

Self-similarity, conservation of entropy/bits and the black hole information puzzle¹

Douglas Singleton,^{a,b} Elias C. Vagenas^c and Tao Zhu^{d,e}

^a*Department of Physics, California State University Fresno,
2345 East San Ramon Avenue M/S MH37, Fresno, CA 93740-8031, U.S.A.*

^b*Department of Physics, Institut Teknologi Bandung,
Jalan Ganesha 10, Bandung 40132, Indonesia*

^c*Theoretical Physics Group, Department of Physics, Kuwait University,
P.O. Box 5969, Safat 13060, Kuwait*

^d*GCAP-CASPER, Physics Department, Baylor University,
One Bear Place, # 97316, Waco, TX 76798-7316, U.S.A.*

^e*Institute for Advanced Physics & Mathematics, Zhejiang University of Technology,
18 Chao-Wang Rd, Hangzhou, 310032, China*

E-mail: dougs@csufresno.edu, elias.vagenas@ku.edu.kw,
Tao_Zhu@baylor.edu

ABSTRACT: John Wheeler coined the phrase “it from bit” or “bit from it” in the 1980s. However, much of the interest in the connection between information, i.e. “bits”, and physical objects, i.e. “its”, stems from the discovery that black holes have characteristics of thermodynamic systems having entropies and temperatures. This insight led to the information loss problem — what happens to the “bits” when the black hole has evaporated away due to the energy loss from Hawking radiation? In this essay we speculate on a radical answer to this question using the assumption of self-similarity of quantum correction to the gravitational action and the requirement that the quantum corrected entropy be well behaved in the limit when the black hole mass goes to zero.

KEYWORDS: Models of Quantum Gravity, Black Holes, Spacetime Singularities

ARXIV EPRINT: [1311.2015](https://arxiv.org/abs/1311.2015)

¹Awarded a fourth prize in the 2013 FQXi’s Essay Contest “It From Bit, or Bit From It”.

Contents

1	Self-similarity and order-\hbar^n quantum gravity corrections	1
2	Black hole entropy to all orders in \hbar	2
3	Conservation of energy, entropy/information and solution to the information loss puzzle	5

1 Self-similarity and order- \hbar^n quantum gravity corrections

In this essay we look at the connection between physical objects, i.e. “its”, and information/entropy, i.e “bits”,¹ in the context of black hole physics. In particular, we focus on the relationship between the initial information/entropy contained in the horizon of a Schwarzschild black hole and the final entropy carried by the outgoing, *correlated* photons of Hawking radiation. The correlation of the photons comes from taking into account conservation of energy and the back reaction of the radiation on the structure of the Schwarzschild space-time in the tunneling picture [2, 3] of Hawking radiation. Since, in the first approximation, Hawking radiation is thermal there are no correlations between the outgoing Hawking radiated photons. This leads to the information loss puzzle of black holes which can be put as follows: the original black hole has an entropy given by $S_{\text{BH}} = \frac{4\pi k_B G M^2}{c\hbar}$ which can be written as $S_{\text{BH}} = \frac{k_B A}{4l_{\text{Pl}}^2}$ where $A = 4\pi r_H^2$ is the horizon area of the black hole and $r_H = \frac{2GM}{c^2}$ is the location of the horizon [4]. One can think of this areal entropy as being composed of Planck sized area “bits”, $A_{\text{Pl}} = l_{\text{Pl}}^2$, where the Planck length is defined as $l_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}}$. If Hawking radiation were truly thermal, then the entropy of the outgoing thermal radiation would be larger than this Bekenstein area entropy. Since entropy increases, some information is lost. But this violates the prime directive of quantum mechanics that quantum evolution should be unitary and, thus, information and entropy should be conserved.

