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Quantum periods for $\mathcal{N} = 2$ $SU(2)$ SQCD around the superconformal point

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Abstract

We study the Argyres–Douglas theories realized at the superconformal point in the Coulomb moduli space of $\mathcal{N} = 2$ supersymmetric $SU(2)$ QCD with $N_f = 1, 2, 3$ hypermultiplets in the Nekrasov–Shatashvili limit of the Omega-background. The Seiberg–Witten curve of the theory is quantized in this limit and the periods receive the quantum corrections. By applying the WKB method for the quantum Seiberg–Witten curve, we calculate the quantum corrections to the Seiberg–Witten periods around the superconformal point up to the fourth order in the parameter of the Omega background.

1. Introduction

A large class of $\mathcal{N} = 2$ supersymmetric gauge theories has a superconformal fixed point at strong coupling in the Coulomb moduli space, where mutually non-local BPS states become massless. This theory becomes an interacting $\mathcal{N} = 2$ superconformal field theory, which is called the Argyres–Douglas (AD) theory [1,2]. The BPS spectrum of the AD theory can be studied by the Seiberg–Witten (SW) curve, which are obtained from degeneration of the curve of $\mathcal{N} = 2$ gauge theories [1–3]. The dynamics of AD theories is an interesting subject of recent studies from the viewpoint of M5-branes compactified on a punctured Riemann surface [4–6] and its relation to two-dimensional conformal field theories [7–10].

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In the weak coupling region, one can compute the partition function of $\mathcal{N}=2$ gauge theories based on the microscopic Lagrangian in the Ω-background, which deforms four-dimensional spacetime by the torus action with two parameters (ϵ_1, ϵ_2) [11,12]. The partition function is related to conformal blocks of two-dimensional conformal field theories [13,14], the partition functions of topological strings [15,16], and the solutions of the Painléve equations [17], where the Ω-deformation parameters enter into the formulas of the central charges and the string coupling. It would be interesting to study the effects of the Ω-deformations in the strong coupling region. However in the strong coupling region such as the superconformal point, we have no appropriate microscopic Lagrangian. In the case of the self dual Ω-background with $\epsilon_1 = -\epsilon_2$, the Argyres–Douglas theories have been studied by using the holomorphic anomaly equation [15,18] and the E-strings [19].

The purpose of this paper is to study the Argyres–Douglas theories in the Ω-background realized at the superconformal point of $\mathcal{N}=2$ supersymmetric gauge theories. In particular, we consider the Nekrasov–Shatashvili (NS) limit [20] of the Ω background where one of the deformation parameters ϵ_2 is set to be zero. In this limit the SW curve becomes a differential equation which is obtained by the canonical quantization procedure of the symplectic structure induced by the SW differential. The Planck constant $\hbar$ corresponds to the remaining deformation parameter ϵ_1. The WKB solution of the differential equation gives the Ω-deformation of the SW periods which is the main subject of this paper.

The quantum SW curve has been studied for $\mathcal{N}=2$ theories in the weak coupling regions. A simple example is $SU(2)$ pure Yang–Mills theory where the quantum SW curve becomes the Schrödinger equation with the sine-Gordon potential [21] and the WKB solution is shown to agree with that obtained from the NS limit of the Nekrasov function. The expansion of the periods around the massless monopole point in the Coulomb moduli space has been studied in [22]. For $\mathcal{N}=2$ $SU(2)$ SQCD with $N_f \leq 4$ hypermultiplets, the WKB solutions of the quantum SW curves have been studied in [23] in the weak coupling region, while in the strong coupling region the solutions around the massless monopole point have been studied in [24]. Generalization to other $\mathcal{N}=2$ theories and their relations to the Nekrasov partition functions have been studied extensively [25,23,26–28].

In this paper we will study the quantum SW periods around the superconformal point of the moduli space of $\mathcal{N}=2$ $SU(2)$ SQCD with $N_f = 1, 2, 3$ hypermultiplets. The SW curve degenerates into a simpler curve which represents the SW curve of the Argyres–Douglas theory. We will calculate the WKB solution of the quantum SW curve of the AD theory and compute the quantum corrections up to the fourth order in $\hbar$.

This paper is organized as follows: In Section 2, we review the SW curve and the SW differential near the superconformal point of the $\mathcal{N}=2$ $SU(2)$ SQCD. In Section 3, we quantize the SW curve of the AD theories and derive the differential equations satisfied by quantum periods. In Section 4, we calculate the quantum corrections to the SW periods near the superconformal point, which are expressed in terms of the hypergeometric function. Section 5 is devoted to conclusions and discussion. In the Appendix, we present detailed analysis of the fourth order terms in the quantum SW periods for the $N_f = 3$ AD theory.

2. Seiberg–Witten curve at the superconformal point

In this section we study the Argyres–Douglas theory which appears at the superconformal point in the moduli space of $\mathcal{N}=2$ $SU(2)$ SQCD with $N_f = 1, 2, 3$ hypermultiplets. We begin with the Seiberg–Witten curve for the $\mathcal{N}=2$ $SU(2)$ gauge theory with $N_f(=1,2,3)$ hypermultiplets which is given by

$$C(p)-\frac{\Lambda_{N_f}^{2-\frac{N_f}{2}}}{2}\left(z+\frac{G(p)}{z}\right)=0, \tag{2.1}$$

where Λ_{N_f} is the QCD scale parameter. $C(p)$ and $G(p)$ are defined by

$$C(p)=\begin{cases} p^2-u, & N_f=1,\\ p^2-u+\frac{\Lambda_2^2}{8}, & N_f=2,\\ p^2-u+\frac{\Lambda_3}{4}\left(p+\frac{m_1+m_2+m_3}{2}\right), & N_f=3,\end{cases} \tag{2.2}$$

$$G(p)=\prod_{i=1}^{N_f}(p+m_i), \tag{2.3}$$

where u is the Coulomb moduli parameter and $m_1,\ldots,m_{N_f}$ are the mass parameters of the hypermultiplets. The SW differential is defined by

$$\lambda_{\mathrm{SW}}=p\left(d\log G(p)-2d\log z\right). \tag{2.4}$$

The SW periods $\Pi^{(0)}:=(a^{(0)},a_D^{(0)})$ are

$$a^{(0)}(u)=\oint_\alpha \lambda_{\mathrm{SW}},\qquad a_D^{(0)}(u)=\oint_\beta \lambda_{\mathrm{SW}} \tag{2.5}$$

where α and β are the canonical one-cycles on the curve. Here the superscript (0) refers the “undeformed” (or classical) period. The SW curve (2.1) can be written into the standard form [29]

$$y^2=C(p)^2-\Lambda_{N_f}^{4-N_f}G(p) \tag{2.6}$$

by introducing

$$y=\Lambda_{N_f}^{2-\frac{N_f}{2}}z-C(p). \tag{2.7}$$

The SW differential (2.4) is expressed as

$$\lambda_{SW}=pd\log\left(\frac{C(p)-y}{C(p)+y}\right). \tag{2.8}$$

The u-derivative of the SW differential becomes the holomorphic differential:

$$\frac{\partial\lambda_{\mathrm{SW}}}{\partial u}=\frac{2\partial_u z}{z}dp+d(*)=\frac{2dp}{y}+d(*) \tag{2.9}$$

where $\partial_u:=\frac{\partial}{\partial u}$. Differentiating the SW period $\Pi^{(0)}$ with respect to u, one obtains the periods for the curve:

$$\partial_u a^{(0)}(u)=\oint_\alpha\frac{2\partial_u z}{z}dp=\oint_\alpha\frac{2}{y}dp,\qquad \partial_u a_D^{(0)}(u)=\oint_\beta\frac{2\partial_u z}{z}dp=\oint_\beta\frac{2}{y}dp. \tag{2.10}$$

The period $\partial_u \Pi^{(0)}$ is evaluated as the elliptic integral. For the curve of the form $y^2 = \prod_{i=1}^{4}(x-e_i)$, it is convenient to introduce the variables

$$D = \sum_{i<j} e_i^2 e_j^2 - 6\prod_{i=1}^{4} e_i - \sum_{i<j<k} (e_i^2 e_j e_k + e_i e_j^2 e_k + e_i e_j e_k^2), \tag{2.11}$$

$$w = -\frac{27\Delta}{4D^3}, \tag{2.12}$$

where Δ is of the discriminant

$$\Delta = \prod_{i<j}(e_i - e_j)^2, \tag{2.13}$$

and w is inverse of the modular J-function of the curve [30]. Then it is shown that the integral $F = (-D)^{\frac{1}{4}} \int \frac{dx}{y}$ obeys the hypergeometric differential equation

$$w(1-w)\frac{d^2F}{dw^2} + (\gamma - (\alpha+\beta+1)w)\frac{dF}{dw} - \alpha\beta F = 0 \tag{2.14}$$

with $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $\gamma = 1$. For the SW curve (2.6) this leads to the Picard–Fuchs equation for $\Pi^{(0)}$ [31–33,24] as the third order differential equation with respect to u.

