, from the perfect black body, one expects also deviations from Bose-Einstein and Fermi-Dirac distributions [9]. How can one attack this problem? It is by analogy with other various fields of Science, also beyond BHs, for example, the case of planets and stars [9, 32]. One defines the effective temperature of a body such as a star or planet as the temperature of a black body that would emit the same total amount of electromagnetic radiation [9, 32]. The importance of the effective temperature in a star is stressed by the issue that the effective temperature and the bolometric luminosity are the two fundamental physical parameters needed to place a star on the Hertzsprung-Russell diagram [9, 32]. Both effective temperature and bolometric luminosity actually depend on the chemical composition of a star; see again [9, 32]. The concept of effective temperature has been introduced by the author in BH physics in [11, 12] for the Schwarzschild BH and generalized in [14] to the Kerr BH and in [15] to the nonextremal Reissner-Nordström BH by the author and collaborators. $T_E(\omega)$ depends on the energy-frequency of the emitted radiation while the ratio $T_E(\omega)/T_H = 2M/(2M - \omega)$ represents the deviation of the BH radiation spectrum from the strictly thermal behavior [8, 9, 11–15, 21].

After the introduction of $T_E(\omega)$ one can introduce other *effective quantities*. Considering the initial BH mass *before* the emission, M, and the final BH mass *after* the emission, $M - \omega$, respectively [8, 9, 11–15, 21], the *effective BH mass* and the *effective BH horizon during* its contraction, that is, *during* the emission of the particle, are defined as [8, 9, 11–15, 21]

$$M_E \equiv M - \frac{\omega}{2}, \qquad r_E \equiv 2M_E. \tag{14}$$

These effective quantities are average quantities [8, 9, 11–15, 21]. The effective horizon r_E is the average of the initial and final horizons and the effective mass M_E is the average of the initial and final masses [8, 9, 11–15, 21]. *Before* the emission the Hawking temperature it is $T_{H\,\text{initial}} = 1/8\pi M$; *after* the emission one gets $T_{H\,\text{final}} = 1/8\pi(M - \omega)$. Then, T_E results to be the inverse of the average value of the inverses of the initial and final Hawking temperatures [8, 9, 11–15, 21]. This implies that the Hawking temperature *has a discrete character in time* [8, 11–15]. We stress that the introduction of the effective temperature does not degrade the importance of the Hawking temperature [9]. In fact, as T_H changes with a discrete behavior in time, let us ask: which value of the Hawking temperature has to be associated with the emission of the particle? Has one to consider the value of the Hawking temperature *before* the *emission* or the value of the Hawking temperature after the emission? The answer is that one has to consider an *intermediate* value, the effective temperature, which is the inverse of the average value of the inverses of the initial and final Hawking temperatures [9]. In some way, $T_E(\omega)$ represents the value of the Hawking temperature *during* the emission of the quantum [8, 9, 11–15, 21]. Then, the effective temperature takes into account the nonstrictly thermal character of the radiation spectrum and the nonstrictly continuous character of subsequent emissions of Hawking quanta.

Now, let us rewrite (13) as [8]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)} = \beta_H\left(1 - \frac{\omega}{2M}\right), \tag{15}$$

where $\beta_H \equiv 1/T_H$. Using Hawking's arguments [8, 33, 34], we write down the euclidean form of the metric as [8]

$$ds_E^2 = x^2\left[\frac{d\tau}{4M\left(1 - \omega/2M\right)}\right]^2 + \left(\frac{r}{r_E}\right)^2 dx^2 \\ + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right), \tag{16}$$

which is regular at $x = 0$ and $r = r_E$. τ is treated as an angular variable with period $\beta_E(\omega)$ [8, 33, 34]. Replacing the quantity $\sum_i \beta_i(\hbar^i/M^{2i})$ in [33] with the quantity $-\omega/2M$ [8], one follows step by step the detailed analysis in [33] obtaining the *effective Schwarzschild line element* [8] as follows:

$$ds_E^2 \equiv -\left(1 - \frac{2M_E}{r}\right)dt^2 + \frac{dr^2}{1 - 2M_E/r} \\ + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right). \tag{17}$$

It is also simple to check that r_E in (16) is the same as in (14) [8]. Therefore, the *effective surface gravity* can be defined as $1/4M_E$. Thus, the effective line element (17) takes into account the BH *dynamical* geometry during the emission of the particle. Following step by step the analysis in [7], at the end the correct physical states for boson and fermions read [8]

$$|\Psi\rangle_{\text{boson}} = \left(1 - \exp\left(-8\pi M_E\omega\right)\right)^{1/2} \\ \cdot \sum_n \exp\left(-4\pi n M_E\omega\right)\left|n_{\text{out}}^{(L)}\right\rangle \otimes \left|n_{\text{out}}^{(R)}\right\rangle \\ |\Psi\rangle_{\text{fermion}} = \left(1 + \exp\left(-8\pi M_E\omega\right)\right)^{-1/2} \\ \cdot \sum_n \exp\left(-4\pi n M_E\omega\right)\left|n_{\text{out}}^{(L)}\right\rangle \otimes \left|n_{\text{out}}^{(R)}\right\rangle. \tag{18}$$

Then, one immediately finds that the correct, nonstrictly thermal, distributions are given exactly by (1). Those equations represent the distributions associated with probability of emission (10).

Resuming, in [7] the tunnelling approach on Hawking radiation has been improved by explicitly finding a black body spectrum associated with BHs. This result has been obtained by using a reformulation of the tunnelling mechanism. On the other hand, the result in [7] had a problem because it was in contrast with the result in [2, 3] that found a probability of emission compatible with a nonstrictly black body spectrum instead. Using the recent introduction of a BH effective state [8, 9, 11–15, 21], such a contradiction has been solved in [8] through a slight modification of the analysis in [7]. In this section the analysis in [8] has been reviewed, showing that the final result consists in a nonstrictly black body spectrum from the tunnelling mechanism with the two associated nonstrictly distributions (1).

3. Quasi-Normal Modes as "Electron States" in a Bohr-Like Black Hole Model

In this section we review the results in [11–13] for the Schwarzschild BH showing that the correction to the thermal spectrum in [2, 3] is also very important for the physical interpretation of BH QNMs and, in turn, it results important for realizing the underlying theory of quantum gravity. It is indeed a general conviction that BHs are theoretical laboratories for developing a quantum gravity theory and in this paper BH QNMs are naturally interpreted in terms of quantum levels, the "electron states" of a Bohr-like BH model [11–13].

BH QNMs are modes of radial perturbations obeying the time independent Schrödinger-like equation [11–13, 35]

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \omega^2 \right) \phi. \tag{19}$$

Working in strictly thermal approximation the Regge-Wheeler potential is introduced as [11–13, 35]

$$V(x) = V[x(r)] = \left(1 - \frac{2M}{r} \right) \left(\frac{l(l+1)}{r^2} + 2\frac{(1-j^2)M}{r^3} \right), \tag{20}$$

where $j = 0, 1, 2$ for scalar, vector, and gravitational perturbation, respectively.

