

Nilpotent orbits and the Coulomb branch of $T^\sigma(G)$ theories: special orthogonal vs orthogonal gauge group factors

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ABSTRACT: Coulomb branches of a set of $3d \mathcal{N} = 4$ supersymmetric gauge theories are closures of nilpotent orbits of the algebra $\mathfrak{so}(n)$. From the point of view of string theory, these quantum field theories can be understood as effective gauge theories describing the low energy dynamics of a brane configuration with the presence of orientifold planes [1]. The presence of the orientifold planes raises the question to whether the orthogonal factors of a the gauge group are indeed *orthogonal* $O(N)$ or *special orthogonal* $SO(N)$. In order to investigate this problem, we compute the Hilbert series for the Coulomb branch of $T^\sigma(SO(n)^\vee)$ theories, utilizing the *monopole formula*. The results for all nilpotent orbits from $\mathfrak{so}(3)$ to $\mathfrak{so}(10)$ which are *special* and *normal* are presented. A new relationship between the choice of $SO/O(N)$ factors in the gauge group and the *Lusztig's Canonical Quotient* $\bar{A}(\mathcal{O}_\lambda)$ of the corresponding nilpotent orbit is observed. We also provide a new way of projecting several magnetic lattices of different $SO(N)$ gauge group factors by the diagonal action of a \mathbb{Z}_2 group.

KEYWORDS: Field Theories in Lower Dimensions, Gauge Symmetry, Supersymmetric Gauge Theory, Brane Dynamics in Gauge Theories

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1 Introduction

In [2] $3d \mathcal{N} = 4$ quiver gauge theories known as $T_\rho^\sigma(G)$ theories [1, 3] were studied for G a classical Lie group of type A , B , C and D . The Hilbert series¹ for the Coulomb branch and for the Higgs branch of these theories were computed for the case when either σ or ρ are trivial partitions. For $T^\sigma(G)$ on the B , C and D cases (where ρ is trivial) an interesting phenomenon occurs: there are several possible quivers for a single choice of partition σ and the Coulomb branches of these quivers differ by a projection by a discrete group, like for example \mathbb{Z}_2 .

On the other hand, Coulomb branches of $T^\sigma(G)$ theories are predicted to be closures of nilpotent orbits of the algebra $\text{Lie}(G^\vee)$ [3, 5–8], where G^\vee is the Langlands or GNO dual group of G [9]. Therefore the question arises as to which of the different quivers obtained

¹See [4] for a review of Hilbert series and moduli spaces of $3d \mathcal{N} = 4$ theories.

for a single theory has the closure of the nilpotent orbit as its Coulomb branch, rather than a discrete cover of the same.

The present note constitutes a systematic approach to this problem. The Hilbert series of the Coulomb branches of all the candidate quivers are computed utilizing the *monopole formula* [4, 10, 11]. Analyzing the Hilbert series of each moduli space one can discern which Coulomb branch corresponds to the closure of a nilpotent orbit. The following pages contain both the quivers that resulted to have the closure of the nilpotent orbit as their Coulomb branch and the corresponding Hilbert series.

Section 2 of the paper reviews the construction of $T^\sigma(G^\vee)$ *orthosymplectic quivers* given in [2]. Section 3 overviews the *monopole formula*, used to compute the Hilbert series of the Coulomb branch of the quivers. Section 4 tabulates the $T^\sigma(G^\vee)$ quivers with the choice of $\mathrm{SO}(N)/\mathrm{O}(N)$ gauge nodes such that their Coulomb branch is the closure of the corresponding nilpotent orbit. Results for $\mathfrak{so}(3)$ to $\mathfrak{so}(10)$ are presented along with their Hilbert series and plethystic logarithms. Section 5 contains an analysis of the results and provides an answer to the problem. In section 6 a new method to project several magnetic lattices by the diagonal action of a \mathbb{Z}_2 group is introduced. This mechanism is a key part of the answer to the problem and opens the door to a new range of computations that can be performed utilizing the *monopole formula*.

2 Constructing quiver diagrams for $T^\sigma(G)$ theories

This section reviews the method of [2] to obtain the quiver of the $3d \mathcal{N} = 4$ superconformal field theory known as $T^\sigma(G)$. This quiver is determined by a partition σ and a classical Lie group G . Please see [4] for a review on $3d \mathcal{N} = 4$ quiver gauge theories and their moduli spaces.

Two cases are considered here: $T^\sigma(\mathrm{SO}(2n))$ and $T^\sigma(\mathrm{USp}(2n))$.

2.1 $T^\sigma(\mathrm{SO}(2n))$ quiver

For $G = \mathrm{SO}(2n)$ one considers partitions $\sigma \in \mathcal{P}_D(2n)$. $\mathcal{P}_D(2n)$ is the set of all partitions of the integer number $2n$ such that all even parts occur with even multiplicity. For example $\sigma = [4, 4, 4, 4, 3, 2, 2, 1]$ is a partition of $\mathcal{P}_D(24)$.

We only consider partitions such that

$$(d_{\mathrm{LS}})^2(\sigma) = \sigma \tag{2.1}$$

where d_{LS} is the Lusztig-Spaltenstein map [12] that transposes and D-collapses the partition, see the discussion in² [13]. Partitions $\sigma \in \mathcal{P}_D(2n)$ that satisfy equation (2.1) are called *special*.

The orthosymplectic quiver has a chain with a number of $2n - 1$ gauge nodes that alternate between the groups $\mathrm{SO}(N_j)$ and $\mathrm{USp}(N_j)$, starting with $\mathrm{USp}(N_1)$ (we label the nodes from right to left, i.e. $\mathrm{USp}(N_1)$ is the rightmost node). Note that in principle $\mathrm{O}(N_j)$

²In [13] the Lusztig-Spaltenstein map is simply called *Spaltenstein map*.

groups can be chosen instead of $\text{SO}(N_j)$. The parameter N_j can be obtained from the partition data:

$$N_j = \left[\sum_{k=j+1}^{l'} \rho_k \right]_{+,-} - \left(\sum_{i=j+1}^l \sigma_i^T \right), \tag{2.2}$$

where $\sigma^T = [\sigma_1^T, \sigma_2^T, \dots, \sigma_l^T]$ is the transpose partition of σ and $\rho = [1, 1, \dots, 1]$ is the trivial partition of $2n$. The symbol $[p]_+$ means the smallest even integer that is larger or equal to p and $[p]_-$ is the largest even integer smaller or equal to p . Also $l' = 2n$ is the length of ρ and l is the length of σ^T . “+” needs to be used when computing $\text{SO}(N_j)$ gauge nodes whereas “-” is for $\text{USp}(N_j)$ gauge nodes.

A single flavor node can be attached to each gauge node. The flavor nodes also alternate: $\text{SO}(M_j)$ are attached to symplectic gauge nodes and $\text{USp}(M_j)$ are attached to orthogonal gauge nodes. The parameters M_j are also determined by the partition data:

$$M_j = \sigma_j^T - \sigma_{j+1}^T \tag{2.3}$$

where zero parts $\sigma_{l+1}^T = 0$, $\sigma_{l+2}^T = 0$, etc. can be added at the end of partition σ^T if necessary.

2.2 $T^\sigma(\text{USp}(2n))$ quiver

For $G = \text{USp}(2n)$ we consider partitions $\sigma \in \mathcal{P}_C(2n)$. $\mathcal{P}_C(2n)$ is the set of all partitions of the integer number $2n$ such that all odd parts occur with even multiplicity. For example $\sigma = [4, 4, 4, 3, 3, 2, 2, 1, 1]$ is a partition of $\mathcal{P}_C(24)$. Again, only special partitions such that $(d_{\text{LS}})^2(\sigma) = \sigma$ are considered.

The orthosymplectic quiver has now a chain with a number of $2n$ gauge nodes that alternates between the groups $\text{SO}(N_j)$ and $\text{USp}(N_j)$, starting with $\text{SO}(N_1)$. Note that in principle $\text{O}(N_j)$ groups can be chosen instead of $\text{SO}(N_j)$. The parameter N_j can be obtained from the partition data with equation (2.2) (with the difference that in this case $\rho = [1, 1, \dots, 1]$ is the trivial partition of $2n + 1$).

A single flavor node can be attached to each gauge node. The flavor nodes also alternate: $\text{SO}(M_j)$ are attached to symplectic gauge nodes and $\text{USp}(M_j)$ are attached to orthogonal gauge nodes. The parameters M_j are also determined by the partition data as indicated in equation (2.3).

2.3 Coulomb branches

[7, 8] established that the Coulomb branch of the $T^\sigma(G)$ theory should be the closure of the nilpotent orbit³ $\bar{\mathcal{O}}_\lambda \subset \mathfrak{g}^\vee$ with $\mathfrak{g}^\vee = \text{Lie}(G^\vee)$ and

$$\lambda = d_{\text{BV}}(\sigma) \tag{2.4}$$

³For an introduction to nilpotent orbits see [14], we follow the same notation. A standard book on the topic is [13].