There are singularities on the u-plane where some BPS particles become massless and the discriminant Δ (2.13) becomes zero. We consider the superconformal or Argyres–Douglas (AD) point on the u-plane where mutually nonlocal BPS particles become massless [1,2]. For the $SU(2)$ theory with N_f hypermultiplets, the squark and monopole/dyon are both massless at the AD point, where the SW curve degenerates and has higher order zero. For the $SU(2)$ theories with $N_f = 1, 2, 3$ hypermultiplets, the AD points are given as follows: For $N_f = 1$, the Coulomb moduli and the mass are chosen as

$$u = \frac{3}{4}\Lambda_1^2, \qquad m_1 = \frac{3}{4}\Lambda_1. \tag{2.15}$$

The SW curve (2.6) becomes

$$y^2 = \left(p - \frac{2}{3}\Lambda_1\right)\left(p + \frac{1}{2}\Lambda_1\right)^3. \tag{2.16}$$

For $N_f = 2$, we have

$$u = \frac{3}{8}\Lambda_2^2, \qquad m_1 = m_2 = \frac{\Lambda_2}{2}, \tag{2.17}$$

so that the SW curve (2.6) becomes

$$y^2 = \left(p - \frac{3}{2}\Lambda_2\right)\left(p + \frac{\Lambda_2}{2}\right)^3. \tag{2.18}$$

For $N_f = 3$, the superconformal point is given by

$$u = \frac{1}{32}\Lambda_3^2, \qquad m_1 = m_2 = m_3 = \frac{\Lambda_3}{8}, \tag{2.19}$$

where the SW curve (2.6) becomes

$$y^2 = \left(p - \frac{7}{8}\Lambda_3\right)\left(p + \frac{\Lambda_3}{8}\right)^3. \tag{2.20}$$

Let us study the SW curve and the SW differential around the superconformal point. By taking the scaling limit, we identify the operators and couplings which deform the superconformal point. Their scaling dimensions are determined by the SW curve and the fact that the SW differential has the scaling dimension one. We first consider in the $N_f = 1$ theory. The branch point $p = -\frac{\Lambda_1}{2}$ of the curve (2.16) corresponds to $z = \pm\frac{\Lambda_1^{\frac{1}{2}}}{2}$. We expand the curve (2.1) around $z = -\frac{\Lambda_1^{\frac{1}{2}}}{2}$ by introducing

$$\begin{aligned} p =& \epsilon\tilde{p} - \frac{\Lambda_1}{2}, \qquad z = \frac{i2^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}{\Lambda_1}\tilde{z} - \frac{\epsilon^2\tilde{M}}{\Lambda_1^{\frac{1}{2}}} - \frac{\epsilon\tilde{p}}{\Lambda_1^{\frac{1}{2}}} - \frac{\Lambda_1^{\frac{1}{2}}}{2}, \\ u =& \epsilon^3\tilde{u} + \epsilon^2\tilde{M}\Lambda_1 + \frac{3}{4}\Lambda_1^2, \qquad m_1 = \epsilon^2\tilde{M} + \frac{3}{4}\Lambda_1, \end{aligned} \tag{2.21}$$

and consider the scaling limit $\epsilon \to 0$ with fixed $\tilde{u}$ and $\tilde{M}$. At the leading order in ϵ we obtain the curve for the AD theory of (A_1, A_2)-type:

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{M}\Lambda_1\tilde{p} - \frac{\Lambda_1}{2}\tilde{u}. \tag{2.22}$$

Substituting (2.21) into the SW differential (2.4) and expanding around $\epsilon = 0$, the SW differential becomes

$$\lambda_{SW} = \frac{i\epsilon^{\frac{5}{2}}}{2^{\frac{1}{2}}\Lambda_1^{\frac{1}{2}}}\tilde{\lambda}_{SW} + \dots, \tag{2.23}$$

$$\tilde{\lambda}_{SW} := -\frac{8}{\Lambda_1}\tilde{z}d\tilde{p}. \tag{2.24}$$

We read off the scaling dimension of $\tilde{u}$ and $\tilde{M}$ as $\frac{6}{5}$ and $\frac{4}{5}$, respectively, from the curve (2.22). Here $\tilde{u}$ is the operator and $\tilde{M}$ is the corresponding coupling parameter.

For $N_f = 2$, defining the new variables as

$$\begin{aligned} p =& \epsilon\tilde{p} - \frac{\epsilon\tilde{M}}{3} - \frac{\Lambda_2}{2}, \qquad z = \frac{i2^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}{\Lambda_2^{\frac{1}{2}}}\tilde{z} - \epsilon\tilde{p} - \frac{2\epsilon\tilde{M}}{3}, \\ u =& \epsilon^2\tilde{u} - \frac{(\epsilon\tilde{M})^2}{3} + \Lambda_2\epsilon\tilde{M} + \frac{3\Lambda_2^2}{8}, \\ m_1 =& \frac{\Lambda_2}{2} + \epsilon\tilde{M} + \epsilon^{\frac{3}{2}}\tilde{a}, \qquad m_2 = \frac{\Lambda_2}{2} + \epsilon\tilde{M} - \epsilon^{\frac{3}{2}}\tilde{a}, \end{aligned} \tag{2.25}$$

and expanding the curve around $\epsilon = 0$, we find that the curve (2.1) become

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{u}\tilde{p} - \frac{2}{3}\tilde{M}\tilde{u} + \frac{8}{27}\tilde{M}^3 - \frac{\tilde{C}_2\Lambda_2}{4}. \tag{2.26}$$

Here $\tilde{u}$ is the operator, $\tilde{M}$ is the coupling and $\tilde{C}_2 := 2\tilde{a}^2$ is the Casimir invariant of the $U(2)$ flavor symmetry. The corresponding AD theory is of (A_1, A_3)-type.

Substituting (2.25) into (2.4), the SW differential around the superconformal point is

$$\lambda_{\rm SW} = \frac{i\epsilon^{\frac{3}{2}}}{2^{\frac{1}{2}}\Lambda_2^{\frac{1}{2}}}\tilde{\lambda}_{\rm SW} + \cdots \tag{2.27}$$

up to the total derivatives where

$$\tilde{\lambda}_{\rm SW} = -4\tilde{z}\, d\log\left(\tilde{p} + \frac{2}{3}\tilde{M}\right). \tag{2.28}$$

The scaling dimension of $\tilde{u}$, $\tilde{M}$ and $\tilde{C}_2$ are $\frac{4}{3}$, $\frac{2}{3}$ and 2, respectively.

For $N_f = 3$, we define the scaling variables as

$$\begin{aligned}
p &= \epsilon^2\tilde{p} - \epsilon\tilde{M} + \frac{4\left((\epsilon\tilde{M})^2 + \epsilon^3\tilde{u}\right)}{3\Lambda_3} + \frac{16(\epsilon\tilde{M})^3}{9\Lambda_3^2} - \frac{\Lambda_3}{8},\\
z &= \epsilon^3 i\tilde{z} - \frac{4(\epsilon\tilde{M})^3}{3\Lambda_3^{\frac{3}{2}}} - \frac{2(\epsilon\tilde{M})(\epsilon^2\tilde{p})}{\Lambda_3^{\frac{1}{2}}} - \frac{\epsilon^3\tilde{u}}{\Lambda_3^{\frac{1}{2}}},\\
u &= \epsilon^3\tilde{u} - \frac{4(\epsilon\tilde{M})^3}{3\Lambda_3} + (\epsilon\tilde{M})^2 + \frac{3\Lambda_3\epsilon\tilde{M}}{8} + \frac{\Lambda_3^2}{32},\\
m_1 &= \frac{\Lambda_3}{8} + \epsilon\tilde{M} + \epsilon^2\tilde{c}_1, \qquad m_2 = \frac{\Lambda_3}{8} + \epsilon\tilde{M} + \epsilon^2\tilde{c}_2,\\
m_3 &= \frac{\Lambda_3}{8} + \epsilon\tilde{M} - \epsilon^2(\tilde{c}_1 + \tilde{c}_2),
\end{aligned} \tag{2.29}$$

and then consider the limit $\epsilon \to 0$ limit with keeping $\tilde{u}$, $\tilde{M}$, $\tilde{c}_1$ and $\tilde{c}_2$ finite. Rescaling the curve (2.1) we obtain the curve of the AD theory of (A_1, D_4) type:

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{p}\left(\frac{\tilde{C}_2}{2} + \frac{4\tilde{M}\tilde{u}}{\Lambda_3}\right) - \frac{\tilde{u}^2}{\Lambda_3} - \frac{8\tilde{M}^3\tilde{u}}{3\Lambda_3^2} + \frac{16\tilde{M}^6}{27\Lambda_3^3} - \frac{2\tilde{C}_2\tilde{M}^2}{3\Lambda_3} + \frac{\tilde{C}_3}{3} \tag{2.30}$$

where

$$\tilde{C}_2 := 2\left(\tilde{c}_1^2 + \tilde{c}_1\tilde{c}_2 + \tilde{c}_2^2\right), \qquad \tilde{C}_3 := -3\left(\tilde{c}_1^2\tilde{c}_2 + \tilde{c}_1\tilde{c}_2^2\right). \tag{2.31}$$

Here $\tilde{u}$ is the operator and $\tilde{M}$ is the coupling. $\tilde{C}_2$ and $\tilde{C}_3$ are the Casimir invariants associated with the $U(3)$ flavor symmetry. Then the SW differential (2.4) at the superconformal point becomes

$$\lambda_{\rm SW} = \frac{i\epsilon^2}{\Lambda_3^{\frac{1}{2}}}\tilde{\lambda}_{\rm SW} + \cdots \tag{2.32}$$

up to the total derivatives where

$$\tilde{\lambda}_{\rm SW} = i\Lambda_3^{\frac{1}{2}}\left\{2\tilde{p}\, d\log\left(i\tilde{z} - \frac{2\tilde{M}\tilde{p}}{\Lambda_3^{\frac{1}{2}}} - \frac{4\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} - \frac{\tilde{u}}{\Lambda_3^{\frac{1}{2}}}\right) - \sum_{i=1}^{3}\tilde{p}\, d\log(\tilde{p} + \tilde{m}_i)\right\}. \tag{2.33}$$

$\tilde{m}_i$ $(i = 1, \dots, 3)$ are defined by

$$\tilde{m}_1 = \frac{4\tilde{M}^2}{3\Lambda_3} + \tilde{c}_1, \qquad \tilde{m}_2 = \frac{4\tilde{M}^2}{3\Lambda_3} + \tilde{c}_2, \qquad \tilde{m}_3 = \frac{4\tilde{M}^2}{3\Lambda_3} - (\tilde{c}_1 + \tilde{c}_2). \tag{2.34}$$

These parameters are interpreted as the mass parameters at the superconformal point. We see that the scaling dimensions of $\tilde{u}$, $\tilde{M}$, $\tilde{C}_2$, $\tilde{C}_3$ are $\frac{3}{2}$, $\frac{1}{2}$, 2 and 3, respectively.