The relation between the Regge-Wheeler "tortoise" coordinate x and the radial coordinate r is [11–13, 35]

$$x = r + 2M \ln \left(\frac{r}{2M} - 1 \right)$$
$$\frac{\partial}{\partial x} = \left(1 - \frac{2M}{r} \right) \frac{\partial}{\partial r}. \tag{21}$$

BH quasi-normal frequencies are analogous to quasi-stationary states in quantum mechanics [13, 35]. In that way, their frequency can be complex [13, 35]. Purely outgoing boundary conditions are requested both at the horizon ($r = 2M$) and in the asymptotic region ($r \to \infty$) [13, 35]:

$$\phi(x) \sim c_{\pm} \exp(\mp i\omega x) \quad \text{for } x = \pm\infty. \tag{22}$$

The intriguing idea to model the quantum BH in terms of QNMs was pioneered by York in [36]. In that work, using the statistical mechanics of the QNMs, the value of $((0.27654))$ $(16\pi M^2)$ was obtained for the BH entropy, which is near the value usually assumed of the standard Bekenstein-Hawking entropy $4\pi M^2$. *Considering Bohr's correspondence principle* [24], which states that "*transition frequencies at large quantum numbers should equal classical oscillation frequencies*," in [37, 38] the important result that QNMs release information about the area quantization was obtained by Hod. The idea in [37, 38] was to consider the QNMs as being associated with absorption of particles. Some important problems arising from this approach were solved in [25] by Maggiore. An important issue is that QNMs are *countable* frequencies. This was in contrast with ideas on the *continuous* character of Hawking radiation, preventing attempts to

interpret QNMs in terms of emitted quanta and, in turn, to associate QNMs modes with Hawking radiation [35]. On the other hand, the authors of [16–20] and ourselves and collaborators [9, 11–15, 21] made the important observation that the nonthermal spectrum in [2, 3] also implies the countable character of subsequent emissions of Hawking quanta. This key point enables a natural correspondence between Hawking radiation and BH QNMs [9, 11–15, 21] and permits, in turn, interpreting QNMs also in terms of emitted energies [9, 11–15, 21]. BH QNMs represent indeed the BH reaction to small, discrete perturbations in terms of damped oscillations [9, 11–15, 21]. If the capture of a particle causing an increase in the horizon area is a type of discrete perturbation [25, 37, 38], it is obvious and natural to consider the emission of a particle causing a decrease in the horizon area also a perturbation which generates a reaction in terms of countable QNMs, a process which is discrete rather than continuous [9, 11–15, 21]. This can be immediately understood if one considers that it is the particle having negative energy and tunnelling inwards which is captured by the hole in the mechanism of pair creation. Thus, the absorbed Hawking quanta having negative energies generate subsequent perturbations "triggering" the QNMs. Concerning this key point, it is important to emphasize that the correspondence existing between emitted radiation and proper oscillation of the emitting body is well known as a fundamental behavior of every radiation process in Science. This natural correspondence between Hawking radiation and BH QNMs permits immediately and naturally interpreting BH QNMs in terms of quantum levels also for emitted energies [9, 11–15, 21]. This is also in agreement with the general idea that BHs can be considered in terms of highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in an underlying theory of quantum gravity [13, 37, 38].

Considering a strictly thermal approximation, BH QNMs are usually labelled as ω_{nl}, with l being the angular momentum quantum number [11–13, 25, 35, 37, 38]. For each l, one finds a countable sequence of BH QNMs, labelled by the principal quantum number n ($n = 1, 2, \ldots$) [11–13, 25, 35, 37, 38]. For large n the Schwarzschild BH QNMs result in being independent of l. They have the following structure [11–13, 25, 35, 37, 38]:

$$\omega_n = \ln 3 \times T_H + 2\pi i \left(n + \frac{1}{2} \right) \times T_H + \mathcal{O}\left(n^{-1/2} \right)$$
$$= \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M} \left(n + \frac{1}{2} \right) + \mathcal{O}\left(n^{-1/2} \right). \tag{23}$$

Equation (23) has been originally obtained numerically in [39, 40]. Later, it has been also derived in analytical way in [35, 41]. On the other hand, (23) is strictly correct only for scalar and gravitational perturbations [13, 25, 35]. In any case, as we work in the large n approximation, the leading term in the imaginary part of the complex frequencies becomes dominant [13, 25, 35], in full agreement with Bohr correspondence principle [24]. Then, (23) is well approximated by [13, 25, 35]

$$\omega_n \simeq \frac{2\pi i n}{8\pi M}. \tag{24}$$

In [35] it has been shown that (24) holds also for vector and half integer spin perturbations, again in agreement with Bohr correspondence principle. A key point is that (24) works only in strictly thermal approximation. If one wants to take into due account the deviation from the perfect black body spectrum, the Hawking temperature T_H must be replaced by the effective temperature T_E in (24) obtaining [11–13]

$$\omega_n \simeq \frac{2\pi i n}{4\pi \left[2M - |\omega_n|\right]}. \tag{25}$$

An intuitive derivation of (25) can be found in [11, 12]. We rigorously derived such an equation in the Appendix of [13]. Further details on that derivation can be analysed as follows. The BH *dynamical* geometry during the emission of the particle is taken into account by the effective line element (17). Although this does not mean that one can immediately replace T_H with T_E in (24), the effective line element (17) permits introducing the following *effective equations* [11–13]:

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \omega^2\right)\phi, \tag{26}$$

$$V(x) = V[x(r)]$$

$$= \left(1 - \frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2} + 2\frac{(1-j^2)M_E}{r^3}\right), \tag{27}$$

where $V[x(r)]$ is the *effective Regge-Wheeler potential*, and

$$x = r + 2M_E \ln\left(\frac{r}{2M_E} - 1\right)$$

$$\frac{\partial}{\partial x} = \left(1 - \frac{2M_E}{r}\right)\frac{\partial}{\partial r}. \tag{28}$$

In order to simplify the following equations, here we also set [13]

$$2M_E = r_E \equiv 1, \qquad m \equiv n + 1. \tag{29}$$

We stress that the *Planck mass* m_p is equal to 1 in Planck units. Then, one rewrites (23) as [13]

$$\frac{\omega_m}{m_p^2} = \frac{\ln 3}{4\pi} + \frac{i}{2}\left(m - \frac{1}{2}\right) + \mathcal{O}\left(m^{-1/2}\right), \quad \text{for } m \gg 1, \tag{30}$$

where now $m_p \neq 1$. Setting [13]

$$\widetilde{\omega}_m \equiv \frac{\omega_m}{m_p^2}, \tag{31}$$

(30), (26), (27), and (28) read [13]

$$\widetilde{\omega}_m = \frac{\ln 3}{4\pi} + \frac{i}{2}\left(m - \frac{1}{2}\right) + \mathcal{O}\left(m^{-1/2}\right), \quad \text{for } m \gg 1, \tag{32}$$

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \widetilde{\omega}^2\right)\phi, \tag{33}$$

$$V(x) = V[x(r)] = \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{3(1-j^2)}{r^3}\right), \tag{34}$$

$$x = r + \ln(r-1)$$

$$\frac{\partial}{\partial x} = \left(1 - \frac{1}{r}\right)\frac{\partial}{\partial r}, \tag{35}$$

respectively. Now, if one replies the same rigorous analytical calculation in the Appendix of [13] or the analogous calculation in [35], but starting from (33), (34), and (35) and satisfying purely outgoing boundary conditions both at the effective horizon ($r_E = 2M_E$) and in the asymptotic region ($r = \infty$), the final result in the leading term in the imaginary part of the complex frequencies will be, obviously and rigorously, (25). An important issue has to be clarified. One could take position against the above analysis claiming that M_E and r_E (and consequently the *effective tortoise coordinate* and the effective Regge-Wheeler potential) are frequency dependent. But we note that (29) translates such a frequency dependence into a continually rescaled mass unit in the discussion in the Appendix of [13]. It is simple to show that such a rescaling is extremely slow and is always included within factor 2. Thus, it does not influence the analysis in the Appendix of [13]. In fact, we note that although $\widetilde{\omega}$ in the analysis in the Appendix of [13] can be very large because of definition (31), ω must instead be always minor than the BH initial mass as BHs cannot emit more energy than their total mass. Thus, the analysis in the Appendix of [13] can be considered a "quasi-asymptotic" analysis; that is, the ω can be extremely large but not infinity; see also the below discussion on the maximum value for the overtone number n. Inserting this constraint in (14), one gets the range of permitted values of $M_E(|\omega_n|)$ as [13]

