Three-Loop Quark Jet Function

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We calculate the massless quark jet function to three-loop order. The quark jet function is a universal ingredient in SCET factorization for many collider and decay processes with quark initiated final state jets. Our three-loop result contributes to the resummation for observables probing the invariant mass of final state quark jets at primed next-to-next-to-next-to-leading-logarithmic accuracy. It represents the first complete three-loop result for a factorization ingredient describing collinear radiation. Furthermore it constitutes a major component of the $N$-jettiness subtraction method at next-to-next-to-next-to-leading order accuracy, which eventually may enable the calculation of fully differential cross sections with a colorful final state at this order.

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Introduction.—In QCD processes involving highly energetic partons factorization plays a crucial role. Most importantly it provides a means to disentangle perturbative physics from nonperturbative physics. Ideally, the nonperturbative effects can thus be absorbed into a universal (process-independent) functions as, e.g., the parton distribution functions. More generally, whenever there is a strong hierarchy of scales one can hope to establish a factorization formula at leading order in the small scale ratio(s) that separates the physics happening at the different scales. Besides a considerable simplification, this usually allows us to resum large logarithms of the small scale ratio(s) to all orders in perturbation theory via renormalization group equations (RGEs) for the individual factorization ingredients. Such factorization formulas are conveniently derived in soft-collinear effective theory (SCET) [1–6].

In the following we will consider decay or scattering processes with final state jets and a large scale hierarchy between the jet invariant mass ($\sim \tau$) and the total center of mass energy ($\sim Q$). The cross section differential in a generic observable $\tau$ that constrains the jet invariant mass then schematically takes the factorized form

$$\frac{d\sigma}{d\tau} = H(Q) \times [B_a \otimes B_b \otimes J_{i_1} \otimes \cdots \otimes J_{i_n} \otimes S](\tau). \quad (1)$$

at leading order in $\tau/Q$ and all orders in $\alpha_s$. The $\otimes$ symbol denotes a convolution of the type

$$A(\tau) \otimes B(\tau) \equiv \int d\tau' A'(\tau - \tau') B(\tau'). \quad (2)$$

For concreteness, we assume here a process with two incoming $(a, b)$ and $N$ outgoing partons involved in the hard interaction, which is described by the hard function $H(Q)$, as, e.g., observed in proton-proton collisions at the LHC. For $\tau \ll Q$ the initial state radiation is then collimated along the two incoming beam directions and the final state radiation is collimated along $N$ different jet directions. Wide-angle soft emissions are taken into account by the soft function $S(\tau)$. The beam functions $B_i(\tau)$ and the jet functions $J_i(\tau)$ describe the effects of collinear radiation in the beam and final state jets, respectively. The functions $S$, $B_i$, and $J_i$ are universal in the sense that they are independent of the details of the hard process (e.g., the colorless final state). The collinear functions $B_i$ and $J_i$ are furthermore equal for any observable that in the collinear limit effectively reduces to a measurement of the jet invariant mass ($\tau \to \sqrt{s}$). A prime example for such an observable obeys factorization [7] as in Eq. (1) is the $N$-jettiness event shape [13] including the special cases beam thrust [14] ($0$-jettiness) and thrust [15] ($2$-jettiness).

In this Letter we focus on the jet function $J_q(s)$ for the case that the corresponding hard parton initiating the jet is a massless (anti-)quark ($i = q$). The SCET (quark) jet function was introduced in Ref. [4]. It can be defined in terms of standard QCD fields as [16]

$$J_q(s) = \frac{1}{\pi N_c} \text{Im} \left[ \frac{i}{\hat{n} \cdot p} \int d^4x e^{-ip \cdot x} \times \langle 0 | T \text{Tr} \left[ \frac{g}{4} W^\dagger(0) \psi(0) \bar{\psi}(x) W(x) \right] | 0 \rangle \right], \quad (3)$$