Set \mathcal{P}	$d_{\text{BV}}(\sigma)$	Set \mathcal{P}^\vee
$\mathcal{P}_A(n)$	σ^T	$\mathcal{P}_A(n)$
$\mathcal{P}_B(2n+1)$	$(\sigma^T)^-{}_C$	$\mathcal{P}_C(2n)$
$\mathcal{P}_C(2n)$	$(\sigma^+)^T{}_B$	$\mathcal{P}_B(2n+1)$
$\mathcal{P}_D(2n)$	$(\sigma^T)_D$	$\mathcal{P}_D(2n)$

Table 1. Barbasch-Vogan map d_{BV} . The first column is the set of partitions \mathcal{P} such that $\sigma \in \mathcal{P}$. The last column is the set of partitions such that $d_{\text{BV}}(\sigma) \in \mathcal{P}^\vee$.

The map d_{BV} is the Barbasch-Vogan map [15] defined by equation⁴ (2.8) of [7], which is taken from equation (5) in [16]. This is summarized in table 1, where σ^T is the transpose partition, σ^\pm is defined as $\sigma^+ := [\sigma_1, \sigma_2, \dots, \sigma_s, 1]$ and $\sigma^- := [\sigma_1, \sigma_2, \dots, \sigma_s - 1]$. σ_X , with $X = B, C, D$, is the X -collapse of partition σ as defined in [13]. $\mathcal{P}_A(n)$ is the set of all partitions of n ; $\mathcal{P}_B(2n+1)$ is the set of partitions of $2n+1$ where even parts have even multiplicity; $\mathcal{P}_C(2n)$ is the set of partitions of $2n$ where odd parts have even multiplicity and $\mathcal{P}_D(2n)$ is the set of partitions of $2n$ where odd parts have even multiplicity.

This means that for $G = \text{SO}(2n)$ the Coulomb branches of $T^\sigma(\text{SO}(2n))$ for special partitions σ cover the whole set of closures of special nilpotent orbits of $\mathfrak{so}(2n)$. For $G = \text{USp}(2n)$ Coulomb branches of $T^\sigma(\text{USp}(2n))$ for special partitions σ cover the whole set of closures of special nilpotent orbits of $\mathfrak{so}(2n+1)$. There are some exceptions for this statement: closures of nilpotent orbits that are non-normal have not been found as Coulomb branches of any theory.⁵

2.4 Example: $\lambda = [3, 3, 1]$ orbit of $\mathfrak{so}(7)$

Let us look at an example for the closure of the nilpotent orbit $\bar{\mathcal{O}}_{[3,3,1]} \subset \mathfrak{so}(7)$. This means that $\lambda = [3, 3, 1]$ and $\mathfrak{g}^\vee = \mathfrak{so}(7)$. Therefore the relevant quiver is $T^\sigma(G)$ with:

$$\begin{aligned}
 G &= \text{SO}(7)^\vee = \text{USp}(6) \\
 \sigma &= d_{\text{BV}}[3, 3, 1] = [2, 2, 2]
 \end{aligned}
 \tag{2.5}$$

Note that $\lambda = d_{\text{BV}}(\sigma)$ implies $\sigma = d_{\text{BV}}(\lambda)$ because we are restricting σ to the set of *special* partitions. The quiver theory is $T^{[2,2,2]}(\text{USp}(6))$. The necessary partition data in order to obtain the quiver is:

$$\begin{aligned}
 \sigma^T &= [2, 2, 2]^T = [3, 3] \\
 \rho &= [1, 1, 1, 1, 1, 1]
 \end{aligned}
 \tag{2.6}$$

⁴Note that the name *Spaltenstein map* is adopted in [7] for the Barbasch-Vogan map. Also note that there is a mistake in [2] where the map is defined by equation (6.2): for a partition of C-type the map is defined as transposition, addition of a new part and B-collapsing; the correct order is addition of a new part, then transposition and then B-collapsing.

⁵It remains a challenge to figure out if Coulomb branches can be *non-normal*, as their Higgs branch counterparts. The challenge has recently gained importance since Nakajima’s mathematical construction for the Coulomb branch [17] has been shown to always be a normal variety [18].

The N_j parameters for the gauge nodes are:

$$\begin{aligned}
 N_1 &= [6]_+ - 3 = 3 \\
 N_2 &= [5]_- = 4 \\
 N_3 &= [4]_+ = 4 \\
 N_4 &= [3]_- = 2 \\
 N_5 &= [2]_+ = 2 \\
 N_6 &= [1]_- = 0
 \end{aligned}
 \tag{2.7}$$

The M_j parameters for the flavor nodes are:

$$\begin{aligned}
 M_1 &= 3 - 3 = 0 \\
 M_2 &= 3 - 0 = 3 \\
 M_3 &= 0 - 0 = 0 \\
 M_4 &= 0 - 0 = 0 \\
 M_5 &= 0 - 0 = 0 \\
 M_6 &= 0 - 0 = 0
 \end{aligned}
 \tag{2.8}$$

This gives four possible quivers with different Coulomb branch \mathcal{C} depending on the choice $\text{SO}/\text{O}(N_j)$ for N_j even,⁶ depicted in figure 1. The aim of this paper is to compute the Coulomb branch of the four quivers in order to determine which of them has the closure of the nilpotent orbit $\mathcal{C} = \bar{\mathcal{O}}_{[3,3,1]}$ as its Coulomb branch, and therefore corresponds to $T^{[2,2,2]}(\text{USp}(6))$. In this case the answer is the quiver in figure 1(a), as is shown in the third row of table 6.

3 The monopole formula

3.1 The Hilbert series

A Hilbert series $H(t)$ can be computed in order to determine the Coulomb branch of each quiver, see [4]. In this case the Hilbert series can be obtained via the *monopole formula* introduced in [10], which counts dressed monopole operators in the chiral ring. We shall only consider the unrefined Hilbert series $H(t)$, given as:

$$H(t) = \sum_{\mathbf{m} \in \Gamma_{\mathcal{G}^\vee}^*/W_{\mathcal{G}^\vee}} t^{\Delta(\mathbf{m})} P_{\mathcal{G}}(\mathbf{m}; t),
 \tag{3.1}$$

where \mathcal{G} is the gauge group and the magnetic fluxes \mathbf{m} are summed over the dominant Weyl chamber of the weight lattice of \mathcal{G}^\vee , the GNO dual group of \mathcal{G} [9]. \mathbf{m} breaks \mathcal{G} into

⁶Note that only orthogonal groups with $\text{O}(N_j)$ with N_j even are considered. This is a straight forward exercise since a description of their magnetic lattices and dressing factors has been provided in [2]. The analysis of N_j odd is left for future work.

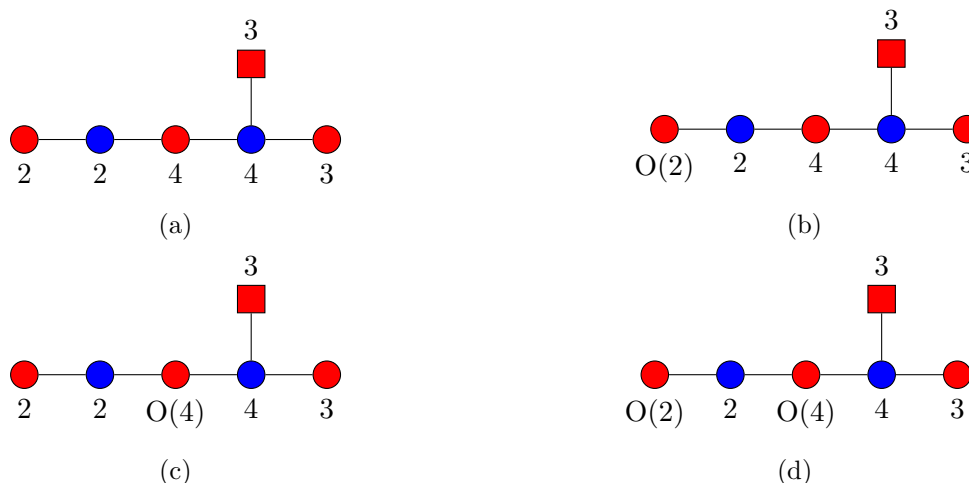


Figure 1. Candidate quivers for $T^{[2,2,2]}(\text{USp}(6))$ theory. Gauge nodes are denoted by a circle and flavor nodes are represented by a square. Red gauge nodes with label N_j denote $\text{SO}(N_j)$ groups and blue gauge nodes with label N_j denote $\text{USp}(N_j)$ gauge groups. The red square node with label $M_2 = 3$ denotes an $\text{SO}(3)$ flavor node. When the gauge group $\text{O}(N_j)$ is chosen instead of $\text{SO}(N_j)$ the group is written explicitly.

a residual gauge group $\mathcal{G}_{\mathbf{m}}$. In $3d \mathcal{N} = 4$, the dressing factors $P_{\mathcal{G}}(\mathbf{m}; t)$ are given by the dimensions d_i of the Casimir invariants of $\mathcal{G}_{\mathbf{m}}$:

$$P_{\mathcal{G}}(\mathbf{m}; t) = \prod_{i=1}^r \frac{1}{(1 - t^{d_i(\mathcal{G}_{\mathbf{m}})})} \quad (3.2)$$

where $r = \text{rank}(\mathcal{G})$. $\Delta(\mathbf{m})$ is the *conformal dimension* of the bare monopole operators:

$$\Delta(\mathbf{m}) = - \sum_{\alpha \in \Delta_+} |\alpha(\mathbf{m})| + \frac{1}{2} \sum_{i=1}^n \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(\mathbf{m})|, \quad (3.3)$$

where the first sum is over positive roots of the gauge group \mathcal{G} and it is the contribution from the vector multiplets of the theory. The second sum is over weights of the representation \mathcal{R}_i under which the hypermultiplets of the theory transform (do not confuse the symbol ρ_i for the weights with the previous symbol ρ for the trivial partitions).