$$\frac{M}{2} \leq M_E\left(|\omega_n|\right) \leq M. \tag{36}$$

Thus, setting $2M_E(|\omega_n|) = r_E(|\omega_n|) \equiv 1(|\omega_n|)$ one sees that the range of permitted values of the continually rescaled mass unit is always included within factor 2 [13]. On the other hand, the countable sequence of QNMs is very large; see the below discussion on the maximum value for the overtone number n and [11, 13]. This implies that the mass unit's rescaling is extremely slow [13]. Therefore, the reader can easily check, by reviewing the discussion in the Appendix of [13] step by step, that the continually rescaled mass unit did not influence the analysis.

Let us discuss another argument which emphasizes the correctness of the analysis in the Appendix of [13]. We can choose to consider M_E as being constant within the range

(36) [13]. In that case, we show that such an approximation is very good [13]. Equation (36) implies indeed that the range of permitted values of $T_E(|\omega_n|)$ is [13]

$$T_H = T_E(0) \le T_E(|\omega_n|) \le 2T_H = T_E\left(\left|\omega_{n_{\max}}\right|\right), \quad (37)$$

where T_H is the initial Hawking temperature. Then, if we fix $M_E = M/2$ in the analysis, the approximate result is [13]

$$\omega_n \simeq 2\pi i n \times 2T_H. \quad (38)$$

On the other hand, if one fixes $M_E = M$ as in thermal approximation, the approximate result is [13]

$$\omega_n \simeq 2\pi i n \times T_H. \quad (39)$$

We see that both of the approximate results in correspondence with the extreme values in range (36) have the same order of magnitude [13]. Thus, fixing $2M_E = r_E \equiv 1$ does not change the order of magnitude of the final (approximated) result with respect to the exact result [13]. In particular, setting $T_E = (3/2)T_H$ the uncertainty in the final result is 0.33, while in the result of the thermal approximation (39) the uncertainty is 2 [13]. Hence, even if one considers M_E as constant, the result in the Appendix of [13] is more precise than the thermal approximation of previous literature. Thus, the derivation of (25) is surely correct.

Equation (25) has the following elegant interpretation [11, 12]. QNMs determine the position of poles of Green's function on the given background and the Euclidean BH solution converges to a *nonstrictly* thermal circle at infinity with the inverse temperature $\beta_E(\omega_n) = 1/T_E(|\omega_n|)$ [11, 12]. Then, the spacing of the poles in (25) coincides with the spacing $2\pi i T_E(|\omega_n|) = 2\pi i T_H(2M/(2M - |\omega_n|))$, which is expected for a *nonstrictly* thermal Green's function [11, 12].

As BHs cannot emit more energy than their total mass, the physical solution for the absolute values of the frequencies (25), that is, the one for which it is $|\omega_n| \le M$, is [11–13]

$$E_n \equiv |\omega_n| = M - \sqrt{M^2 - \frac{n}{2}}. \quad (40)$$

E_n is interpreted like the total energy emitted at level n [11–13]. As the square root in (40) must be a real nonnegative number, we need also [11, 13]

$$M^2 - \frac{n}{2} \ge 0. \quad (41)$$

Solving expression (41), one gets a maximum value for the overtone number n as follows:

$$n \le n_{\max} = 2M^2, \quad (42)$$

corresponding to $E_n = M$. This means that $n_{\max}$ is a finite number, although it can be very very large. Result (42) is correct if one assumes a total BH evaporation. But in [42], it has been shown that the Generalized Uncertainty Principle prevents the total BH evaporation in exactly the same way that the Uncertainty Principle prevents the hydrogen atom from total collapse. In fact, the collapse is prevented by

dynamics rather than by symmetry, when the *Planck distance* and the *Planck mass* are approached [42]. Then, one has to slightly modify (41) obtaining (the *Planck mass* is equal to 1 in Planck units)

$$M^2 - \frac{n}{2} \ge 1. \quad (43)$$

The solution of (43) is

$$n \le n_{\max} = 2\left(M^2 - 1\right), \quad (44)$$

which gives a different value of the maximum value for the overtone number n. Let us consider, for example, a BH mass of 10 solar masses. One obtains $n_{\max} \sim 10^{78}$ from both (42) and (44). Thus, we understand that although the total number of QNMs for emitted energies is not infinity, our "quasi-asymptotic" analysis for large n is an excellent approximation.

Considering an emission from the ground state (i.e., a BH which is not excited) to a state with large $n = n_1$ and using (40), the BH mass changes from M to

$$M_{n_1} \equiv M - E_{n_1} = \sqrt{M^2 - \frac{n_1}{2}}. \quad (45)$$

In the transition from the state with $n = n_1$ to a state with $n = n_2$, where $n_2 > n_1$, the BH mass changes again from M_{n_1} to

$$M_{n_2} \equiv M_{n_1} - \Delta E_{n_1 \to n_2} = M - E_{n_2}$$
$$= \sqrt{M^2 - \frac{n_2}{2}}, \quad (46)$$

where

$$\Delta E_{n_1 \to n_2} \equiv E_{n_2} - E_{n_1} = M_{n_1} - M_{n_2}$$
$$= \sqrt{M^2 - \frac{n_1}{2}} - \sqrt{M^2 - \frac{n_2}{2}} \quad (47)$$

is the jump between the two levels due to the emission of a particle having frequency $\Delta E_{n_1 \to n_2}$. Thus, we have found the intriguing result that the BH mass varies in function of the initial mass M and of the BH quantum level [13].

Now, following [11–13], important consequences on BHs quantum physics, which arise from the correspondence between Hawking radiation and BH QNMs, will be discussed, starting from the *area quantization*.

Bekenstein [28] proposed that the area of the BH horizon is quantized in units of the Planck length in quantum gravity (the *Planck length* $l_p = 1.616 \times 10^{-33}$ cm is equal to one in Planck units). His famous result was that the Schwarzschild BH area quantum is $\Delta A = 8\pi$ [28]. In [37, 38] Hod had the intriguing idea to consider BH QNMs like quantum levels for absorption of particles. In that way, he found a different numerical coefficient [37, 38]. It is important to recall that the Hod rule $\Delta A = 4\ln 3$ is actually a special case of one suggested by Mukhanov in [43]. Mukhanov proposed $\Delta A = 4\ln k$ with $k = 2, 3, \ldots$. This entered into the joint paper of Bekenstein and Mukhanov [44] and then into the review

of Bekenstein [29]. In any case, Hod's work was reanalyzed in [25] by Maggiore, who reobtained the original result of Bekenstein. We further improved the result in [25] taking into account the deviation from the perfect thermal spectrum in [11–13] and adding to the analysis the perturbations due to the subsequent emissions of Hawking quanta. In fact, in our approach we used (25) instead of (24) [11–13]. Setting $n_1 = n - 1$, $n_2 = n$ in (47), one gets the emitted energy for a jump among two neighboring levels [11–13] as

$$\Delta E_{n-1 \to n} = E_n - E_{n-1} = \sqrt{M^2 - \frac{n-1}{2}} - \sqrt{M^2 - \frac{n}{2}}. \quad (48)$$

