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Matrix model of Chern–Simons matter theories beyond the spherical limit

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A class of matrix models that arises as a partition function in U(N) Chern–Simons matter theories on the three-sphere is investigated. Employing the standard technique of $1/N$ expansion we solve the system beyond the planar limit. In particular, we study a case where the matrix model potential has $1/N$ correction and give a general solution thereof up to the order of $1/N^2$. We confirm that the general solution correctly reproduces the past exact result of the free energy up to the order in the case of pure Chern–Simons theory. We also apply to the matrix model of $\mathcal{N} = 2$ Chern–Simons theory with arbitrary numbers of fundamental chiral multiplets and anti-fundamental ones, which does not admit Fermi gas analysis in general.

Subject Index B04, B35, B83

1. Introduction

Recent progress in supersymmetric Chern–Simons matter theories has been made on the basis of the exact result by means of supersymmetric localization. This technique allows one to compute the partition function of supersymmetric theories exactly by the steepest descent method, which reduces the path integral calculation to that of a certain matrix model [1]. This calculation was done generically on $\mathbf{S}^3$ [2–4] (see also Refs. [5,6]) and on $\mathbf{S}^2 \times \mathbf{S}^1$—called the superconformal index [7,8] (see also Refs. [9–11]). These exact results were in precise agreement with the prediction from AdS_4/CFT_3 duality [12,13] (see also Refs. [14,15]). See Ref. [16] for a review and further references.

On the other hand, progress in non-supersymmetric Chern–Simons matter theories has also been made not relying on the localization technique but on the $1/N$ expansion technique by restricting the class of matter fields to vector fields. It was conjectured that such a system is exactly soluble in the 't Hooft large-N limit [17,18]. The thermal partition function in Chern–Simons vector models on $\mathbf{S}^2 \times \mathbf{S}^1$ was determined exactly in the leading order of the $1/N$ expansion near the critical high temperature [19] (see also Refs. [20,21]), which was used to show the three-dimensional bosonization duality [22–24].

In contrast, the exact large-N analysis of the three-sphere partition function for any non-supersymmetric Chern–Simons matter theory has not been performed due to its technical difficulty. So far, analysis of the three-sphere partition function has been done perturbatively near the weak coupling limit of the Chern–Simons coupling constant [25] to confirm that the system obeys the F-theorem [3]. Perturbative analysis is, however, not enough to provide evidence for duality, and exact large-N analysis is keenly awaited.

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In this situation we change gears to study a class of matrix models that is to be obtained as the three-sphere partition function of Chern–Simons matter theories in order to capture a generic feature of such a class of matrix models toward the bigger goal of showing bosonization duality on the three-sphere. We are also interested in analyzing such a class of matrix models beyond the planar limit because the bosonization duality is expected to hold at the subleading order in the $1/N$ expansion [21]. (See Refs. [26–28] for recent arguments.)

To illustrate the kind of matrix models that are to be studied, let us consider the partition function of generic $\mathrm{U}(N)_k$ Chern–Simons matter theories on the three-sphere with unit radius:

$$Z = \int \mathcal{D}A\mathcal{D}\Phi \exp\left\{-\frac{ik}{4\pi}\int_{\mathbf{S}^3}\left(A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right) - S[\Phi, A]\right\}, \tag{1}$$

where Φ denotes all the matter fields collectively and $S[\Phi, A]$ is the action for the matter fields. This may be computed perturbatively as follows (see Ref. [29] for details). We first expand the gauge field by the vector spherical harmonics

$$A_\mu(x) = \sum_{s\in\frac{1}{2}\mathbf{N}}\left(\sum_{\substack{|l|\le s\\ |r|\le s+1}} a^{ss+1}_{l\,r}Y^{ss+1}_{l\,r}{}_\mu(x) + \sum_{\substack{|l|\le s+1\\ |r|\le s}} a^{s+1s}_{l\quad r}Y^{s+1s}_{l\quad r\mu}(x) + \sum_{|l|,|r|\le s} a^{s,s}_{l,r}Y^{ss}_{l\,r\mu}(x)\right). \tag{2}$$

We take the Lorenz gauge $\nabla_\mu A^\mu = 0$, which kills the modes $a^{s,s}_{l,r}$ with $s > 0$. The residual gauge can kill the mode $a^{0,0}_{0,0}$ except for its Cartan part, which we denote by σ. We expand the matter fields in a similar manner. Taking into account the Faddeev–Popov determinant, we integrate out all the massive modes such as $a^{s,s+1}_{l,r}, a^{s+1,s}_{l,r}$ and all the modes coming from the matter fields, which are massive on the three-sphere. Then the partition function reduces to the finite-dimensional integration of the effective action over σ:

$$Z = \int d^N\sigma \exp\left\{-i\frac{k}{4\pi}\sum_{s=1}^{N}\sigma_s^2 + \cdots\right\}, \tag{3}$$

where the ellipsis is some function of σ_s generated by integrating out the massive modes. When the matter fields vanish, the effective action can be exactly computed by, for example, supersymmetric localization or cohomological localization [30] by adding the auxiliary fields to complete the gauge field in the $\mathcal{N} = 2$ vector multiplet. The result is [31]

$$Z \sim \int_{\mathbf{R}^N} d^N\sigma \exp\left\{-i\frac{k}{4\pi}\sum_{s=1}^{N}\sigma_s^2 + \sum_{t\ne s}^{N}\log 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right)\right\}. \tag{4}$$

Then, by denoting the correction of the matter fields to the effective action by $V[\sigma]$, the partition function is such that

$$Z \sim \int_{\mathbf{R}^N} d^N\sigma \prod_{t\ne s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right)e^{-V[\sigma]}. \tag{5}$$

In this note we analyze this class of matrix models by restricting the form of the potential to consist of single trace operators: $V[\sigma] = N\sum_{s=1}^{N} W_\sigma(\sigma_s)$.[1] The goal of this paper is to solve this class

[1] This restriction corresponds to the representation of the matter fields excluding higher-dimensional representations such as the adjoint one.

of matrix models incorporating the standard technique of $1/N$ expansion developed in the study of ordinary Hermitian matrix models [32] beyond the spherical limit [33,34]. See Refs. [35,36] for reviews and further references.

The rest of this paper is written as follows. In Sect. 2 we perform some preliminary analysis of the matrix models. In Sect. 3 we derive the loop equation for this class of matrix models. In Sect. 4 we solve the loop equation by using the $1/N$ expansion. We give a general solution for the planar limit (Sect. 4.1) and for the genus one (Sect. 4.3). Then we apply this solution to a few examples in Sect. 5. We first apply it to pure Chern–Simons theory and compare with the known exact result to test the validity of the presented framework (Sect. 5.1). We then apply it to $\mathcal{N}=2$ Chern–Simons theory with arbitrary numbers of fundamental and anti-fundamental chiral multiplets (Sect. 5.2). Sect. 6 is devoted to discussion and future direction. In the appendix we give a brief review of the partition function of pure Chern–Simons theory on the three-sphere in order for this paper to be self-contained.

2. Matrix model of Chern–Simons matter theories

Throughout this paper we investigate a class of matrix models such that

$$Z = \mathfrak{N} \int_{\mathbf{R}^N} d^N\sigma \prod_{t\neq s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right) \exp\left\{-N\sum_{s=1}^{N} W_\sigma(\sigma_s)\right\}, \tag{6}$$

where $\mathfrak{N}$ is a normalized constant and $W_\sigma(\sigma)$ is a non-singular function of σ, which can be determined at least perturbatively by integrating out the massive modes for the original Chern–Simons matter theory. When the original theory has $\mathcal{N}=2$ supersymmetry, the matrix model potential can be determined exactly by using the localization method [2]. For example, for $\mathcal{N}=2$ $\mathrm{U}(N)_k$ Chern–Simons theory with N_F fundamental chiral multiplets with the canonical R-charge, the partition function is of the form [2,3]

$$Z_{\Box}^{\mathcal{N}=2} = \mathfrak{N} \int_{\mathbf{R}^N} d^N\sigma \prod_{t\neq s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right) \exp\left\{\sum_{s=1}^{N} -\left(\frac{ik}{4\pi}\sigma_s^2+\frac{1}{2}i\zeta\sigma_s\right)+N_F\,\ell\left(\frac{-i\sigma_s}{2\pi}+\frac{1}{2}\right)\right\}, \tag{7}$$

where $\ell(x) = -x\log(1-e^{2\pi ix})+\frac{i}{2}(\pi x^2+\frac{1}{\pi}\mathrm{Li}_2(e^{2\pi ix}))-\frac{i\pi}{12}$, and ζ is the FI parameter, which is equivalent to the real mass parameter. Adding the same number of anti-fundamental chiral multiplets to this system, the partition function becomes

$$Z_{\Box,\bar{\Box}}^{\mathcal{N}=2} = \mathfrak{N} \int_{\mathbf{R}^N} d^N\sigma \prod_{t\neq s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right) \exp\left\{\sum_{s=1}^{N} -\left(\frac{ik}{4\pi}\sigma_s^2+\frac{1}{2}i\zeta\sigma_s+N_F\log\cosh\frac{\sigma_s}{2}\right)\right\}. \tag{8}$$

See Refs. [37,38] for detailed analysis of this type of matrix model.

For explicit computation we write the form of the potential in Eq. (6) as

$$W_\sigma(\sigma) = \frac{1}{2\widetilde{\lambda}}\sigma^2 + \sum_{p=0}^{\infty} t_p e^{\sigma p} + \delta W_\sigma(\sigma), \tag{9}$$

where $\widetilde{\lambda}, t_p$ are parameters. In this parametrization the partition function of pure Chern–Simons theory on $\mathbf{S}^3$, which we briefly review in the appendix, is given by

$$\widetilde{\lambda} = -2\pi i \frac{N}{k}, \quad \mathfrak{N} = \frac{(-)^{\frac{N(N-1)}{2}} \exp\left\{\frac{-\pi(N-1)N(N+1)}{6ik}\right\} i^{\frac{N^2}{2}}}{(2\pi)^N N!}, \quad t_p = 0, \quad \delta W_\sigma = i\frac{\zeta}{2N}. \tag{10}$$

See Eq. (A4) for the normalization. The first example in Eq. (7) is formally given by

$$\begin{aligned} &\frac{1}{\widetilde{\lambda}} = \frac{ik}{2\pi N} + \frac{iN_F}{4\pi N}, \quad t_0 = -\frac{N_F}{N}\frac{i\pi}{24}, \quad t_p = \frac{iN_F}{2N}\frac{(-)^{p-1}}{p^2}, \\ &\delta W_\sigma(\sigma) = \sum_{p=0}^{\infty} u_p \sigma e^{\sigma p}, \quad u_0 = i\frac{\zeta}{2N} - \frac{N_F}{4N}, \quad u_p = \frac{iN_F}{2\pi N}\frac{(-)^p}{p}, \end{aligned} \tag{11}$$

where $p \geq 1$.[2] The second example in Eq. (8) is

$$\widetilde{\lambda} = -2\pi i \frac{N}{k}, \quad t_0 = \frac{N_F}{N}\log 2, \quad t_p = \frac{N_F}{N}\frac{(-)^p}{p} \quad (p \geq 1), \quad \delta W_\sigma = \frac{i}{2N}\zeta + \frac{N_F}{2N}. \tag{12}$$

The matrix model of Eq. (6) can be recast in the same form as Hermitian matrix models with positive eigenvalues. By changing the integration variables so that $\phi_s = e^{\sigma_s}$, the partition function becomes

$$Z = \mathfrak{N} \int_{\mathbf{R}_+^N} d^N\phi \prod_{t\neq s}^{N} (\phi_s - \phi_t) \exp\left\{-N \sum_{s=1}^{N} W(\phi_s)\right\}, \tag{13}$$

where $\mathbf{R}_+$ represents the positive real axis and

$$W(\phi_s) = \frac{1}{2\widetilde{\lambda}}(\log\phi_s)^2 + \log\phi_s + \sum_{p=0}^{\infty} t_p \phi_s^p + \delta W(\phi_s), \tag{14}$$

where $\delta W(\phi_s)$ is analytic on the positive real axis. Note that the matrix model potential has the logarithmic cut on the negative real axis.

As a result, the partition function can be written by using a positive definite Hermitian matrix Φ as

$$Z \propto \int \mathcal{D}\Phi e^{-N \mathrm{Tr} W(\Phi)}, \tag{15}$$

with $W(\Phi) = \frac{1}{2\widetilde{\lambda}}(\log\Phi)^2 + \log\Phi + \sum_{p=0}^{\infty} t_p \Phi^p + \delta W(\Phi)$. This suggests that this class of matrix model can be analyzed by using the standard technique employed in ordinary Hermitian matrix models. In what follows we show that the free energy and correlators of some sector can be determined in order in the $1/N$ expansion.

3. Loop equation

There is a well-known method to determine the free energy in matrix models by using the so-called resolvent, which is in the current situation defined by the vacuum expectation value of the generating

[2] This expansion may not be useful for practical computation, though.

function of regular single trace operators:

$$\omega(z) := \frac{1}{N}\sum_{s=1}^{N}\left\langle\frac{1}{z-\phi_s}\right\rangle = \frac{1}{N}\mathrm{Tr}\left\langle\frac{1}{z-\Phi}\right\rangle = \frac{1}{N}\sum_{p\geq 0}\frac{\langle \mathrm{Tr}\Phi^p\rangle}{z^{p+1}}. \tag{16}$$

We remark that the resolvent is well defined around infinity and formally behaves as $\omega(z) \sim \frac{1}{z}$ in the vicinity of infinity, though this asymptotic behavior is not guaranteed due to the fact that the potential in Eq. (14) has the logarithmic cut, which ends at infinity. This suggests that the behavior of the resolvent around infinity generally gets a logarithmic correction. Still, we can expect that there exists a limit approaching infinity such that the resolvent behaves as $\omega(z) \sim \frac{1}{z}$ on a certain patch. This will be important in determining the resolvent later.