We now study the SW periods for the AD theories associated with $SU(2)$ theory with N_f hypermultiplets. We write the SW curves in the form of

$$\tilde{z}^2 = \tilde{p}^3 - \rho_{N_f}\tilde{p} - \sigma_{N_f} \tag{2.35}$$

for the N_f AD theory. Here ρ_{N_f} and σ_{N_f} are read off from (2.22), (2.26) and (2.30). We have normalized the SW differential $\tilde{\lambda}_{SW}$ (2.24), (2.28) and (2.33) such that

$$\frac{\partial}{\partial\tilde{u}}\tilde{\lambda}_{SW} = \frac{2d\tilde{p}}{\tilde{z}}. \tag{2.36}$$

The SW periods are defined by

$$\tilde{\Pi}^{(0)} = (\tilde{a}^{(0)}, \tilde{a}_D^{(0)}) = \left(\int_{\tilde{\alpha}} \tilde{\lambda}_{SW}, \int_{\tilde{\beta}} \tilde{\lambda}_{SW}\right), \tag{2.37}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are canonical 1-cycles on the curve (2.35). Differentiating the SW periods with respect to $\tilde{u}$, we have the period integral $\int \frac{d\tilde{p}}{\tilde{z}}$ of the holomorphic differential $\frac{d\tilde{p}}{\tilde{z}}$:

$$\omega = \int_{\tilde{\alpha}} \frac{d\tilde{p}}{\tilde{z}}, \quad \omega_D = \int_{\tilde{\beta}} \frac{d\tilde{p}}{\tilde{z}}. \tag{2.38}$$

As in the case of $SU(2)$ SQCD, the period integral is expressed in terms of the hypergeometric functions of the argument:

$$\tilde{w}_{N_f} := -\frac{27\tilde{\Delta}_{N_f}}{4\tilde{D}_{N_f}^3} = 1 - \frac{27\sigma_{N_f}^2}{4\rho_{N_f}^3}. \tag{2.39}$$

Here $\tilde{\Delta}_{N_f}$ and $\tilde{D}_{N_f}$ correspond to Δ in (2.13) and D in (2.11), respectively, which are defined by

$$\tilde{\Delta}_{N_f} = 4\rho_{N_f}^3 - 27\sigma_{N_f}^2, \tag{2.40}$$

$$\tilde{D}_{N_f} = -3\rho_{N_f}. \tag{2.41}$$

For example, we will evaluate the integrals (2.38) around the point $\tilde{w}_{N_f} = 0$, where the $\tilde{\alpha}$-cycle is chosen as a vanishing cycle. Using the quadratic and cubic transformation [34,35], the periods are given by

$$\omega^0(\tilde{w}, \tilde{D}) = 2\pi\left(-\tilde{D}\right)^{-\frac{1}{4}} F\left(\frac{1}{12}, \frac{5}{12}; 1; \tilde{w}\right), \tag{2.42}$$

$$\omega_D^0(\tilde{w}, \tilde{D}) = -2i\pi\left(-\tilde{D}\right)^{-\frac{1}{4}}\left(\frac{3\log 12}{2\pi}F\left(\frac{1}{12}, \frac{5}{12}; 1; \tilde{w}\right) - \frac{1}{2\pi}F_*\left(\frac{1}{12}, \frac{5}{12}; 1; \tilde{w}\right)\right), \tag{2.43}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. $F_*(\alpha, \beta; 1; z)$ is defined by

$$F_*(\alpha, \beta; 1; z) = F(\alpha, \beta; 1; z)\log z + F_1(\alpha, \beta; 1; z) \tag{2.44}$$

and

$$F_1(\alpha, \beta; 1; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n!)^2} \sum_{r=0}^{n-1} \left(\frac{1}{\alpha + r} + \frac{1}{\beta + r} - \frac{2}{1 + r} \right) z^n. \tag{2.45}$$

We have omitted the subscript N_f of $\tilde{w}$ and $\tilde{D}$ for brevity. Since the dual period has logarithmic divergence around $\tilde{w} = 0$, it does not represent the expansion around the superconformal point, where $\tilde{u}$ and $\tilde{M}$ have fractional scaling dimensions.

We will perform the analytic continuation of the solutions around $\tilde{w} = 0$ to those of $\tilde{w} = \infty$ by using the connection formula [34]

$$\begin{aligned} F(\alpha, \beta; \gamma; z) =& \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (1 - z)^{-\alpha} F\left(\alpha, \gamma - \beta; \alpha - \beta + 1; \frac{1}{1 - z}\right) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (1 - z)^{-\beta} F\left(\beta, \gamma - \alpha; -\alpha + \beta + 1; \frac{1}{1 - z}\right), \end{aligned} \tag{2.46}$$

where $|\arg(1 - z)| < \pi$. We then find that the periods (2.42) and (2.43) become

$$\begin{aligned} \omega^{\infty}(\tilde{w}, \tilde{D}) =& 2\pi(-\tilde{D})^{-\frac{1}{4}} \left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)} (1 - \tilde{w})^{-\frac{1}{12}} F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{1}{1 - \tilde{w}}\right) \right. \\ &\left. + \frac{\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)} (1 - \tilde{w})^{-\frac{5}{12}} F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \frac{1}{1 - \tilde{w}}\right) \right), \end{aligned} \tag{2.47}$$

$$\begin{aligned} \omega_D^{\infty}(\tilde{w}, \tilde{D}) =& 2i\pi(-\tilde{D})^{-\frac{1}{4}} \left(\frac{(-1)^{\frac{5}{6}}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)} (1 - \tilde{w})^{-\frac{1}{12}} F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{1}{1 - \tilde{w}}\right) \right. \\ &\left. + \frac{(-1)^{\frac{1}{6}}\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)} (1 - \tilde{w})^{-\frac{5}{12}} F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \frac{1}{1 - \tilde{w}}\right) \right), \end{aligned} \tag{2.48}$$

respectively. Similarly we can perform the analytic continuation to the solutions around $\tilde{w} = 1$. By using the connection formula

$$\begin{aligned} F(\alpha, \beta; \gamma; z) =& \frac{(1 - z)^{-\alpha-\beta+\gamma}\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\ &\times F(\gamma - \alpha, \gamma - \beta; -\alpha - \beta + \gamma + 1; 1 - z) \\ &+ \frac{\Gamma(\gamma)\Gamma(-\alpha - \beta + \gamma)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z), \end{aligned} \tag{2.49}$$

we obtain expansion around $\tilde{w} = 1$:

$$\begin{aligned} \omega^1(\tilde{w}, \tilde{D}) =& \pi^{-\frac{1}{2}}(-\tilde{D})^{-\frac{1}{4}} \left(6\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right) F\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; 1 - \tilde{w}\right) \right. \\ &\left. -(1 - \tilde{w})^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right) F\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}; 1 - \tilde{w}\right) \right), \end{aligned} \tag{2.50}$$

$$\begin{aligned} \omega_D^1(\tilde{w}, \tilde{D}) =& -i\pi^{-\frac{1}{2}}(-\tilde{D})^{-\frac{1}{4}} \left(6\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right) F\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; 1 - \tilde{w}\right) \right. \\ &\left. +(1 - \tilde{w})^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right) F\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}; 1 - \tilde{w}\right) \right). \end{aligned} \tag{2.51}$$

Based on these formulas, we discuss the SW periods for the AD theories. For the $N_f = 1$ theory, $\tilde{w}_1$ and $\tilde{D}_1$ are given by

$$\tilde{w}_1 = 1 - \frac{27\tilde{u}^2}{16\Lambda_1 \tilde{M}^3}, \tag{2.52}$$

$$\tilde{D}_1 = -3\Lambda_1 \tilde{M}. \tag{2.53}$$

The superconformal point corresponds to $\tilde{w}'_1 := \frac{1}{1-\tilde{w}_1} = 0$. Therefore eqs. (2.47) and (2.48) give the expansion around the superconformal point:

$$\frac{\partial \tilde{a}}{\partial \tilde{u}} = 2\omega^{\infty}(\tilde{w}_1, \tilde{D}_1), \qquad \frac{\partial \tilde{a}_D}{\partial \tilde{u}} = 2\omega_D^{\infty}(\tilde{w}_1, \tilde{D}_1). \tag{2.54}$$

By integrating them over $\tilde{u}$, we obtain the SW periods

$$\begin{aligned}\tilde{a}^{(0)} =& \frac{3^{\frac{1}{2}}\Lambda_1^{\frac{3}{2}}}{2^{\frac{1}{2}}\cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(2^{\frac{8}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(-\frac{5}{12},\frac{1}{12};\frac{2}{3};\tilde{w}'_1\right)\right.\\ &\left.+15\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{3}\right)F\left(-\frac{1}{12},\frac{5}{12};\frac{4}{3};\tilde{w}'_1\right)\right),\end{aligned} \tag{2.55}$$

$$\begin{aligned}\tilde{a}_D^{(0)} =& \frac{3^{\frac{1}{2}}\Lambda_1^{\frac{3}{2}}}{2^{\frac{1}{2}}\cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(-2^{\frac{8}{3}}(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(-\frac{5}{12},\frac{1}{12};\frac{2}{3};\tilde{w}'_1\right)\right.\\ &\left.+15(-1)^{\frac{2}{3}}\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{3}\right)F\left(-\frac{1}{12},\frac{5}{12};\frac{4}{3};\tilde{w}'_1\right)\right).\end{aligned} \tag{2.56}$$

We note that the SW periods $\tilde{\Pi}^{(0)}$ satisfy the Picard–Fuchs equation [36]

$$\left(1-\tilde{w}'_1\right)\tilde{w}'_1\frac{\partial^2}{\partial \tilde{w}_1'^2}\tilde{\Pi}^{(0)} + \frac{2}{3}\left(1-\tilde{w}'_1\right)\frac{\partial}{\partial \tilde{w}'_1}\tilde{\Pi}^{(0)} + \frac{5}{144}\tilde{\Pi}^{(0)} = 0. \tag{2.57}$$

From (2.55) and (2.56) we see that the SW periods scale as $\tilde{u}^{\frac{5}{6}}$. Since the SW periods $a^{(0)}$ and $a_D^{(0)}$ have the scaling dimension one, the scaling dimension of $\tilde{u}$ and $\tilde{M}$ is given by $\frac{6}{5}$ and $\frac{4}{5}$, respectively [2]. The expansion of the coupling constant $\tau^{(0)} := \frac{\partial_{\tilde{u}}\tilde{a}_D^{(0)}}{\partial_{\tilde{u}}\tilde{a}^{(0)}}$ in $\tilde{w}'_1$ does not contain logarithmic terms, which implies that the theory is around the superconformal point. The SW periods (2.55) and (2.56) represent the expansions in the coupling $\tilde{M}$ with fixed $\tilde{u}$ in the scaling limit. We note that the present expansions for N_f theories are different from the results in the previous literatures [35,15], where the coupling and the Casimir invariants are chosen to be zero, $\tilde{u}$ is small without taking the scaling limit. In [37] the expansion of the SW periods without taking the scaling limit has been presented.