Result (48) holds for Kerr BHs too, when $M^2 \gg J$, where J is the angular momentum of the BH [14]. In the Schwarzschild BH, the *horizon area A* is related to the mass by the relation $A = 16\pi M^2$. Thus, a variation ΔM of the mass implies a variation

$$\Delta A = 32\pi M \Delta M \quad (49)$$

of the area. On the other hand, after a high number of emissions (and potential absorptions because neighboring particles can be captured by the BH), the BH mass changes from M to [13]

$$M_{n-1} \equiv M - E_{n-1}, \quad (50)$$

where E_{n-1} is the total energy emitted by the BH at that time (the BH is excited at a level $n - 1$ [13]). A further emission causes a transition from the state with $n - 1$ to the state with n [13], and the BH mass changes again from M_{n-1} to [13]

$$M_n \equiv M - E_{n-1} - \Delta E_{n-1 \to n}. \quad (51)$$

If one uses (48), one gets [13]

$$M_n = M - E_{n-1} + \sqrt{M^2 - \frac{n}{2}} - \sqrt{M^2 - \frac{n-1}{2}}$$
$$= M - E_{n-1} + E_{n-1} - E_n = M - E_n, \quad (52)$$

and now the BH is excited at the level n [13]. Using (40), (50) and (52) become [13]

$$M_{n-1} = \sqrt{M^2 - \frac{n-1}{2}}, \quad (53)$$
$$M_n = \sqrt{M^2 - \frac{n}{2}}.$$

Then, using (49) and (48) we get [13]

$$\Delta A_{n-1} \equiv 32\pi M_{n-1} \Delta E_{n-1 \to n}. \quad (54)$$

Equation (54) should give the area quantum of an excited BH when one considers an emission from the level $n - 1$ to the level n in function of the principal quantum number n and of the initial BH mass. Actually, there is a problem. In fact, an absorption from the level n to the level $n - 1$ is now possible and the correspondent absorbed energy is [12, 13]

$$E_{n-1} - E_n = -\Delta E_{n-1 \to n} \equiv \Delta E_{n \to n-1}. \quad (55)$$

Then, the area quantum for the transition (55) should be

$$\Delta A_n \equiv 32\pi M_n \Delta E_{n \to n-1} \quad (56)$$

and one gets the strange result that the absolute value of the area quantum for an emission from the level $n - 1$ to the level n is different from the absolute value of the area quantum for an absorption from the level n to the level $n - 1$ because it is $M_{n-1} \neq M_n$ [13]. One expects the area spectrum to be the same for absorption and emission instead [13]. In order to solve this inconsistency, one considers again the *effective mass* corresponding to the transitions between the two levels n and $n - 1$. In fact, that effective mass is the same for emission and absorption [13]:

$$M_{E(n, \, n-1)} \equiv \frac{1}{2}\left(M_{n-1} + M_n\right)$$
$$\frac{1}{2}\left(\sqrt{M^2 - \frac{n-1}{2}} + \sqrt{M^2 - \frac{n}{2}}\right). \quad (57)$$

If one replaces M_{n-1} with $M_{E(n, \, n-1)}$ in (54) and M_n with $M_{E(n, \, n-1)}$ in (56), one obtains

$$\Delta A_{n-1} \equiv 32\pi M_{E(n, \, n-1)} \Delta E_{n-1 \to n} \quad \text{emission}$$
$$\Delta A_n \equiv 32\pi M_{E(n, \, n-1)} \Delta E_{n \to n-1} \quad \text{absorption} \quad (58)$$

and now it is $|\Delta A_n| = |\Delta A_{n-1}|$. By using (48) and (57) one finds

$$\left|\Delta A_n\right| = \left|\Delta A_{n-1}\right| = 8\pi, \quad (59)$$

which is exactly the original famous result by Bekenstein [28], which is spin independent and in full agreement with Bohr's correspondence principle [13]. Thus, one takes result (59) as the quantization of the area of the horizon of a Schwarzschild BH. This is a strong confirmation of the correctness of our analysis. Putting $A_{n-1} \equiv 16\pi M_{n-1}^2$ and $A_n \equiv 16\pi M_n^2$, the formulas of the number of quanta of area are [13]

$$N_{n-1} \equiv \frac{A_{n-1}}{\left|\Delta A_{n-1}\right|} = \frac{16\pi M_{n-1}^2}{32\pi M_{E(n, \, n-1)} \cdot \Delta E_{n-1 \to n}}$$
$$= \frac{M_{n-1}^2}{2M_{E(n, \, n-1)} \cdot \Delta E_{n-1 \to n}} \quad (60)$$

before the emission, and [13]

$$N_n \equiv \frac{A_n}{\left|\Delta A_n\right|} = \frac{16\pi M_n^2}{32\pi M_{E(n, \, n-1)} \cdot \Delta M_n}$$
$$= \frac{M_n^2}{2M_{E(n, \, n-1)} \cdot \Delta E_{n-1 \to n}} \quad (61)$$

after the emission, respectively. It is easy to check that [13]

$$N_{n-1} - N_n = \frac{M_{n-1}^2 - M_n^2}{2M_{E(n, \, n-1)} \cdot \Delta E_{n-1 \to n}}$$
$$= \frac{\Delta M_n \left(M_{n-1} + M_n\right)}{2M_{E(n, \, n-1)} \cdot \Delta E_{n-1 \to n}} = 1 \quad (62)$$

as one expects. Then, the formulas of the famous Bekenstein-Hawking entropy are [13]

$$
(S_{\text{BH}})_{n-1} \equiv \frac{A_{n-1}}{4} = 8\pi N_{n-1} M_{n-1} \cdot \Delta E_{n-1 \to n}
$$
$$
= 4\pi \left(M^2 - \frac{n-1}{2} \right) \tag{63}
$$

before the emission and [13]

$$
(S_{\text{BH}})_n \equiv \frac{A_n}{4} = 8\pi N_n M_n \cdot \Delta E_{n-1 \to n} = 4\pi \left(M^2 - \frac{n}{2} \right) \tag{64}
$$

after the emission, respectively. Notice that as $n \gg 1$, one obtains $(S_{\text{BH}})_n \simeq (S_{\text{BH}})_{n-1}$ [13]. Formulas (63) and (64) are extremely important. In fact, they give the Bekenstein-Hawking entropy as function of the BH original mass and of the BH quantum level n. They work for all $j = 0, 1, 2$, in total agreement with Bohr's correspondence principle.