Once the resolvent is determined, the coupling dependence on $\{t_p\}$ of the free energy given by $F = -\log Z$ is determined by

$$\frac{d}{dT_z}F = N^2\left(\frac{1}{z} - \omega(z)\right), \tag{17}$$

where $\frac{d}{dT_z} = \sum_{p\geq 1}\frac{-1}{z^{p+1}}\frac{\partial}{\partial t_p}$. This can be seen from the fact that $\frac{\partial F}{\partial t_p} = N\,\langle \mathrm{Tr}\Phi^p\rangle$.

In order to determine the resolvent systematically, we first derive the Schwinger–Dyson equation for a generic operator $\mathcal{O}[\phi]$. The vacuum expectation value of $\mathcal{O}[\phi]$ is defined by

$$\langle\mathcal{O}[\phi]\rangle = \frac{\mathfrak{N}}{Z}\int_{\mathbf{R}_+^N} d^N\phi\,\mathcal{O}[\phi]\left(\prod_{t\neq s}^{N}(\phi_s-\phi_t)\exp\left\{-N\sum_{s=1}^{N}W(\phi_s)\right\}\right). \tag{18}$$

Consider a one-to-one transformation on $\mathbf{R}_+$ denoted by $\phi_s \to \phi_s'$. Then we obtain the identity such that

$$\frac{\mathfrak{N}}{Z}\int_{\mathbf{R}_+^N} d^N\phi\,\mathcal{O}[\phi]\left(\prod_{t\neq s}^{N}(\phi_s-\phi_t)\exp\left\{-N\sum_{s=1}^{N}W(\phi_s)\right\}\right)$$
$$= \frac{\mathfrak{N}}{Z}\int_{\mathbf{R}_+^N} d^N\phi'\,\mathcal{O}[\phi']\left(\prod_{t\neq s}^{N}(\phi_s'-\phi_t')\exp\left\{-N\sum_{s=1}^{N}W(\phi_s')\right\}\right).$$

Suppose the (infinitesimal) transformation $\phi_s' = \phi_s + a\delta\phi_s$. Expanding the right-hand side in terms of a, we find that the zeroth-order term cancels the left-hand side and the equation at the linear order gives the Schwinger–Dyson equation

$$\left\langle\sum_{s=1}^{N}\left(\mathcal{O}[\phi]\frac{\partial\delta\phi_s}{\partial\phi_s} + \frac{\partial\mathcal{O}[\phi]}{\partial\phi_s}\delta\phi_s\right) + 2\sum_{s>t}\frac{\mathcal{O}[\phi]}{\phi_s-\phi_t}(\delta\phi_s-\delta\phi_t) - \sum_{s=1}^{N}N\mathcal{O}[\phi]\frac{\partial W(\phi_s)}{\partial\phi_s}\delta\phi_s\right\rangle = 0. \tag{19}$$

To derive the loop equation from the Schwinger–Dyson equation let us choose $\mathcal{O} = 1$, $\delta\phi_s = \frac{\phi_s}{z-\phi_s}$. Then the transformation $\phi_s \to \phi_s'$ becomes one-to-one on $\mathbf{R}_+$, because $\delta\phi_s|_{\phi_s=0} = 0$, and $a\frac{\partial\delta\phi_s}{\partial\phi_s} = \frac{az}{(z-\phi_s)^2} > 0$ for $az > 0$. Then the left-hand side in the Schwinger–Dyson equation in

Eq. (19) is computed as

$$z\left\langle\sum_{s=1}^{N}\frac{1}{(z-\phi_s)^2}+2\sum_{s>t}\frac{1}{(z-\phi_s)(z-\phi_t)}-\sum_{s=1}^{N}N\frac{\partial W(\phi_s)}{\partial\phi_s}\frac{1}{z-\phi_s}\right\rangle+\left\langle\sum_{s=1}^{N}N\frac{\partial W(\phi_s)}{\partial\phi_s}\right\rangle.$$

The first two terms are computed as

$$\left\langle\sum_{s=1}^{N}\frac{1}{(z-\phi_s)^2}+2\sum_{s>t}\frac{1}{(z-\phi_s)(z-\phi_t)}\right\rangle=\sum_{s,t=1}^{N}\left\langle\frac{1}{(z-\phi_s)(z-\phi_t)}\right\rangle=\frac{d}{dT_z}\omega(z)+N^2\omega(z)^2.$$

The third term is

$$\left\langle -N\sum_{s=1}^{N}\frac{W'(\phi_s)}{z-\phi_s}\right\rangle=\left\langle -N\sum_{s=1}^{N}\int_{\mathbf{R}_+}dx\delta(x-\phi_s)\frac{W'(x)}{z-x}\right\rangle=-N^2\int_{\mathbf{R}_+}dx\rho(x)\frac{W'(x)}{z-x},\tag{20}$$

where we define the density function

$$\rho(x):=\frac{1}{N}\mathrm{Tr}\,\langle\delta(x-\Phi)\rangle=\frac{1}{N}\sum_{s=1}^{N}\langle\delta(x-\phi_s)\rangle.\tag{21}$$

Hereafter we assume that z is outside the support of the density function in order to exclude the case where Eq. (20) is divergent. The density function satisfies $\int_{\mathbf{R}_+}dx\rho(x)=1$, and can be computed by evaluating the discontinuity of the resolvent across the real axis. Indeed, by using the formula

$$\frac{1}{x\mp i\epsilon}=\mathcal{P}\frac{1}{x}\pm\pi i\delta(x),\tag{22}$$

where x is a real number, ϵ is an infinitely small positive number, and $\mathcal{P}$ denotes the principal value, the discontinuity of the resolvent between $x\pm i\epsilon$ is computed as

$$\omega(x-i\epsilon)-\omega(x+i\epsilon)=2\pi i\rho(x).\tag{23}$$

By using this relation, Eq. (20) can be rewritten as

$$\left\langle -N\sum_{s=1}^{N}\frac{W'(\phi_s)}{z-\phi_s}\right\rangle=-N^2\oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\frac{W'(w)}{z-w}\omega(w),\tag{24}$$

where $\mathcal{C}_{\mathbf{R}_+}$ denotes a circle encircling $\mathbf{R}_+$ counterclockwise. The fourth term vanishes because

$$\begin{aligned}\left\langle\sum_n NW'(\phi_n)\right\rangle&=\frac{\mathfrak{N}}{Z}\int_{\mathbf{R}_+^N}d^N\phi\sum_n NW'(\phi_n)\prod_{s\neq t}(\phi_s-\phi_t)\exp\left\{-\sum_{s=1}^{N}NW(\phi_s)\right\}\\&=\frac{\mathfrak{N}}{Z}\sum_n\int_{\mathbf{R}_+^N}d^N\phi\prod_{s\neq t}(\phi_s-\phi_t)\left(-\frac{\partial}{\partial\phi_n}\exp\left\{-\sum_{s=1}^{N}NW(\phi_s)\right\}\right)\\&=\frac{\mathfrak{N}}{Z}\sum_n\int_{\mathbf{R}_+^N}d^N\phi\left(\frac{\partial}{\partial\phi_n}\prod_{s\neq t}(\phi_s-\phi_t)\exp\left\{-\sum_{s=1}^{N}NW(\phi_s)\right\}\right)\\&=\sum_n\left\langle\sum_{t\neq n}\frac{2}{\phi_n-\phi_t}\right\rangle=0.\end{aligned}\tag{25}$$

Collecting these, we obtain the loop equation

$$\omega(z)^2 - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W'(w)}{z-w}\omega(w) + \frac{1}{N^2}\frac{d}{dT_z}\omega(z) = 0. \tag{26}$$

Note that this is of the same form as that of the ordinary matrix models except for the integration region.

Once the resolvent is determined by the loop equation, so are the density function by Eq. (23) and the coupling dependence on $\{t_p\}$ of the free energy by Eq. (17), and similarly for the other coupling dependence. For example, when $\delta W(\Phi) = \sum_{p\geq 0} u_p \Phi^p \log \Phi$, the other coupling dependence on $\{u_p\}$ of the free energy is determined by

$$\frac{d}{dU_z}F = -N^2 \upsilon(z), \tag{27}$$

where $\frac{d}{dU_z} = \sum_{p\geq 0} \frac{-1}{z^{p+1}} \frac{\partial}{\partial u_p}$ and $\upsilon(z)$ is the vacuum expectation value of the generating function of singlet operators of the form $\mathrm{Tr}(\Phi^p \log \Phi)$:

$$\upsilon(z) := \frac{1}{N}\mathrm{Tr}\left\langle \frac{\log \Phi}{z-\Phi}\right\rangle = \frac{1}{N}\sum_{s=1}^{N}\left\langle \frac{\log \phi_s}{z-\phi_s}\right\rangle, \tag{28}$$

which is computed by using the resolvent $\omega(z)$ as

$$\upsilon(z) = \int_{\mathbf{R}_+} dx \rho(x) \frac{\log x}{z-x} = \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i}\omega(w)\frac{\log w}{z-w}. \tag{29}$$

Similarly, the $\widetilde{\lambda}$ dependence of the free energy is determined as

$$\begin{aligned}\frac{\partial F}{\partial \widetilde{\lambda}} &= -\frac{\partial \log \mathfrak{N}}{\partial \widetilde{\lambda}} - \frac{N}{2\widetilde{\lambda}^2}\left\langle \sum_{s=1}^{N} (\log \phi_s)^2 \right\rangle = -\frac{\partial \log \mathfrak{N}}{\partial \widetilde{\lambda}} - \frac{N^2}{2\widetilde{\lambda}^2}\int_{\mathbf{R}_+} dx \rho(x)(\log x)^2 \\ &= -\frac{\partial \log \mathfrak{N}}{\partial \widetilde{\lambda}} - \frac{N^2}{2\widetilde{\lambda}^2}\oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i}\omega(w)(\log w)^2. \end{aligned} \tag{30}$$

Acting $\frac{d}{dT_z}$ and $\frac{d}{dU_z}$ on the free energy gives correlators of singlet operators such that

$$\left(\prod_{l=1}^{m} \frac{d}{dU_{w_l}}\right)\left(\prod_{k=1}^{n} \frac{d}{dT_{z_k}}\right)(-F) = N^{n+m}\left\langle \prod_{l=1}^{m} \mathrm{Tr}\left(\frac{\log \Phi}{w_l - \Phi}\right)\prod_{k=1}^{n}\mathrm{Tr}\left(\frac{1}{z_k-\Phi}\right)\right\rangle_{\mathrm{conn}} \tag{31}$$

with $n > 1$, where the subscript conn means the connected part of the correlator. It is also possible to compute correlators including some number of the operator $\mathrm{Tr}(\log \Phi)^2$ by differentiating the free energy several times with respect to $\widetilde{\lambda}$.

We remark that it is not guaranteed and has to be confirmed that correlators computed in this way agree with those computed from the original theory by using the path integral.[3] In other words, the potential of the matrix model generally depends on operators inserted in the path integral. This can easily be seen by considering a partition function of some supersymmetric theory computed by using the localization method. Since correlators of non-supersymmetric operators cannot be computed by

[3] The author would like to thank S. Sugimoto for discussion on this point.

the exact method, the potential of the matrix model cannot be reused to compute correlators of non-supersymmetric operators. In this context the matrix model potential is available when operators inserted in the path integral or parameters deforming the original theory maintain supersymmetry.