For the $N_f = 2$ theory, we have

$$\tilde{w}_2 = 1 - \frac{(\frac{27}{2}\tilde{C}_2\Lambda_2 - 16\tilde{M}^3 + 36\tilde{M}\tilde{u})^2}{432\tilde{u}^3}, \tag{2.58}$$

$$\tilde{D}_2 = -3\tilde{u}. \tag{2.59}$$

The superconformal point corresponds to $\tilde{w}_2 = 1$ or $\tilde{w}'_2 := 1 - \tilde{w}_2 = 0$. Eqs. (2.50) and (2.51) provide the expansion around the superconformal point:

$$\frac{\partial\tilde{a}}{\partial\tilde{u}} = 2\omega^1(\tilde{w}_2,\tilde{D}_2), \qquad \frac{\partial\tilde{a}_D}{\partial\tilde{u}} = 2\omega^1_D(\tilde{w}_2,\tilde{D}_2). \tag{2.60}$$

Expanding them around $\tilde{w}'_2 = 0$, where $\frac{\tilde{M}^2}{\tilde{u}} \ll 1$ and $\frac{\tilde{C}_2\Lambda_2}{\tilde{u}^{\frac{3}{2}}} \ll 1$, and integrating over $\tilde{u}$, one obtains the SW periods, which are given by

$$\begin{aligned}\tilde{a}^{(0)} =& \Lambda_2^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{\frac{3}{4}}\left(\frac{2^4\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}{3^{\frac{1}{4}}\pi^{\frac{1}{2}}} - \frac{2^3\cdot 3^{\frac{1}{4}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\right.\\ &\left. - \frac{3^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}} + \cdots\right),\end{aligned} \tag{2.61}$$

$$\begin{aligned}\tilde{a}_D^{(0)} =& \Lambda_2^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{\frac{3}{4}}\left(-\frac{2^4 i\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}{3^{\frac{1}{4}}\pi^{\frac{1}{2}}} - \frac{2^3\cdot 3^{\frac{1}{4}} i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\right.\\ &\left. - \frac{3^{\frac{1}{2}} i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}} + \cdots\right).\end{aligned} \tag{2.62}$$

We see again that the scaling dimensions of $\tilde{u}$, $\tilde{M}$ and $\tilde{C}_2$ are $\frac{4}{3}$, $\frac{2}{3}$ and 2, respectively. The expansions of the periods (2.61) and (2.62) have no logarithmic behavior.

For the $N_f = 3$ theory, we have

$$\tilde{w}_3 = 1 - \frac{(-9\tilde{C}_3\Lambda_3^3 + 18\tilde{C}_2\Lambda_3^2\tilde{M}^2 - 16\tilde{M}^6 + 72\Lambda_3\tilde{M}^3\tilde{u} + 27\Lambda_3^2\tilde{u}^2)^2}{108\Lambda_3^6\left(\frac{\tilde{C}_2}{2} + 4\frac{\tilde{M}\tilde{u}}{\Lambda_3}\right)^3}, \tag{2.63}$$

$$\tilde{D}_3 = -3\left(\frac{\tilde{C}_2}{2} + \frac{4\tilde{M}\tilde{u}}{\Lambda_3}\right). \tag{2.64}$$

The superconformal point corresponds to $\tilde{w}_3 = \infty$ or $\tilde{w}'_3 := \frac{1}{1-\tilde{w}_3} = 0$. Then (2.47) and (2.48) provides the periods around the superconformal point:

$$\frac{\partial\tilde{a}}{\partial\tilde{u}} = 2\omega^\infty(\tilde{w}_3,\tilde{D}_3), \qquad \frac{\partial\tilde{a}_D}{\partial\tilde{u}} = 2\omega^\infty_D(\tilde{w}_3,\tilde{D}_3). \tag{2.65}$$

Expanding these in $\tilde{w}'_3$, where $\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \ll 1$, $\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4} \ll 1$ and $\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2} \ll 1$, and integrating (2.65) over $\tilde{u}$, we obtain the SW periods:

$$\begin{aligned}\tilde{a}^{(0)} =& \Lambda_3^{\frac{3}{2}}(-1)^{\frac{5}{6}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{\frac{2}{3}}\left(\frac{5\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2\cdot 3^{\frac{1}{2}}\pi^{\frac{1}{2}}} - \frac{2^3\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}}\right.\\ &\left. + \frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2\cdot\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4}\right)^{\frac{1}{3}} + \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2}\right) + \cdots\right),\end{aligned} \tag{2.66}$$

$$\tilde{a}_D^{(0)} = \Lambda_3^{\frac{3}{2}} (-1)^{\frac{1}{6}} \left(\frac{\tilde{u}}{\Lambda_3^{\frac{2}{3}}} \right)^{\frac{2}{3}} \left(\frac{5\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2 \cdot 3^{\frac{1}{2}} \pi^{\frac{1}{2}}} - \frac{2^3 \Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{3^{\frac{1}{2}} \pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{1}{3}} \right.$$
$$\left. + \frac{i\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot \pi^{\frac{1}{2}}} \left(\frac{\tilde{C}_2^3 \Lambda_3^2}{\tilde{u}^4} \right)^{\frac{1}{3}} + \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2 \cdot 3^{\frac{3}{2}} \pi^{\frac{1}{2}}} \left(\frac{\tilde{C}_3 \Lambda_3}{\tilde{u}^2} \right) + \cdots \right). \tag{2.67}$$

It turns out that the scaling dimensions of $\tilde{u}$, $\tilde{M}$, $\tilde{C}_2$ and $\tilde{C}_3$ are $\frac{3}{2}$, $\frac{1}{2}$, 2 and 3, respectively. As in the case of $N_f = 1$ and 2 theories, the expansion of the SW periods has no logarithmic term.

Although the SW curves for N_f theories become a common cubic form, their SW differentials take different forms due to the flavor symmetry. This means that we need to introduce different quantization conditions for each N_f as we will discuss in the next section.

3. Quantum Seiberg–Witten curves and periods

In this section we study the deformation of the SW periods in the Ω-background at the superconformal point for the $SU(2)$ gauge theory with $N_f(= 1, 2, 3)$ hypermultiplets. We take the Nekrasov–Shatashvili (NS) limit such that one of the two deformation parameters (ϵ_1, ϵ_2) of the Ω background is going to be zero. The other parameter plays a role of the Planck constant $\hbar$. From the analysis of the Ω-deformed low-energy effective action, the deformed periods in the NS limit are shown to satisfy the Bohr–Sommerfeld quantization condition [20]:

$$\oint \lambda_{\mathrm{SW}} = in\hbar, \qquad (n \in \mathbf{Z}). \tag{3.1}$$

This condition also follows from the quantization of the SW curve, which is introduced by the canonical quantization of the holomorphic symplectic structure defined by $d\lambda_{SW}$. The quantum SW curve becomes the ordinary differential equation. Its WKB solution gives the quantum correction to the SW periods, which can be represented in the form $\hat{\mathcal{O}}_k \Pi^{(0)}$ for some differential operator $\hat{\mathcal{O}}_k$ with respect to the moduli parameters. In the following we will construct $\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_4$ explicitly and compute the second and fourth order corrections to the SW periods in $\hbar$ around the superconformal point.

3.1. $N_f = 1$ theory

We start with the $N_f = 1$ theory. The SW differential (2.24) defines a symplectic form $d\tilde{\lambda}_{\mathrm{SW}} = d\tilde{z} \wedge d\tilde{p}$ on the $(\tilde{z}, \tilde{p})$ space. We quantize the system by replacing the coordinate $\tilde{z}$ by the differential operator:

$$\tilde{z} = -i\hbar \frac{\partial}{\partial \tilde{p}}. \tag{3.2}$$

Then the SW curve becomes the Schrödinger type equation:

$$\left(-\hbar^2 \frac{\partial^2}{\partial \tilde{p}^2} + Q(\tilde{p}) \right) \Psi(\tilde{p}) = 0, \tag{3.3}$$

where

$$Q(\tilde{p}) = -\left(\tilde{p}^3 - \tilde{M}\Lambda_1 \tilde{p} - \frac{\Lambda_1}{2}\tilde{u} \right). \tag{3.4}$$

We study the WKB solution to the equation (3.3):

$$\Psi(\tilde{p}) = \exp\left(\frac{i}{\hbar}\int^{\tilde{p}} \Phi(y)dy\right), \tag{3.5}$$

where

$$\Phi(y) = \sum_{n=0}^{\infty} \hbar^n \phi_n(y). \tag{3.6}$$

Substituting the expansion (3.6) into (3.3), one obtains the recursion relations for $\phi_n(\tilde{p})$'s. Note that $\phi_n(\tilde{p})$ for odd n becomes a total derivative and only $\phi_n(\tilde{p})$ for even n contributes to the period integrals. The first three ϕ_{2n}'s are given by

$$\phi_0(\tilde{p}) = iQ^{\frac{1}{2}}, \tag{3.7}$$

$$\phi_2(\tilde{p}) = \frac{i}{48}\frac{\partial_{\tilde{p}}^2 Q}{Q^{\frac{3}{2}}}, \tag{3.8}$$

$$\phi_4(\tilde{p}) = -\frac{7i}{1356}\frac{(\partial_{\tilde{p}}^2 Q)^2}{Q^{\frac{7}{2}}} + \frac{i}{768}\frac{\partial_{\tilde{p}}^4 Q}{Q^{\frac{5}{2}}}, \tag{3.9}$$

up to total derivatives where $\partial_{\tilde{p}} := \frac{\partial}{\partial \tilde{p}}$. We define the quantum SW periods

$$\tilde{\Pi} = (\tilde{a}, \tilde{a}_D) = \left(\oint_{\tilde{\alpha}} \Phi(\tilde{p})d\tilde{p}, \int_{\tilde{\beta}} \Phi(\tilde{p})d\tilde{p}\right) \tag{3.10}$$

along the canonical 1-cycles $\tilde{\alpha}$ and $\tilde{\beta}$. The periods are expanded in $\hbar$ as

$$\tilde{\Pi} = \tilde{\Pi}^{(0)} + \hbar^2\tilde{\Pi}^{(2)} + \hbar^4\tilde{\Pi}^{(4)} + \cdots \tag{3.11}$$

where $\tilde{\Pi}^{(2n)} := \oint \phi_{2n}(\tilde{p})d\tilde{p}$. $\tilde{\Pi}^{(0)}$ is the classical SW period. Similarly, we define $\tilde{a}^{(2n)}$ and $\tilde{a}_D^{(2n)}$ by

$$\tilde{a} = \tilde{a}^{(0)} + \hbar^2\tilde{a}^{(2)} + \hbar^4\tilde{a}^{(4)} + \cdots, \tag{3.12}$$

$$\tilde{a}_D = \tilde{a}_D^{(0)} + \hbar^2\tilde{a}_D^{(2)} + \hbar^4\tilde{a}_D^{(4)} + \cdots. \tag{3.13}$$