On the other hand, the total BH entropy contains at least three parts which are necessary to realize the underlying theory of quantum gravity [11–13, 26, 27]. They are the usual Bekenstein-Hawking entropy and two subleading corrections: the logarithmic term and the inverse area term [11–13, 26, 27]:

$$
S_{\text{total}} = S_{\text{BH}} - \ln S_{\text{BH}} + \frac{3}{2A}. \tag{65}
$$

Then, one gets [13]

$$
(S_{\text{total}})_{n-1} = 4\pi \left(M^2 - \frac{n-1}{2} \right)
$$
$$
- \ln \left[4\pi \left(M^2 - \frac{n-1}{2} \right) \right] \tag{66}
$$
$$
+ \frac{3}{32\pi \left(M^2 - (n-1)/2 \right)}
$$

before the emission, and [13]

$$
(S_{\text{total}})_n = 4\pi \left(M^2 - \frac{n}{2} \right)
$$
$$
- \ln \left[4\pi \left(M^2 - \frac{n}{2} \right) \right] + \frac{3}{32\pi \left(M^2 - n/2 \right)} \tag{67}
$$

after the emission, respectively. As a consequence, at level $n - 1$, the BH has a number of microstates as follows:

$$
g(N_{n-1})
$$
$$
\propto \exp \left\{ 4\pi \left(M^2 - \frac{n-1}{2} \right) \right.
$$
$$
- \ln \left[4\pi \left(M^2 - \frac{n-1}{2} \right) \right] \tag{68}
$$
$$
\left. + \frac{3}{32\pi \left(M^2 - (n-1)/2 \right)} \right\}
$$

and, at level n, after the emission, the number of microstates is

$$
g(N_n)
$$
$$
\propto \exp \left\{ 4\pi \left(M^2 - \frac{n}{2} \right) \right.
$$
$$
- \ln \left[4\pi \left(M^2 - \frac{n}{2} \right) \right] \tag{69}
$$
$$
\left. + \frac{3}{32\pi \left(M^2 - n/2 \right)} \right\}.
$$

We note that when n is large, but not large enough, it is also $E_n \ll M_n \simeq M$ and one gets [11, 13]

$$
\Delta A = 32\pi M \Delta E_{n-1 \to n}, \tag{70}
$$

$$
N = \frac{A}{|\Delta A|} = \frac{16\pi M^2}{32\pi M \cdot \Delta E_{n-1 \to n}} = \frac{M}{2\Delta E_{n-1 \to n}}, \tag{71}
$$

$$
S_{\text{BH}} = \frac{A}{4} = 8\pi N M \cdot \Delta E_{n-1 \to n}, \tag{72}
$$

$$
S_{\text{total}} = 8\pi N M \cdot \Delta E_{n-1 \to n} - \ln \left[8\pi N M \cdot \Delta E_{n-1 \to n} \right]
$$
$$
+ \frac{3}{64\pi N M \cdot \Delta E_{n-1 \to n}}, \tag{73}
$$

$$
g(N) \propto \exp \left\{ 8\pi N M \cdot \Delta E_{n-1 \to n} \right.
$$
$$
- \ln \left[8\pi N M \cdot \Delta E_{n-1 \to n} \right] \tag{74}
$$
$$
\left. + \frac{3}{64\pi N M \cdot \Delta E_{n-1 \to n}} \right\}.
$$

On the other hand, for large n it is also $\Delta E_{n-1 \to n} \approx 1/4M$ [11, 13] and equations from (70) to (74) become

$$
|\Delta A| \approx 8\pi, \tag{75}
$$

$$
N \approx 2M^2, \tag{76}
$$

$$
S_{\text{BH}} \approx 2\pi N, \tag{77}
$$

$$
S_{\text{total}} \simeq 2\pi N - \ln 2\pi N + \frac{3}{16\pi N}, \tag{78}
$$

$$
g(N) \propto \exp \left[2\pi N - \ln (2\pi N) + \frac{3}{16\pi N} \right], \tag{79}
$$

which are in agreement with previous literature [25–27], where the strictly thermal approximation has been used.

Let us resume the way in which the BH model analysed in this section works. If M is the original BH mass (in quantum terms the BH is in its *ground state*), after a high number of emissions (and potential absorptions as the BH can capture neighboring particles), the BH arrives at an excited level $n - 1$ and its mass is now $M_{n-1} \equiv M - E_{n-1}$, where E_{n-1} is the total energy emitted at that time and also the absolute value of the frequency of the QNM associated with the excited level

$n - 1$ [13]. Now, the BH emits an energy $\Delta E_{n-1 \to n} = E_n - E_{n-1}$ to jump to the subsequent level n. Thus, the BH mass decreases again [13]:

$$
\begin{aligned}
M_n &\equiv M - E_{n-1} - \Delta E_{n-1 \to n} \\
&= M - E_{n-1} + E_{n-1} - E_n = M - E_n.
\end{aligned}
\tag{80}
$$

Notice that, in principle, the BH can bring back to the level $n - 1$ absorbing an energy $-\Delta E_{n-1 \to n} = \Delta E_{n \to n-1} = E_{n-1} - E_n$. The quantum of area is *the same* for both absorption and emission, given by (59), as one expects.

One finds three different physical situations for excited BHs ($n \gg 1$) [13]:

(1) n is large, but not large enough as we have also $E_n \ll M_n \simeq M$ and we can use (70), (72), (73), and (74) which results in a better approximation than (77), (78), and (79) which were used in previous literature in strictly thermal approximation; see [25–27], for example. This is indeed the approximation that we used in our pioneering works [11, 12].

(2) n is very much larger than in previous point, but the Planck scale has not yet been approached. In that case, as it can be $E_n \lesssim M$, $M_n \simeq M$ does not hold. One has to use the equations from (63) to (69).

(3) At the Planck scale, n is larger also than in previous point 2 and one needs a full quantum gravity theory, which is not yet known.

We stress that, in our BH model, during a quantum jump a discrete amount of energy is radiated and, for large values of the principal quantum number n, the analysis becomes independent of the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another and the absolute values of the QNMs frequencies represent the energy "shells." In Bohr model [22, 23] electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation (in standard units) $E = hf$, where h is the Planck constant and f is the transition frequency. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to (47). The similarity is completed if one notes that the interpretation of (40) is of a particle, the "electron," quantized on a circle of length [9, 11, 12] as

$$
L = \frac{1}{T_E(E_n)} = 4\pi \left(M + \sqrt{M^2 - \frac{n}{2}} \right),
\tag{81}
$$

which is analogous to the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr model [22, 23]. On the other hand, Bohr model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. In the same way, the Bohr-like BH model should

be an approximated model with respect to the definitive, but at the present time unknown, BH model arising from a full quantum gravity theory.

Another key point is as follows. In Hawking's original computation [1] if an emission can occur for a quantum of energy E, then it can also occur for any other quantum of energy bE, where b is a continuous real parameter between 0 and M/E, where M is the BH mass. After the emission of a quantum of energy bE, the BH radial coordinate is determined continuously by the continuous parameter b. In other words, emissions of Hawking quanta look completely random. The situation looks to be similar within the semiclassical context in which Parikh-Wilczek perform their calculation [2, 3]. But here there is an important difference. The discrete behavior in time of the radiation spectrum, in the sense that we stressed in Section 2, implies the countable character of the subsequent emitted Hawking quanta and, in turn, the correspondence between the countable perturbations generated by the absorbed negative energies and the BH QNMs. The fundamental consequence is that, differently from Hawking's original computation [1], now emissions of Hawking quanta are *not* completely random. They are indeed governed by (47). In fact, let us consider an emission from the BH ground state to a state with large n. After that, using (42) (although we recall that the last area quantum corresponds to the final Planck mass which is prevented from evaporating by the Generalized Uncertainty Principle [42]), one sees that the BH will have a finite and discrete number of potential emissions given by

$$
n_{\max} - n = 2M^2 - n.
\tag{82}
$$

It is enlightening to observe that such a number of potential residual emissions, which is equal to the residual number of QNMs, is also equal to the residual number of area quanta. In fact, by using (50) and recalling that $r_H = 2M$, one easily compute the area of the BH excited at level n as

$$
A_n = 16\pi M_n^2 = 16\pi \left(M^2 - \frac{n}{2} \right),
\tag{83}
$$

which, dividing for Bekenstein's area quantum $|\Delta A_n| = 8\pi$ [28] that we retrieved in (59), gives the number of area quanta for the BH excited at level n:

$$
N_n = 2M^2 - n.
\tag{84}
$$

Thus, we understand that the key point is exactly Bekenstein's idea on area quantization [28]; that is, as for large n the BH area is quantized, and the BH can emit *only* energies which are consistent with such a quantization. In other words, emissions of Hawking quanta are not completely random because the BH can emit only energies which corresponds to reductions of its areas which are multiples of Bekenstein's area quantum $|\Delta A_n| = 8\pi$ given by (59). Hence, our results are completely consistent with the idea that the Schwarzschild spacetime is quantized around the BH core.