3.2 The plethystic logarithm

Once a Hilbert series $H(t)$ is obtained, it is useful to analyze its plethystic logarithm [19], $PL(H(t))$ defined as:

$$PL(H(t)) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(H(t^k)) \quad (3.4)$$

where $\mu(k)$ is the Möbius function. The $PL(H(t))$ is a generating function that counts generators of the chiral ring and their relations. Since a moduli space that is the closure of a nilpotent orbit restricts the dimension of the generators of the chiral ring to $\Delta = 1$ (see discussion in sections 3.1 and 3.2 of [14]) the $PL(H(t))$ provides invaluable information for the present analysis.

4 Quivers for closures of nilpotent orbits

This section contains the main results of the paper. The quiver diagrams of theories $T^\sigma(G)$ with the right choice of $\text{SO}/\text{O}(N_j)$ gauge nodes so their Coulomb branch \mathcal{C} is the closure of nilpotent orbit $\bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(n)$ are tabulated for $3 \leq n \leq 10$. Remember that $\sigma = d_{\text{BV}}(\lambda)$ and vice versa. The Hilbert series and plethystic logarithms of the Coulomb branch of these quivers are also tabulated.

4.1 Tables: quivers

For the tables in this section:

- All red nodes with label N denote $\text{SO}(N)$ groups and all blue nodes with label N denote $\text{USp}(N)$ groups. Red nodes corresponding to $\text{O}(N)$ groups are explicitly labeled as $\text{O}(N)$.
- A single quiver is provided for each closure of a special nilpotent orbit $\bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(n)$ that is normal.
- The partition λ of the nilpotent orbit is given in exponential notation, i.e $[2^3]$ is the same as $[2, 2, 2]$.
- Partitions λ corresponding to non-special orbits and non-normal orbits are indicated explicitly. No quivers are provided for these orbits as we cannot obtain quivers whose Coulomb branch is the closure of the nilpotent orbit using the prescriptions in sections 2 and 3.
- Trivial partitions $\lambda = [1^n]$ have been omitted from the analysis since the corresponding Coulomb branch is trivial.
- The *very even* partition $\lambda = [2^2]$ corresponds to two different orbits $\mathcal{O}_{[2^2]}^I$ and $\mathcal{O}_{[2^2]}^{II}$ under the adjoint action of $\text{SO}(4)$. The corresponding Coulomb branch is isomorphic to the closure of only one of these orbits. This is also the case for the remaining very even partitions $\lambda = [2^4]$ and $\lambda = [4^2]$.
- The symbol (\dagger) denotes a quiver where the GNO magnetic lattice of two gauge groups is projected by a diagonal action of a \mathbb{Z}_2 group. This action is described in section 6.
- The Lusztig's Canonical Quotient⁷ $\bar{A}(\mathcal{O}_\lambda)$ corresponding to each orbit is shown. The choice of $\text{SO}/\text{O}(N)$ gauge factors in the quiver seems to be beautifully encoded in this group. This is discussed at length in section 5.

It is worthwhile to highlight that the resulting quivers with Coulomb branch \mathcal{C}_1 being the closure of the minimal orbit of $\mathfrak{so}(2n)$ (with partition $\lambda = [2^2, 1^{2n-4}]$) and the resulting quivers with Coulomb branch \mathcal{C}_2 being the closure of the next to minimal orbit of $\mathfrak{so}(2n-1)$ (with partition $\lambda = [3, 1^{2n-4}]$) differ only by a \mathbb{Z}_2 factor on the first node (this is the rightmost node in the tables). This is a compelling physical realization of the mathematical result by Brylinski and Kostant, that states that the two varieties are related by a \mathbb{Z}_2 quotient [22]: $\mathcal{C}_2 = \mathcal{C}_1/\mathbb{Z}_2$.

⁷This is a finite discrete group associated to each nilpotent orbit. It was defined in [20] for special orbits and in [21] extended to all orbits. See also [7, 16].

Partition (λ)	$T^\sigma(\text{SO}(3)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[3]		1

Table 2. $T^\sigma(\text{SO}(3)^\vee)$ quivers with Coulomb branch the closure of a nilpotent orbit $\bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(3)$. The partition $\sigma = d_{\text{BV}}(\lambda)$ is the dual partition of λ under the Barbasch-Vogan map. $\bar{A}(\mathcal{O}_\lambda)$ is the Lusztig's Canonical Quotient of each orbit.

Partition (λ)	$T^\sigma(\text{SO}(4)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[3, 1]		1
[2 ²]		1

Table 3. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(4)$. The dual partition is $\sigma = d_{\text{BV}}(\lambda)$.

Partition (λ)	$T^\sigma(\text{SO}(5)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[5]		1
[3, 1 ²]		\mathbb{Z}_2
[2 ² , 1]	Non-special	1

Table 4. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(5)$. The dual partition is $\sigma = d_{\text{BV}}(\lambda)$.

Partition (λ)	$T^\sigma(\text{SO}(6)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[5, 1]		1
[3^2]		1
[3, 1^3]		1
[2^2, 1^2]		1

Table 5. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(6)$. The dual partition is $\sigma = d_{\text{BV}}(\lambda)$.

Partition (λ)	$T^\sigma(\text{SO}(7)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[7]		1
[5, 1^2]		\mathbb{Z}_2
[3^2, 1]		1
[3, 2^2]	Non-normal	1
[3, 1^4]		\mathbb{Z}_2
[2^2, 1^3]	Non-special	1

Table 6. $T^\sigma(\text{SO}(7)^\vee)$ quivers with Coulomb branch the closure of a nilpotent orbit $\bar{\mathcal{O}}_\lambda$ of $\mathfrak{so}(7)$. The partition $\sigma = d_{\text{BV}}(\lambda)$ is the dual partition of λ under the Barbasch-Vogan map. $\bar{A}(\mathcal{O}_\lambda)$ is the Lusztig's Canonical Quotient of each orbit.

Partition (λ)	$T^\sigma(\text{SO}(8)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[7, 1]		1
[5, 3]		1
[5, 1^3]		1
[4^2]		1
[3^2, 1^2]		\mathbb{Z}_2
[3, 2^2, 1]	Non-special	1
[3, 1^5]		1
[2^4]		1
[2^2, 1^4]		1

Table 7. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(8)$. The dual partition is $\sigma = d_{BV}(\lambda)$.

Partition (λ)	$T^\sigma(\text{SO}(9)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[9]		1
[7, 1 ²]		\mathbb{Z}_2
[5, 3, 1]		\mathbb{Z}_2
[5, 2 ²]	Non-normal	1
[4 ² , 1]	Non-special	1
[5, 1 ⁴]		\mathbb{Z}_2

Table 8. $T^\sigma(\text{SO}(9)^\vee)$ quivers with Coulomb branch the closure of a nilpotent orbit $\bar{\mathcal{O}}_\lambda$ of $\mathfrak{so}(9)$. The partition $\sigma = d_{\text{BV}}(\lambda)$ is the dual partition of λ under the Barbasch-Vogan map. $\bar{A}(\mathcal{O}_\lambda)$ is the Lusztig's Canonical Quotient of each orbit.

Partition (λ)	$T^\sigma(\text{SO}(9)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
[3 ³]		1
[3 ² , 1 ³]		1
[3, 2 ² , 1 ²] [†]		\mathbb{Z}_2
[2 ⁴ , 1]	Non-special	1
[3, 1 ⁶]		\mathbb{Z}_2
[2 ² , 1 ⁵]	Non-special	1

Table 9. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(9)$. The dual partition is $\sigma = d_{\text{BV}}(\lambda)$. The brackets in row $[3, 2^2, 1^2]^\dagger$ correspond to a choice of gauge nodes discussed in section 6.

Partition (λ)	$T^\sigma(\text{SO}(10)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
$[9, 1]$		1
$[7, 3]$		1
$[5, 5]$		1
$[7, 1^3]$		1
$[5, 3, 1^2]$		\mathbb{Z}_2
$[4^2, 1^2]$		1
$[5, 2^2, 1]$	Non-special	
$[3^3, 1]$		1

Table 10. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(10)$. The dual partition is $\sigma = d_{BV}(\lambda)$.