Substituting (3.4) into (3.8) and (3.9), one finds that

$$\phi_2(\tilde{p}) = \frac{1}{\Lambda_1^2}\frac{\partial}{\partial \tilde{M}}\frac{\partial}{\partial \tilde{u}}\phi_0(\tilde{p}),$$

$$\phi_4(\tilde{p}) = \frac{7}{10\Lambda_1^4}\frac{\partial^2}{\partial \tilde{M}^2}\frac{\partial^2}{\partial \tilde{u}^2}\phi_0(\tilde{p}). \tag{3.14}$$

The classical SW periods $\tilde{\Pi}^{(0)}$ satisfy the Picard–Fuchs equation (2.57). It is also found to satisfy the differential equation with respect to $\tilde{M}$ and $\tilde{u}$:

$$\frac{\partial^2}{\partial \tilde{M}\partial \tilde{u}}\tilde{\Pi}^{(0)} = -\frac{3\tilde{u}}{2\tilde{M}}\frac{\partial^2}{\partial \tilde{u}^2}\tilde{\Pi}^{(0)} - \frac{1}{4\tilde{M}}\frac{\partial}{\partial \tilde{u}}\tilde{\Pi}^{(0)}. \tag{3.15}$$

From (3.14), the second and fourth order terms satisfy

$$\tilde{\Pi}^{(2)} = \frac{1}{\Lambda_1^2}\frac{\partial}{\partial \tilde{M}}\frac{\partial}{\partial \tilde{u}}\tilde{\Pi}^{(0)}, \tag{3.16}$$

$$\tilde{\Pi}^{(4)} = \frac{7}{10\Lambda_1^4}\frac{\partial^2}{\partial \tilde{M}^2}\frac{\partial^2}{\partial \tilde{u}^2}\tilde{\Pi}^{(0)}. \tag{3.17}$$

We note that the higher order corrections can be calculated by taking the scaling limit of those of the $N_f = 1$ $SU(2)$ theory. The second and fourth order corrections to the SW periods for the $N_f = 1$ theory are given as [24]. We can show that the formulas in [24] reduces to (3.16) and (3.17) in the scaling limit (2.21). The quantization conditions for the AD theories become different although they take the same form for the SQCDs. Therefore it is nontrivial to check that the scaling limit of the quantum SW periods of the SQCDs gives those of the AD theories. In Section 4, we will calculate the deformed SW periods around the superconformal point by using the relations (3.16) and (3.17) up to fourth order.

3.2. $N_f = 2$ theory

Next we discuss the quantum SW curve for the $N_f = 2$ theory. We introduce a new variable ξ by

$$\tilde{p} = e^{\xi} - \frac{2}{3}\tilde{M}, \tag{3.18}$$

so that the SW differential (2.27) becomes a canonical form

$$\tilde{\lambda}_{\mathrm{SW}} = \tilde{z}d\xi. \tag{3.19}$$

The SW curve (2.26) takes the form:

$$\tilde{z}^2 - \left(e^{3\xi} - 2\tilde{M}e^{2\xi} + e^{\xi}\left(\frac{4\tilde{M}^2}{3} - \tilde{u}\right) - \frac{\Lambda_2\tilde{C}_2}{4}\right) = 0. \tag{3.20}$$

Replacing $\tilde{z}$ by the differential operator

$$\tilde{z} = -i\hbar\frac{\partial}{\partial \xi}, \tag{3.21}$$

we obtain the quantum SW curve:

$$\left(-\hbar^2\frac{\partial^2}{\partial \xi^2} + Q(\xi)\right)\Psi(\xi) = 0 \tag{3.22}$$

where

$$Q(\xi) = -\left(e^{3\xi} - 2\tilde{M}e^{2\xi} + e^{\xi}\left(\frac{4\tilde{M}^2}{3} - \tilde{u}\right) - \frac{\Lambda_2\tilde{C}_2}{4}\right). \tag{3.23}$$

We consider the WKB solution to the wave function $\Psi(\xi)$ which is defined by (3.5). The leading term $\phi_0(\xi)$ in the expansion (3.6) in $\hbar$ is given by $\phi_0(\xi) = \tilde{z}(\xi)$, which gives the classical SW periods $\tilde{\Pi}^{(0)} = \int \phi_0(\xi)d\xi$. One can show that $(-\tilde{D}_2)^{\frac{1}{4}}\partial_{\tilde{u}}\tilde{\Pi}^{(0)}$ satisfies the Picard–Fuchs equation (2.14). $\tilde{\Pi}^{(0)}$ also satisfies the differential equation

$$\frac{\partial^2}{\partial\tilde{M}\partial\tilde{u}}\tilde{\Pi}^{(0)} = L_2\left(4\tilde{u}\frac{\partial^2}{\partial\tilde{u}^2}\tilde{\Pi}^{(0)} + \frac{\partial}{\partial\tilde{u}}\tilde{\Pi}^{(0)}\right) \tag{3.24}$$

where

$$L_2 := \frac{4\left(4\tilde{M}^2 - 3\tilde{u}\right)}{27\Lambda_2\tilde{C}_2 + 24\tilde{M}\tilde{u} - 32\tilde{M}^3}. \tag{3.25}$$

From (3.8) and (3.9), we find that the second and fourth order corrections are related to the classical SW period as

$$\tilde{\Pi}^{(2)} = \left(\frac{1}{4}\frac{\partial}{\partial\tilde{M}}\frac{\partial}{\partial\tilde{u}} + \frac{\tilde{M}}{3}\frac{\partial^2}{\partial\tilde{u}^2}\right)\tilde{\Pi}^{(0)}, \tag{3.26}$$

$$\tilde{\Pi}^{(4)} = \left(\frac{7\tilde{M}^2}{90}\frac{\partial^4}{\partial\tilde{u}^4} + \frac{1}{20}\frac{\partial^3}{\partial\tilde{u}^3} + \frac{7}{160}\frac{\partial^2}{\partial\tilde{u}^2}\frac{\partial^2}{\partial\tilde{M}^2} + \frac{7\tilde{M}}{60}\frac{\partial^3}{\partial\tilde{u}^3}\frac{\partial}{\partial\tilde{M}}\right)\tilde{\Pi}^{(0)}. \tag{3.27}$$

Note that (3.26) and (3.27) are defined up to the Picard–Fuchs equations. We also note that one can derive these relations from those of $N_f = 2$ $SU(2)$ theory, which are given by [24]. We find that the second and fourth order formulas of the $N_f = 2$ theory [24] lead to (3.26) and (3.27) after taking the scaling limit (2.25).

3.3. $N_f = 3$ theory

Finally we study the quantum SW curve for the $N_f = 3$ theory. We introduce a new coordinate ξ by

$$\tilde{z} = -i\left(e^{\xi} + \frac{2\tilde{M}\tilde{p}}{\Lambda_3^{\frac{1}{2}}} + \frac{4\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} + \frac{\tilde{u}}{\Lambda_3^{\frac{1}{2}}}\right), \tag{3.28}$$

so that the SW differential (2.32) becomes the canonical form

$$\tilde{\lambda}_{SW} = i\Lambda_3\left(\tilde{p}d\tilde{\xi} + \sum_{i=1}^{3}\tilde{p}d\log(\tilde{p} + \tilde{m}_i)\right). \tag{3.29}$$

Then the SW curve (2.30) can be written as

$$e^{2\xi} + (f_0\tilde{p} + f_1)e^{\xi} + g(\tilde{p}) = 0, \tag{3.30}$$

where

$$\begin{aligned} f_0 &= \frac{4\tilde{M}}{\Lambda_3^{\frac{1}{2}}}, \qquad f_1 = \frac{8\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} + \frac{2\tilde{u}}{\Lambda_3^{\frac{1}{2}}}, \\ g(\tilde{p}) &= \tilde{p}^3 - \rho_3\tilde{p} - \sigma_3 + \left(\frac{2\tilde{M}\tilde{p}}{\Lambda_3^{\frac{1}{2}}} + \frac{4\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} + \frac{\tilde{u}}{\Lambda_3^{\frac{1}{2}}}\right)^2. \end{aligned} \tag{3.31}$$

Replacing the coordinate ξ by the differential operator

$$\xi = -i\hbar\frac{\partial}{\partial\tilde{p}}, \tag{3.32}$$

one obtains the quantum SW curve. But we need to consider the ordering of the operators. In general we can define the ordering of the operators by

$$t\tilde{p}e^{-i\hbar\partial_{\tilde{p}}}\Psi(\tilde{p})+e^{-i\hbar\partial_{\tilde{p}}}((1-t)\tilde{p}\Psi(\tilde{p}))=(\tilde{p}-i(1-t)\hbar)e^{-i\hbar\partial_{\tilde{p}}}\Psi(\tilde{p}), \tag{3.33}$$

parametrized by t $(0\le t\le 1)$. We will use the $t=\frac{1}{2}$ prescription as in [23]. Then the quantum SW curve (3.30) takes the form

$$\left(\exp(-2i\hbar\partial_{\tilde{p}})+\left(\frac{1}{2}f_0\tilde{p}+f_1\right)\exp(-i\hbar\partial_{\tilde{p}})+\exp(-i\hbar\partial_{\tilde{p}})\frac{1}{2}f_0\tilde{p}+g(\tilde{p})\right)\Psi(\tilde{p})=0. \tag{3.34}$$

We consider the WKB solution (3.5) to the quantum curve. The leading term is given by $\phi_0(\tilde{p}):=\xi(\tilde{p})$. To discuss the higher order terms in $\hbar$, we rewrite the quantum curve by introducing

$$J(\alpha):=\exp\left(-\frac{i}{\hbar}\int^{\tilde{p}}\Phi(y)dy\right)\exp\left(-i\hbar\alpha\partial_{\tilde{p}}\right)\exp\left(\frac{i}{\hbar}\int^{\tilde{p}}\Phi(y)dy\right).$$