To begin our examination of these issues of the thermodynamics of black holes and the loss versus conservation of information, we lay out our basic framework. We will consider a massless scalar field $\phi(\mathbf{x}, t)$ in the background of a Schwarzschild black hole whose metric is given by

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r} \right)} dr^2 + r^2 d\Omega^2, \tag{1.1}$$

in units with $G = c = 1$. From here onward in the essay we will set $G = c = 1$ but will keep \hbar explicitly. The horizon is located by setting $1 - \frac{2M}{r_H} = 0$ or $r_H = 2M$. Into this

¹There is an equivalence or connection between information, entropy and bits and we will use these terms somewhat interchangeably throughout this essay. A nice overview of the close relationship between information, entropy and bits can be found in reference [1].

space-time, we place a massless scalar field obeying the Klein-Gordon equation

$$-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)\phi = 0 . \tag{1.2}$$

By the radial symmetry of the Schwarzschild space-time as given by eq. (1.1), the scalar field only depends on r and t . Expanding $\phi(r, t)$ in a WKB form gives

$$\phi(r, t) = \exp\left[\frac{i}{\hbar}I(r, t)\right] \tag{1.3}$$

where $I(r, t)$ is the one-particle action which can be expanded in powers of \hbar via the general expression

$$I(r, t) = I_0(r, t) + \sum_{j=1}^{\infty} \hbar^j I_j(r, t). \tag{1.4}$$

Here, $I_0(r, t)$ is the classical action and $I_j(r, t)$ are order \hbar^j quantum corrections. We now make the assumption that quantum gravity is *self-similar*² in the following sense: the higher order corrections to the action, $I_j(r, t)$, are proportional to $I_0(r, t)$, i.e. $I_j(r, t) = \gamma_j I_0(r, t)$ where γ_j are constants. With this assumption, eq. (1.4) becomes

$$I(r, t) = \left(1 + \sum_{j=1}^{\infty} \gamma_j \hbar^j\right) I_0(r, t) . \tag{1.5}$$

From eq. (1.5), one sees that $\gamma_j \hbar^j$ is dimensionless. In the units we are using, i.e. $G = c = 1$, \hbar has units of the Planck length squared, i.e. l_{Pl}^2 , thus γ_j should have units of an inverse distance squared to the j^{th} power. The natural distance scale defined by eq. (1.1) is the horizon distance $r_H = 2M$, thus

$$\gamma_j = \frac{\alpha_j}{r_H^{2j}} \tag{1.6}$$

with α_j dimensionless constants which we will fix via the *requirement* that information/entropy be well behaved in the $M \rightarrow 0$ limit. Thus, in this way we will obtain an explicit, all orders in \hbar correction to the entropy and show how this gives a potential solution to the black hole information puzzle.

2 Black hole entropy to all orders in \hbar

In [6] the set-up of the previous section was used to obtain an expression for the quantum corrected temperature of Hawking radiation [7] to all orders in \hbar . This was done by applying the tunneling method introduced in [2, 3] to the WKB-like expression given

²Broadly speaking, self-similarity means that a system “looks the same” at different scales. A standard example is the Koch snowflake [5] where any small segment of the curve has the same shape as a larger segment. Here, self-similarity is applied in the sense that as one goes to smaller distance scales/higher energy scales by going to successive orders in \hbar that the form of the quantum corrections remains the same.

by eqs. (1.3), (1.5), and (1.6). From [6], the quantum corrected Hawking temperature is given as

$$T = \frac{\hbar}{8\pi M} \left(1 + \sum_{j=1}^{\infty} \frac{\alpha_j \hbar^j}{r_H^{2j}} \right)^{-1}. \tag{2.1}$$

In this expression, $\frac{\hbar}{8\pi M}$ is the semi-classical Hawking temperature and the other terms are higher order quantum corrections. At this point, since the α_j 's are completely undetermined, the expression in eq. (2.1) does not have much physical content but is simply a parameterizing of the quantum corrections. However, by requiring that the quantum corrected black hole entropy be well behaved in the limit $M \rightarrow 0$, we will fix α_j 's and show how this leads to conservation of information/entropy, thus providing an answer to the black hole information loss puzzle.