where $T$ is the time-ordering operator, $n^n$ is the lightlike jet direction ($\hat{n} \cdot n = 2$, $n^2 = \hat{n}^2 = 0$), $p^n$ is the jet momentum.
\[ W(x) = P \exp \left[ ig \int_{-\infty}^{0} ds\tilde{n} \cdot A(x + s\tilde{n}) \right] \]  

\[ e^\frac{\alpha}{\pi} \]  

\[ C \]  

\[ R = \frac{d}{d} \]  

\[ d = n - 2e, \quad n \in \mathbb{Z}, \quad n > 4 \]
render it IR (quasi-)finite. For our MIs $n = 6$ turned out to be sufficient. One can then carefully increase the power of some propagators (by one) to decrease the degree of UV divergence without generating new IR singularities. [49] Once a quasifinite integral is found in this way it can be related to the original MI plus integrals with less propagators in $d = 4 - 2\epsilon$ dimensions by dimensional recurrence and another IBP reduction. In some (exceptional) cases the quasifinite integral does not reduce to the original MI and one has to try another quasifinite candidate. An algorithm that for a given integral automatically determines a desired number of proper quasifinite integrals in shifted spacetime dimensions is implemented in the public program REDuze [50].

To perform the remaining convergent Feynman parameter integrals of the quasifinite MIs we first expand the integrands to sufficiently high order in $\epsilon$. After that we integrate them using the sector decomposition programs FIESTA [52] and PYSecDec [53]. For many of the MIs we also have obtained analytic results with the Mellin Barnes technique [54,55] employing the REDuze package [56,57] as well as the PSLQ algorithm [58]. We found perfect agreement in all cases.

To complete the calculation of the bare three-loop contribution to $J_q(s)$ we have to take the imaginary part according to Eq. (3) and consistently expand in $\epsilon$. To this end we use

$$\text{Im}[(s - i0)^{-1-\epsilon}] = -\sin(\pi \epsilon)\theta(s) s^{-1-\epsilon}$$

(5)

and

$$\mu^{2\epsilon}\theta(s) s^{-1-\epsilon} = -\frac{\delta(s)}{ae} + \sum_{n=0}^{\infty} \frac{(-ae)^n}{n!} \frac{1}{\mu^2} L_n\left(\frac{s}{\mu^2}\right)$$

(6)

with the usual plus distributions defined as

$$L_n(x) = \left[\frac{\theta(x)\ln^n x}{x}\right] = \lim_{\epsilon \to 0} \frac{d}{dx} \left[\theta(x - \epsilon) \frac{\ln^{n+1} x}{n+1}\right].$$

(7)

Convolutions among the $L_n(s)$ take the form

$$(L_m \otimes L_n(s)) = V^{mn}_k \delta(s) + \sum_{k=0}^{m+n+1} V^{mn}_k L_k(s).$$

(8)

A generic expression for $V^{mn}_k$ is given in Ref. [59].

Result.—Bare and renormalized jet functions are related by ($i = q, g$)

$$J_i^{\text{bare}}(s) = Z_j(s,\mu) \otimes J_i(s,\mu).$$

(9)

Throughout this work we employ the $\overline{\text{MS}}$ renormalization scheme. The RGE of the jet function reads

$$\mu \frac{d}{d\mu} J_i(s,\mu) = \gamma_j(s,\mu) \otimes J_i(s,\mu),$$

(10)

with the anomalous dimension

$$\gamma_j(s,\mu) = - (\gamma^{(i)}_j)^{-1} (s,\mu) \otimes \mu \frac{d}{d\mu} (\gamma^{(i)}_j)(s,\mu)$$

$$= - 2 \Gamma^i_{\text{cusp}}(\alpha_s) \frac{1}{\mu^2} \mathcal{L}_0\left(\frac{s}{\mu^2}\right) + \gamma^{(i)}_j(\alpha_s) \delta(s).$$

(11)

(12)