4. Time Evolution of Bohr-Like Black Hole Governed by a Time Dependent Schrödinger Equation: Independent Solution to the Black Hole Information Paradox

In his famous paper [10] Hawking verbatim stated that "*Because part of the information about the state of the system is lost down the hole, the final situation is represented by a density matrix rather than a pure quantum state.*" This statement was the starting point of the popular "*BH information paradox.*"

In Section 3, we naturally interpreted BH QNMs in terms of quantum levels in a Bohr-like BH model. In this section, following [9], we explicitly write down a *time dependent Schrödinger equation* for the system composed by Hawking radiation and BH QNMs. We show that the physical state and the correspondent *wave function* are written in terms of a *unitary* evolution matrix instead of a density matrix [9]. As a consequence, the final state results in a *pure* quantum state instead of mixed one [9]. Hence, Hawking's claim is falsified by the time evolution of the Bohr-like BH model [9]. BH evaporation results indeed in a unitary and time dependent process in which information comes out [9]. This is in full agreement with the assumption by 't Hooft that Schrödinger equations can be used universally for all dynamics in the universe [45]. The final conclusion is that BH evaporation must be information preserving [9].

In addition, it will be shown that the present approach permits also solving the entanglement problem connected with the information paradox [9].

The BH information paradox is considered one of the most famous and intriguing scientific controversies in the whole history of Science [9]. In classical general relativity, a BH is the ultimate prison. Nothing can escape from it. As a consequence, when matter falls into a BH, one can consider the information encoded preserved inside it, although being inaccessible to outside observers. The celebrated Hawking's discovery that quantum effects cause the BH to emit radiation radically changed this situation [1]. Hawking made a further analysis [10], showing that the detailed form of the radiation emitted by a BH should be thermal and independent of the structure and composition of matter that collapsed to form the BH. In that way, the radiation state results in a completely mixed one which cannot carry information on the BH's formation. After Hawking's original claim that we verbatim rewrote above, enormous time and effort have been and are currently devoted to solve the information puzzle. In fact, consequences of the BH information paradox are not trivial. As pure quantum states arising from collapsed matter would decay into mixed states if information is lost in BH evaporation, quantum gravity should not be unitary [46]! Various researchers worked and currently work on the information puzzle. Some people think that quantum information is destroyed by BH evaporation. Other people claim that the above cited Hawking's statement was not correct and information is, instead, preserved. An interesting and a popular Science book on the so-called "Black Hole War" has been written by Susskind [47] and the paradox became introduced into physics folklore [47, 48]. Two famous

bets have been made by Hawking that BH does destroy information [47]. The first one is having Thorne like cosigner with Preskill and the latter with Page [47]. In 2004-2005 Hawking reversed his opinion claiming that information would probably be recovered [46, 47]. Various attempts to solve the information puzzle and historical notes on the controversy can be found in [45–49]. Recently, Hawking reversed his opinion again, with a couple of ambiguous statements verbatim claiming that "*The chaotic collapsed object will radiate deterministically but chaotically. It will be like weather forecasting on Earth. That is unitary, but chaotic, so there is effective information loss. One can't predict the weather more than a few days in advance*" [50].

As we previously recalled, a key point, concerning not only the BH information paradox, but the whole BH quantum physics, is that the BH radiation spectrum is not strictly thermal [2, 3], differently from Hawking's original computations [1, 10]. Now, we show that the time evolution of the Bohr-like BH model is governed by a time dependent Schrödinger equation which enables pure quantum states to evolve to pure quantum states in a unitary evolution which preserves quantum information and, in turn, falsifies the above cited statement by Hawking [9]. It will be also shown that, in addition, the following approach solves the entanglement problem connected with the information paradox [9].

Let us start by recalling that E_n in (40) is interpreted like the total energy emitted by the BH at that time, that is, when the BH is excited at a level n; see Section 3 and [9, 13]. If one considers an emission from the ground state to a state with large n and uses (40), the BH mass changes from M to [9]

$$M_n \equiv M - E_n = \sqrt{M^2 - \frac{n}{2}}. \tag{85}$$

In the transition from the state with n to a state with $m > n$, the BH mass changes again from M_n to [9]

$$M_m \equiv M_n - \Delta E_{n \to m} = M - E_m$$
$$= \sqrt{M^2 - \frac{m}{2}}, \tag{86}$$

where $\Delta E_{n \to m} \equiv E_m - E_n = M_n - M_m$ is the jump between the two levels due to the emission of a particle having frequency $\omega_{n,m} = \Delta E_{n \to m}$ [9]. Let us show that the energy emitted in an arbitrary transition from n to m, where $m > n$ (we are considering an emission), is proportional to the effective temperature $[T_E]_{n \to m}$ associated with the transition [9]. Putting [9]

$$\Delta E_{n \to m} \equiv E_m - E_n = M_n - M_m = K [T_E]_{n \to m}, \tag{87}$$

where M_n and M_m are given by (85) and (86), we search values of the constant K for which (87) is satisfied. As discussed in [8, 9, 11–15, 21] and in Section 2, the effective temperature is the inverse of the average value of the inverses of the initial and final Hawking temperatures:

$$[T_E]_{n \to m} = \frac{1}{4\pi (M_n + M_m)}. \tag{88}$$

Hence, one rewrites (87) as [9]

$$\Delta E_{n \to m} = M_n^2 - M_m^2 = \frac{K}{4\pi}. \quad (89)$$

By using (85) and (86), (89) becomes [9]

$$\frac{1}{2}(m - n) = \frac{K}{4\pi}. \quad (90)$$

Thus, (87) is satisfied for $K = 2\pi(m - n)$, and we find [9]

$$\Delta E_{n \to m} = 2\pi(m - n)\left[T_E\right]_{n \to m}. \quad (91)$$

Considering (12), we can write the probability of emission between the two levels n and m in an elegant form [9] as follows:

$$\Gamma_{n \to m} = \alpha \exp - \left\{ \frac{\Delta E_{n \to m}}{\left[T_E\right]_{n \to m}} \right\} = \alpha \exp\left[-2\pi(m - n)\right], \quad (92)$$

where the prefactor is $\alpha \sim 1$. Then, one finds that the probability of emission between two arbitrary levels n and m is proportional to $\exp[-2\pi(m - n)]$. We observe that the probability of emission has its maximum value $\sim \exp(-2\pi)$ for $m = n + 1$, that is, for two adjacent levels, as we intuitively expect [9].