4. Solution of the loop equation

In this section we solve the loop equation in Eq. (26) in the $1/N$ expansion. The analysis will depend on the large-N behavior of the potential. Generically it can be written as

$$W(\phi) = W_0(\phi) + \frac{1}{N} W_1(\phi), \tag{32}$$

where $W_0(\phi), W_1(\phi)$ do not depend on N. For explicit calculation maintaining a certain extent of the generality we study the case where the potential is given by Eq. (14) with $\widetilde{\lambda}, t_p$ of order one:

$$W_0(\phi) = \frac{1}{2\widetilde{\lambda}} (\log \phi)^2 + \log \phi + \sum_{p=0}^{\infty} t_p \phi^p + \cdots . \tag{33}$$

In the examples of supersymmetric Chern–Simons theory in Sect. 2, this case corresponds to the 't Hooft limit with the number of flavors N_F of order N. Accordingly, the consistent $1/N$ expansion of the resolvent will be such that

$$\omega(z) = \sum_{g=0}^{\infty} (N^{-2g} \omega_g(z) + N^{-2g-1} \omega_{g+\frac{1}{2}}(z)) = \sum_{\bar{g} \in \frac{1}{2}\mathbf{N}} N^{-2\bar{g}} \omega_{\bar{g}}(z). \tag{34}$$

Plugging this into the loop equation and expanding with respect to $1/N$, the loop equation is decomposed as follows. For $g = 0$,

$$\omega_0^2(z) = \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_0'(w)\omega_0(w)}{z - w}, \tag{35}$$

$$\hat{K}\omega_{\frac{1}{2}}(z) = - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_0(w)}{z - w}, \tag{36}$$

which we call the genus-zero and genus-half loop equations respectively for convenience, and for $g \geq 1$,

$$\begin{aligned} \hat{K}\omega_g(z) &= \sum_{g'=1}^{g-1} \omega_{g'}(z)\omega_{g-g'}(z) \\ &\quad + \sum_{g'=0}^{g-1} \omega_{g'+\frac{1}{2}}(z)\omega_{g-g'-\frac{1}{2}}(z) - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_{g-\frac{1}{2}}(w)}{z - w} + \frac{d}{dT_z}\omega_{g-1}(z), \\ \hat{K}\omega_{g+\frac{1}{2}}(z) &= 2\omega_g(z)\omega_{\frac{1}{2}}(z) + \sum_{g'=1}^{g-1} 2\omega_{g'}(z)\omega_{g-g'+\frac{1}{2}}(z) - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_g(w)}{z - w} + \frac{d}{dT_z}\omega_{g-\frac{1}{2}}(z), \end{aligned} \tag{37}$$

where we define

$$\hat{K}f(z) := \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_0'(w)}{z - w} f(w) - 2\omega_0(z)f(z). \tag{38}$$

From these equations, $\omega_{\bar{g}}(z)$ can be determined in order from $\bar{g}=0$. Once the resolvent is determined at the order $\bar{g}$ in the $1/N$ expansion, so is the density function from Eq. (23) as

$$\rho_{\bar{g}}(x)=\frac{\omega_{\bar{g}}(x-i\epsilon)-\omega_{\bar{g}}(x+i\epsilon)}{2\pi i}, \tag{39}$$

where $\rho(z)=\sum_{g=0}^{\infty}(N^{-2g}\rho_g(z)+N^{-2g-1}\rho_{g+\frac{1}{2}}(z))=\sum_{\bar{g}\in\frac{1}{2}\mathbf{N}}N^{-2\bar{g}}\rho_{\bar{g}}(z)$. The coupling dependence of the free energy on t_p is determined from Eq. (17) as

$$\frac{d}{dT_z}F_0=\frac{1}{z}-\omega_0(z),\quad \frac{d}{dT_z}F_{\bar{g}}=-\omega_{\bar{g}}(z), \tag{40}$$

where $\bar{g}\geq\frac{1}{2}$ and $F=\sum_{g=0}^{\infty}(N^{2-2g}F_g+N^{1-2g}F_{g+\frac{1}{2}})=\sum_{\bar{g}\in\frac{1}{2}\mathbf{N}}N^{2-2\bar{g}}F_{\bar{g}}$.

4.1. Planar solution

Let us solve the planar loop equation of Eq. (35). First we show that the planar loop equation contains the saddle point equation of the starting matrix model in the large-N limit. For this purpose we compute the discontinuity of both sides in Eq. (35) between $x-i\epsilon$ and $x+i\epsilon$. The discontinuity of the left-hand side is

$$(\omega_0(x-i\epsilon)-\omega_0(x+i\epsilon))(\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon))=2\pi i\rho_0(x)(\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon)),$$

where we used Eq. (39). That of the right-hand side is

$$\int_{\mathbf{R}_+}dyW_0'(y)\left(\frac{\rho_0(y)}{x-i\epsilon-y}-\frac{\rho_0(y)}{x+i\epsilon-y}\right)=\int_{\mathbf{R}_+}dyW_0'(y)\rho_0(y)2\pi i\delta(x-y)=W_0'(x)2\pi i\rho_0(x).$$

Therefore we obtain

$$\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon)=W_0'(x), \tag{41}$$

with x in the support of the density function in the leading order of the $1/N$ expansion. This is the same as the saddle point equation derived from the starting matrix model in Eq. (13) in the large-N limit.

Suppose that the support of the density function consists of s distinct connected intervals, $\mathrm{supp}(\rho_0)=\cup_{i=1}^{s}[a_{2i-1},a_{2i}]$, where $0<a_1<\cdots<a_{2s}$. Taking account of the fact that the loop planar equation in Eq. (35) is quadratic, we make each interval correspond to a square root cut of the solution. Under this ansatz we solve Eq. (41). Let us consider a trial function $H(z)$ that sees the deviation of the resolvent from the s-cut square root function $h(z)=\sqrt{\prod_{i=1}^{2s}(z-a_i)}$:

$$\omega_0(z)=h(z)H(z). \tag{42}$$

As mentioned in the previous section, we solve the loop equation so that the resolvent behaves as $\omega_0(z)\sim\frac{1}{z}$ in the limit approaching infinity. This suggests that the trial function behaves as $H(z)\sim\frac{1}{z^{s+1}}$ up to the signature, and thus is analytic around infinity. Therefore, using the Cauchy theorem we find that[4]

$$\oint_{\mathcal{C}_\infty} \frac{dw}{2\pi i} \frac{H(w)}{w-z} = 0, \tag{43}$$

where z is a complex number outside $\mathrm{supp}(\rho_0)$ and $\mathcal{C}_\infty$ is an infinitely large circle. Assuming further that the trial function is analytic except for the support of the leading density function, we can compute the left-hand side by deforming the contour to the non-analytic region:

$$H(z) + \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{H(w)}{w-z} = H(z) + \fint_{\mathrm{supp}(\rho_0)} \frac{dy}{2\pi i} \frac{W_0'(y)}{(y-z)h(y)}, \tag{44}$$

where we used Eq. (41) in advance and $\mathcal{C}_{\mathrm{supp}(\rho_0)}$ denotes a circle encircling the intervals $\mathrm{supp}(\rho_0)$ counterclockwise. Therefore the trial function is determined as

$$H(z) = -\fint_{\mathrm{supp}(\rho_0)} \frac{dy}{2\pi i} \frac{W_0'(y)}{(y-z)h(y)} = -\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{W_0'(w)}{(w-z)h(w)} \frac{1}{2}, \tag{45}$$

so is the planar resolvent:

$$\omega_0(z) = \frac{-h(z)}{2} \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{W_0'(w)}{(w-z)h(w)}. \tag{46}$$

Then the planar density function is computed from Eq. (39) as

$$\rho_0(x) = \frac{h(x)}{\pi i} \fint_{\mathrm{supp}(\rho_0)} \frac{dy}{2\pi i} \frac{W_0'(y)}{(x-y)h(y)}, \tag{47}$$

with $x \in \mathrm{supp}(\rho_0) = \cup_{i=1}^{s}[a_{2i-1}, a_{2i}]$ and $h(x) := h(x - i\epsilon)$ for $x \in \mathbf{R}_+$.

The endpoints of the cuts are determined in the following way. Assume that the solution obtained above behaves asymptotically as $\omega_0(z) = \frac{1}{z} + \cdots$ approaching infinity. This is satisfied if and only if

$$\frac{1}{2} \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{w^k W_0'(w)}{h(w)} = \pm\delta_{k,s} \qquad \forall k = 0, \dots, s, \tag{48}$$

where the signature is chosen suitably. These give $s+1$ constraints for $2s$ endpoints of the cuts, which is not sufficient unless $s = 1$. For the $s \geq 2$ case, the residual conditions are provided by stability against the tunneling of eigenvalues between different cuts [39]. We demonstrate the residual condition following Ref. [40]. First we write the total matrix model potential in terms of the density function in the large-N limit:

$$\frac{V_{\mathrm{tot}}}{N^2} = \int_{\mathbf{R}_+} dx \varrho_0(x) W_0(x) - \fint_{\mathbf{R}_+} dx dy \varrho_0(x) \varrho_0(y) \log|x-y| - \mu \left(\int_{\mathbf{R}_+} dx \varrho_0(x) - 1 \right), \tag{49}$$

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[4] One may more generally conclude that, for example, $\oint_{\mathcal{C}_\infty} \frac{dw}{2\pi i} \frac{(w^s + P_1(z) w^{s-1} + \cdots + P_{s-1}(z)) H(w)}{w-z} = 0$, where $P_i(z)$ are polynomials of z. The resolvent obtained from this form in the same way as described below at first looks different from Eq. (46), but reduces to the same form by using the boundary condition, which is the same as Eq. (48). We give a comment on this point below.

where μ is a Lagrange multiplier and ϱ_0 is the dynamical planar density function determined by the saddle point equation

$$W_0(x) - 2\fint_{\mathbf{R}_+} dy\varrho_0(y)\log|x-y| - \mu = 0. \tag{50}$$

Differentiating this with respect to x leads to Eq. (41), which we solved as Eq. (46). This suggests that integrating Eq. (41) with respect to x does not get back to Eq. (50), because the density function is not analytic on the edges of the cuts so integration of Eq. (41) takes different values on each interval in general. Requiring those values to be the same (as μ) gives a non-trivial condition.[5] To write down the condition we define a function $\widetilde{\mu}$ on $\mathbf{R}_+$ by

$$\widetilde{\mu}(x) := \mathrm{Re}\left[W_0(x) - 2\fint_{\mathbf{R}_+} dy\widetilde{\rho}_0(y)\log(x-y)\right], \tag{51}$$

where $\widetilde{\rho}_0(y)$ is defined by the analytic continuation of $\rho_0(y)$ from an interval to the whole positive real axis. Then the function $\widetilde{\rho}_0(x)$ takes pure imaginary values outside $\mathrm{supp}(\rho_0)$. Differentiating with respect to x gives $\widetilde{\mu}(x)' = \mathrm{Re}[-2\pi i\widetilde{\rho}_0(x)]$, which suggests that the derivative of $\widetilde{\mu}(x)$ vanishes on each cut and thus $\widetilde{\mu}(x)$ is constant on each cut, as expected. The condition for all of these constants to be equal can be written as $\widetilde{\mu}(a_{2i}-\epsilon) = \widetilde{\mu}(a_{2i+1}+\epsilon)$ $\forall i = 1, 2, \ldots, s-1$. Since

$$\widetilde{\mu}(a_{2i+1}+\epsilon) - \widetilde{\mu}(a_{2i}-\epsilon) = \int_{a_{2i}-\epsilon}^{a_{2i+1}+\epsilon} dx\widetilde{\mu}(x)' = \int_{a_{2i}-\epsilon}^{a_{2i+1}+\epsilon} dx(-2\pi i\widetilde{\rho}_0(x)) = -\oint_{\beta_i} dw\omega_0(w),$$

where $\beta_i = \mathcal{C}_{[a_{2i},a_{2i+1}]}$ is a circle encircling the interval $[a_{2i}, a_{2i+1}]$ counterclockwise, we obtain[6]

$$\int_{a_{2i}-\epsilon}^{a_{2i+1}+\epsilon} dx\widetilde{\rho}_0(x) = 0 \qquad \text{or} \qquad \oint_{\beta_i} dw\omega_0(w) = 0 \tag{52}$$

$\forall i = 1, 2, \ldots, s-1$. These yield the residual $s-1$ constraint equations to fix the $2s$ endpoints of the cuts.

There is a comment on the solution in Eq. (46). By using the condition in Eq. (48), the solution can be rewritten in a different form such as

$$\omega_0(z) = \frac{-h(z)}{2z^k}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i}\frac{w^k W_0'(w)}{(w-z)h(w)}, \tag{53}$$

where $k = 0, 1, \ldots, s$. On the other hand, this form of solution can be obtained directly by starting with a different equation from Eq. (43) as mentioned in footnote 4. Then the cuts can be determined not only by the asymptotic condition $\omega_0(z) = \frac{1}{z} + \cdots$ with $z \sim \infty$, but also the fact that $\omega_0(z)$ is non-singular around $z \sim 0$. These two conditions give the condition in Eq. (48).

The free energy in the leading order of the $1/N$ expansion is easily determined as

$$F_0 = V_{\mathrm{tot}} - \lim_{N\to\infty}\frac{\log\mathfrak{N}}{N^2} \tag{54}$$

[5] It is possible to consider a case where the Lagrange multiplier in Eq. (49) takes different values on each interval. In this case their values become parameters of the theory and play the role of a kind of chemical potential.

[6] The chemical potentials mentioned in footnote 5 can be added such that $\oint_{\beta_i} dw\omega_0(w) = \mu_i$ for $i = 1, 2, \ldots, s-1$.

with Eq. (50). From this expression, we can reproduce the derivatives of the free energy with respect to the coupling constants obtained in the previous section. For example, acting $\frac{d}{dT_z}$ on Eq. (54) yields

$$\frac{dF_0}{dT_z} = \int_{\mathbf{R}_+} dx \rho_0(x) \frac{dW_0(x)}{dT_z} = \int_{\mathbf{R}_+} dx \rho_0(x) \frac{-x}{z(z-x)} = \frac{1}{z} - \int_{\mathbf{R}_+} dx \frac{\rho_0(x)}{z-x}, \tag{55}$$

where we used Eq. (50), $\int_{\mathbf{R}_+} dx \rho_0(x) = 1$, and Eq. (33). This is nothing but the equation in Eq. (40).