The following we use the expansions

$$\Gamma^i_{\text{cusp}}(\alpha_s) = \sum_{n=0}^{\infty} \frac{\alpha_s}{4\pi} \gamma^{[n+1]}_i(\alpha_s) \gamma^{(n)}_{\alpha_i}(s,\mu).$$

(13)

The coefficients of the (lightlike) cusp anomalous dimension ($\gamma^i$) [61–63] and the collinear jet anomalous dimension ($\gamma^i_0$) [21] for $n = 0, 1, 2$ are, e.g., listed in Ref. [60]. The jet function coefficients have the form

$$J^{(m)}_{i-1}(s,\mu) = J^{(m)}_{i-1}(s) + \sum_{n=0}^{2m-1} J^{(m)}_{i-n} \frac{1}{\mu^2} \mathcal{L}_n\left(\frac{s}{\mu^2}\right).$$

(14)

The coefficients of the $\mu$-dependent plus distributions in Eq. (14) can be expressed in terms of lower-loop coefficients and anomalous dimensions by iteratively solving the RGE in Eq. (10). Up to three loops we find

$$J^{(1)}_{i,1} = \Gamma^{(1)}_i,$$

$$J^{(1)}_{i,0} = -\frac{\gamma^{(1)}_0}{2},$$

$$J^{(2)}_{i,3} = \frac{(\Gamma^{(2)}_i)^2}{2},$$

$$J^{(2)}_{i,2} = -\frac{\Gamma^{(2)}_i}{2} \left(\frac{3\gamma^{(3)}_0}{2} + 3\gamma^{(2)}_0\right),$$

$$J^{(2)}_{i,1} = \Gamma^{(2)}_i - (\Gamma^{(2)}_i)^2 \frac{\pi^2}{6} + \frac{\gamma^{(2)}_0}{2} \left(\frac{\gamma^{(2)}_0}{2} + \beta_0\right) + \Gamma^{(1)}_i J^{(1)}_{i-1},$$

$$J^{(2)}_{i,0} = (\Gamma^{(2)}_i)^2 \frac{3 C_3}{12} + \Gamma^{(2)}_i \frac{\gamma^{(3)}_0}{12} - \frac{\gamma^{(2)}_0}{2} \left(\frac{\gamma^{(2)}_0}{2} + \beta_0\right) J^{(1)}_{i-1},$$