Partition (λ)	$T^\sigma(\text{SO}(10)^\vee)$	$\bar{A}(\mathcal{O}_\lambda)$
$[5, 1^5]$		1
$[3^2, 2^2]$	Non-normal	1
$[3^2, 1^4]$		\mathbb{Z}_2
$[3, 2^2, 1^3]$	Non-special	1
$[2^4, 1^2]$		1
$[3, 1^7]$		1
$[2^2, 1^6]$		1

Table 11. Quivers with $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(10)$. The dual partition is $\sigma = d_{\text{BV}}(\lambda)$.

4.2 Tables: Hilbert series and plethystic logarithms

The Hilbert series of the Coulomb branch of the set of $T^\sigma(G^\vee)$ quivers presented in the previous section are computed using the *monopole formula* (see section 3).

The computation of the Hilbert series $H(t)$ is done perturbatively. $H(t)$ is expanded in powers of t around $t = 0$:

$$H(t) = \sum_{i=0}^{\infty} a_i t^i \tag{4.1}$$

and the coefficients a_i are computed starting with $a_0 = 1$. The highest order a_k that can be computed is restricted by the computing power, obtaining an expression:

$$H(t) = \sum_{i=0}^k a_i t^i + O(t^{k+1}) \tag{4.2}$$

One can then use properties of the nilpotent orbit such as its complex dimension (which in this case equals twice the rank of the gauge group of the corresponding $T^\sigma(G^\vee)$ quiver).

Closures of nilpotent orbits that are normal have a Hilbert series with rational form:

$$H(t) = \frac{P(t)}{(1-t)^p} \tag{4.3}$$

where p is the complex dimension and $P(t)$ is a palindromic polynomial of degree $p/2$ with $P(1)$ non zero. This allows to reconstruct the rational form of $H(t)$ after computing a finite set of coefficients a_i .

For nilpotent orbits of $\mathfrak{so}(n)$ with $n < 10$ the Hilbert series of closures of nilpotent orbits can be compared with the results in [23]. For nilpotent orbits of $\mathfrak{so}(10)$ we turn our attention to the plethystic logarithm defined in section 3, which provides the degrees of generators and the degrees of relations between them. This information can be compared with the constrains (such as *rank* or *degree of nilpotency*) on the nilpotent orbit when viewed as a set of matrices, i.e. linear operators acting on the fundamental representation of the Lie algebra.⁸

The following tables include the Hilbert series and the plethystic logarithms of the quivers displayed in the previous section. The plethystic logarithm is given up to an order so it at least includes the relations that show nilpotency.

Partition (λ)	Hilbert series	Plethystic logarithm
[3]	$\frac{(1+t)}{(1-t)^2}$	$3t - t^2$

Table 12. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 2.

Partition (λ)	Hilbert series	Plethystic logarithm
[3, 1]	$\frac{(1-t^2)^2}{(1-t)^6}$	$6t - 2t^2$
[2 ²]	$\frac{1+t}{(1-t)^2}$	$3t - t^2$

Table 13. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 3.

Partition (λ)	Hilbert series	Plethystic logarithm
[5]	$\frac{(1+t)^2(1+t^2)}{(1-t)^8}$	$10t - t^2 - t^4$
[3, 1 ²]	$\frac{(1+t)(1+3t+t^2)}{(1-t)^6}$	$10t - 6t^2 + 5t^3 - 10t^4 + 24t^5 + O(t^6)$
[2 ² , 1]	Non-special	

Table 14. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 4.

⁸We have also compared the Hilbert series obtained for $\mathfrak{so}(10)$ with Hilbert series for closures of nilpotent orbits computed via localization techniques by Rudolph Kalveks (these results are described in an upcoming publication [24]).

Partition (λ)	Hilbert series	Plethystic logarithm
$[5, 1]$	$\frac{(1-t^2)(1-t^3)(1-t^4)}{(1-t)^{15}}$	$15t - t^2 - t^3 - t^4$
$[3^2]$	$\frac{1+4t+10t^2+4t^3+t^4}{(1-t)^{10}(1+t)^{-1}}$	$15t - t^2 - 16t^3 + 30t^4 - 16t^5 + O(t^6)$
$[3, 1^3]$	$\frac{1+5t+t^2}{(1-t)^8(1+t)^{-2}}$	$15t - 16t^2 + 35t^3 - 126t^4 + 504t^5 + O(t^6)$
$[2^2, 1^2]$	$\frac{1+8t+t^2}{(1-t)^6(1+t)^{-1}}$	$15t - 36t^2 + 160t^3 - 945t^4 + 6048t^5 + O(t^6)$

Table 15. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 5.

Partition (λ)	Hilbert series	Plethystic logarithm
$[7]$	$\frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t)^{21}}$	$21t - t^2 - t^4 - t^6$
$[5, 1^2]$	$\frac{1+3t+6t^2+3t^3+t^4}{(1-t)^{16}(1+t)^{-2}(1+t^2)^{-1}}$	$21t - t^2 - 7t^3 + 6t^4 + O(t^6)$
$[3^2, 1]$	$\frac{1+6t+21t^2+28t^3+21t^4+6t^5+t^6}{(1-t)^{14}(1+t)^{-1}}$	$21t - t^2 - 28t^3 + 63t^4 - 36t^5 + O(t^6)$
$[3, 2^2]$	Non-normal	
$[3, 1^4]$	$\frac{1+10t+20t^2+10t^3+t^4}{(1-t)^{10}(1+t)^{-1}}$	$21t - 36t^2 + 140t^3 - 784t^4 + 4788t^5 + O(t^6)$
$[2^2, 1^3]$	Non-special	

Table 16. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 6.

Partition (λ)	Hilbert series	Plethystic logarithm
[7, 1]	$\frac{(1-t^2)(1-t^4)^2(1-t^6)}{(1-t)^{28}}$	$28t - t^2 - 2t^4 - t^6$
[5, 3]	$\frac{1 + 3t + 8t^2 + 16t^3 + 28t^4 + 16t^5 + 8t^6 + 3t^7 + t^8}{(1-t)^{22}(1+t)^{-3}}$	$28t - t^2 - 2t^4 - 28t^5 + O(t^6)$
[5, 1 ³]	$\frac{1 + 6t + 21t^2 + 28t^3 + 21t^4 + 6t^5 + t^6}{(1-t)^{20}(1+t)^{-2}(1+t^2)^{-1}}$	$28t - t^2 - 28t^3 + 62t^4 - 36t^5 + O(t^6)$
[4, 4]	$\frac{1 + 6t + 21t^2 + 28t^3 + 21t^4 + 6t^5 + t^6}{(1-t)^{20}(1+t)^{-2}(1+t^2)^{-1}}$	$28t - t^2 - 28t^3 + 62t^4 - 36t^5 + O(t^6)$
[3 ² , 1 ²]	$\frac{1 + 9t + 45t^2 + 109t^3 + 152t^4 + 109t^5 + 45t^6 + 9t^7 + t^8}{(1-t)^{18}(1+t)^{-1}}$	$28t - t^2 - 56t^3 + 161t^4 - 107t^5 + O(t^6)$
[3, 2 ² , 1]	Non-special	
[3, 1 ⁵]	$\frac{1 + 14t + 36t^2 + 14t^3 + t^4}{(1-t)^{12}(1+t)^{-2}}$	$28t - 71t^2 + 420t^3 - 3360t^4 + 29148t^5 + O(t^6)$
[2 ⁴]	$\frac{1 + 14t + 36t^2 + 14t^3 + t^4}{(1-t)^{12}(1+t)^{-2}}$	$28t - 71t^2 + 420t^3 - 3360t^4 + 29148t^5 + O(t^6)$
[2 ² , 1 ⁴]	$\frac{1 + 17t + 48t^2 + 17t^3 + t^4}{(1-t)^{10}(1+t)^{-1}}$	$28t - 106t^2 + 833t^3 - 8400t^4 + 91392t^5 + O(t^6)$

Table 17. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 7.