The quantum SW curve (3.34) is written as

$$J(2)+\left(f_0\left(\tilde{p}-\frac{i}{2}\hbar\right)+f_1\right)J(1)+g(x)=0. \tag{3.35}$$

Substituting (3.6) into (3.35), we can determine $\phi_n(\tilde{p})$ in a recursive way. $\phi_0(\tilde{p})$ is expressed as

$$\phi_0(\tilde{p})=\log\left(\frac{1}{2}(-f_0\tilde{p}-f_1+2\tilde{y})\right) \tag{3.36}$$

which is equal to $\tilde{\xi}(\tilde{p})$. Here $\tilde{y}$ is defined by

$$\tilde{y}^2=\frac{1}{4}(f_0\tilde{p}+f_1)^2-g(\tilde{p}). \tag{3.37}$$

$\phi_1(\tilde{p})$ is shown to be the total derivative:

$$\phi_1(\tilde{p})=\frac{\partial}{\partial\tilde{p}}\left(\frac{i}{2}\phi_0(\tilde{p})+\frac{i}{4}\log 4\tilde{y}\right). \tag{3.38}$$

We can show that $\phi_3(\tilde{p})$ is also a total derivative. ϕ_2 and ϕ_4 are found to be

$$\phi_2(\tilde{p})=\frac{(-f_0\tilde{p}-f_1)\,g''(\tilde{p})}{96\tilde{y}^3}+\frac{f_0^2\,(f_0\tilde{p}+f_1)}{192\tilde{y}^3}, \tag{3.39}$$

$$\begin{aligned}\phi_4(\tilde{p})=&g^{(4)}(\tilde{p})\left(\frac{(f_0\tilde{p}+f_1)\,g(\tilde{p})}{1536\tilde{y}^5}+\frac{-f_0\tilde{p}-f_1}{5760\tilde{y}^3}\right)+g^{(3)}(\tilde{p})\left(\frac{f_0 g(\tilde{p})}{480\tilde{y}^5}+\frac{f_0}{720\tilde{y}^3}\right)\\&+g''(\tilde{p})\left(-\frac{7f_0^2\,(f_0\tilde{p}+f_1)\,g(\tilde{p})}{3072\tilde{y}^7}-\frac{7f_0^2\,(f_0\tilde{p}+f_1)}{7680\tilde{y}^5}\right)\\&+g''(\tilde{p})^2\left(\frac{7\,(f_0\tilde{p}+f_1)\,g(\tilde{p})}{3072\tilde{y}^7}+\frac{7\,(f_0\tilde{p}+f_1)}{7680\tilde{y}^5}\right)+\frac{7f_0^4\,(f_0\tilde{p}+f_1)\,g(\tilde{p})}{12288\tilde{y}^7}\\&+\frac{7f_0^4\,(f_0\tilde{p}+f_1)}{30720\tilde{y}^5},\end{aligned} \tag{3.40}$$

up to the total derivative.

For the classical SW periods $\tilde{\Pi}^{(0)}$, $(-\tilde{D}_3)^{\frac{1}{4}}\partial_{\tilde{u}}\tilde{\Pi}^{(0)}$ satisfies the Picard–Fuchs equation (2.14). $\tilde{\Pi}^{(0)}$ also satisfies the differential equation with respect to $\tilde{M}$ and $\tilde{u}$:

$$\frac{\partial^2}{\partial\tilde{M}\partial\tilde{u}}\tilde{\Pi}^{(0)} = b_3\frac{\partial^2}{\partial\tilde{u}^2}\tilde{\Pi}^{(0)} + c_3\frac{\partial}{\partial\tilde{u}}\tilde{\Pi}^{(0)} \tag{3.41}$$

where

$$b_3 = \frac{4\tilde{M}\left(3\Lambda_3\tilde{M}\tilde{u} + 4\tilde{M}^4 - 3\Lambda_3^2\rho_3\right)\rho_3 + 27\Lambda_3^2\tilde{u}\sigma_3}{3\Lambda_3\left(9\Lambda_3\tilde{M}\sigma_3 - 4\tilde{M}^3\rho_3 - 3\Lambda_3\tilde{u}\rho_3\right)}, \tag{3.42}$$

$$c_3 = \frac{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2 - 12\Lambda_3^2\tilde{M}^2\rho_3}{3\Lambda_3\left(9\Lambda_3\tilde{M}\sigma_3 - 4\tilde{M}^3\rho_3 - 3\Lambda_3\tilde{u}\rho_3\right)}. \tag{3.43}$$

ρ_3 and σ_3 are read off from (2.30). Substituting (3.31) into (3.39) and (3.40) we find that formulas for the second and fourth order corrections in $\hbar$:

$$\tilde{\Pi}^{(2)} = \left(-\frac{\tilde{M}^2}{12}\frac{\partial^2}{\partial\tilde{u}^2} - \frac{\Lambda_3}{16}\frac{\partial}{\partial\tilde{u}}\frac{\partial}{\partial\tilde{M}}\right)\tilde{\Pi}^{(0)}, \tag{3.44}$$

$$\tilde{\Pi}^{(4)} = \left(\frac{7\tilde{M}^4}{1440}\frac{\partial^4}{\partial\tilde{u}^4} + \frac{\Lambda_3\tilde{M}}{192}\frac{\partial^3}{\partial\tilde{u}^3} + \frac{7\Lambda_3^2}{2560}\frac{\partial^2}{\partial\tilde{u}^2}\frac{\partial^2}{\partial\tilde{M}^2} + \frac{7\Lambda_3\tilde{M}^2}{960}\frac{\partial^3}{\partial\tilde{u}^3}\frac{\partial}{\partial\tilde{M}}\right)\tilde{\Pi}^{(0)}. \tag{3.45}$$

These formulas can be also obtained by taking scaling limit (2.29) of those in $N_f = 3$ $SU(2)$ SQCD [24].

In the next section we will calculate the quantum corrections to the SW periods as expansions in coupling constant and the mass parameters.

4. Quantum SW periods around the superconformal point

In the previous section we have constructed the quantum SW curves and the quantum SW periods of the AD theory, which are obtained by acting the differential operators on the classical SW periods. In this section we will calculate an explicit form of the quantum SW periods around the superconformal point up to the fourth order in $\hbar$. We will consider the expansion in the coupling constant and the mass parameters of the AD theory.

4.1. $N_f = 1$ theory

We first discuss the $N_f = 1$ theory around the superconformal point. Substituting (2.55) and (2.56) into (3.16) and changing the variables $(\tilde{u}, \tilde{M})$ to $(\tilde{u}, \tilde{w}_1')$, the second order corrections to the SW periods are expressed in terms of hypergeometric function as

$$\tilde{a}^{(2)} = \frac{1}{2^{\frac{5}{2}}\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left(F_1^{(2)}(\tilde{w}_1') - F_2^{(2)}(\tilde{w}_1', \tilde{u})\right), \tag{4.1}$$

$$\tilde{a}_D^{(2)} = \frac{1}{2^{\frac{5}{2}}\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left((-1)^{\frac{2}{3}}F_1^{(2)}(\tilde{w}_1') + (-1)^{\frac{1}{3}}F_2^{(2)}(\tilde{w}_1')\right), \tag{4.2}$$

where

$$F_1^{(2)}(\tilde{w}_1') = 2^{\frac{7}{3}} \cdot 3\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\left(F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \tilde{w}_1'\right) - 5F\left(\frac{11}{12}, \frac{17}{12}; \frac{4}{3}; \tilde{w}_1'\right)\right), \tag{4.3}$$

$$F_2^{(2)}(\tilde{w}_1') = -7\tilde{w}_1'^{\frac{2}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(\frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \tilde{w}_1'\right). \tag{4.4}$$

Similarly, substituting (2.55) and (2.56) into (3.17) and changing the variables $(\tilde{u}, \tilde{M})$ to $(\tilde{u}, \tilde{w}_1')$, we find that the fourth order corrections to the SW periods (3.13) become

$$\tilde{a}^{(4)} = -\frac{7}{2^{\frac{43}{6}} \cdot 3^{\frac{5}{2}} \cdot 5\pi^{\frac{1}{2}}\Lambda_1^{\frac{17}{2}}} \frac{\tilde{w}_1'^{\frac{1}{3}}}{(\tilde{w}_1' - 1)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(-F_1^{(4)}(\tilde{w}_1') + F_2^{(4)}(\tilde{w}_1')\right), \tag{4.5}$$

$$\tilde{a}_D^{(4)} = -\frac{7}{2^{\frac{43}{6}} \cdot 3^{\frac{5}{2}} \cdot 5\pi^{\frac{1}{2}}\Lambda_1^{\frac{17}{2}}} \frac{\tilde{w}_1'^{\frac{1}{3}}}{(\tilde{w}_1' - 1)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left((-1)^{\frac{1}{3}}F_1^{(4)}(\tilde{w}_1') + (-1)^{\frac{2}{3}}F_2^{(4)}(\tilde{w}_1')\right), \tag{4.6}$$

where

$$F_1^{(4)}(\tilde{w}_1') = 2^3 \cdot 7 \cdot 13\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)\left((11\tilde{w}_1' + 13)F\left(\frac{19}{12}, \frac{25}{12}; \frac{5}{3}; \tilde{w}_1'\right)\right.$$
$$\left. - 5F\left(\frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \tilde{w}_1'\right)\right), \tag{4.7}$$

$$F_2^{(4)}(\tilde{w}_1') = 2^{\frac{1}{3}} \cdot 5 \cdot 11 \cdot 17\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\left((7\tilde{w}_1' + 17)F\left(\frac{23}{12}, \frac{29}{12}; \frac{7}{3}; \tilde{w}_1'\right)\right.$$
$$\left. - F\left(\frac{17}{12}, \frac{23}{12}; \frac{7}{3}; \tilde{w}_1'\right)\right). \tag{4.8}$$