4.2. *Hole correction*

The hole correction of the resolvent is determined by the genus-half loop equation in Eq. (36). Let us derive the saddle point equation at this order from Eq. (36). For this purpose let us rewrite Eq. (36) as

$$\oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_0(w) + W_0'(w)\omega_{\frac{1}{2}}(w)}{z-w} - 2\omega_0(z)\omega_{\frac{1}{2}}(z) = 0. \tag{56}$$

As in the planar case, we compute the discontinuity of the left-hand side between $x - i\epsilon$ and $x + i\epsilon$. The discontinuity of the first term is computed as

$$\int_{\mathbf{R}_+} dy (W_1'(y)\rho_0(y) + W_0'(y)\rho_{\frac{1}{2}}(y)) \left(\frac{1}{x - i\epsilon - y} - \frac{1}{x + i\epsilon - y} \right) = 2\pi i (W_1'(x)\rho_0(x) + W_0'(x)\rho_{\frac{1}{2}}(x)).$$

That of the second term is

$$-2\pi i (\rho_0(x)(\omega_{\frac{1}{2}}(x - i\epsilon) + \omega_{\frac{1}{2}}(x + i\epsilon)) + W_0'(x)\rho_{\frac{1}{2}}(x)).$$

Therefore we obtain

$$\omega_{\frac{1}{2}}(x - i\epsilon) + \omega_{\frac{1}{2}}(x + i\epsilon) = W_1'(x) \tag{57}$$

with $x \in \mathrm{supp}(\rho_0)$. Combining this with the planar saddle point equation in Eq. (35) we obtain

$$\omega_{0,\frac{1}{2}}(x - i\epsilon) + \omega_{0,\frac{1}{2}}(x + i\epsilon) = W'(x) \tag{58}$$

with $x \in \mathrm{supp}(\rho_0)$, where we set $\omega_{0,\frac{1}{2}} := \omega_0 + N^{-1}\omega_{\frac{1}{2}}$. This is the same form as the planar saddle point equation in Eq. (35), replacing $W_0(x)$ with $W(x)$, and the previous argument to solve this equation holds without any modification. Therefore *a solution of the usual saddle point equation is correct up to the order of hole correction!* This is a nice simplification, while the caveat is the region that Eq. (58) holds. That is, the region where we need to solve Eq. (58) is on $\mathrm{supp}(\rho_0)$. However, when we solve Eq. (58) as done in Sect. 4 the cut appears as the support of the density function including the hole correction, $\mathrm{supp}(\rho_{0,\frac{1}{2}})$. This small discrepancy may imply that the loop equation can be solved by assuming that *the support of the planar density function matches the one including the hole correction*. This assumption may be important to separate out the genus-half one from $\omega_{0,\frac{1}{2}}(z)$. We expect that the discussion above will hold in more general matrix models such as two-matrix models. We leave the proof of this conjecture to future work.

4.3. Genus-one correction

Let us determine the genus-one correction of the resolvent. For simplicity we first study the case where $W_1 = 0$, so $\omega_{\frac{1}{2}} = 0$. In this case the genus-one loop equation, Eq. (37) with $g = 1$, reduces to

$$\hat{K}\omega_1(z) = \frac{d}{dT_z}\omega_0(z). \tag{59}$$

This can be solved in the same manner as in the ordinary Hermitian matrix model [33,34]. Let us first compute the right-hand side:

$$\begin{aligned}\frac{d\omega_0(z)}{dT_z} &= \frac{d\log h(z)}{dT_z}\omega_0(z) + \frac{h(z)}{2}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{\frac{dW_0'(w)}{dT_z} - W_0'(w)\frac{d\log h(w)}{dT_z}}{(z-w)h(w)}\\ &= \frac{d\log h(z)}{dT_z}\omega_0(z) + \frac{h(z)}{2}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{-1}{(z-w)^3 h(w)} + \frac{h(z)}{2}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{-W_0'(w)\frac{d\log h(w)}{dT_z}}{(z-w)h(w)}.\end{aligned}$$

Then the second term is computed as

$$\frac{h(z)}{2}\oint_{\mathcal{C}_z}\frac{dw}{2\pi i}\frac{1}{(z-w)^3 h(w)} = -\frac{h(z)}{2}\frac{1}{2}\left(\frac{1}{h(z)}\right)'' = \frac{-1}{4}\left(\frac{3}{4}\sum_{i=1}^{2s}\frac{1}{(z-a_i)^2} + \frac{1}{2}\sum_{i<j}\frac{1}{(z-a_i)(z-a_j)}\right).$$

The third term is

$$\frac{h(z)}{2}\sum_{i=1}^{2s}\frac{d(-a_i)}{dT_z}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{-W_0'(w)}{2(z-w)h(w)(w-a_i)} = -\frac{d\log h(z)}{dT_z}\omega_0(z) + \sum_{i=1}^{2s}\frac{da_i}{dT_z}\frac{1}{z-a_i}\frac{1}{4}h(z)M_i^{(1)},$$

where we set

$$M_i^{(k)} := \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{W_0'(w)}{h(w)(w-a_i)^k}. \tag{60}$$

Therefore we obtain

$$\frac{d\omega_0(z)}{dT_z} = -\left(\frac{3}{16}\sum_{i=1}^{2s}\frac{1}{(z-a_i)^2} + \frac{1}{8}\sum_{i<j}\frac{1}{(z-a_i)(z-a_j)}\right) + \sum_{i=1}^{2s}\frac{da_i}{dT_z}\frac{1}{z-a_i}\frac{1}{4}h(z)M_i^{(1)}. \tag{61}$$

In order to compute $\frac{da_i}{dT_z}$, we act $\frac{d}{dT_z}$ on the constraint equations of the edges of the cuts, Eqs. (48) and (52):

$$\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}w^k\frac{\frac{d}{dT_z}W_0'(w) - W_0'(w)\frac{d}{dT_z}\log h(w)}{h(w)} = 0, \qquad k = 0, 1, \ldots, s, \tag{62}$$

$$\int_{a_{2l}}^{a_{2l+1}}dx\frac{d}{dT_z}\left(W_0'(x) - 2\omega_0(x)\right) = 0, \qquad l = 1, 2, \ldots, s-1, \tag{63}$$

where we used $2\pi i\widetilde{\rho}_0(x) = W_0'(x) - 2\omega_0(x)$. These are computed as

$$\frac{kz^{k-1}}{h(z)} + \frac{1}{2}\sum_{i=1}^{2s}\left(a_i^k \frac{da_i}{dT_z}M_i^{(1)} - \frac{z^k}{h(z)(z-a_i)}\right) = 0, \tag{64}$$

$$\frac{1}{2}\sum_{i=1}^{2s}\frac{K_{l,i}}{h(z)(z-a_i)} + \frac{1}{2}\sum_{i=1}^{2s}K_{l,i}\frac{d(-a_i)}{dT_z}M_i^{(1)} = 0, \tag{65}$$

where $K_{l,i} := \int_{a_{2l}}^{a_{2l+1}} dx \frac{h(x)}{(x-a_i)}$. The solution can be written as

$$\frac{da_i}{dT_z} = \frac{1}{M_i^{(1)}}\left(\frac{1}{h(z)(z-a_i)} + \sum_{l'=0}^{s-2}\alpha_{i,l'}\frac{z^{l'}}{h(z)}\right), \tag{66}$$

where $\alpha_{i,l}$ are determined by plugging this back in and setting the coefficients of polynomials with respect to z to zero. The determining equations of $\alpha_{i,l}$ are:

$$\sum_{i=1}^{2s}(a_i^k\alpha_{i,l'} - a_i^{k-1-l'}) = 0 \quad (0 \le l' \le k-2), \quad \frac{1}{2}\sum_{i=1}^{2s}a_i^k\alpha_{i,k-1} + k - \frac{1}{2} = 0, \tag{67}$$

$$\sum_{i=1}^{2s}a_i^k\alpha_{i,l'} = 0 \quad (k \le l' \le s-2), \tag{68}$$

$$\sum_{i=1}^{2s}K_{l,i}\alpha_{i,l'}{=}0 \quad (1 \le l \le s-1,\ 0 \le l' \le s-2). \tag{69}$$

Substituting Eq. (66) into Eq. (61), we find

$$\frac{d\omega_0(z)}{dT_z} = \frac{1}{16}\sum_i \frac{1}{(z-a_i)^2} - \frac{1}{8}\sum_{i<j}\frac{1}{(z-a_i)(z-a_j)} + \frac{1}{4}\sum_{i=1}^{2s}\sum_{l'=0}^{s-2}\frac{1}{z-a_i}\alpha_{i,l'}a_i^{l'}. \tag{70}$$

This can be rewritten as the image of the linear operator $\hat{K}$ in such a way that

$$\frac{d\omega_0(z)}{dT_z} = \hat{K}\left[\frac{1}{16}\sum_i \chi_i^{(2)}(z) - \frac{1}{8}\sum_{i<j}\frac{\chi_i^{(1)}(z) - \chi_j^{(1)}(z)}{a_i - a_j} + \frac{1}{4}\sum_{i=1}^{2s}\sum_{l'=0}^{s-2}\chi_i^{(1)}(z)\alpha_{i,l'}a_i^{l'}\right], \tag{71}$$

where $\chi_i^{(n)}(z)$ satisfies $\hat{K}\chi_i^{(n)}(z) = \frac{1}{(z-a_i)^n}$ for $n \ge 1$. $\chi_i^{(n)}(z)$ is constructed inductively as follows. Start with the identity

$$\frac{1}{(z-w)(w-a_i)^{\mathfrak{n}}} = \frac{1}{(z-w)(z-a_i)^{\mathfrak{n}}} + \sum_{k=1}^{\mathfrak{n}}\frac{1}{(z-a_i)^k(w-a_i)^{\mathfrak{n}-k+1}} \tag{72}$$

$\forall \mathfrak{n} \ge 1$. Acting $\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{W_0'(w)}{h(w)}$ on both sides and computing the right-hand side results in

$$\int_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{W_0'(w)}{h(w)}\frac{1}{(z-w)(w-a_i)^{\mathfrak{n}}} = 2\omega_0(z)\frac{1}{h(z)(z-a_i)^{\mathfrak{n}}} + \sum_{k=1}^{\mathfrak{n}}\frac{M_i^{(\mathfrak{n}-k+1)}}{(z-a_i)^k}. \tag{73}$$

Equivalently,

$$\hat{K}\left(\frac{1}{h(z)(z-a_i)^{\mathfrak{n}}}\right)=\sum_{k=1}^{\mathfrak{n}}\frac{M_i^{(\mathfrak{n}-k+1)}}{(z-a_i)^k}. \tag{74}$$

The case with $\mathfrak{n}=1$ implies that $\chi_i^{(1)}(z)=\frac{1}{M_i^{(1)}}\frac{1}{h(z)(z-a_i)}$. Assuming that $\chi_i^{(n)}(z)$ is constructed so that $\hat{K}\chi_i^{(n)}(z)=\frac{1}{(z-a_i)^n}$ is valid for $n\leq\mathfrak{n}-1$, we can rewrite the right-hand side as $\frac{M_i^{(1)}}{(z-a_i)^{\mathfrak{n}}}+\sum_{k=1}^{\mathfrak{n}-1}M_i^{(\mathfrak{n}-k+1)}\hat{K}\chi_i^{(k)}(z)$. Therefore we find that

$$\hat{K}\left(\frac{1}{M_i^{(1)}}\left(\frac{1}{h(z)(z-a_i)^{\mathfrak{n}}}-\sum_{k=1}^{\mathfrak{n}-1}M_i^{(\mathfrak{n}-k+1)}\chi_i^{(k)}(z)\right)\right)=\frac{1}{(z-a_i)^{\mathfrak{n}}}. \tag{75}$$

Hence, if we define $\chi_i^{(\mathfrak{n})}(z)$ by

$$\chi_i^{(\mathfrak{n})}(z)=\frac{1}{M_i^{(1)}}\left(\frac{1}{h(z)(z-a_i)^{\mathfrak{n}}}-\sum_{k=1}^{\mathfrak{n}-1}M_i^{(\mathfrak{n}-k+1)}\chi_i^{(k)}(z)\right), \tag{76}$$

then $\hat{K}\chi_i^{(n)}(z)=\frac{1}{(z-a_i)^n}$ is valid for $n=\mathfrak{n}$. By using this function we finally solve the genus-one loop equation as

$$\omega_1(z)=\frac{1}{16}\sum_{i=1}^{2s}\chi_i^{(2)}(z)-\frac{1}{8}\sum_{i<j}\frac{\chi_i^{(1)}(z)-\chi_j^{(1)}(z)}{a_i-a_j}+\frac{1}{4}\sum_{i=1}^{2s}\sum_{l'=0}^{s-2}\chi_i^{(1)}(z)\alpha_{i,l'}a_i^{l'} \tag{77}$$

up to terms in the kernel of the operator $\hat{K}$ such as $\frac{z^m}{h(z)}$ with $m=0,\ldots,s$. Note that $\omega_1(z)$ behaves at most as $\frac{1}{z^{s+1}}$, and thus the leading asymptotic behavior of the total resolvent $\omega(z)$ is unchanged; so is the cut. From Eq. (77), the coupling dependence of the genus-one free energy can be determined. For example, the dependence on t_p is determined by Eq. (40), and that on $\widetilde{\lambda}$ is by Eq. (30).[7]

Next we consider the case where the matrix model potential contains the $1/N$ correction: $W_1(w)\neq 0$. In this case, as derived in Eq. (37), the genus-one loop equation is corrected by the genus-half resolvent so that $\hat{K}\omega_1(z)=\omega_{\frac{1}{2}}(z)^2-\oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\frac{W_1'(w)\omega_{\frac{1}{2}}(w)}{z-w}+\frac{d}{dT_z}\omega_0(z)$. This equation is more involved and there may be some simplification in the way that the planar resolvent and the genus-half one can be determined at the same time as shown in Sect. 4.2. To see this, consider the deviation of the resolvent from the solution $\omega(z)=\omega_{0,\frac{1}{2}}(z)+\delta\omega(z)$ and substitute this into the original loop equation of Eq. (26). We obtain

$$2\omega_{0,\frac{1}{2}}(z)\delta\omega(z)+\delta\omega(z)^2-\oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\frac{W'(w)}{z-w}\delta\omega(w)+\frac{1}{N^2}\frac{d}{dT_z}(\omega_{0,\frac{1}{2}}(z)+\delta\omega(z))=0. \tag{78}$$

Since $\omega_{0,\frac{1}{2}}$ is of order one, $\delta\omega(z)$ is of order $1/N^2$: $\delta\omega=N^{-2}\tilde{\omega}_1+\mathcal{O}(N^{-3})$. Therefore, at the leading order of the $1/N$ expansion this reduces to

$$\hat{\mathcal{K}}\tilde{\omega}_1(z)=\frac{d}{dT_z}\omega_{0,\frac{1}{2}}(z), \tag{79}$$

[7] The differential equation in Eq. (40) will be solved as in the original Hermitian matrix model for the one-cut case [33] and the two-cut case [34], though such explicit solutions of the genus-one free energy do not contain the information about the dependence on other coupling constants such as $\widetilde{\lambda}$.

where we define $\hat{\mathcal{K}}$ by

$$\hat{\mathcal{K}}f(z) := \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i}\frac{W'(w)}{z-w}f(w) - 2\omega_{0,\frac{1}{2}}(z)f(z). \tag{80}$$

This equation is of the same form as the one without the hole correction by replacing ω_0, W_0 with $\omega_{0,\frac{1}{2}}, W$, respectively. Since the above argument to solve this equation holds as it is by performing the replacement, a solution of Eq. (79) is given in the same form as Eq. (77), where a_i and $M_i^{(n)}$ are replaced with the ones including the hole correction. We emphasize that this simplification happens only at the genus-one order, and at higher order one may need to solve Eq. (37) in general.