Partition (λ)	Hilbert series	Plethystic logarithm
[9]	$\frac{(1-t^2)(1-t^4)(1-t^6)(1-t^8)}{(1-t)^{36}}$	$36t - t^2 - t^4 - t^6$
[7, 1 ²]	$\frac{1 + 5t + 15t^2 + 35t^3 + 60t^4 + 85t^5 + 104t^6 + 110t^7 + 104t^8 + 85t^9 + 60t^{10} + 35t^{11} + 15t^{12} + 5t^{13} + t^{14}}{(1-t)^{30}(1+t)^{-1}}$	$36t - t^2 - 10t^4 + 9t^5 - t^6 - 36t^8 + O(t^9)$
[5, 3, 1]	$\frac{1 + 6t + 22t^2 + 62t^3 + 138t^4 + 227t^5 + 264t^6 + 227t^7 + 138t^8 + 62t^9 + 22t^{10} + 6t^{11} + t^{12}}{(1-t)^{28}(1+t)^{-2}}$	$36t - t^2 - 10t^4 - 27t^5 + 36t^6 + O(t^7)$
[5, 2 ²]	Non-normal	
[5, 1 ⁴]	$\frac{1 + 10t + 55t^2 + 136t^3 + 190t^4 + 136t^5 + 55t^6 + 10t^7 + t^8}{(1-t)^{24}(1+t)^{-2}(1+t^2)^{-1}}$	$36t - t^2 - 84t^3 + 314t^4 - 396t^5 + O(t^6)$
[3 ³]	$\frac{1 + 10t + 56t^2 + 194t^3 + 438t^4 + 578t^5 + 438t^6 + 194t^7 + 56t^8 + 10t^9 + t^{10}}{(1-t)^{24}(1+t)^{-2}}$	$36t - t^2 - 36t^3 + 27t^4 + 92t^5 + O(t^6)$
[3 ² , 1 ³]	$\frac{1 + 13t + 91t^2 + 335t^3 + 737t^4 + 946t^5 + 737t^6 + 335t^7 + 91t^8 + 13t^9 + t^{10}}{(1-t)^{22}(1+t)^{-1}}$	$36t - t^2 - 120t^3 + 477t^4 - 523t^5 + O(t^6)$
[3, 2 ² , 1]	Non-special	
[3, 2 ² , 1 ²] [†]	$\frac{1 + 14t + 106t^2 + 454t^3 + 788t^4 + 454t^5 + 106t^6 + 14t^7 + t^8}{(1-t)^{20}(1+t)^{-2}}$	$36t - t^2 - 120t^3 - 18t^4 + 4284t^5 + O(t^6)$
[3, 1 ⁶]	$\frac{1 + 21t + 105t^2 + 175t^3 + 105t^4 + 21t^5 + t^6}{(1-t)^{14}(1+t)^{-1}}$	$36t - 127t^2 + 1050t^3 - 11340t^4 + 132552t^5 + O(t^6)$
[2 ² , 1 ⁵]	Non-special	

Table 18. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 8 and table 9.

Partition (λ)	Hilbert series	Plethystic logarithm
[9, 1]	$\frac{(1-t^2)(1-t^4)(1-t^5)(1-t^6)(1-t^8)}{(1-t)^{45}}$	$45t - t^2 - t^4 - t^5 - t^6 - t^8$
[7, 3]	$\frac{1 + 4t + 12t^2 + 28t^3 + 57t^4 + 103t^5 + 171t^6 + 219t^7 + 250t^8 + 219t^9 + 171t^{10} + 103t^{11} + 57t^{12} + 28t^{13} + 12t^{14} + 4t^{15} + t^{16}}{(1-t)^{38}(1+t)^{-3}}$	$45t - t^2 - t^4 - t^5 - t^6 - 45t^7 + O(t^8)$
[5, 5]	$\frac{1 + 7t + 28t^2 + 84t^3 + 210t^4 + 416t^5 + 647t^6 + 742t^7 + 647t^8 + 416t^9 + 210t^{10} + 84t^{11} + 28t^{12} + 7t^{13} + t^{14}}{(1-t)^{36}(1+t)^{-2}(1+t^2)^{-1}}$	$45t - t^2 - t^4 - 46t^5 + O(t^6)$
[7, 1 ³]	$\frac{1 + 5t + 18t^2 + 50t^3 + 75t^4 + 130t^5 + 131t^6 + 170t^7 + 131t^8 + 130t^9 + 75t^{10} + 50t^{11} + 18t^{12} + 5t^{13} + t^{14}}{(1-t)^{36}(1+t)^{-4}}$	$45t - t^2 - 46t^4 + 99t^5 - 56t^6 + O(t^8)$
[5, 3, 1 ²]	$\frac{1 + 8t + 38t^2 + 136t^3 + 359t^4 + 742t^5 + 1162t^6 + 1378t^7 + 1162t^8 + 742t^9 + 359t^{10} + 136t^{11} + 38t^{12} + 8t^{13} + t^{14}}{(1-t)^{34}(1+t)^{-3}}$	$45t - t^2 - 46t^4 + 54t^5 + O(t^6)$
[4 ² , 1 ²]	$\frac{1 + 11t + 67t^2 + 297t^3 + 968t^4 + 2353t^5 + 4004t^6 + 4838t^7 + 4004t^8 + 2353t^9 + 968t^{10} + 297t^{11} + 67t^{12} + 11t^{13} + t^{14}}{(1-t)^{32}(1+t)^{-2}}$	$45t - t^2 - 100t^4 + 153t^5 + O(t^6)$
[5, 2 ² , 1]	Non-special	
[3 ³ , 1]	$\frac{1 + 14t + 105t^2 + 515t^3 + 1750t^4 + 4151t^5 + 6924t^6 + 8200t^7 + 6924t^8 + 4151t^9 + 1750t^{10} + 515t^{11} + 105t^{12} + 14t^{13} + t^{14}}{(1-t)^{30}(1+t)^{-1}}$	$45t - t^2 - 45t^3 + 308t^5 + O(t^6)$

Table 19. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 10.

Partition (λ)	Hilbert series	Plethystic logarithm
$[5, 1^5]$	$\frac{1 + 15t + 121t^2 + 485t^3 + 1185t^4 + 1847t^5 + 2130t^6 + 1847t^7 + 1185t^8 + 485t^9 + 121t^{10} + 15t^{11} + t^{12}}{(1-t)^{28}(1+t)^{-2}}$	$45t - t^2 - 210t^3 + 1154t^4 - 2376t^5 + O(t^6)$
$[3^2, 2^2]$	Non-normal	
$[3^2, 1^4]$	$\frac{1 + 18t + 171t^2 + 885t^3 + 2805t^4 + 5522t^5 + 6936t^6 + 5522t^7 + 2805t^8 + 885t^9 + 171t^{10} + 18t^{11} + t^{12}}{(1-t)^{26}(1+t)^{-1}}$	$45t - t^2 - 255t^3 + 1410t^4 - 2587t^5 + O(t^6)$
$[3, 2^2, 1^3]$	Non-special	
$[2^4, 1^2]$	$\frac{1 + 23t + 223t^2 + 925t^3 + 1532t^4 + 925t^5 + 223t^6 + 23t^7 + t^8}{(1-t)^{20}(1+t)^{-2}}$	$45t - 55t^2 - 156t^3 + 3420t^4 - 33471t^5 + O(t^6)$
$[3, 1^7]$	$\frac{1 + 27t + 169t^2 + 321t^3 + 169t^4 + 27t^5 + t^6}{(1-t)^{16}(1+t)^{-2}}$	$45t - 211t^2 + 2310t^3 - 32340t^4 + 489720t^5 + O(t^6)$
$[2^2, 1^6]$	$\frac{1 + 30t + 201t^2 + 394t^3 + 201t^4 + 30t^5 + t^6}{(1-t)^{14}(1+t)^{-1}}$	$45t - 265t^2 + 3354t^3 - 53295t^4 + 914430t^5 + O(t^6)$

Table 20. Hilbert series and plethystic logarithm for the Coulomb branch of quivers in table 11.

4.3 First case with $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2^2$

The first nilpotent orbit with a Lusztig's Canonical Quotient of the form $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2^2$ is the orbit $\mathcal{O}_\lambda \subset \mathfrak{so}(13)$ with $\lambda = [5, 3^2, 1^2]$. The quiver with Coulomb branch $\mathcal{C} = \bar{\mathcal{O}}_\lambda$ is different from all the ones computed so far since it corresponds to the choice of *two* orthogonal gauge group factors (all previous results in section 4.1 have either *zero* or *one* orthogonal gauge group factor). It is depicted in figure 2 and it corresponds to theory $T^{[4^2, 2^2]}(\text{USp}(12))$.

The computed Hilbert series is:

$$H(t) = 1 + 78t + 3080t^2 + 82082t^3 + 1660658t^4 + 27202734t^5 + O(t^6) \quad (4.4)$$

Its plethystic logarithm:

$$PL(H(t)) = 78t - t^2 - t^4 - 364t^5 + O(t^6) \quad (4.5)$$

5 Analysis of the results

Analyzing the obtained results we are able to produce a prescription that fully determines the orthosymplectic quiver corresponding to $T^\sigma(\text{SO}(n)^\vee)$ theory such that its Coulomb branch is $\mathcal{C} = \bar{\mathcal{O}}_\lambda \subset \mathfrak{so}(n)$ with $\lambda = d_{\text{BV}}(\sigma)$.

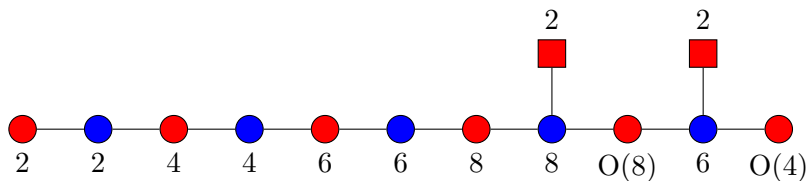


Figure 2. Quiver for the theory $T^{[4^2, 2^2]}(\text{USp}(12))$ with Coulomb branch $\mathcal{C} = \bar{\mathcal{O}}_{[5, 3^2, 1^2]} \subset \mathfrak{so}(13)$.