Expanding in $\tilde{w}_1'$ around $\tilde{w}_1' = 0$, the quantum SW periods become

$$\begin{aligned}\tilde{a} =& \Lambda_1^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(-\frac{2^{\frac{7}{2}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}} - \frac{7\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{6^{\frac{1}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}} + \cdots\right) \\ &+ \frac{\hbar^2}{\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left(\frac{7\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{2^{\frac{1}{6}} \cdot 3^{\frac{19}{2}}\pi^{\frac{1}{2}}} + \cdots\right) \\ &+ \frac{\hbar^4}{\Lambda_1^{\frac{17}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(\frac{7^2 \cdot 13\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^{\frac{19}{6}} \cdot 3^{\frac{9}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}} + \cdots\right) + \cdots,\end{aligned} \tag{4.9}$$

$$\tilde{a}_D = \Lambda_1^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(\frac{2^{\frac{7}{2}}(-1)^{\frac{1}{3}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}} - \frac{7(-1)^{\frac{2}{3}}\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{6^{\frac{1}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}} + \cdots\right)$$

$$
+\frac{\hbar^2}{\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left(\frac{7(-1)^{\frac{2}{3}}\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{2^{\frac{1}{6}}\cdot 3^{\frac{19}{2}}\pi^{\frac{1}{2}}}+\cdots\right)
$$

$$
+\frac{\hbar^4}{\Lambda_1^{\frac{17}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(-\frac{7^2\cdot 13(-1)^{\frac{1}{3}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^{\frac{19}{6}}\cdot 3^{\frac{9}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)+\cdots. \tag{4.10}
$$

We define the effective coupling constant[1] $\tilde{\tau}$ of the deformed theory by

$$
\tilde{\tau}:=\frac{\partial_{\tilde{u}}\tilde{a}_D}{\partial_{\tilde{u}}\tilde{a}}, \tag{4.11}
$$

which is expanded in $\hbar$ as

$$
\tilde{\tau}=\tilde{\tau}^{(0)}+\hbar^2\tilde{\tau}^{(2)}+\hbar^4\tilde{\tau}^{(4)}+\cdots. \tag{4.12}
$$

Substituting (4.9) and (4.10) into (4.11) and expanding in $\hbar$, we find

$$
\tilde{\tau}=\left(-(-1)^{\frac{1}{3}}+\frac{3^{\frac{1}{2}}\cdot 7i\pi^{\frac{1}{2}}\Gamma\left(-\frac{7}{6}\right)}{10\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{6}\right)}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)
$$

$$
+\frac{\hbar^2}{\Lambda_1^5}\left(-\frac{2^{\frac{4}{3}}\cdot 3^{\frac{1}{2}}i\pi^{\frac{1}{2}}\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(-\frac{5}{6}\right)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{3}}+\cdots\right)
$$

$$
+\frac{\hbar^4}{\Lambda_1^{10}}\left(-\frac{2\cdot 3^{\frac{3}{2}}i\pi^{\frac{1}{2}}\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{5}{6}\right)^2\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{10}{3}}+\cdots\right)+\cdots. \tag{4.13}
$$

We can express $\tilde{\tau}$ as a function of $\tilde{a}$ by solving (4.9). Then integrating it over $\tilde{a}$ twice, we obtain the free energy. We find that the free energy at $\tilde{M}=0$ agrees with the one obtained from the E-string theory [38]. We note that the present expansions for N_f theories in the coupling parameter are different from those in the self-dual Ω-background [15], where the expansions in the operator have been done with the zero coupling and without taking the scaling limit.

4.2. $N_f=2$ theory

We next compute the quantum corrections to the SW periods for the $N_f=2$ theory. From (3.26) and (2.60) we find that the second order corrections are given by

$$
\tilde{a}^{(2)}=-\frac{1}{2^4\cdot 3^{\frac{15}{4}}\pi^{\frac{1}{2}}\Lambda_2^{\frac{3}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}}\left(F_1^{(2)}(\tilde{w}_2')-F_2^{(2)}(\tilde{w}_2')\right), \tag{4.14}
$$

$$
\tilde{a}_D^{(2)}=\frac{i}{2^4\cdot 3^{\frac{15}{4}}\pi^{\frac{1}{2}}\Lambda_2^{\frac{3}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}}\left(F_1^{(2)}(\tilde{w}_2')+F_2^{(2)}(\tilde{w}_2')\right), \tag{4.15}
$$

[1] Note that the present definition of the effective coupling constant is inverse of the one in [1].

where we have defined $\tilde{w}_2' = 1 - \tilde{w}_2$. Here $F_1^{(2)}(\tilde{w}_2')$ and $F_2^{(2)}(\tilde{w}_2')$ are defined by

$$F_1^{(2)}(\tilde{w}_2') = 3^2\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)\left(2^2\cdot 3^{\frac{1}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} F\left(\frac{5}{12},\frac{13}{12};\frac{1}{2};\tilde{w}_2'\right)\right.$$
$$\left. - 5\tilde{w}_2'^{\frac{1}{2}} F\left(\frac{13}{12},\frac{17}{12};\frac{3}{2};\tilde{w}_2'\right)\right), \tag{4.16}$$

$$F_2^{(2)}(\tilde{w}_2') = 6^2\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)\left(3F\left(\frac{7}{12},\frac{11}{12};\frac{3}{2};\tilde{w}_2'\right) + 7X^{(2)}F\left(\frac{11}{12},\frac{19}{12};\frac{3}{2};\tilde{w}_2'\right)\right), \tag{4.17}$$

where

$$X^{(2)} = -3 + 2\cdot 3^{\frac{1}{2}}\tilde{w}_2'^{\frac{1}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}. \tag{4.18}$$

Expanding the second order terms in $\tilde{w}_2'$ around $\tilde{w}_2' = 0$, where $\frac{\tilde{M}^2}{\tilde{u}} \ll 1$ and $\frac{\tilde{C}_2\Lambda_2}{\tilde{u}^{\frac{3}{2}}} \ll 1$, we obtain

$$\tilde{a}^{(2)} = \frac{1}{\Lambda_2^{\frac{3}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}}\left(-\frac{3^{\frac{1}{4}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2\pi^{\frac{1}{2}}} + \frac{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^4\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} + \cdots\right), \tag{4.19}$$

$$\tilde{a}_D^{(2)} = \frac{1}{\Lambda_2^{\frac{3}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}}\left(-\frac{3^{\frac{1}{4}}i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2\pi^{\frac{1}{2}}} - \frac{i\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^4\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} + \cdots\right). \tag{4.20}$$

The fourth order corrections can be obtained in a similar manner. We find that

$$\tilde{a}^{(4)} = \frac{1}{2^9\cdot 3^{\frac{11}{4}}\cdot 5\pi^{\frac{1}{2}}\Lambda_2^{\frac{9}{2}}}\frac{1}{\tilde{w}_2'^{\frac{1}{2}}(\tilde{w}_2'-1)^2}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(F_1^{(4)}(\tilde{w}_2') - F_2^{(4)}(\tilde{w}_2')\right), \tag{4.21}$$

$$\tilde{a}_D^{(4)} = -\frac{i}{2^9\cdot 3^{\frac{11}{4}}\cdot 5\pi^{\frac{1}{2}}\Lambda_2^{\frac{9}{2}}}\frac{1}{\tilde{w}_2'^{\frac{1}{2}}(\tilde{w}_2'-1)^2}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(F_1^{(4)}(\tilde{w}_2') + F_2^{(4)}(\tilde{w}_2')\right), \tag{4.22}$$

where

$$F_1^{(4)}(\tilde{w}_2') = \Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)\left(-14X_1^{(4)}F\left(\frac{1}{12},\frac{5}{12};\frac{1}{2};\tilde{w}_2'\right) + X_2^{(4)}F\left(\frac{5}{12},\frac{13}{12};\frac{1}{2};\tilde{w}_2'\right)\right), \tag{4.23}$$

$$F_2^{(4)}(\tilde{w}_2') = 14\tilde{w}_2'^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)\left(-2X_1^{(4)}F\left(\frac{7}{12},\frac{11}{12};\frac{3}{2};\tilde{w}_2'\right)\right.$$
$$\left. + X_2^{(4)}F\left(\frac{11}{12},\frac{19}{12};\frac{3}{2};\tilde{w}_2'\right)\right). \tag{4.24}$$

Here the coefficients X_1 and X_2 are defined by

$$\begin{aligned}X_1^{(4)} =& -2^2\cdot 3^{\frac{3}{2}}\tilde{w}_2'^{\frac{1}{2}}\left(10\tilde{w}_2'+11\right)+3\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\left(377\tilde{w}_2'+127\right)\\ &-2^3\cdot 3^{\frac{1}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)\tilde{w}_2'^{\frac{1}{2}}\left(13\tilde{w}_2'+113\right)+28\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{3}{2}}\left(13\tilde{w}_2'+11\right),\end{aligned} \tag{4.25}$$

$$\begin{aligned}X_2^{(4)} =& -3^{\frac{3}{2}}\tilde{w}_2'^{\frac{1}{2}}\left(1345\tilde{w}_2'+671\right)+6\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\left(520\tilde{w}_2'^2+4639\tilde{w}_2'+889\right)\\ &-2^2\cdot 3^{\frac{3}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)\tilde{w}_2'^{\frac{1}{2}}\left(593\tilde{w}_2'+1423\right)+56\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{3}{2}}\left(211\tilde{w}_2'+77\right).\end{aligned} \tag{4.26}$$

Expanding the fourth order corrections to the SW periods in $\tilde{w}_2'$ around $\tilde{w}_2'=0$, where $\frac{\tilde{M}^2}{\tilde{u}} \ll 1$ and $\frac{\tilde{C}_2\Lambda_2}{\tilde{u}^{\frac{3}{2}}} \ll 1$, we get

$$\tilde{a}^{(4)} = \frac{1}{\Lambda_2^{\frac{9}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(-\frac{11\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^9\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}-\frac{3^{\frac{1}{4}}\cdot 5\cdot 7\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^8\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right), \tag{4.27}$$

$$\tilde{a}_D^{(4)} = \frac{1}{\Lambda_2^{\frac{9}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(\frac{11i\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^9\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}-\frac{3^{\frac{1}{4}}\cdot 5\cdot 7i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^8\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right). \tag{4.28}$$

The effective coupling constant $\tilde{\tau}$ is expanded in $\hbar$ as

$$\begin{aligned}\tilde{\tau} =& \left(-i-\frac{i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{3^{\frac{1}{2}}\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\frac{i3^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^3\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}\left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}}+\cdots\right)\\ &+\frac{\hbar^2}{\Lambda_2^3}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{2}}\left(\frac{3^{\frac{1}{2}}i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^4\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}+\frac{3^2i\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{11}{12}\right)^2}{\Gamma\left(\frac{1}{12}\right)^2\Gamma\left(\frac{5}{12}\right)^2}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right)\\ &+\frac{\hbar^4}{\Lambda_2^6}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-3}\left(-\frac{3i\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{11}{12}\right)^2}{2^9\Gamma\left(\frac{5}{12}\right)^2\Gamma\left(\frac{13}{12}\right)^2}\right.\\ &\left.-\frac{3^{\frac{1}{2}}i\left(3\Gamma\left(\frac{7}{12}\right)^3\Gamma\left(\frac{11}{12}\right)^3+19\pi^2\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)\right)}{2^{10}\Gamma\left(\frac{5}{12}\right)^3\Gamma\left(\frac{13}{12}\right)^3}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right)+\cdots.\end{aligned} \tag{4.29}$$

It would be interesting to compare the free energy with that of the E-string theory, which is left for future work.