5. Applications

In this section we apply the presented formulation developed in the previous section to a few examples. First we apply it to the three-sphere partition function in $\mathrm{U}(N)_k$ pure Chern–Simons theory in order to test the presented framework by comparing with the exact result known for pure Chern–Simons theory as reviewed in the appendix. Secondly, we apply it to $\mathcal{N}=2$ $\mathrm{U}(N)_k$ Chern–Simons theory with n_F fundamental chiral multiplets and $\bar{n}_F$ anti-fundamental ones. This system does not admit the Fermi gas analysis in general, and there may be no systematic way to study the system beyond the spherical limit except for our formulation at present.

5.1. *Pure Chern–Simons theory*

The matrix model potential for pure Chern–Simons theory is given by Eq. (13) with Eq. (10). For simplicity we first study the case where there is no hole correction. Then $W'(w) = W_0'(w) = \frac{\log w}{w\tilde{\lambda}} + \frac{1}{w}$.

Let us first determine the planar resolvent. For $\tilde{\lambda} > 0$, the potential has only one stable minimum so we have only to consider a solution with one cut: $\mathrm{supp}(\rho_0) = [a_-, a_+]$ with $0 < a_- < a_+$. In order to simplify the integration we start with a solution of the form of Eq. (53) with $k=1$:

$$\omega_0(z) = \frac{-h(z)}{2z}\oint_{\mathcal{C}_{[a_-,a_+]}} \frac{dw}{2\pi i}\frac{\frac{\log w}{\tilde{\lambda}}+1}{(w-z)h(w)}, \tag{81}$$

where $h(z) = \sqrt{(z-a_-)(z-a_+)}$. Inflating the contour we can compute the right-hand side as

$$\begin{aligned}\omega_0(z) &= \frac{h(z)}{2z}\oint_{\mathcal{C}_{(-\infty,0]}} \frac{dw}{2\pi i}\frac{\frac{\log w}{\tilde{\lambda}}}{(w-z)h(w)} + \frac{h(z)}{2z}\frac{\frac{\log z}{\tilde{\lambda}}+1}{h(z)} \\ &= \frac{\log\left(\frac{(a_-+a_+)z-2a_-a_+-2\sqrt{a_-a_+}h(z)}{(-a_--a_+-2h(z)+2z)}\right)}{2\tilde{\lambda}z} + \frac{1}{2z},\end{aligned} \tag{82}$$

where we computed the first term as

$$\frac{h(z)}{2\tilde{\lambda}z}\int_{-\infty}^{0}\frac{dw}{2\pi i}\frac{(\log|w|-\pi i)-(\log|w|+\pi i)}{(w-z)h(w)} = \frac{h(z)}{2\tilde{\lambda}z}\frac{\log\left(\frac{(a_-+a_+)z-2a_-a_+-2\sqrt{a_-a_+}h(z)}{z(-a_--a_+-2h(z)+2z)}\right)}{h(z)}. \tag{83}$$

The edges of the cut, a_-, a_+, are determined by the asymptotic behavior around infinity and the regularity around the origin. $\omega_0(z)$ can approach $\frac{1}{z}$ when $z \to -\infty$, which is achieved if and only if $\log\left(\frac{a_-+a_++2\sqrt{a_-a_+}}{2+2}\right) = \tilde{\lambda}$. ω_0 is regular at the origin if and only if $\log\left(\frac{-4a_-a_+}{-a_--2\sqrt{a_-a_+}-a_+}\right) = -\tilde{\lambda}$.

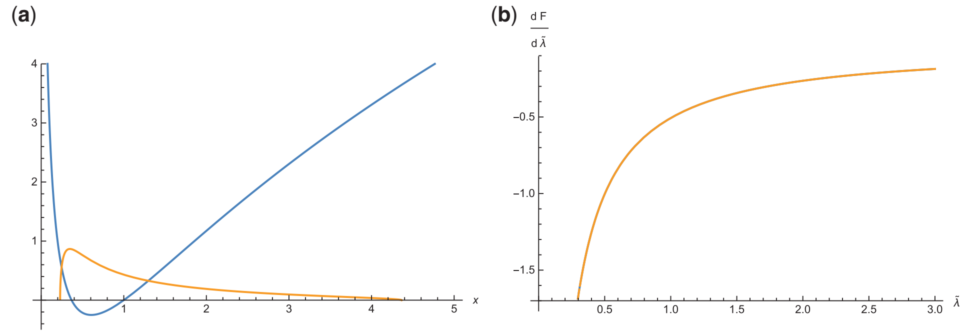

Fig. 1. (a) The blue and yellow curves depict the matrix model potential and the planar density function, respectively, when $\widetilde{\lambda} = 0.5$. The eigenvalues tend to clump around the potential minimum. (b) The differentiation of the planar free energy with respect to $\widetilde{\lambda}$ is plotted. The blue curve depicts the result obtained by the resolvent method and the yellow one is from the past exact result. They almost coincide.

These can be solved as $a_\pm = (e^{\frac{\widetilde{\lambda}}{2}} \pm \sqrt{e^{\widetilde{\lambda}} - 1})^2$. Note that $a_- a_+ = 1$. Then the planar resolvent can be simplified as

$$\omega_0(z) = \frac{1}{\widetilde{\lambda} z} \log\left(\frac{z + 1 + h(z)}{2}\right). \tag{84}$$

The planar density function is computed as

$$\rho_0(x) = \frac{\tan^{-1}\left(\frac{\sqrt{(x-a_-)(a_+-x)}}{1+x}\right)}{\pi \widetilde{\lambda} x}. \tag{85}$$

This solution matches the one given in Ref. [36], where the solution is expressed in the original coordinates. We plot the density function as well as the potential in Fig. 1(a).

The planar free energy is given by Eq. (54) and its $\widetilde{\lambda}$ derivative is Eq. (30):

$$\frac{\partial F_0}{\partial \widetilde{\lambda}} = \frac{1}{12} - \frac{1}{2\widetilde{\lambda}^2} \int_{a_-}^{a_+} dx \rho_0(x)(\log x)^2. \tag{86}$$

We could not perform the integration on the right-hand side analytically, so instead we evaluated it numerically. The numerical result is in good agreement with the past exact result of Eq. (A19), as can be seen in Fig. 1(b).

Next we study the genus-one correction. Now we consider the one-cut solution, so the genus-one correction of the resolvent is given by

$$\omega_1(z) = \frac{1}{16}(\chi_-^{(2)}(z) + \chi_+^{(2)}(z)) - \frac{1}{8}\frac{1}{a_- - a_+}(\chi_-^{(1)}(z) - \chi_+^{(1)}(z)), \tag{87}$$

where $\chi_\pm$ are defined by Eq. (76). $M_\pm^{(1)}, M_\pm^{(2)}$ are computed by Eq. (60). As in the computation of the resolvent, the integration can be simplified by using Eq. (48):

$$\begin{aligned} M_\pm^{(1)} &= \frac{1}{a_\pm} \oint_{\mathcal{C}_{[a_-,a_+]}} \frac{dw}{2\pi i} \frac{w W_0'(w)}{h(w)(w - a_\pm)} \\ &= \frac{1}{\widetilde{\lambda} a_\pm} \oint_{\mathcal{C}_{[a_-,a_+]}} \frac{dw}{2\pi i} \frac{\log w}{h(w)(w - a_\pm)} + \frac{1}{a_\pm} \oint_{\mathcal{C}_{[a_-,a_+]}} \frac{dw}{2\pi i} \frac{1}{h(w)(w - a_\pm)}. \end{aligned}$$

By inflating the contour to infinity, the first term is computed as

$$\frac{-1}{\tilde{\lambda}a_\pm}\int_{-\infty}^{0} dy \frac{-1}{h(y)(y-a_\pm)} = \frac{-1}{\tilde{\lambda}a_\pm}\left(\frac{2\sqrt{w-a_\mp}}{(a_\pm - a_\mp)\sqrt{w-a_\pm}}\right)\Bigg|_{-\infty}^{0} = \frac{-2}{\tilde{\lambda}a_\pm\sqrt{a_\pm}(\sqrt{a}_\pm + \sqrt{a}_\mp)},$$

and the second term vanishes. Therefore $M_\pm^{(1)} = \frac{-2}{\tilde{\lambda}a_\pm^{3/2}(\sqrt{a}_\pm+\sqrt{a}_\mp)}$. In the same way, $M_\pm^{(2)}$ are computed as $M_\pm^{(2)} = \frac{2(5\sqrt{a_\pm}+4\sqrt{a_\mp})}{3a_\pm^{5/2}\tilde{\lambda}(\sqrt{a_\pm}+\sqrt{a_\mp})^2}$.

The genus-one correction of the free energy is computed by using Eq. (30):

$$\frac{\partial F_1}{\partial \tilde{\lambda}} = \frac{1}{12} - \frac{1}{2\tilde{\lambda}^2}\int_{a_-}^{a_+} dx \rho_1(x)(\log x)^2 = \frac{1}{12} - \frac{1}{\tilde{\lambda}^2}\int_{-\infty}^{0} dx \omega_1(x)\log(-x). \tag{88}$$

This time we could perform the integral analytically:

$$\frac{\partial F_1}{\partial \tilde{\lambda}} = \frac{e^{\tilde{\lambda}}(\tilde{\lambda}-2)+\tilde{\lambda}+2}{24(e^{\tilde{\lambda}}-1)\tilde{\lambda}} = \frac{\tilde{\lambda}\coth\left(\frac{\tilde{\lambda}}{2}\right)-2}{24\tilde{\lambda}}. \tag{89}$$

This result is in precise agreement with the past exact result of Eq. (A20) with $\zeta = 0$.

Next we consider the case where the matrix model potential has a hole correction by the "FI parameter":[8] $W'(w) = \frac{\log w}{w\tilde{\lambda}} + \frac{1+\frac{\tilde{\zeta}}{N}}{w}$ with $\tilde{\zeta} = i\frac{\zeta}{2}$, which is still integrable as shown in the appendix. As discussed in Sect. 4.2, we solve the usual saddle point equation $\omega(x-i\epsilon)+\omega(x+i\epsilon) = W'(x)$, which is correct up to the hole order. To emphasize the difference from the previous computation we denote the cut with a prime, so that $\mathrm{supp}(\rho_{0,\frac{1}{2}}) = [a'_-, a'_+]$. Then a solution of this saddle point equation, $\omega_{0,\frac{1}{2}}(x)$, is given by

$$\omega_{0,\frac{1}{2}}(z) = \frac{-\acute{h}(z)}{2z}\oint_{\mathcal{C}_{[a'_-,a'_+]}} \frac{dw}{2\pi i}\frac{\frac{\log w}{\tilde{\lambda}} + 1 + \frac{\tilde{\zeta}}{N}}{(w-z)\acute{h}(w)}, \tag{90}$$

where $\acute{h}(z) = \sqrt{(z-a'_-)(z-a'_+)}$. This can be computed in the same way as previously and we obtain[9]

$$\omega_{0,\frac{1}{2}}(z) = \frac{\log\left(\frac{(a'_-+a'_+)z-2a'_-a'_+-2\sqrt{a'_-a'_+}\acute{h}(z)}{\left(-a'_--a'_+-2\acute{h}(z)+2z\right)}\right)}{2\tilde{\lambda}z} + \frac{1+\frac{\tilde{\zeta}}{N}}{2z}. \tag{91}$$

The edges of the cut are also determined in the same way. The result is $a'_\pm = ca_\pm$, where $c = e^{-\tilde{\lambda}\frac{\tilde{\zeta}}{N}}$ and $a_\pm$ are the same as previously. By using this, the resolvent up to the hole order is simplified as

[8] The usage of this terminology can be justified by adding some auxiliary fields into the pure Chern–Simons theory so that the theory has $\mathcal{N} = 2$ supersymmetry.