Using eq. (2.1), we can calculate the Bekenstein entropy to all orders in \hbar . In particular, the Bekenstein entropy of black holes can be obtained by integrating the first law of thermodynamics, $dM = TdS$ with the temperature T given by eq. (2.1), i.e. $S = \int \frac{dM}{T}$. Integrating this over the mass, M , of the black hole (and recalling that $r_H = 2M$) gives the modified entropy as a function of M

$$S_{\text{BH}}(M) = \frac{4\pi}{\hbar} M^2 + \pi\alpha_1 \ln \left(\frac{M^2}{\hbar} \right) - \pi \sum_{j=1}^{\infty} \frac{\alpha_{j+1}}{4^j j} \left(\frac{\hbar}{M^2} \right)^j. \tag{2.2}$$

To lowest order $S_0(M) = \frac{4\pi}{\hbar} M^2$ for which the limit $M \rightarrow 0$ is well behaved, i.e. $S_0(M \rightarrow 0) \rightarrow 0$, as expected since as the mass vanishes so should the entropy. On the other hand, for the first, logarithmic correction as well as the other higher corrections, the quantum corrected entropy diverges. One way to fix these logarithmic and power divergences in $S_{\text{BH}}(M)$ as $M \rightarrow 0$ is to postulate that the Hawking radiation and resulting evaporation turn off when the black hole reaches some small, “remnant” mass m_R [8]. Here, we take a different path — by assuming that quantum corrected black hole entropy should not diverge in the $M \rightarrow 0$ limit we will obtain a condition that fixes almost all the unknown α_j 's. To accomplish this, the third term in eq. (2.2) should sum up to a logarithm which can then be combined with the second logarithmic term to give a non-divergent entropy, i.e. $S(M \rightarrow 0) \neq \pm\infty$. This condition can be achieved by taking the α_j 's as

$$\alpha_{j+1} = \alpha_1 (-4)^j \text{ for } j = 1, 2, 3, \dots \tag{2.3}$$

This again shows self-similarity since all the α_j 's are proportional to each other. For this choice in eq. (2.3), the sum in eq. (2.2), i.e. the third term, becomes $+\alpha_1 \pi \ln(1 + \hbar/M^2)$. Combining this term with the second, logarithmic quantum correction, the entropy takes the form

$$S_{\text{BH}}(M) = \frac{4\pi}{\hbar} M^2 + \pi\alpha_1 \ln \left(1 + \frac{M^2}{\hbar} \right). \tag{2.4}$$

As $M \rightarrow 0$, this “all orders in \hbar ” entropy tends to zero, i.e. $S_{\text{BH}}(M) \rightarrow 0$. There is a subtle issue with identifying the sum in eq. (2.2) with $\alpha_1 \pi \ln(1 + \hbar/M^2)$ — strictly this is only valid for $\sqrt{\hbar} < M$, i.e. when the mass, M , is larger than the Planck mass. However, we

can use analytic continuation to define the sum via $\alpha_1 \pi \ln(1 + \hbar/M^2)$ even for $\sqrt{\hbar} > M$. This is analogous to the trick in String Theory [14] where the sum $\sum_{j=1}^{\infty} j$ is defined as $\zeta(-1) = -\frac{1}{12}$ using analytic continuation of the zeta function, i.e. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Other works [11–13] have investigated quantum corrections to the entropy beyond the classical level. These expressions, in general, involve logarithmic and higher order divergences as $M \rightarrow 0$ as we also find to be the case for our generic expression in eq. (2.2). However, here, as a result of our assumption of self-similarity of the \hbar^n corrections, we find an expression for $S_{\text{BH}}(M)$ which has a well behaved $M \rightarrow 0$ limit.

This “lucky” choice of α_j ’s in eq. (2.3) which gave the all orders in \hbar expression for $S_{\text{BH}}(M)$ in eq. (2.4) was motivated by making the primary physical requirement that the entropy of the black hole be well behaved and finite. Usually, the focus in black hole physics is to find some way to tame the divergent Hawking temperature in the $M \rightarrow 0$ limit whereas here the primary physical requirement has been on making sure that the entropy/information content of the black hole is well behaved to all orders in \hbar .