4.3. $N_f = 3$ theory

We now discuss the $N_f = 3$ case. Using (3.44) and (2.65) we find that the second order corrections to the SW periods are given by

$$\tilde{a}^{(2)} = \frac{1}{2^{\frac{10}{3}} \cdot 3^{\frac{7}{2}} \pi^{\frac{1}{2}} \tilde{w}'_3 \Lambda_3^3} \left(\frac{\tilde{u}}{\Lambda_3^2} \right) (-\sigma_3)^{-\frac{5}{6}} \left(1 + \frac{4}{3} \frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right) \left(F_1^{(2)}(\tilde{w}'_3) + F_2^{(2)}(\tilde{w}'_3) \right), \quad (4.30)$$

$$\begin{aligned} \tilde{a}_D^{(2)} =& \frac{i}{2^{\frac{10}{3}} \cdot 3^{\frac{7}{2}} \pi^{\frac{1}{2}} \tilde{w}'_3 \Lambda_3^3} \left(\frac{\tilde{u}}{\Lambda_3^2} \right) (-\sigma_3)^{-\frac{5}{6}} \left(1 + \frac{4}{3} \frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right) \\ & \times \left((-1)^{\frac{5}{6}} F_1^{(2)}(\tilde{w}'_3) + (-1)^{\frac{1}{6}} F_2^{(2)}(\tilde{w}'_3) \right), \end{aligned} \quad (4.31)$$

where $\tilde{w}'_3 := \frac{1}{1-\tilde{w}_3}$. $F_1^{(2)}(\tilde{w}'_3)$ and $F_2^{(2)}(\tilde{w}'_3)$ are defined by

$$F_1^{(2)}(\tilde{w}'_3) = 18\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\left(F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \tilde{w}'_3\right) - X^{(2)} F\left(\frac{7}{12}, \frac{13}{12}; \frac{2}{3}; \tilde{w}'_3\right)\right), \quad (4.32)$$

$$\begin{aligned} F_2^{(2)}(\tilde{w}'_3) =& -\frac{3\tilde{w}'_3}{2^{\frac{2}{3}}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(-\frac{1}{3}\right)\left(F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \tilde{w}'_3\right) \right. \\ & \left. - 5X^{(2)} F\left(\frac{11}{12}, \frac{17}{12}; \frac{4}{3}; \tilde{w}'_3\right)\right). \end{aligned} \quad (4.33)$$

Here $X^{(2)}$ is given by

$$X^{(2)} = 1 + \frac{2^{\frac{2}{3}} \cdot 3\tilde{M}\Lambda_3}{(3\tilde{u}\Lambda_3 + 4\tilde{M}^3)} (-\sigma_3)^{\frac{1}{3}} \tilde{w}'^{\frac{2}{3}}_3. \quad (4.34)$$

Expanding the second order corrections to the SW periods in $\tilde{w}'_3$, where $\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \ll 1$, $\frac{\tilde{C}_2^3 \Lambda_3^2}{\tilde{u}^4} \ll 1$ and $\frac{\tilde{C}_3 \Lambda_3}{\tilde{u}^2} \ll 1$, we obtain

$$\begin{aligned} \tilde{a}^{(2)} =& \frac{1}{\Lambda_3^{\frac{1}{2}}} \left(\frac{\tilde{u}}{\Lambda_3^2} \right)^{-\frac{2}{3}} \\ & \times \left(-\frac{(-1)^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^{\frac{1}{2}} \pi^{\frac{1}{2}}} + \frac{\left(1 + 19(-1)^{\frac{1}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^4 \pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{2}{3}} + \cdots \right), \end{aligned} \quad (4.35)$$

$$\begin{aligned} \tilde{a}_D^{(2)} =& \frac{1}{\Lambda_3^{\frac{1}{2}}} \left(\frac{\tilde{u}}{\Lambda_3^2} \right)^{-\frac{2}{3}} \\ & \times \left(-\frac{(-1)^{\frac{5}{6}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^{\frac{1}{2}} \pi^{\frac{1}{2}}} + \frac{(-1)^{\frac{2}{3}}\left(1 + 19(-1)^{\frac{1}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^4 \pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{2}{3}} + \cdots \right). \end{aligned} \quad (4.36)$$

The effective coupling constant is found to be

$$\begin{aligned}\tilde{\tau} =& \left(-(-1)^{\frac{1}{3}} - \frac{2^4 i\pi^2}{\Gamma\left(\frac{1}{6}\right)^2 \Gamma\left(\frac{1}{3}\right)^2} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{1}{3}} - \frac{2i\pi^2}{\Gamma\left(\frac{1}{6}\right)^2 \Gamma\left(\frac{1}{3}\right)^2} \left(\frac{\tilde{C}_2^3 \Lambda_3^2}{\tilde{u}^4} \right)^{\frac{1}{3}} + \cdots \right) \\ &+ \frac{\hbar^2}{\Lambda_3^2} (-1)^{\frac{5}{6}} \left(\frac{\tilde{u}}{\Lambda_3^2} \right)^{-\frac{4}{3}} \\ &\times \left(-\frac{3^{\frac{1}{2}} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{5}{6}\right)}{5\Gamma\left(-\frac{5}{6}\right) \Gamma\left(\frac{1}{3}\right)} - \frac{2^{\frac{10}{3}} \cdot 3^{\frac{1}{2}} \Gamma\left(-\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right)^2}{5^2 \Gamma\left(-\frac{5}{6}\right)^2 \Gamma\left(\frac{1}{6}\right)} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{1}{3}} + \cdots \right) \\ &+ \cdots . \end{aligned} \tag{4.37}$$

We can calculate the $\hbar^4$-order correction to the effective coupling constant in a similar way. The result is

$$\begin{aligned}\tilde{\tau}^{(4)} =& \frac{(-1)^{\frac{1}{6}}}{\Lambda_3^4} \left(\frac{\tilde{u}}{\Lambda_3^2} \right)^{-\frac{8}{3}} \\ &\times \left(\frac{2^{\frac{4}{3}} \cdot 3^{\frac{1}{2}} \pi \Gamma\left(\frac{5}{6}\right)^2}{5^2 \Gamma\left(-\frac{5}{6}\right)^2 \Gamma\left(\frac{1}{6}\right)^2} + \frac{2^3 \cdot 3^{\frac{3}{2}} \pi^{\frac{1}{2}} \Gamma\left(-\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right)^3}{5^3 \Gamma\left(-\frac{5}{6}\right)^3 \Gamma\left(\frac{1}{6}\right)^2} \left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \right)^{\frac{1}{3}} + \cdots \right). \end{aligned} \tag{4.38}$$

In summary, we have explicitly calculated the quantum corrections to the SW periods in terms of the hypergeometric functions up to the fourth orders in $\hbar$ for the AD theories of the (A_1, A_2), (A_1, A_3) and (A_1, D_4)-types.

5. Conclusions and discussions

In this paper we studied the quantum SW periods around the superconformal point of $\mathcal{N} = 2$ $SU(2)$ SQCD with $N_f = 1, 2, 3$ hypermultiplets, which is deformed in the Nekrasov–Shatashvili limit of the Ω-background. The scaling limit around the superconformal point gives the SW curves of the corresponding Argyres–Douglas theories. The SW curves take the form of cubic elliptic curve for all N_f. But the SW differentials take the different form, which introduce the different quantization condition. We have computed the quantum corrections to the SW periods up to the fourth order in $\hbar$, which are obtained from the classical periods by acting the differential operators with respect to the moduli parameters. They are shown to agree with the scaling limit of the SW periods of the original SQCD. We wrote down the explicit form of the quantum corrections in terms of hypergeometric functions. It is interesting to explore the higher order corrections in $\hbar$. In particular the resurgence method helps us to understand non-perturbative structure of the $\hbar$-corrections [39–42]. The SW curve for N_f theory at the superconformal point are given by the scaling limit of the SW curve for the original SQCD. The $SU(2)$ theory with the N_f hypermultiplets are obtained by the decoupling limit for the $SU(2)$ theory with $N_f = 4$ hypermultiplets. It is interesting to see the SW curve at the various superconformal points by combining both the scaling limit and the decoupling limit of the SW curve of the $SU(2)$ theory with $N_f = 4$ hypermultiplets.

So far we have studied the AD theories around the superconformal fixed point, where the SW periods and the effective coupling constant are expanded in the Coulomb moduli parameter with fractional power. It would be interesting to study the $\hbar$-corrections to the beta functions around the conformal point [43]. Note that the moduli space of these AD theories contains the point, where one of the periods shows the logarithmic behavior around the point. It would be interesting to describe the theory around the point by the Nekrasov partition function.

It is known that the four-dimensional theories in the NS limit are described by certain quantum integrable systems. The quantum corrections to the periods provide some data of the integrable systems. For $N_f = 1$ case, the curve describes the same AD theory as $SU(3)$ $\mathcal{N} = 2$ super Yang–Mills theory [1], whose quantum curve is the Schrödinger equation with cubic polynomial potential. In [44], using the ODE/IM correspondence (for a review see [45]), it is shown that the exponential of the quantum period can be regarded as the Y-function of the quantum integrable model associated with the Yang–Lee edge singularity. It is interesting to study this relation further by computing further higher order corrections by using the ODE/IM correspondence. It is also interesting to generalize the quantum SW curve for the AD theories associated with higher rank gauge theories [37].

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