In order to present the prescription let us define a *chain* Ξ_i of $\text{SO}/\text{O}(N_j)$ groups as a subset of the quiver constituted by consecutive $\text{SO}/\text{O}(N_j)$ gauge nodes (starting with $\text{SO}/\text{O}(N_i)$, following with $\text{SO}/\text{O}(N_{i+2})$, etc. and ending with $\text{SO}/\text{O}(N_{i+2m-2})$, where the *length* or number of nodes in the chain is m) such that:

- N_{i+2p} is even for $0 \leq p \leq m - 1$.
- If the quiver has gauge nodes $\text{USp}(N_{i-1})$ and/or $\text{USp}(N_{i+2m-1})$, they have flavor nodes attached to them.
- Gauge nodes $\text{USp}(N_{i+2p+1})$ with $0 \leq p \leq m - 2$ do not have flavor node attached to them, i.e.

$$M_{i+2p+1} = \sigma_{i+2p+1}^T - \sigma_{i+2p+2}^T = 0, \quad 0 \leq p \leq m - 2 \tag{5.1}$$

When faced with the choice $\text{SO}/\text{O}(N_j)$: the maximal magnetic lattice is the lattice for the gauge group where all *chains* Ξ_i are chosen to have $\text{SO}(N_{i+2p})$ as their groups for all p . Let us define this as the *default* lattice. This magnetic lattice is chosen when $\bar{A}(\mathcal{O}_\lambda)$ is the trivial group.

When the Lusztig’s Canonical Quotient is $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2$ the chain Ξ_i with the lowest i changes. If the chain has length one, i.e. it constitutes a single node $\text{SO}(N_i)$, this node is changed to $\text{O}(N_i)$. From the point of view of the magnetic lattice this translates into projecting the magnetic lattice of $\text{SO}(N_i)$ by a \mathbb{Z}_2 action. If the chain has length m bigger than one, the magnetic lattice of the groups $\text{SO}(N_i) \times \text{SO}(N_{i+2}) \times \dots \times \text{SO}(N_{i+2m-2})$ needs to be projected by the diagonal \mathbb{Z}_2 acting diagonally on the individual lattices of all the groups $\text{SO}(N_{i+2p})$. This is the case of the quiver marked with the symbol (\dagger) in table 9. How to perform this diagonal \mathbb{Z}_2 action is a new result that has never been discussed before as far as the authors are aware. This process is explained in detail in the next section.

When the Lusztig’s Canonical Group has the form $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2^k$ the k chains Ξ_i with lowest i are changed in the same fashion:⁹ in each chain Ξ_i all the lattices of the groups $\text{SO}(N_{i+2p})$ are projected by a diagonal \mathbb{Z}_2 action. This is for example the case of the quiver in figure 2: $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2^2$, the first chain Ξ_1 has length one, the magnetic lattice of its gauge group $\text{SO}(4)$ is acted by a single \mathbb{Z}_2 , so the group changes to $\text{O}(4)$; the second chain Ξ_3 has also length one, its gauge group $\text{SO}(8)$ changes to $\text{O}(8)$, i.e. its magnetic lattice is halved by a different \mathbb{Z}_2 action.

⁹We have observed that in the case of nilpotent orbits of $\mathfrak{so}(2n + 1)$ the total number of chains Ξ_i coincides with $k + 1$. For $\mathfrak{so}(2n)$ the total number of chains is either equal or greater than $k + 1$.

6 Diagonal \mathbb{Z}_2 actions on different magnetic lattices

6.1 Diagonal action on the magnetic lattices of $\text{SO}(2) \times \text{SO}(4)$

For clarity let us discuss the quiver with $\mathcal{C} = \bar{\mathcal{O}}_{[3,2^2,1^2]} \subset \mathfrak{so}(9)$ depicted in table 9. There are two *chains*: Ξ_1 with groups $\text{SO}(N_1 = 2)$ and $\text{SO}(N_3 = 4)$, and Ξ_5 with groups $\text{SO}(N_5 = 4)$ and $\text{SO}(N_7 = 2)$. The aim is to divide the magnetic lattice of the groups in Ξ_1 by a single \mathbb{Z}_2 action.

Let the weights and dressing factors of the magnetic lattices of gauge group factors $\text{SO}(2)$, $\text{O}(2)$, $\text{SO}(4)$ and $\text{O}(4)$, as they were discussed in [2], be described by tables 21, 22, 23 and 24. Note that for the groups $\text{O}(2)$ and $\text{O}(4)$ some magnetic fluxes are defined to have vanishing dressing factor, this is equivalent to removing them from the lattice. The choice to include them in the tables becomes justified in the following discussion.

A general description of the magnetic lattices and dressing factors of gauge groups $\text{SO}(N)$ and $\text{USp}(2N)$ can be found in appendix A. *Classical Casimir contribution for classical groups* of [10]. For the group $\text{SO}(2N)$ the magnetic weights are the weights of the GNO dual group, $\text{SO}(2N)$. The dominant Weyl chamber of the group can be parametrized by an integer vector:

$$\mathbf{m} = (m_1, m_2, \dots, m_N) \tag{6.1}$$

with $m_1 \geq m_2 \geq \dots \geq |m_N|$. For the gauge group $\text{SO}(2N + 1)$ the magnetic weights are summed over the dominant Weyl chamber in the weight lattice of the GNO dual, $\text{USp}(2N)$. These can be parametrized by the same integer vector \mathbf{m} in equation (6.1) but this time with $m_1 \geq m_2 \geq \dots \geq m_N$. For the gauge group $\text{O}(2N)$ the magnetic weights and dressing factors can be found in appendix A. *Monopole formula for orthogonal and special orthogonal gauge groups* of [2]. Starting with the dominant Weyl chamber of $\text{SO}(2N)$ described by (6.1) the parity symmetry \mathbb{Z}_2 identifies (m_1, m_2, \dots, m_N) with $(m_1, m_2, \dots, -m_N)$. This results in a Weyl chamber isomorphic to the Weyl chamber of $\text{USp}(2N)$ that is employed for the $\text{SO}(2N + 1)$ gauge group. As explained in [2], not only the Weyl chambers of magnetic weights of $\text{O}(2N)$ and $\text{SO}(2N + 1)$ are isomorphic, but their dressing factors are also identical.

The combination of the magnetic lattices of gauge groups $\text{SO}(2)$, $\text{O}(2)$, $\text{SO}(4)$ and $\text{O}(4)$ gives four possibilities for chain Ξ_1 (the aim of this section is to present a fifth option). Let

$$\Gamma_i := \Gamma_{\mathcal{G}_i^\vee}^* / \mathcal{W}_{\mathcal{G}_i^\vee} \tag{6.2}$$

where $\Gamma_{\mathcal{G}_i^\vee}^*$ is the weight lattice of \mathcal{G}_i^\vee , the GNO dual group of \mathcal{G}_i , and $\mathcal{W}_{\mathcal{G}_i^\vee}$ is the Weyl group of \mathcal{G}_i^\vee .

The four options are:

$$\begin{aligned} \mathcal{G}_1 &= \text{SO}(2) \times \text{SO}(4) \\ \mathcal{G}_2 &= \text{O}(2) \times \text{SO}(4) \\ \mathcal{G}_3 &= \text{SO}(2) \times \text{O}(4) \\ \mathcal{G}_4 &= \text{O}(2) \times \text{O}(4) \end{aligned} \tag{6.3}$$

Monopole fluxes \mathbf{m}_1	$P_{\text{SO}(2)}(\mathbf{m}_1; t)$
(0)	$\frac{1}{1-t}$
(± 1)	$\frac{1}{1-t}$

Table 21. Data for SO(2) group. \mathbf{m}_1 runs from $-\infty$ to $+\infty$.

Monopole fluxes \mathbf{m}_1	$P_{\text{O}(2)}(\mathbf{m}_1; t)$
(0)	$\frac{1}{1-t^2}$
(+1)	$\frac{1}{1-t}$
(-1)	0

Table 22. Data for O(2) group. \mathbf{m}_1 runs from $-\infty$ to $+\infty$.

Monopole fluxes \mathbf{m}_2	$P_{\text{SO}(4)}(\mathbf{m}_2; t)$
(0, 0)	$\frac{1}{(1-t^2)^2}$
(1, ± 1)	$\frac{1}{(1-t)(1-t^2)}$
(1, 0)	$\frac{1}{(1-t)^2}$
(2, ± 1)	$\frac{1}{(1-t)^2}$

Table 23. Data for SO(4) group. $\mathbf{m}_2 = (m_{2,1}, m_{2,2})$ with $+\infty > m_{2,1} \geq |m_{2,2}| \geq 0$.

Monopole fluxes \mathbf{m}_2	$P_{\text{O}(4)}(\mathbf{m}_2; t)$
(0, 0)	$\frac{1}{(1-t^2)(1-t^4)}$
(1, +1)	$\frac{1}{(1-t)(1-t^2)}$
(1, -1)	0
(1, 0)	$\frac{1}{(1-t)(1-t^2)}$
(2, +1)	$\frac{1}{(1-t)^2}$
(2, -1)	0

Table 24. Data for O(4) group. $\mathbf{m}_2 = (m_{2,1}, m_{2,2})$ with $+\infty > m_{2,1} \geq |m_{2,2}| \geq 0$.