The expression for $S_{\text{BH}}(M)$ still contains an arbitrary constant, namely α_1 , which is the first order quantum correction. This first order correction has been calculated in some theories of quantum gravity. For example, in Loop Quantum Gravity one finds that $\alpha_1 = -\frac{1}{2}$ [15]. Once α_1 is known, our assumption of self-similarity and the requirement that information/entropy be well behaved fixes the second and higher order quantum corrections. One can ask how unique is the choice in eq. (2.3)? Are there other choices which would yield $S_{\text{BH}}(M=0) \rightarrow 0$? As far as we have been able to determine, there are no other choices of α_j ’s that give $S(M=0) \rightarrow 0$, and also conserves entropy/information as we will demonstrate in the next section. However, we have not found a formal proof of the uniqueness of the choice of α_j ’s.

If one leaves α_1 as a free parameter — does not fix it to the Loop Quantum Gravity value, i.e. $\alpha_1 = -\frac{1}{2}$ —, then there is an interesting dividing point in the behavior of the entropy in eq. (2.4) at $\alpha_1 = -4$. For $\alpha_1 \geq -4$, the entropy in eq. (2.4) goes to zero, i.e. $S_{\text{BH}} = 0$, only at $M = 0$. For $\alpha_1 < -4$, the entropy in eq. (2.4) goes to zero, i.e. $S_{\text{BH}} = 0$, at $M = 0$ and also at some other value $M = M^* > 0$ where M^* satisfies the equation $\frac{4\pi}{\hbar}(M^*)^2 + \pi\alpha_1 \ln\left(1 + \frac{(M^*)^2}{\hbar}\right) = 0$. Thus, depending on the first quantum correction α_1 the black hole mass can vanish if $\alpha_1 \geq -4$, or one can be left with a “remnant” of mass M^* if $\alpha_1 < -4$. It might appear that one could rule out this last possibility since for $M^* > 0$ the black hole would still have a non-zero temperature via eq. (2.1) and, thus, the black hole should continue to lose mass via evaporation leading to masses $M < M^*$ which would give $S < 0$ for the case when $\alpha_1 < -4$. However, if the Universe has a positive cosmological constant, i.e. space-time is de Sitter, then the Universe will be in a thermal state at the Hawking-Gibbons temperature, i.e. $T_{GH} = \frac{\hbar\sqrt{\Lambda}}{2\pi}$ [16] where $\Lambda > 0$ is the cosmological constant. Thus, if the quantum corrected black hole temperature from eq. (2.1) becomes equal to T_{GH} the evaporation process can stop at this finite temperature and still consistently have $S = 0$. This situation would give some interesting and non-trivial connection between the Universal parameter Λ and the final fate of every black hole (in the case when $\alpha_1 < -4$).

3 Conservation of energy, entropy/information and solution to the information loss puzzle

We now want to show that the initial (quantum corrected) entropy of the black hole given in eq. (2.4) can be exactly accounted for by the entropy of the emitted radiation so that entropy/information, i.e. “bits”, is conserved. The fact that this happens depends crucially on the specific, logarithmic form of the quantum corrected entropy in eq. (2.4). This, retrospectively, puts an additional constraint on the α_j 's from eq. (2.3) — other choices of α_j 's would not in general lead to both a well behaved S in the $M \rightarrow 0$ limit *and* to entropy/information conservation. As we will see, this conservation of information/entropy is connected with the conservation of energy.

To start our analysis, we note that in the picture of Hawking radiation as a tunneling phenomenon the tunneling rate, i.e. Γ , and the change in entropy are related by [2]

$$\Gamma = e^{\Delta S_{\text{BH}}} . \tag{3.1}$$

When the black hole of mass M emits a quanta of energy ω energy conservation tells us that the mass of the black hole is reduced to $M - \omega$. Connected with this, the entropy of the black hole will change according to $\Delta S_{\text{BH}} = S_{\text{BH}}(M - \omega) - S_{\text{BH}}(M)$ [9, 10]. Using eq. (2.4) for the quantum corrected entropy, one obtains for the change in entropy

$$\Delta S_{\text{BH}} = -\frac{8\pi}{\hbar}\omega \left(M - \frac{\omega}{2}\right) + \pi\alpha_1 \ln \left[\frac{\hbar + (M - \omega)^2}{\hbar + M^2}\right] . \tag{3.2}$$