[9] The planar resolvent and the hole one are determined from this by

$$\acute{\omega}_0(z) = \frac{\log\left(\frac{(a'_-+a'_+)z-2a'_-a'_+-2\sqrt{a'_-a'_+}\acute{h}(z)}{\left(-a'_--a'_+-2\acute{h}(z)+2z\right)}\right)}{2\tilde{\lambda}z} + \frac{1}{2z}, \quad \omega_{\frac{1}{2}}(z) = \frac{\tilde{\zeta}}{2z},$$

which should be done before the edges of the cuts are determined.

$\omega_{0,\frac{1}{2}}(z) = c^{-1}\omega_0(\acute{z})$, where $\acute{z} = c^{-1}z$. As argued in Sect. 4.3, the genus-one resolvent is given by

$$\tilde{\omega}_1(z) = \frac{1}{16}(\acute{\chi}_-^{(2)}(z) + \acute{\chi}_+^{(2)}(z)) - \frac{1}{8}\frac{1}{a'_- - a'_+}(\acute{\chi}_-^{(1)}(z) - \acute{\chi}_+^{(1)}(z)), \tag{92}$$

where $\acute{\chi}_i^{(n)}(z)$ is given by Eq. (76) with a_i replaced by a'_i. By using $a'_\pm = ca_\pm$, the genus-one resolvent including the FI term can be written as $\tilde{\omega}_1(z) = c^{-2}\omega_1(\acute{z})$. Finally, we compute the differentiation of the total free energy with respect to $\widetilde{\lambda}$:

$$\frac{\partial \acute{F}}{\partial \widetilde{\lambda}} = \frac{7N^2-1}{12} - \frac{N^2}{2\widetilde{\lambda}^2}\int_{a'_-}^{a'_+} dx\rho_{0,\frac{1}{2}}(x)(\log x)^2 - \frac{1}{2\widetilde{\lambda}^2}\int_{a'_-}^{a'_+} dx\acute{\rho}_1(x)(\log x)^2 + \cdots, \tag{93}$$

where the ellipsis represents the terms of order N^{-3}. Then the second term is computed as

$$-\frac{N^2}{2\widetilde{\lambda}^2}\int_{a'_-}^{a'_+} dx\rho_{0,\frac{1}{2}}(x)(\log x)^2 = -\frac{N^2}{2\widetilde{\lambda}^2}\left(\int_{a_-}^{a_+} d\acute{x}\rho_0(\acute{x})(\log \acute{x})^2 + (\log c)^2\right).$$

The third term is

$$-\frac{1}{2\widetilde{\lambda}^2}\int_{a'_-}^{a'_+} dx\acute{\rho}_1(x)(\log x)^2 = -\frac{1}{\widetilde{\lambda}^2}\int_{-\infty}^{0} dx\tilde{\omega}_1(x)\log(-x) = -\frac{c^{-1}}{\widetilde{\lambda}^2}\int_{-\infty}^{0} d\acute{x}\omega_1(\acute{x})\log(-\acute{x}).$$

As a result, we obtain

$$\frac{\partial \acute{F}}{\partial \widetilde{\lambda}} = \frac{\partial F}{\partial \widetilde{\lambda}} - \frac{N^2}{2\widetilde{\lambda}^2}(\log c)^2 + \cdots = \frac{\partial F}{\partial \widetilde{\lambda}} - \frac{\widetilde{\zeta}^2}{2} + \cdots. \tag{94}$$

This is in perfect agreement with the past exact result of Eq. (A20).

5.2. *$\mathcal{N} = 2$ Chern–Simons theory with arbitrary numbers of fundamental and anti-fundamental chiral multiplets*

As another example we consider the matrix model of $\mathcal{N} = 2$ Chern–Simons theory with n_f fundamental chiral multiplets and $\bar{n}_f$ anti-fundamental ones with the canonical R-charge. Let us set

$$n_f^{(\pm)} = n_f \pm \bar{n}_f. \tag{95}$$

Without losing generality, we can assume that $n_f \geq \bar{n}_f$. The matrix model potential of this system is given by combining Eqs. (7) and (8):

$$W(\phi_s) = \frac{1}{2\widetilde{\lambda}}(\log\phi_s)^2 + \left(1 + \frac{\widetilde{\zeta}}{N}\right)\log\phi_s + \frac{\bar{n}_f}{N}\log\left(\frac{\sqrt{\phi_s} + 1/\sqrt{\phi_s}}{2}\right) - \frac{n_f^{(-)}}{N}\ell\left(\frac{-i\log\phi_s}{2\pi} + \frac{1}{2}\right). \tag{96}$$

Its derivative is

$$W'(w) = \frac{\log w}{w\widetilde{\lambda}} + \frac{1 + \frac{\widetilde{\zeta}}{N}}{w} + \frac{1-w}{2w(1+w)}\frac{1}{N}\left(n_f^{(-)}\frac{i\log w}{2\pi} - n_f^{(+)}\frac{1}{2}\right), \tag{97}$$

where we used $\ell'(z) = -\pi z\cot(\pi z)$. Thus it is clear that the matrix model potential takes complex values for a general number of chiral multiplets.

In order to determine the number of cuts in the resolvent by identifying that of the potential minimum, we regard the matrix model potential as an analytic function with all the parameters. When $n_f^{(-)}$ is pure imaginary and $\tilde{\lambda}$, $\tilde{\zeta}$, and $n_f^{(+)}$ are real, the potential becomes real. We fix the number of cuts of the resolvent in this situation.

We consider the large-k, N limit, holding its ratio and the other parameters $\tilde{\zeta}, n_f^{(\pm)}$ fixed. In this limit the potential has only one stable minimum, as in the case of pure Chern–Simons theory, so we have only to consider a solution with one cut: $\mathrm{supp}(\rho_0) = [\mathfrak{a}_-, \mathfrak{a}_+]$ with $0 < \mathfrak{a}_- < \mathfrak{a}_+$.

We compute the resolvent up to the hole correction by Eq. (53) with $k = 1$:

$$\omega_{0,\frac{1}{2}}(z) = \frac{-h(z)}{2z} \oint_{\mathcal{C}_{[\mathfrak{a}_-,\mathfrak{a}_+]}} \frac{dw}{2\pi i} \frac{\frac{\log w}{\tilde{\lambda}} + 1 + \frac{\tilde{\zeta}}{N} + \frac{1-w}{2(1+w)} \frac{1}{N} \left(n_f^{(-)} \frac{i \log w}{2\pi} - n_f^{(+)} \frac{1}{2} \right)}{(w-z)h(w)}, \tag{98}$$

where $h(z) = \sqrt{(z-\mathfrak{a}_-)(z-\mathfrak{a}_+)}$. Inflating the contour we can compute the right-hand side by picking up the pole as

$$\omega_{0,\frac{1}{2}}(z) = \frac{h(z)}{2z} \oint_{\mathcal{C}_{(-\infty,0]}} \frac{dw}{2\pi i} \frac{\frac{\log w}{\tilde{\lambda}} + \frac{n_f^{(-)}}{N} \frac{i \log w}{4\pi} \frac{1-w}{1+w}}{(w-z)h(w)} + \frac{h(z)}{2z} \frac{\frac{1}{N}\left(-n_f^{(+)} \frac{1}{2}\right)}{(-1-z)h(-1)}$$
$$+ \frac{h(z)}{2z} \frac{\frac{\log z}{\tilde{\lambda}} + 1 + \frac{\tilde{\zeta}}{N} + \frac{1-z}{2(1+z)} \frac{1}{N} \left(n_f^{(-)} \frac{i \log z}{2\pi} - n_f^{(+)} \frac{1}{2} \right)}{h(z)}, \tag{99}$$

where the first term is the contribution of the logarithmic branch cut $(-\infty, 0]$, the second one is that of the pole at $w = -1$, and the third one is at $w = z$. We compute the integrations such that

$$\oint_{\mathcal{C}_{(-\infty,0]}} \frac{dw}{2\pi i} \frac{\log w}{(w-z)h(w)} = \int_{-\infty}^{0} dw \frac{-1}{(w-z)h(w)} = \frac{f(z) - \log(z)}{h(z)}, \tag{100}$$

$$\oint_{\mathcal{C}_{(-\infty,0]}} \frac{dw}{2\pi i} \frac{(1-w)\log w}{(1+w)(w-z)h(w)} = \int_{-\infty}^{0} dw \mathcal{P} \frac{-(1-w)}{(w+1)(w-z)h(w)} = \frac{F(z) - F(-1) + \frac{(z-1)\log(z)}{h(z)}}{z+1}, \tag{101}$$

where

$$f(z) = \log\left(\frac{(\mathfrak{a}_- + \mathfrak{a}_+)z - 2\mathfrak{a}_-\mathfrak{a}_+ - 2\sqrt{\mathfrak{a}_-\mathfrak{a}_+} h(z)}{-\mathfrak{a}_- - \mathfrak{a}_+ - 2h(z) + 2z} \right), \quad F(z) := \frac{(1-z)f(z)}{h(z)}. \tag{102}$$

Then the resolvent becomes

$$\omega_{0,\frac{1}{2}}(z) = \frac{1}{2z} \left[f(z) \left(\frac{1}{\tilde{\lambda}} + \frac{n_f^{(-)}}{N} \frac{i}{4\pi} \frac{1-z}{1+z} \right) + 1 + \frac{\tilde{\zeta}}{N} \right.$$
$$\left. + \frac{1}{(z+1)} \frac{1}{N} \left(-n_f^{(-)} \frac{i}{4\pi} h(z) \frac{2f(-1)}{h(-1)} + n_f^{(+)} \frac{1}{2} \frac{h(z)}{h(-1)} - n_f^{(+)} \frac{1-z}{4} \right) \right]. \tag{103}$$

The edges of the cut $\mathfrak{a}_-, \mathfrak{a}_+$ are determined by the asymptotic behavior around infinity and the regularity around the origin. $\omega_{0,\frac{1}{2}}(z)$ can approach $\frac{1}{z}$ when $z \to -\infty$, which is achieved if and only if

$$\frac{1}{2}\Bigg[\log\left(\frac{\mathfrak{a}_- + \mathfrak{a}_+ + 2\sqrt{\mathfrak{a}_-\mathfrak{a}_+}}{2+2}\right)\left(\frac{1}{\tilde{\lambda}} - \frac{n_f^{(-)}}{N}\frac{i}{4\pi}\right) + 1 + \frac{1}{N}\left(\tilde{\zeta} + n_f^{(-)}\frac{i}{4\pi}\frac{2f(-1)}{h(-1)} + n_f^{(+)}\frac{1}{2}\frac{-1}{h(-1)} - n_f^{(+)}\frac{-1}{4}\right)\Bigg] = 1. \quad (104)$$

$\omega_{0,\frac{1}{2}}$ is regular at the origin if and only if

$$\log\left(\frac{-4\mathfrak{a}_-\mathfrak{a}_+}{-\mathfrak{a}_- - 2\sqrt{\mathfrak{a}_-\mathfrak{a}_+} - \mathfrak{a}_+}\right)\left(\frac{1}{\tilde{\lambda}} + \frac{n_f^{(-)}}{N}\frac{i}{4\pi}\right) + 1 + \frac{1}{N}\left(\tilde{\zeta} - n_f^{(-)}\frac{i}{4\pi}h(0)\frac{2f(-1)}{h(-1)} + n_f^{(+)}\frac{1}{2}\frac{h(0)}{h(-1)} - n_f^{(+)}\frac{1}{4}\right) = 0. \quad (105)$$

From these equations the edges of the cut are determined order by order in $1/N$.

As argued in Sect. 4.2, the planar resolvent should be determined so as to have the same cut as that of $\omega_{0,\frac{1}{2}}(z)$. Since the leading part of the potential in the large-N limit is unchanged, the form of the planar resolvent is unchanged except for the edges of the cut: Eq. (82) with $a_i \to \mathfrak{a}_i$. The genus-half resolvent is determined before the edges of the cut are expanded in the $1/N$ power series and given by

$$\omega_{\frac{1}{2}}(z) = \frac{1}{2z}\Bigg[\tilde{\zeta} + f(z)\left(n_f^{(-)}\frac{i}{4\pi}\frac{1-z}{1+z}\right) + \frac{1}{(z+1)}\left(-n_f^{(-)}\frac{i}{4\pi}h(z)\frac{2f(-1)}{h(-1)} + n_f^{(+)}\frac{1}{2}\frac{h(z)}{h(-1)} - n_f^{(+)}\frac{1-z}{4}\right)\Bigg]. \quad (106)$$

Then, as argued in Sect. 4.3, the genus-one resolvent is given by

$$\tilde{\omega}_1(z) = \frac{1}{16}(\chi_-^{(2)}(z) + \chi_+^{(2)}(z)) - \frac{1}{8}\frac{1}{\mathfrak{a}_- - \mathfrak{a}_+}(\chi_-^{(1)}(z) - \chi_+^{(1)}(z)), \quad (107)$$

where the $\chi_i^{(n)}(z)$ are given by Eq. (76) with $a_\pm$ replaced by $\mathfrak{a}_\pm$. The $\tilde{\lambda}$ derivative of the free energy up to the genus-one order is given by

$$\frac{\partial F}{\partial\tilde{\lambda}} = -\frac{\partial \log \mathfrak{N}}{\partial\tilde{\lambda}} - \frac{N^2}{2\tilde{\lambda}^2}\int_{\mathfrak{a}_-}^{\mathfrak{a}_+} dx\rho_{0,\frac{1}{2}}(x)(\log x)^2 - \frac{1}{\tilde{\lambda}^2}\int_{-\infty}^{0} dx\tilde{\omega}_1(x)\log(-x). \quad (108)$$

It is known that this system has the dual description known as Seiberg-like duality [41]. The dual theory is $U(N')_{-k}$ Chern–Simons theory with n_f fundamental and $\bar{n}_f$ anti-fundamental chiral multiplets, with $n_f\bar{n}_f$ mesonic operators as well as some monopole operators with a suitable superpotential, where N' depends generally on N, k, n_f, and $\bar{n}_f$, which is still in the class investigated in this paper. It would be interesting to test the duality from our general solution. We hope to come back to this problem in a future publication.