In a quantum mechanical framework, we physically interpret emissions of Hawking quanta like quantum jumps among the unperturbed levels (40) [9, 11–13]. Following [9, 51], the time evolution of perturbations can be described by the operator

$$U(t) = \begin{cases} W(t) & \text{for } 0 \le t \le \tau \\ 0 & \text{for } t < 0, \ t > \tau, \end{cases} \quad (93)$$

and the complete (time dependent) Hamiltonian is described by the operator [9, 51]

$$H(x, t) \equiv V(x) + U(t), \quad (94)$$

where $V(x)$ is the *effective Regge-Wheeler* potential (27) of the time independent *effective Schröedinger-like equation* (26). Then, considering a wave function $\psi(x, t)$, we can write the correspondent *time dependent Schrödinger equation* for the system as [9, 51]

$$i \frac{d\left|\psi(x, t)\right\rangle}{dt} = \left[V(x) + U(t)\right]\left|\psi(x, t)\right\rangle \\ = H(x, t)\left|\psi(x, t)\right\rangle. \quad (95)$$

If $\varphi_m(x)$ and ω_m are the eigenfunctions of the time independent Schröedinger-like equation (26) and the correspondent eigenvalues, respectively, the state satisfying (95) is [9, 51]

$$\left|\psi(x, t)\right\rangle = \sum_m a_m(t) \exp\left(-i\omega_m t\right)\left|\varphi_m(x)\right\rangle. \quad (96)$$

We consider Dirac delta perturbations [9, 11–13, 25] which represent subsequent absorptions of particles having negative energies being associated with emissions of Hawking quanta

in the mechanism of particle pair creation. In the basis $\left|\varphi_m(x)\right\rangle$, the matrix elements of $W(t)$ are [9, 51]

$$W_{ij}(t) \equiv A_{ij}\delta(t), \quad (97)$$

with $W_{ij}(t) = \left\langle\varphi_i(x)|W(t)|\varphi_j(x)\right\rangle$ [9, 51] and the A_{ij} are real. As we want to solve the complete quantum mechanical problem described by operator (94), we need to know the probability amplitudes $a_m(t)$ due to the application of the perturbation described by the time dependent operator (93) [9, 51], representing the perturbation associated with the emission of a Hawking quantum [9, 51]. For $t < 0$, that is, before the perturbation operator (93) starts to work, the system is in a stationary state $\left|\varphi_n(t, x)\right\rangle$, at the quantum level n, with energy $E_n = |\omega_n|$ given by (40) [9, 51]. Thus, only the term

$$\left|\psi_n(x, t)\right\rangle = \exp\left(-i\omega_n t\right)\left|\varphi_n(x)\right\rangle \quad (98)$$

in (96) is not null for $t < 0$. This implies $a_m(t) = \delta_{mn}$ for $t < 0$. After the emission the perturbation operator (93) stops to work and for $t > \tau$ the probability amplitudes $a_m(t)$ return to be time independent, having the value $a_{n \to m}(\tau)$ [9, 51]. In other words, for $t > \tau$, the system is in the state [9, 51]:

$$\left|\psi_{\text{final}}(x, t)\right\rangle = \sum_{m=n}^{m_{\max}} a_{n \to m}(\tau) \exp\left(-i\omega_m t\right)\left|\varphi_m(x)\right\rangle, \quad (99)$$

described by the *wave function* $\psi_{\text{final}}(x, t)$ [9, 51], and one sees that the probability to find the system in an eigenstate having energy $E_m = |\omega_m|$ is [9, 51]

$$\Gamma_{n \to m}(\tau) = \left|a_{n \to m}(\tau)\right|^2. \quad (100)$$

A standard analysis will give the following differential equation from (99) [9, 51]:

$$i \frac{d}{dt} a_{n \to m}(t) = \sum_{l=m}^{m_{\max}} W_{ml} a_{n \to l}(t) \exp\left[i\left(\Delta E_{l \to m}\right)t\right]. \quad (101)$$

The Dayson series permits obtaining the solution [9, 51]

$$a_{n \to m} = -i \int_0^t \left\{W_{mn}\left(t'\right) \exp\left[i\left(\Delta E_{n \to m}\right)t'\right]\right\} dt' \quad (102)$$

to first order in $U(t)$. Inserting (97) in (102) we get [9, 51]

$$a_{n \to m} = iA_{mn} \int_0^t \left\{\delta\left(t'\right) \exp\left[i\left(\Delta E_{n \to m}\right)t'\right]\right\} dt' = \frac{i}{2} A_{mn}. \quad (103)$$

Combining (103) with (92) and (100) at the end we obtain [9, 51]

$$\alpha \exp\left[-2\pi(m - n)\right] = \frac{1}{4} A_{mn}^2$$

$$A_{mn} = 2\sqrt{\alpha} \exp\left[-\pi(m - n)\right] \quad (104)$$

$$a_{n \to m} = -i\sqrt{\alpha} \exp\left[-\pi(m - n)\right].$$

We recall that it is $\sqrt{\alpha} \sim 1$. Then we get $A_{mn} \sim 10^{-2}$ for $m = n + 1$, that is, when the probability of emission has its maximum value [9, 51]. Therefore, second order terms in $U(t)$ are $\sim 10^{-4}$, which means that our approximate result to first order in $U(t)$ is very good [9, 51]. We note that for $m > n + 1$ the approximation is better because the A_{mn} are even smaller than 10^{-2}. Then, the final form of the ket representing the state is [9, 51]

$$\left| \psi_{\text{final}} (x, t) \right\rangle = \sum_{m=n}^{m_{\text{max}}} - i \sqrt{\alpha} \exp \left[-\pi (m - n) - i\omega_m t \right] \cdot \left| \varphi_m (x) \right\rangle . \tag{105}$$

State (105) represents a *pure final state instead of a mixed final state* and the states are written in terms of a *unitary* evolution matrix instead of a density matrix [9]. Therefore, one finds that information is not a loss in BH evaporation [9]. The result is in full agreement with the assumption by 't Hooft that Schrödinger equations can be used universally for all dynamics in the universe [45].

We observe that the final state of (105) is due to potential emissions of Hawking quanta having negative energies which perturb the BH and "trigger" the QNMs corresponding to potential arbitrary transitions $n \rightarrow m$, with $m > n$ [9]. Then, the subsequent *collapse of the wave function* to a new stationary state [9]

$$\left| \psi_m (x, t) \right\rangle = \exp (-i\omega_m t) \left| \varphi_m (x) \right\rangle , \tag{106}$$

at the quantum level m, implies that the wave function of the particle having negative energy $-\Delta E_{n \rightarrow m} = \omega_n - \omega_m$ has been transferred to the QNM and it is given by [9]

$$\left| \psi_{-(m-n)} (x, t) \right\rangle$$
$$\equiv - \exp \left[i (\omega_m - \omega_n) t \right] \left[\left| \varphi_m (x) \right\rangle - \left| \varphi_n (x) \right\rangle \right] . \tag{107}$$

Wave function (107) is entangled with the wave function of the particle with positive energy propagating towards infinity in the mechanism of particle creation by BHs. Below it will be shown that this key point solves the entanglement problem connected with the information paradox [9].

Our analysis is strictly correct only for excited BHs, that is, for $n \gg 1$ [9]. For this reason we assumed an emission from the ground state to a state with large n in the discussion [9]. On the other hand, as we have seen in Section 4, a state with large n is always reached at late times, maybe not through a sole emission from the ground state but through various subsequent emissions and potential absorptions [9].