Combining eqs. (3.1) and (3.2), the corrected tunneling rate takes the form

$$\Gamma(M; \omega) = \left(\frac{\hbar + (M - \omega)^2}{\hbar + M^2}\right)^{\pi\alpha_1} \exp \left[-\frac{8\pi}{\hbar}\omega \left(M - \frac{\omega}{2}\right)\right] . \tag{3.3}$$

The term $\exp \left[-\frac{8\pi}{\hbar}\omega \left(M - \frac{\omega}{2}\right)\right]$ represents the result of energy conservation and back reaction on the tunneling rate [9, 10]; the term to the power $\pi\alpha_1$ represents the quantum corrections to all orders in \hbar . This result of being able to write the tunneling rate as the product of these two effects, namely back reaction and quantum corrections, depended crucially on the specific form of $S_{\text{BH}}(M)$ and ΔS_{BH} from eqs. (2.4) and (3.2), respectively, which in turn was crucially tied to our specific choice of α_j 's in eq. (2.3). Note that even in the classical limit, where one ignores the quantum corrections by setting $\pi\alpha_1 = 0$, there is a deviation from a thermal spectrum due to the ω^2 term in the exponent in eq. (3.3).

We now find the connection between the tunneling rate given by eq. (3.3) and the entropy of the emitted radiation, i.e. S_{rad} . Assuming that the black hole mass is completely radiated away, we have the relationship $M = \omega_1 + \omega_2 + \dots + \omega_n = \sum_{j=1}^n \omega_j$ between the mass of the black hole and the sum of the energies, i.e. ω_j , of the emitted field quanta. The probability for this radiation to occur is given by the following product of Γ 's [17–19] which is defined in eq. (3.3)

$$P_{\text{rad}} = \Gamma(M; \omega_1) \times \Gamma(M - \omega_1; \omega_2) \times \dots \times \Gamma \left(M - \sum_{j=1}^{n-1} \omega_j; \omega_n\right) . \tag{3.4}$$

The probability of emission of the individual field quanta of energy ω_j is given by

$$\begin{aligned} \Gamma(M; \omega_1) &= \left(\frac{\hbar + (M - \omega_1)^2}{\hbar + M^2} \right)^{\pi\alpha_1} \exp \left[-\frac{8\pi}{\hbar} \omega_1 \left(M - \frac{\omega_1}{2} \right) \right], \\ \Gamma(M - \omega_1; \omega_2) &= \left(\frac{\hbar + (M - \omega_1 - \omega_2)^2}{\hbar + (M - \omega_1)^2} \right)^{\pi\alpha_1} \exp \left[-\frac{8\pi}{\hbar} \omega_2 \left(M - \omega_1 - \frac{\omega_2}{2} \right) \right], \\ &\dots\dots, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \Gamma \left(M - \sum_{j=1}^{n-1} \omega_j; \omega_n \right) &= \left(\frac{\hbar + (M - \sum_{j=1}^{n-1} \omega_j - \omega_n)^2}{\hbar + (M - \sum_{j=1}^{n-1} \omega_j)^2} \right)^{\pi\alpha_1} \exp \left[-\frac{8\pi}{\hbar} \omega_n \left(M - \sum_{j=1}^{n-1} \omega_j - \frac{\omega_n}{2} \right) \right] \\ &= \left(\frac{\hbar}{\hbar + (M - \sum_{j=1}^{n-1} \omega_j)^2} \right)^{\pi\alpha_1} \exp(-4\pi\omega_n^2/\hbar). \end{aligned}$$

The Γ 's of the form $\Gamma(M - \omega_1 - \omega_2 - \dots - \omega_{j-1}; \omega_j)$ represent the probability for the emission of a field quantum of energy ω_j with the condition that first the field quanta of energy $\omega_1 + \omega_2 + \dots + \omega_{j-1}$ have been emitted in sequential order.