6. Discussion

In this paper we have performed a general analysis of a class of matrix models describing Chern–Simons matter theories on the three-sphere incorporating the standard technique of $1/N$ expansion developed in the study of ordinary Hermitian matrix models. We have derived the loop equation for all orders in the $1/N$ expansion and presented its explicit solution up to the genus-one order when the potential has a $1/N$ correction. We have applied the formulation to pure Chern–Simons theory and confirmed that the presented solution reproduces the exact result known in the past. We have also applied the framework to $\mathcal{N} = 2$ Chern–Simons theory with arbitrary numbers of fundamental and anti-fundamental chiral multiplets, and obtained a formal expression for the solution up to the genus-one order in the $1/N$ expansion.

This paper mainly focused on the construction of the framework to solve a class of matrix models. We are very much interested in applying the formula obtained in this paper to a duality pair of Chern–Simons matter systems and testing that the bosonization duality holds at the next leading order in the $1/N$ expansion. In particular, it would be interesting to develop the presented large-N technique in a class of unitary matrix models which arises as a partition function of Chern–Simons matter theories on $\mathbf{S}^2 \times \mathbf{S}^1$. For a class of Chern–Simons vector models, the effective matrix model potential was determined exactly in the leading order of the large-N limit [19], and the three-dimensional bosonization was confirmed at that order. We hope that the formulation developed in this paper is useful for future study in this direction.

In this paper, in order to study beyond the planar limit we adopted the iterative procedure given in Refs. [33,34]. Another iterative approach has been proposed, using the Feynman graph of the trivalent vertexes [42,43]. It would be interesting to reformulate the formula presented in this note in terms of the different approach.

Another interesting question is whether this class of matrix models has the equivalent description of some two-dimensional CFTs as ordinary Hermitian matrix models [44–46] (see also Ref. [47]). Naively, the answer seems to be no due to the fact that the degrees of freedom in a three-dimensional system are much bigger than those of a two-dimensional one in a generic situation. However, we have a suspicion that the answer could be yes for a certain matrix model of this kind, intuitively because vector models coupling to Chern–Simons theory appear as an effective field theory of an anyonic system [48,49], and the wave function describing a quantum Hall state known as the Laughlin wave function [50] is given by a correlator of certain two-dimensional (rational) CFTs [51]. In fact, it was shown that this answer becomes yes for a similar class of matrix models to the one studied in this paper [52], where the corresponding CFT is identified with a q-deformed one. Exploring this question is left for future work.

There is a straightforward generalization of the presented formulation to a different gauge group [53] or two matrices. This generalization to two matrices is important for the application to higher supersymmetric Chern–Simons matter theories such as the ABJM theory [54]. The $1/N$ correction of the free energy in the ABJM theory was computed in Refs. [55–57]. In this development, a new technique called the Fermi gas approach was invented [56]. This approach is powerful for studying non-perturbative aspects of the ABJM theory from the $\mathbf{S}^3$ partition function [58,59]. It is an important problem to test whether the traditional techniques of matrix models can reproduce the results obtained by new ones in recent developments beyond the spherical limit.

We hope to come back to these issues in the near future.

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Appendix. Partition function of pure Chern–Simons theory on $\mathbf{S}^3$

In this appendix we give a brief overview of the three-sphere partition function in $\mathrm{U}(N)_k$ pure Chern–Simons theory, and a derivation of its large-N expansion as used in the main text.

The partition function is defined formally by a path integral over the gauge field on $\mathbf{S}^3$ such that[10]

$$Z_{\mathrm{CS}} = \int \mathcal{D}A \exp\left\{-\frac{ik}{2\pi}\int_{\mathbf{S}^3}\left(\frac{1}{2}A\wedge dA - \frac{i}{3}A\wedge A\wedge A\right)\right\}. \tag{A1}$$

The classic paper Ref. [60] demonstrated explicitly in the case of SU(2) that this can be exactly determined as a function of the Chern–Simons level without performing the path integral by clarifying its relation to a modular transformation matrix of the characters in the corresponding affine Lie algebra. Generalization to an arbitrary gauge group is straightforward. Since modular transformation matrices had already been determined in general affine Lie algebras [61], the exact result of the partition function for $\mathrm{U}(N)_k$ pure Chern–Simons theory was given by

$$Z_{\mathrm{CS}} = k^{-\frac{N}{2}}\prod_{I=1}^{N-1}\left(2\sin\frac{\pi I}{k}\right)^{N-I}. \tag{A2}$$

After this exact result was studied in terms of the $1/N$ and $1/k$ expansions [62,63], it was insightfully observed that the Chern–Simons partition function of Eq. (A2) exactly matches that of the topological string theory on a Calabi–Yau three-fold background by identifying the string coupling constant with the pure imaginary Chern–Simons level [64]. This led to the conjecture of gauge/geometry duality [65] between Chern–Simons and topological string theories [66,67].

It was subsequently pointed out that the partition function in Eq. (A1) reduces to a matrix model such that [31]

$$Z_{\mathrm{CS}} = \frac{(-)^{\frac{N(N-1)}{2}}\exp\left\{\frac{-\pi(N-1)N(N+1)}{6ik}\right\}i^{\frac{N^2}{2}}}{(2\pi)^N N!}\int_{\mathbf{R}^N} d^N\sigma \exp\left\{-i\frac{k}{4\pi}\sum_{s=1}^{N}\sigma_s^2\right\}\prod_{t\neq s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right). \tag{A3}$$

This matrix model was extensively studied in relation to topological string theory [68,69]. With the help of the Weyl denominator formula, the matrix integral was explicitly performed by Gaussian integration in perfect agreement with Eq. (A2) [2]. This matrix model was also evaluated exactly by the orthogonal (or characteristic) polynomial method in accordance with Eq. (A2) [70]. The orthogonal polynomials associated with this matrix model were found to be Stiltjes–Wigert polynomials.

Let us compute the matrix model in a Fermi-gas-like approach [56] including the "FI term":

$$\begin{aligned} Z_{\mathrm{CS}} = {} & \frac{(-)^{\frac{N(N-1)}{2}}\exp\left\{\frac{-\pi(N-1)N(N+1)}{6ik}\right\}i^{\frac{N^2}{2}}}{(2\pi)^N N!} \\ & \times \int_{\mathbf{R}^N} d^N\sigma \exp\left\{-i\frac{k}{4\pi}\sum_{s=1}^{N}\sigma_s^2 - i\frac{1}{2}\zeta\sum_{s=1}^{N}\sigma_s\right\}\prod_{t\neq s}^{N} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right). \end{aligned} \tag{A4}$$

[10] Here, k is the renormalized Chern–Simons coupling constant so that $k = \kappa + \mathrm{sgn}(\kappa)N$, where κ is the level of the corresponding WZW model.

For this purpose we rewrite the partition function as a determinant by using the Weyl denominator formula

$$\prod_{s>t} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right) = \det_{s,t}\left[\exp\left\{\sigma_s\left(t-\frac{N+1}{2}\right)\right\}\right]. \tag{A5}$$

From this formula we can show that

$$\prod_{s\neq t} 2\sinh\left(\frac{\sigma_s-\sigma_t}{2}\right) = N!\det_{s,t}\left[e^{\sigma_s(-s+t)}\right], \tag{A6}$$

which enables us to rewrite the partition function as

$$Z_{\rm CS} = (-)^{\frac{N(N-1)}{2}}\exp\left\{\frac{-\pi(N-1)N(N+1)}{6ik}\right\} i^{\frac{N^2}{2}}\det_{s,t}\left[\int\frac{d\sigma_s}{2\pi}\exp\left\{-(i\frac{k}{4\pi}\sigma_s^2+i\frac{1}{2}\zeta\sigma_s)\right\}e^{\sigma_s(-s+t)}\right]. \tag{A7}$$

The inside of the determinant is computed by Gaussian integration as follows:

$$\int\frac{d\sigma_s}{2\pi}\exp\left\{-(i\frac{k}{4\pi}\sigma_s^2+i\frac{1}{2}\zeta\sigma_s)\right\}e^{\sigma_s(-s+t)} = \sqrt{\frac{1}{ik}}\exp\left\{\frac{-\pi i(-i\frac{1}{2}\zeta-s+t)^2}{k}\right\}. \tag{A8}$$

Plugging this back in gives

$$\begin{aligned} Z_{\rm CS} &= (-)^{\frac{N(N-1)}{2}}\exp\left\{\frac{-\pi(N-1)N(N+1)}{6ik}\right\} i^{\frac{N^2}{2}}\det_{s,t}\left[\sqrt{\frac{1}{ik}}\exp\left\{\frac{-\pi i(-i\frac{1}{2}\zeta-s+t)^2}{k}\right\}\right] \\ &= k^{\frac{-N}{2}}\exp\left\{\frac{\pi i N\zeta^2}{4k}\right\}\prod_{s>t}2\sin\left(\frac{\pi(s-t)}{k}\right), \end{aligned}$$

where in the second equation we used $\det_{s,t}[f_sM_{s,t}] = (\prod_s f_s)\det_{s,t}[M_{s,t}]$ and the Weyl denominator formula in Eq. (A5). By using the formula $\prod_{s>t}2\sin\left(\frac{\pi(s-t)}{k}\right) = \prod_{I=1}^{N}(2\sin\frac{\pi I}{k})^{N-I}$, we obtain

$$Z_{\rm CS} = k^{-\frac{N}{2}}\exp\left\{\frac{\pi i N\zeta^2}{4k}\right\}\prod_{I=1}^{N-1}\left(2\sin\frac{\pi I}{k}\right)^{N-I}. \tag{A9}$$

Then the free energy is computed as

$$F_{\rm CS} = -\log Z_{\rm CS} = \frac{N}{2}\log k - \frac{\pi i N\zeta^2}{4k} - \sum_{I=1}^{N-1}(N-I)\log\left(2\sin\frac{\pi I}{k}\right). \tag{A10}$$

The $1/N$ expansion was done as follows [64,66]. The expansion coefficients are determined as functions of $\lambda = \frac{N}{k}$:

$$F_{\rm CS} = \sum_{g=0}^{\infty} N^{2-2g}F_g(\lambda). \tag{A11}$$

By using

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{\pi n}\right)^2\right), \quad \log(1-x) = -\sum_{m=1}^{\infty}\frac{x^m}{m}, \tag{A12}$$

the second term in Eq. (A10) can be expanded as

$$-\sum_{I=1}^{N-1}(N-I)\log\left(2\sin\frac{\pi I}{k}\right)=\sum_{I=1}^{N-1}(I-N)\log\left(2\frac{\pi I\lambda}{N}\prod_{n=1}^{\infty}\left(1-\left(\frac{\pi I\lambda/N}{\pi n}\right)^2\right)\right)$$
$$=-\frac{N(N-1)}{2}\log\frac{2\pi\lambda}{N}+\sum_{I=1}^{N-1}(I-N)\log I+\sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\left(\frac{\lambda}{N}\right)^{2m}\sum_{I=1}^{N-1}(N-I)I^{2m}, \quad \text{(A13)}$$

where $\zeta(m)$ is the zeta function defined by $\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}$. The second term in Eq. (A13) can be expressed by using Barn's function as $\sum_{I=1}^{N-1}(I-N)\log I=-\log G(N+1)$, whose large-$N$ expansion is known:

$$\log G(N+1)=N^2\left(\frac{1}{2}\log N-\frac{3}{4}\right)+\frac{N}{2}\log 2\pi-\frac{B_2}{2}\log N+\zeta'(-1)+\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}.$$

Here, B_n is the nth Bernoulli number defined by[11]

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n. \quad \text{(A14)}$$

The summation in the last term in Eq. (A13) can be done by using a formula such that[12]

$$\sum_{I=1}^{N-1}(N-I)I^{2m}=\frac{N^{2m+2}}{(2m+1)(2m+2)}+\sum_{g=1}^{m}\binom{2m}{2g-2}\frac{-B_{2g}}{2g}N^{2m+2-2g}. \quad \text{(A15)}$$

Plugging these back in, we obtain the free energy as

$$\begin{aligned}F_{\rm CS}=&\frac{N}{2}\log k-\frac{N(N-1)}{2}\log\frac{2\pi\lambda}{N}\\&+\sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\lambda^{2m}\left[\frac{N^2}{(2m+1)(2m+2)}+\sum_{g=1}^{m}\binom{2m}{2g-2}\frac{-B_{2g}}{2g}N^{2-2g}\right]\\&-\frac{\pi iN\zeta^2}{4k}-\left(N^2\left(\frac{1}{2}\log N-\frac{3}{4}\right)+\frac{N}{2}\log 2\pi-\frac{B_2}{2}\log N+\zeta'(-1)+\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}\right)\\=&N^2\left[-\frac{1}{2}\log 2\pi\lambda+\frac{3}{4}+\sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\frac{\lambda^{2m}}{(2m+1)(2m+2)}\right]+\frac{B_2}{2}\log N-\frac{\pi iN\zeta^2}{4k}\\&-\zeta'(-1)+\sum_{m=1}^{\infty}\zeta(2m)\lambda^{2m}\frac{-B_2}{2m}+\sum_{g=2}^{\infty}N^{2-2g}\frac{-B_{2g}}{2g(2g-2)}\left[1+\sum_{m=g}^{\infty}\zeta(2m)\lambda^{2m}2\binom{2m-1}{2g-3}\right].\end{aligned}$$

[11] Our definition of the Bernoulli number is different from the one adopted in several past works such as Refs. [63,64,66]. The difference is $B_g^{\rm (there)}=(-)^{g-1}B_{2g}^{\rm (here)}$.