Now, let us discuss another key point, which concerns quantum entanglement. We could think that although previous analysis discusses a very natural model of Hawking radiation and BH evaporation, there is no reference to the BH spacetime, where information is assumed to be conserved. There are indeed authors who claim that the real challenge in solving the information paradox is to reconcile models of Hawking radiation with the spacetime structure within the BH horizon, where the quantum information falling into the singularity is causally separated from the outgoing

Hawking quanta; see the work by Mathur [49], for example. In any case, this kind of criticism does not work for the analysis in this section. In the above analysis there is indeed a subtle connection between the emitting Hawking quanta and the BH spacetime within the horizon, where information is conserved. This approach to the BH information problem concerns the entanglement structure of the wave function which is associated with the particle pair creation [49]. In fact, in order to solve the information puzzle, we need to know the part of the wave function in the interior of the BH horizon [49], that is, the part of the wave function associated with the particle having negative energy in the tunnelling mechanism. In the emissions of Hawking quanta, this is exactly the part of the wave function which becomes entangled with the part of the wave function outside, that is, the part of the wave function associated with the particle which has positive energy and escapes from the BH [49]. If we ignore such an interior part of the wave function, we miss the entanglement completely, failing to understand the information problem [49]. But when one considers the above discussed correspondence between Hawking radiation and BH QNMs, the particle which has negative energy and falls into the singularity transfers its part of the wave function and, in turn, the information encoded in such a part of the wave function to the QNM. In other terms, the emitted quanta are entangled with BH QNMs, which are the oscillations of the BH horizon. This key point is exactly the subtle connection between the emitted Hawking quanta and the BH spacetime that we need to find. We explain this important issue in detail. Again, we emphasize that the correspondence between emitted radiation and proper oscillation of the emitting body is a fundamental behavior of every radiation process in Nature, and this issue helps to solve the entanglement problem. The mechanism of particles creation by BHs [1] has been described as tunnelling arising from vacuum fluctuations near the BH horizon in [2–8] and in Introduction of this paper. Let us again assume an initial emission from the BH ground state to a state having large n; say $n = n_1 \gg 1$. The absorbed particle, which has negative energy $-|\omega_{n_1}|$, generates a QNM which has an energy-frequency $E_{n_1} = |\omega_{n_1}|$. As a consequence, the BH mass changes from M to

$$M_{n_1} \equiv M - E_{n_1} = \sqrt{M^2 - \frac{n_1}{2}} . \tag{108}$$

Thus, the energy of the first particle absorbed by the BH, which has negative energy, is transferred, together with its part of the wave function, to the QNM which is, in turn, entangled with the emitted particle having positive energy. Let us consider (107). If one sets $n = 0$ and $m = n_1$, one finds that the part of the wave function in the interior of the horizon, that is, the part of the wave function associated with the particle having negative energy (infalling mode) which has been transferred to the QNM, is [9]

$$\left| \psi_{-n_1} (x, t) \right\rangle = - \exp \left(i\omega_{n_1} t \right) \left| \varphi_{n_1} (x) \right\rangle . \tag{109}$$

Now, we consider a second emission. This new emission corresponds to the transition from the state with $n = n_1$ to

another state with, say, $n = n_2 > n_1$. The BH mass changes from M_{n_1} to

$$
\begin{aligned}
M_{n_2} &\equiv M_{n_1} - \Delta E_{n_1 \to n_2} = M - E_{n_2} \\
&= \sqrt{M^2 - \frac{n_2}{2}},
\end{aligned}
\tag{110}
$$

and $\Delta E_{n_1 \to n_2} \equiv E_{n_2} - E_{n_1} = M_{n_1} - M_{n_2}$ is the jump between the two levels. The energy of the second particle absorbed by the BH which has negative energy is transferred, together with its part of the wave function, again to the QNM, which has an increased energy-frequency $E_{n_2} = |\omega_{n_2}|$ and is now entangled with both of the two emitted particles which have positive energy. If one uses again (107) and sets $n = n_1$ and $m = n_2$, one finds that the part of the wave function of the second infalling mode which has been transferred to the QNM is [9]

$$
\begin{aligned}
&\left| \psi_{-(n_2 - n_1)}(x, t) \right\rangle \\
&= -\exp\left[i\left(\omega_{n_2} - \omega_{n_1} \right) t \right] \left[\left| \varphi_{n_2}(x) \right\rangle - \left| \varphi_{n_1}(x) \right\rangle \right].
\end{aligned}
\tag{111}
$$

Let us consider a third emission, corresponding to the transition from the state with $n = n_2$ to a further different state with, say, $n = n_3 > n_2$. Now, the BH mass changes from M_{n_2} to

$$
\begin{aligned}
M_{n_3} &\equiv M_{n_2} - \Delta E_{n_2 \to n_3} = M - E_{n_3} \\
&= \sqrt{M^2 - \frac{n_3}{2}},
\end{aligned}
\tag{112}
$$

where $\Delta E_{n_2 \to n_3} \equiv E_{n_3} - E_{n_2} = M_{n_2} - M_{n_3}$ is the jump between the two levels. Again, the energy of the third particle absorbed by the BH and having negative energy is transferred, together with its part of the wave function, to the QNM which has now a further increased energy-frequency $E_{n_3} = |\omega_{n_3}|$ and is entangled with the three emitted particles which have positive energy. Now, (107) with $n = n_2$ and $m = n_3$ gives the part of the wave function of the third infalling mode which has been transferred to the QNM as

$$
\begin{aligned}
&\left| \psi_{-(n_3 - n_2)}(x, t) \right\rangle \\
&= -\exp\left[i\left(\omega_{n_3} - \omega_{n_2} \right) t \right] \left[\left| \varphi_{n_3}(x) \right\rangle - \left| \varphi_{n_2}(x) \right\rangle \right].
\end{aligned}
\tag{113}
$$

The process will continue again, and again, and again... till the *Planck distance* and the *Planck mass* are approached by the evaporating BH. At that point, the Generalized Uncertainty Principle prevents the total BH evaporation; see Section 3 and [13, 42], and we need a full theory of quantum gravity for the further evolution.

In any case, we emphasize again that the energy E_n of the generic QNM having principal quantum number n is interpreted like the total energy emitted by the BH at that time, that is, when the BH is excited at a level n [9, 13]. As a consequence, such a QNM is entangled with all the Hawking quanta emitted at that time.

Therefore, all the quantum physical information which has fallen into the singularity is not causally separated from

the outgoing Hawking radiation, but is instead recovered and codified in (105) through the correspondence between Hawking radiation and BH QNMs. Following Mathur [49], the solution to the information puzzle is to find a physical effect that we could have missed. In this section, we have shown that the natural correspondence between Hawking radiation and BH QNMs which governs the BH evaporation *is exactly that missed physical effect*.

5. Conclusion Remarks

In this review paper some recent important results in BH quantum physics, which concern the BH effective state and the Bohr-like model for BHs in [9, 11–13], have been reanalyzed. The correspondence between Hawking radiation and BH QNMs permits one indeed to naturally consider QNMs in terms of BH quantum levels in a semiclassical model somewhat similar to the historical semiclassical model of the structure of a hydrogen atom introduced by Bohr in 1913 [22–24]. In the Bohr-like BH in a certain sense, QNMs represent the "electron" jumping from a level to another and the absolute values of the QNMs frequencies "triggered" by emissions (Hawking radiation) and absorption of particles represent the energy "shells" of the "gravitational hydrogen atom."

Again, we stress that Bohr model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. Then, one expects the Bohr-like BH model to be an approximated model with respect to the definitive, but at the present time unknown, BH model arising from a complete theory of quantum gravity.

Important consequences on the BH information puzzle have been also discussed, reviewing the independent solution to the paradox found in [9]. The system Hawking radiation-BH QNMs obeys indeed a time dependent Schrödinger equation which permits the final BH state to be a *pure* quantum state instead of a mixed one in perfect agreement with the assumption by 't Hooft that Schrödinger equations can be used universally for all dynamics in the universe [45]. We have also shown that our approach also solves the entanglement problem connected with the information paradox, an issue raised in [49].

Finally, for the sake of completeness, we recall that in some cases in extended gravity [52–55] the semiclassical effects may lead to instabilities of BHs, with strange effects like antievaporation. These instabilities may qualitatively change QNMs or even make their emergence impossible.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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