Using eq. (3.5) in eq. (3.4), we find the total probability for the sequential radiation process described above

$$P_{\text{rad}} = \left(\frac{\hbar}{\hbar + M^2} \right)^{\pi\alpha_1} \exp(-4\pi M^2/\hbar). \tag{3.6}$$

The black hole mass could also have been radiated away by a different sequence of field quanta energies, e.g. $\omega_2 + \omega_1 + \dots + \omega_{n-1} + \omega_n$. Assuming each of these different processes has the same probability, one can count the number of microstates, i.e. Ω , for the above process as $\Omega = 1/P_{\text{rad}}$. Then, using the Boltzmann definition of entropy as the natural logarithm of the number of microstates, one gets for the entropy of the emitted radiation

$$S_{\text{rad}} = \ln(\Omega) = \ln \left(\frac{1}{P_{\text{rad}}} \right) = \frac{4\pi}{\hbar} M^2 + \pi\alpha_1 \ln \left(1 + \frac{M^2}{\hbar} \right). \tag{3.7}$$

This entropy of the emitted radiation is identical to the original entropy of the black hole (see eq. (2.4)), thus entropy/information/“bits” are conserved between the initial (black hole plus no radiation) and final (no black hole plus radiated field quanta) states. This implies the same number of microstates between the initial and final states and, thus, unitary evolution. This then provides a possible resolution of the information paradox when the specific conditions are imposed.

The above arguments work even in the case where one ignores the quantum corrections [17–19], i.e. if one lets $\alpha_1 = 0$. While interesting, we are not sure how significant this is since almost certainly quantum corrections will become important as the black mass and entropy go to zero.

In this essay, we have examined the interrelationship of “bits” (information/entropy) and “its” (physical objects/systems) in the context of black hole information. By requiring

that the higher order quantum corrections given in eq. (1.4) be self-similar in the sense $I_j(r, t) \propto I_0$, and that the associated entropy/information of the black hole as given in eq. (2.2) be well behaved in the limit when the black hole mass goes to zero, we were able to relate all the higher order quantum corrections as parameterized by the α_j 's in terms of the first quantum correction α_1 . This proportionality of all α_j 's is another level of self-similarity. The final expression for this quantum corrected entropy, namely eq. (2.4), when combined with energy conservation and the tunneling picture of black hole radiation allow us to show how the original “bits” of black hole information encoded in the horizon were transformed into the “its” of the outgoing correlated Hawking photons, thus providing a potential all orders in \hbar solution to the black hole information loss puzzle.

Finally, as a last comment, it should be stressed that the assumption that the higher order corrections are self-similar in the sense given in eq. (5) (where we take $I_j \propto I_0$) and in eq. (9) (where we take $\alpha_{j+1} \propto \alpha_1$) is not at all what one would expect of the quantum corrections in the canonical approach to quantum gravity where the quantum corrections would in general generate any possible terms consistent with diffeomorphism-invariance. However, this is the problematic aspect of the canonical approach to quantum gravity and, thus, it is worth looking into radical suggestions such as the one proposed here, i.e. that the higher order quantum corrections are greatly simplified by the assumption of self-similarity. This simplification might be seen as an extreme form of the holographic principle of quantum gravity as expounded in [1]. In this monograph, it is pointed out that the entropy of a black hole scales with the area of the horizon while for a normal quantum field theory the entropy will scale as the volume. The conclusion of this observation is that “there are vastly fewer degrees of freedom in quantum gravity than in any QFT” (see section 11 of [1]). This assumption of self-similarity of the quantum corrections is in the vein of the holographic principle, since making the assumption of self-similarity means there are vastly fewer types/forms that the quantum corrections can take as compared to canonical quantum gravity.

Acknowledgments

There are two works — one on self-similarity [20] and one on the peculiar relationship between long distance/IR scales and short distance/UV scales in quantum gravity [21] — which helped inspire parts of this work.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] L. Susskind and J. Lindesay, *An introduction to black holes, information and the string theory revolution: the holographic universe*, World Scientific Publishing Co. Pte. Ltd., Danvers U.S.A. (2005).
- [2] M.K. Parikh and F. Wilczek, *Hawking radiation as tunneling*, *Phys. Rev. Lett.* **85** (2000) 5042 [[hep-th/9907001](https://arxiv.org/abs/hep-th/9907001)] [[INSPIRE](https://inspirehep.net/literature/9907001)].