[12] This can be proved by using

$$\sum_{I=1}^{N}I^m=\frac{N^{m+1}}{m+1}+\frac{N^m}{2}+\sum_{g=1}^{[\frac{m}{2}]}\frac{B_{2g}}{2g}\binom{m}{2g-1}N^{m-2g+1},$$

where $[x]$ is the integer part of x.

As a result, the coefficients in the $1/N$ expansion of the form in Eq. (A11) are determined as

$$F_0 = -\frac{1}{2}\log 2\pi\lambda + \frac{3}{4} + \sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\frac{\lambda^{2m}}{(2m+1)(2m+2)}, \tag{A16}$$

$$F_1 = -\zeta'(-1) - \frac{\pi i\lambda\zeta^2}{4} + \sum_{m=1}^{\infty}\zeta(2m)\lambda^{2m}\frac{-B_2}{2m}, \tag{A17}$$

$$F_g = \frac{-B_{2g}}{2g(2g-2)}\left[1 + \sum_{m=g}^{\infty}\zeta(2m)\lambda^{2m}2\binom{2m-1}{2g-3}\right] \qquad (g \geq 2). \tag{A18}$$

A few comments are in order. The leading term in the $1/N$ expansion, F_0, can be obtained directly from Eq. (A10) by taking the large-N limit [62]:

$$F_0 = \lim_{N\to\infty}\frac{F_{\mathrm{CS}}}{N^2} = \lim_{N\to\infty} -\frac{1}{N}\sum_{I=1}^{N-1}\left(1-\frac{I}{N}\right)\log\left(2\sin\frac{\pi\lambda I}{N}\right) = -\int_0^1 d\tau(1-\tau)\log(2\sin\pi\lambda\tau),$$

where in the last equation we used the definition of the Riemann integral. This can be further computed by using Eq. (A12) as

$$F_0 = \frac{\pi i}{4} - \frac{1}{6}i\pi\lambda + \frac{\zeta(2)}{2i\pi\lambda} + \frac{\zeta(3)}{(2\pi\lambda)^2} - \frac{1}{(2\pi\lambda)^2}\mathrm{Li}_3(e^{-2\pi i\lambda}), \tag{A19}$$

where $\mathrm{Li}_s(z)$ is the polylogarithm defined by $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}$. The next-to-leading term F_1 can be simplified by using Eq. (A12) as follows:

$$F_1 = -\zeta'(-1) - \frac{\pi i\lambda\zeta^2}{4} + \frac{1}{12}\log\frac{\sin\pi\lambda}{\pi\lambda}. \tag{A20}$$

References

[1] V. Pestun, Commun. Math. Phys. **313**, 71 (2012) [arXiv:0712.2824 [hep-th]] [Search INSPIRE].

[2] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. **03**, 089 (2010) [arXiv:0909.4559 [hep-th]] [Search INSPIRE].

[3] D. L. Jafferis, J. High Energy Phys. **05**, 159 (2012) [arXiv:1012.3210 [hep-th]] [Search INSPIRE].

[4] N. Hama, K. Hosomichi, and S. Lee, J. High Energy Phys. **03**, 127 (2011) [arXiv:1012.3512 [hep-th]] [Search INSPIRE].

[5] N. Hama, K. Hosomichi, and S. Lee, J. High Energy Phys. **05**, 014 (2011) [arXiv:1102.4716 [hep-th]] [Search INSPIRE].

[6] Y. Imamura and D. Yokoyama, Phys. Rev. D **85**, 025015 (2012) [arXiv:1109.4734 [hep-th]] [Search INSPIRE].

[7] S. Kim, Nucl.Phys. B **821**, 241 (2009) [arXiv:0903.4172 [hep-th]] [Search INSPIRE].

[8] Y. Imamura and S. Yokoyama, J. High Energy Phys. **04**, 007 (2011) [arXiv:1101.0557 [hep-th]] [Search INSPIRE].

[9] J. Bhattacharya and S. Minwalla, J. High Energy Phys. **01**, 014 (2009) [arXiv:0806.3251 [hep-th]] [Search INSPIRE].

[10] J. Choi, S. Lee, and J. Song, J. High Energy Phys. **03**, 099 (2009) [arXiv:0811.2855 [hep-th]] [Search INSPIRE].

[11] Y. Imamura and S. Yokoyama, Nucl. Phys. B **827**, 183 (2010) [arXiv:0908.0988 [hep-th]] [Search INSPIRE].

[12] N. Drukker, M. Mariño, and P. Putrov, Commun. Math. Phys. **306**, 511 (2011) [arXiv:1007.3837 [hep-th]] [Search INSPIRE].

[13] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, Phys. Rev. D **83**, 046001 (2011) [arXiv:1011.5487 [hep-th]] [Search INSPIRE].

[14] T. Suyama, Nucl. Phys. B **834**, 50 (2010) [arXiv:0912.1084 [hep-th]] [Search INSPIRE].
[15] D. Martelli, A. Passias, and J. Sparks, Nucl. Phys. B **864**, 840 (2012) [arXiv:1110.6400 [hep-th]] [Search INSPIRE].
[16] M. Mariño, J. Phys. A **44**, 463001 (2011) [arXiv:1104.0783 [hep-th]] [Search INSPIRE].
[17] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, Eur. Phys. J. C **72**, 2112 (2012) [arXiv:1110.4386 [hep-th]] [Search INSPIRE].
[18] O. Aharony, G. Gur-Ari, and R. Yacoby, J. High Energy Phys. **03**, 037 (2012) [arXiv:1110.4382 [hep-th]] [Search INSPIRE].
[19] S. Jain, S. Minwalla, T. Sharma, T. Takimi, S. R. Wadia, and S. Yokoyama, J. High Energy Phys. **09**, 009 (2013) [arXiv:1301.6169 [hep-th]] [Search INSPIRE].
[20] S. Jain, S. P. Trivedi, S. R. Wadia, and S. Yokoyama, J. High Energy Phys. **10**, 194 (2012) [arXiv:1207.4750 [hep-th]] [Search INSPIRE].
[21] S. Jain, S. Minwalla, and S. Yokoyama, J. High Energy Phys. **11**, 037 (2013) [arXiv:1305.7235 [hep-th]] [Search INSPIRE].
[22] J. Maldacena and A. Zhiboedov, Class. Quant. Grav. **30**, 104003 (2013) [arXiv:1204.3882 [hep-th]] [Search INSPIRE].
[23] O. Aharony, G. Gur-Ari, and R. Yacoby, J. High Energy Phys. **12**, 028 (2012) [arXiv:1207.4593 [hep-th]] [Search INSPIRE].
[24] O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena, and R. Yacoby, J. High Energy Phys. **03**, 121 (2013) [arXiv:1211.4843 [hep-th]] [Search INSPIRE].
[25] I. R. Klebanov, S. S. Pufu, and B. R. Safdi, J. High Energy Phys. **10**, 038 (2011) [arXiv:1105.4598 [hep-th]] [Search INSPIRE].
[26] O. Aharony, J. High Energy Phys. **02**, 093 (2016) [arXiv:1512.00161 [hep-th]] [Search INSPIRE].
[27] P.-S. Hsin and N. Seiberg, J. High Energy Phys. **09**, 095 (2016) [arXiv:1607.07457 [hep-th]] [Search INSPIRE].
[28] Đ. Radičević, D. Tong, and C. Turner, J. High Energy Phys. **12**, 067 (2016) [arXiv:1608.04732 [hep-th]] [Search INSPIRE].
[29] D. H. Adams, Phys. Lett. B **417**, 53 (1998) [arXiv:hep-th/9709147] [Search INSPIRE].
[30] J. Källén, J. High Energy Phys. **08**, 008 (2011) [arXiv:1104.5353 [hep-th]] [Search INSPIRE].
[31] M. Mariño, Commun. Math. Phys. **253**, 25 (2005) [arXiv:hep-th/0207096] [Search INSPIRE].
[32] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[33] J. Ambjørn, L. Chekhov, C. F. Kristjansen, and Yu. Makeenko, Nucl. Phys. B **404**, 127 (1993); 449, 681 (1995) [erratum] [arXiv:hep-th/9302014] [Search INSPIRE].
[34] G. Akemann, Nucl. Phys. B **482**, 403 (1996) [arXiv:hep-th/9606004] [Search INSPIRE].
[35] A. A. Migdal, Phys. Rept. **102**, 199 (1983).
[36] M. Mariño, arXiv:hep-th/0410165 [Search INSPIRE].
[37] G. Giasemidis and M. Tierz, J. High Energy Phys. **01**, 068 (2016) [arXiv:1511.00203 [hep-th]] [Search INSPIRE].
[38] M. Tierz, J. High Energy Phys. **04**, 168 (2016) [arXiv:1601.06277 [hep-th]] [Search INSPIRE].
[39] F. David, Nucl. Phys. B **348**, 507 (1991).
[40] J. Jurkiewicz, Phys. Lett. B **245**, 178 (1990).
[41] F. Benini, C. Closset, and S. Cremonesi, J. High Energy Phys. **10**, 075 (2011) [arXiv:1108.5373 [hep-th]] [Search INSPIRE].
[42] B. Eynard, J. High Energy Phys. **11**, 031 (2004) [arXiv:hep-th/0407261] [Search INSPIRE].
[43] L. Chekhov and B. Eynard, J. High Energy Phys. **12**, 026 (2006) [arXiv:math-ph/0604014] [Search INSPIRE].
[44] A. Mironov and A. Morozov, Phys. Lett. B **252**, 47 (1990).
[45] R. Dijkgraaf, H. Verlinde, and E. Verlinde, Nucl. Phys. B **348**, 435 (1991).
[46] M. Fukuma, H. Kawai, and R. Nakayama, Commun. Math. Phys. **143**, 371 (1992).
[47] I. K. Kostov, [arXiv:hep-th/9907060] [Search INSPIRE].
[48] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[49] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991).
[50] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[51] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[52] A. Nedelin and M. Zabzine, J. High Energy Phys. **03**, 098 (2017) [arXiv:1511.03471 [hep-th]] [Search INSPIRE].

[53] N. Halmagyi and V. Yasnov, J. High Energy Phys. **02**, 002 (2004) [arXiv:hep-th/0305134] [Search INSPIRE].
[54] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. **10**, 091 (2008) [arXiv:0806.1218 [hep-th]] [Search INSPIRE].
[55] H. Fuji, S. Hirano, and S. Moriyama, J. High Energy Phys. **08**, 001 (2011) [arXiv:1106.4631 [hep-th]] [Search INSPIRE].
[56] M. Mariño and P. Putrov, J. Stat. Mech. **03**, P03001 (2012) [arXiv:1110.4066 [hep-th]] [Search INSPIRE].
[57] M. Mezei and S. S. Pufu, J. High Energy Phys. **02**, 037 (2014) [arXiv:1312.0920 [hep-th]] [Search INSPIRE].
[58] Y. Hatsuda, S. Moriyama, and K. Okuyama, J. High Energy Phys. **01**, 158 (2013) [arXiv:1211.1251 [hep-th]] [Search INSPIRE].
[59] Y. Hatsuda, M. Mariño, S. Moriyama, and K. Okuyama, J. High Energy Phys. **09**, 168 (2014) [arXiv:1306.1734 [hep-th]] [Search INSPIRE].
[60] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[61] V. G. Kač and D. H. Peterson, Adv. Math. **53**, 125 (1984).
[62] M. Camperi, F. Levstein, and G. Zemba, Phys. Lett. B **247**, 549 (1990).
[63] V. Periwal, Phys. Rev. Lett. **71**, 1295 (1993) [arXiv:hep-th/9305115] [Search INSPIRE].
[64] R. Gopakumar and C. Vafa, [arXiv:hep-th/9809187] [Search INSPIRE].
[65] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200] [Search INSPIRE].
[66] R. Gopakumar and C. Vafa, Adv. Theor. Math. Phys. **3**, 1415 (1999) [arXiv:hep-th/9811131] [Search INSPIRE].
[67] R. Gopakumar and C. Vafa, [arXiv:hep-th/9812127] [Search INSPIRE].
[68] M. Aganagic, M. Mariño, and C. Vafa, Commun. Math. Phys. **247**, 467 (2004) [arXiv:hep-th/0206164] [Search INSPIRE].
[69] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa, J. High Energy Phys. **02**, 010 (2004) [arXiv:hep-th/0211098] [Search INSPIRE].
[70] M. Tierz, Mod. Phys. Lett. A **19**, 1365 (2004) [arXiv:hep-th/0212128] [Search INSPIRE].