Note that the dressing factors can be obtained with usual multiplication:

$$\begin{aligned}
 P_1(\mathbf{m}_1, \mathbf{m}_2; t) &= P_{\text{SO}(2)}(\mathbf{m}_1; t)P_{\text{SO}(4)}(\mathbf{m}_2; t) \\
 P_2(\mathbf{m}_1, \mathbf{m}_2; t) &= P_{\text{O}(2)}(\mathbf{m}_1; t)P_{\text{SO}(4)}(\mathbf{m}_2; t) \\
 P_3(\mathbf{m}_1, \mathbf{m}_2; t) &= P_{\text{SO}(2)}(\mathbf{m}_1; t)P_{\text{O}(4)}(\mathbf{m}_2; t) \\
 P_4(\mathbf{m}_1, \mathbf{m}_2; t) &= P_{\text{O}(2)}(\mathbf{m}_1; t)P_{\text{O}(4)}(\mathbf{m}_2; t)
 \end{aligned} \tag{6.4}$$

Γ_1 is the maximal lattice of the set, and the remaining ones can be constructed via a

projection by the action of a finite group:

$$\begin{aligned}
 \Gamma_1 & \text{ is the maximal lattice} \\
 \Gamma_2 & = \Gamma_1/\mathbb{Z}_2 \\
 \Gamma_3 & = \Gamma_1/\mathbb{Z}_2 \\
 \Gamma_4 & = \Gamma_1/(\mathbb{Z}_2 \times \mathbb{Z}_2)
 \end{aligned}
 \tag{6.5}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on Γ_1 defines three non equivalent \mathbb{Z}_2 actions on Γ_1 . Let \mathbb{Z}_2^2 have elements $\{e, a, b, ab\}$ where e is the identity, $a^2 = b^2 = e$ and $ab = ba$. Let the \mathbb{Z}_2 generated by a be the one that gives rise to Γ_2 and the \mathbb{Z}_2 generated by b the one that gives rise to Γ_3 . There is a lattice that has not been considered yet, whose dressing factor cannot be obtained as a product of the form of equation (6.4): the projection of Γ_1 by the action of the \mathbb{Z}_2 group generated by ab . Let us denote this lattice as Γ_5 :

$$\Gamma_5 = \Gamma_1/\mathbb{Z}_2
 \tag{6.6}$$

In order to find the lattice Γ_5 it is necessary to find an implementation of the \mathbb{Z}_2 actions generated by a and b that can be extended to obtain the action generated by ab . There are two objects that change after the projection:

- The elements $\mathbf{m} \in \Gamma$.
- The dressing factor $P(\mathbf{m}; t)$.

Let $P_0(\mathbf{m}; t)$ be the dressing factor of the initial lattice Γ_0 before the projection and $P_f(\mathbf{m}; t)$ be the dressing factor of the resulting lattice $\Gamma_f = \Gamma_0/\gamma$, where γ is a finite symmetry group of Γ_0 . In general Γ_f is a sublattice of Γ_0 . However, one can always extend Γ_f to Γ_0 by assigning dressing factors with zero value to the points in Γ_0 which are not in Γ_f . After this it is only necessary to describe the action of the projection on $P_0(\mathbf{m}; t)$ and $P_f(\mathbf{m}; t)$. Let us define the relation between both dressing factors with a sum over the elements in γ :

$$P_f(\mathbf{m}; t) := \frac{1}{|\gamma|} \sum_{g \in \gamma} g [P_0(\mathbf{m}; t)]
 \tag{6.7}$$

The order of γ is denoted by $|\gamma|$. The action of an element of the group $g \in \gamma$ on the original lattice Γ_0 is denoted via $g [P_0(\mathbf{m}; t)]$ which corresponds to a new dressing factor.

First of all, let us see what can be learned from recasting the results of tables 21, 22, 23 and 24 into sums of the form of equation (6.7). Let \mathbf{m}_1 label the weights in the magnetic lattice of $\text{SO}(2)$ and also $\text{O}(2)$. Let the reflexion symmetry of the magnetic lattice of $\text{SO}(2)$ be $\mathbb{Z}_2 = \{e, a\}$ with $a^2 = e$. Then we can write:

$$P_{\text{O}(2)}(\mathbf{m}_1; t) = \frac{1}{2} (e [P_{\text{SO}(2)}(\mathbf{m}_1; t)] + a [P_{\text{SO}(2)}(\mathbf{m}_1; t)])
 \tag{6.8}$$

The action of the identity element leaves the dressing factor unchanged:

$$e [P_{\text{SO}(2)}(\mathbf{m}_1; t)] = P_{\text{SO}(2)}(\mathbf{m}_1; t)
 \tag{6.9}$$

Monopole fluxes \mathbf{m}_1	$a [P_{\text{SO}(2)}(\mathbf{m}_1; t)]$
(0)	$\frac{1}{1+t}$
(1)	$\frac{1}{1-t}$
(-1)	$-\frac{1}{1-t}$

Table 25. Data for $a [P_{\text{SO}(2)}(\mathbf{m}_1; t)]$. \mathbf{m}_1 runs from $-\infty$ to $+\infty$.

Monopole fluxes \mathbf{m}_2	$b [P_{\text{SO}(4)}(\mathbf{m}_2; t)]$
(0, 0)	$\frac{1}{(1-t^4)}$
(1, +1)	$\frac{1}{(1-t)(1-t^2)}$
(1, -1)	$-\frac{1}{(1-t)(1-t^2)}$
(1, 0)	$\frac{1}{(1-t^2)}$
(2, +1)	$\frac{1}{(1-t)^2}$
(2, -1)	$-\frac{1}{(1-t)^2}$

Table 26. Data for $b [P_{\text{SO}(4)}(\mathbf{m}_2; t)]$. $\mathbf{m}_2 = (m_{2,1}, m_{2,2})$ with $+\infty > m_{2,1} \geq |m_{2,2}| \geq 0$.

Since $P_{\text{SO}(2)}(\mathbf{m}_1; t)$ and $P_{\text{O}(2)}(\mathbf{m}_1; t)$ are already known in this case, they can be used in order to obtain the action of the generator $a \in \mathbb{Z}_2$ on the dressing factor:

$$a [P_{\text{SO}(2)}(\mathbf{m}_1; t)] = 2P_{\text{O}(2)}(\mathbf{m}_1; t) - P_{\text{SO}(2)}(\mathbf{m}_1; t) \quad (6.10)$$

Similarly for $\text{SO}(4)$ and $\text{O}(4)$, let the discrete symmetry group be $\mathbb{Z}_2 = \{e, b\}$ with $b^2 = e$ and the magnetic weights be labeled by \mathbf{m}_2 . One can write the sum as:

$$P_{\text{O}(4)}(\mathbf{m}_2; t) = \frac{1}{2} (e [P_{\text{SO}(4)}(\mathbf{m}_2; t)] + b [P_{\text{SO}(4)}(\mathbf{m}_2; t)]) \quad (6.11)$$

Hence:

$$b [P_{\text{SO}(4)}(\mathbf{m}_2; t)] = 2P_{\text{O}(4)}(\mathbf{m}_2; t) - P_{\text{SO}(4)}(\mathbf{m}_2; t) \quad (6.12)$$

The precise values of the *dressing factors* described by equations (6.10) and (6.12) are presented in tables 25 and 26.

Now the dressing factors of lattices $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 can also be rewritten as sums:

$$\begin{aligned}
 P_1(\mathbf{m}_1, \mathbf{m}_2; t) &= P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t) \\
 P_2(\mathbf{m}_1, \mathbf{m}_2; t) &= \frac{1}{2} (P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t) + a [P_{\text{SO}(2)}(\mathbf{m}_1; t)] P_{\text{SO}(4)}(\mathbf{m}_2; t)) \\
 P_3(\mathbf{m}_1, \mathbf{m}_2; t) &= \frac{1}{2} (P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t) + P_{\text{SO}(2)}(\mathbf{m}_1; t) b [P_{\text{SO}(4)}(\mathbf{m}_2; t)]) \quad (6.13) \\
 P_4(\mathbf{m}_1, \mathbf{m}_2; t) &= \frac{1}{4} (P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t) + a [P_{\text{SO}(2)}(\mathbf{m}_1; t)] P_{\text{SO}(4)}(\mathbf{m}_2; t) \\
 &\quad + P_{\text{SO}(2)}(\mathbf{m}_1; t) b [P_{\text{SO}(4)}(\mathbf{m}_2; t)] + a [P_{\text{SO}(2)}(\mathbf{m}_1; t)] b [P_{\text{SO}(4)}(\mathbf{m}_2; t)])
 \end{aligned}$$