- [3] K. Srinivasan and T. Padmanabhan, *Particle production and complex path analysis*, *Phys. Rev. D* **60** (1999) 024007 [[gr-qc/9812028](#)] [[INSPIRE](#)].
- [4] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333 [[INSPIRE](#)].
- [5] H. Von Koch, *On a continuous curve without tangents constructible from elementary geometry*, in *Classics on fractals*, G. Edgar ed., Addison-Wesley, Reading U.S.A. (1993), pg. 25.
- [6] D. Singleton, E.C. Vagenas, T. Zhu and J.-R. Ren, *Insights and possible resolution to the information loss paradox via the tunneling picture*, *JHEP* **08** (2010) 089 [*Erratum ibid.* **01** (2011) 021] [[arXiv:1005.3778](#)] [[INSPIRE](#)].
- [7] S.W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199 [*Erratum ibid.* **46** (1976) 206] [[INSPIRE](#)].
- [8] L. Xiang, *A note on the black hole remnant*, *Phys. Lett. B* **647** (2007) 207 [[gr-qc/0611028](#)] [[INSPIRE](#)].
- [9] M.K. Parikh, *Energy conservation and Hawking radiation*, [hep-th/0402166](#) [[INSPIRE](#)].
- [10] M. Arzano, A.J.M. Medved and E.C. Vagenas, *Hawking radiation as tunneling through the quantum horizon*, *JHEP* **09** (2005) 037 [[hep-th/0505266](#)] [[INSPIRE](#)].
- [11] S.K. Modak, *Corrected entropy of BTZ black hole in tunneling approach*, *Phys. Lett. B* **671** (2009) 167 [[arXiv:0807.0959](#)] [[INSPIRE](#)].
- [12] T. Zhu, J.-R. Ren and M.-F. Li, *Corrected entropy of Friedmann-Robertson-Walker universe in tunneling method*, *JCAP* **08** (2009) 010 [[arXiv:0905.1838](#)] [[INSPIRE](#)].
- [13] T. Zhu, J.-R. Ren and M.-F. Li, *Corrected entropy of high dimensional black holes*, [arXiv:0906.4194](#) [[INSPIRE](#)].
- [14] B. Zwiebach, *A first course in string theory*, Cambridge University Press, New York U.S.A. (2004), pg. 221.
- [15] K.A. Meissner, *Black hole entropy in loop quantum gravity*, *Class. Quant. Grav.* **21** (2004) 5245 [[gr-qc/0407052](#)] [[INSPIRE](#)].
- [16] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics and particle creation*, *Phys. Rev. D* **15** (1977) 2738 [[INSPIRE](#)].
- [17] B. Zhang, Q.-Y. Cai, L. You and M.-S. Zhan, *Hidden messenger revealed in Hawking radiation: a resolution to the paradox of black hole information loss*, *Phys. Lett. B* **675** (2009) 98 [[arXiv:0903.0893](#)] [[INSPIRE](#)].
- [18] B. Zhang, Q.-Y. Cai, M.-S. Zhan and L. You, *Entropy is conserved in Hawking radiation as tunneling: a revisit of the black hole information loss paradox*, *Annals Phys.* **326** (2011) 350 [[arXiv:0906.5033](#)] [[INSPIRE](#)].
- [19] Y.-X. Chen and K.-N. Shao, *Information loss and entropy conservation in quantum corrected Hawking radiation*, *Phys. Lett. B* **678** (2009) 131 [[arXiv:0905.0948](#)] [[INSPIRE](#)].
- [20] P. Nicolini and B. Niedner, *Hausdorff dimension of a particle path in a quantum manifold*, *Phys. Rev. D* **83** (2011) 024017 [[arXiv:1009.3267](#)] [[INSPIRE](#)].
- [21] E. Spallucci and S. Ansoldi, *Regular black holes in UV self-complete quantum gravity*, *Phys. Lett. B* **701** (2011) 471 [[arXiv:1101.2760](#)] [[INSPIRE](#)].