Monopole fluxes $(\mathbf{m}_1, \mathbf{m}_2)$	$P_5(\mathbf{m}_1, \mathbf{m}_2; t)$
$(+1, 0, 0)$	$\frac{1}{(1-t)(1-t^2)(1-t^4)}$
$(+1, 1, +1)$	$\frac{1}{(1-t)^2(1-t^2)}$
$(+1, 1, -1)$	0
$(+1, 1, 0)$	$\frac{1}{(1-t)^2(1-t^2)}$
$(+1, 2, +1)$	$\frac{1}{(1-t)^3}$
$(+1, 2, -1)$	0
$(0, 0, 0)$	$\frac{1-t+t^2}{(1-t)(1-t^2)(1-t^4)}$
$(0, 1, +1)$	$\frac{1}{(1-t)(1-t^2)^2}$
$(0, 1, -1)$	$\frac{t}{(1-t)(1-t^2)^2}$
$(0, 1, 0)$	$\frac{1+t^2}{(1-t)(1-t^2)^2}$
$(0, 2, +1)$	$\frac{1}{(1-t)^2(1-t^2)}$
$(0, 2, -1)$	$\frac{t}{(1-t)^2(1-t^2)}$
$(-1, 0, 0)$	$\frac{t^2}{(1-t)(1-t^2)(1-t^4)}$
$(-1, 1, +1)$	0
$(-1, 1, -1)$	$\frac{1}{(1-t)^2(1-t^2)}$
$(-1, 1, 0)$	$\frac{t}{(1-t)^2(1-t^2)}$
$(-1, 2, +1)$	0
$(-1, 2, -1)$	$\frac{1}{(1-t)^3}$

Table 27. Data for Γ_5 . \mathbf{m}_1 runs from $-\infty$ to $+\infty$ and $\mathbf{m}_2 = (m_{2,1}, m_{2,2})$ with $+\infty > m_{2,1} \geq |m_{2,2}| \geq 0$. The weights $(\mathbf{m}_1, \mathbf{m}_2)$ where the dressing factor vanishes correspond with points in the lattice that are removed by the projection.

After writing the dressing factors in the form (6.13) it is clear that there is a sum missing from this set: the sum over only the elements $\mathbb{Z}_2 = \{e, ab\} \subset \mathbb{Z}_2^2$:

$$\begin{aligned}
 P_5(\mathbf{m}_1, \mathbf{m}_2; t) &= \frac{1}{|\mathbb{Z}_2|} \sum_{g \in \{e, ab\}} g [P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t)] \tag{6.14} \\
 &= \frac{1}{2} (P_{\text{SO}(2)}(\mathbf{m}_1; t) P_{\text{SO}(4)}(\mathbf{m}_2; t) + a [P_{\text{SO}(2)}(\mathbf{m}_1; t)] b [P_{\text{SO}(4)}(\mathbf{m}_2; t)])
 \end{aligned}$$

This defines lattice $\Gamma_5 = \Gamma_1 / \{e, ab\}$. The specific values of the dressing factor obtained with equation (6.14) for this lattice are given in table 27. We draw the quiver related to this lattice as the quiver in table 9 labeled with partition $\lambda = [3, 2^2, 1^2]$. The brackets in the quiver denote that the gauge groups $\text{SO}(2)$ and $\text{SO}(4)$ in the first chain Ξ_1 contribute not with the entire magnetic lattice but with a \mathbb{Z}_2 diagonal projection of the same.

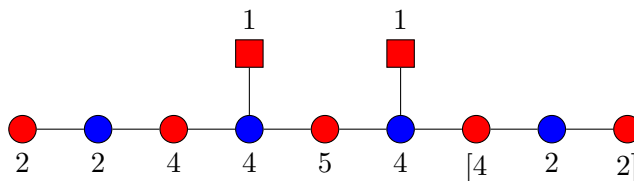


Figure 3. Quiver for the theory $T^{[6,4]}(\text{USp}(10))$ with Coulomb branch $\mathcal{C} = \bar{\mathcal{O}}_{[3,2^2,1^4]} \subset \mathfrak{so}(11)$.

Partition (λ)	Hilbert series	Plethystic logarithm
$[3, 2^2, 1^4]$	$\frac{1 + 25t + 326t^2 + 2433t^3 + 10582t^4 + 26423t^5 + 36250t^6 + 26423t^7 + 10582t^8 + 2433t^9 + 326t^{10} + 25t^{11} + t^{12}}{(1-t)^{28}(1+t)^{-2}}$	$55t - t^2 - 517t^3 + 2706t^4 + 5610t^5 + O(t^6)$

Table 28. Hilbert series and plethystic logarithm for the Coulomb branch of quiver in figure 3.

When this quiver is used on the computation of the Coulomb branch of the corresponding $T^{[4^2]}(\text{SO}(9)^\vee)$ quiver the Hilbert series obtained with the *monopole formula* is indeed the Hilbert series of the closure of nilpotent orbit $\bar{\mathcal{O}}_{[3,2^2,1^2]} \subset \mathfrak{so}(9)$, presented in the 9th row of table 18.

Second example. The Γ_5 lattice can also be used now for partition $\lambda = [3, 2^2, 1^4]$, the next case that requires a diagonal \mathbb{Z}_2 action on $\text{SO}(2)$ and $\text{SO}(4)$. The theory is $T^{[6,4]}(\text{SO}(11)^\vee)$, its quiver is given in figure 3 and its Hilbert series and plethystic logarithm are given in table 28. The Coulomb branch of this quiver is found to be the closure of nilpotent orbit $\mathcal{O}_{[3,2^2,1^4]} \subset \mathfrak{so}(11)$.

6.2 The general case

In general we have the diagonal $\mathbb{Z}_2 = \{e, a_1 a_2 \dots a_m\}$ where $\mathbb{Z}_2 \subset \mathbb{Z}_2^m$ and \mathbb{Z}_2^m is generated by a_k with $k = 1, 2, \dots, m$ acting on a chain Ξ_i of length m with groups $\text{SO}(N_{i+2p})$ and $p = 0, 1, \dots, m-1$. The magnetic lattice of the gauge group $\mathcal{G} = \times_p \text{SO}(N_{i+2p})$, projected by the \mathbb{Z}_2 action, is defined by the dressing factor:

$$P_{(\Xi_i)}(\mathbf{m}_1, \dots, \mathbf{m}_m; t) = \frac{1}{2} \left(\prod_{k=1}^m P_{\text{SO}(N_{i+2k-2})}(\mathbf{m}_k; t) + \prod_{k=1}^m a_k [P_{\text{SO}(N_{i+2k-2})}(\mathbf{m}_k; t)] \right) \quad (6.15)$$

where $(\mathbf{m}_1, \dots, \mathbf{m}_m)$ run over the magnetic weights in the lattice $\Gamma_{\mathcal{G}^\vee}^*/\mathcal{W}_{\mathcal{G}^\vee}$ before the projection. $a_k [P_{\text{SO}(N_{i+2k-2})}(\mathbf{m}_k; t)]$ is defined in terms of the $\text{SO}(N_{i+2k-2})$ and the $\text{O}(N_{i+2k-2})$ dressing factors:

$$a_k [P_{\text{SO}(N_{i+2k-2})}(\mathbf{m}_k; t)] := 2P_{\text{O}(N_{i+2k-2})}(\mathbf{m}_k; t) - P_{\text{SO}(N_{i+2k-2})}(\mathbf{m}_k; t) \quad (6.16)$$

where $P_{\text{O}(N_{i+2k-2})}(\mathbf{m}_k; t)$ is extended over the weights that are in the magnetic lattice of $\text{SO}(N_{i+2k-2})$ but not in the magnetic lattice of $\text{O}(N_{i+2k-2})$ by defining it to be zero at those weights.

7 Conclusion and outlook

The result of this project is an answer to the question *Are the gauge symmetry factors $O(2N_i)$ or $SO(2N_i)$ on the $3d \mathcal{N} = 4$ orthosymplectic quiver of $T^\sigma(\mathrm{SO}(n)^\vee)$ theories?* The answer is given by the Lusztig's Canonical Quotient $\bar{A}(\mathcal{O}_\lambda)$ where $\lambda = d_{VB}(\sigma)$. If $\bar{A}(\mathcal{O}_\lambda)$ is the trivial group all gauge symmetry factors are $SO(2N_i)$. For a group of the form $\bar{A}(\mathcal{O}_\lambda) = \mathbb{Z}_2^k$ the magnetic lattice of the $SO(2N_i)$ gauge factors is projected by k different \mathbb{Z}_2 actions, in the way described in section 5.

Sometimes a single \mathbb{Z}_2 group acts diagonally on multiple $SO(2N_i)$ magnetic lattices. The way of implementing this action is developed here for the first time, section 6. This process widens the current reach of the *monopole formula* and opens the door to many new exciting computations in the field of $3d \mathcal{N} = 4$ gauge theories.

The results of this note can be complemented by analyzing the nature of the Type IIB brane configurations that give rise to the orthosymplectic quivers discussed. This analysis is currently under development and can show different brane configurations corresponding to different choices of $SO/O(N_i)$ gauge group factors.

So far, all the moduli spaces that have been discussed here are closures of special nilpotent orbits. An interesting direction in which this project could be further developed is the analysis of quivers with different choices of $SO/O(N_i)$ gauge factors such that their Coulomb branches are no longer closures of nilpotent orbits. The study of the geometry and the physics behind these moduli spaces is a very rich and challenging problem that remains to be addressed.

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