$$J^{(3)}_{i,5} = \frac{(\Gamma^{(3)}_i)^3}{8},$$

$$J^{(3)}_{i,4} = -\frac{5}{12} (\Gamma^{(3)}_i)^2 \left(\frac{3}{4} \gamma^{(3)}_0 + 2\beta_0\right).$$
\[
J^{(3)}_{i,3} = \frac{\Gamma_0^i}{6} \left( 5\beta_0 \gamma_0^i + 2\beta_0^2 + \frac{3}{2}(\gamma_0^i)^2 - \pi^2(\Gamma_0^i)^2 + 6 \Gamma_1^i \\
+ 3 \Gamma_0^i J^{(1)}_{i,-1} \right),
\]
\[
J^{(3)}_{i,2} = \frac{5}{2} (\Gamma_0^i)^3 \zeta_3 + \frac{\pi^2}{4} (\Gamma_0^i)^2 (\gamma_0^i + \beta_0) - \frac{\Gamma_0^i}{4} (3 \gamma_1 + 2 \beta_1) \\
- \frac{\Gamma_0^i}{4} (3 \gamma_0 + 4 \beta_0) - \frac{\gamma_1^i}{16} (6 \beta_0 \gamma_0^i + 8 \beta_0^2 + (\gamma_0^i)^2) \\
- \frac{\Gamma_0^i}{4} (3 \gamma_0^i + 8 \beta_0^3) J^{(1)}_{i,-1},
\]
\[
J^{(3)}_{i,1} = \frac{\pi^4}{180} (\Gamma_0^i)^3 - \zeta_3 (\Gamma_0^i)^2 (3 \beta_0 + 2 \gamma_0^i) - \frac{\pi^2}{3} \Gamma_0^i \Gamma_1^i \\
- \frac{\pi^2}{12} \Gamma_0^i \gamma_0^i (3 \beta_0 + \gamma_0^i) + \frac{\gamma_1^i}{2} (2 \beta_0 + \gamma_0^i) + \frac{\beta_1 \gamma_0^i}{2} \\
+ \left( \frac{3}{2} \beta_0 \gamma_0^i + 2 \beta_0^2 + \frac{1}{4} (\gamma_0^i)^2 - \frac{\pi^2}{6} (\Gamma_0^i)^2 + \Gamma_1^i \right) J^{(1)}_{i,-1} \\
+ \Gamma_0^i J^{(2)}_{i,-1},
\]
\[
J^{(3)}_{i,0} = \left( 3 \zeta_5 - \frac{\pi^2}{3} \zeta_3 \right) (\Gamma_0^i)^3 + \frac{\pi^4}{90} (\Gamma_0^i)^2 (\beta_0 + \gamma_0^i) + \frac{\pi^2}{4} \\
+ \Gamma_1^i \left( 2 \zeta_3 \Gamma_0^i + \frac{\pi^2}{12} \gamma_0^i \right) \\
+ \Gamma_0^i \left( \frac{\pi^2}{12} \gamma_1^i + \frac{\zeta_3}{2} \beta_0 \gamma_0^i + \frac{\zeta_5}{4} (\gamma_0^i)^2 \right) - \frac{\gamma_1^i}{2} \\
+ \left( \frac{\zeta_3 (\Gamma_0^i)^2}{2} + \frac{\pi^2}{12} \Gamma_0^i (2 \beta_0 + \gamma_0^i) - \beta_1 - \frac{\gamma_1^i}{2} \right) J^{(1)}_{i,-1} \\
- \frac{1}{2} (4 \beta_0 + \gamma_0^i) J^{(2)}_{i,-1},
\]

where \( \beta_0 = \frac{1}{5} C_A - \frac{4}{5} T_F n_f \) and \( \beta_1 = \frac{3}{4} C_A - \frac{2}{3} C_A + 4 C_F T_F n_f \), with \( n_f \) the number of active flavors. Our result for the three-loop quark jet function perfectly reproduces Eq. (15) for \( i = q \). This provides another strong cross check and at the same time represents the first direct calculation of \( \gamma_2^q \), which up to now has been inferred from RG consistency [21] using the three-loop results of Refs. [62,67]. For \( m = 0, 1, 2 \) the constants \( J^{(m)}_{m,-1} \) are, e.g., collected in Ref. [12] in accordance with our conventions. The new result of our work is

\[
J^{(3)}_{q,-1} = C_F^3 \left[ \frac{274 \zeta_3 + 2 \frac{22 \pi^2 \zeta_5}{3} - 400 \zeta_3^2}{3} - 88 \zeta_5 + 1173}{8} \\
- \frac{3505 \pi^2}{72} + \frac{622 \pi^4}{45} - \frac{9871 \pi^6}{8505} \right] \\
+ C_A C_F^2 \left[ \frac{-28241 \zeta_3}{27} + \frac{-2200 \pi^2 \zeta_5}{27} + \frac{424 \pi^2 \zeta_3}{3} + 560 \zeta_5}{9} \\
+ \frac{206197}{324} - \frac{17585 \pi^2}{72} + \frac{18703 \pi^4}{1215} + \frac{1547 \pi^6}{4860} \right] \\
+ C_A^2 C_F \left[ -\frac{187951 \zeta_3}{243} + \frac{394 \pi^2 \zeta_5}{9} + \frac{1528 \pi^2 \zeta_3}{9} - \frac{380 \zeta_5}{9} \\
+ \frac{506020}{52488} - \frac{464665 \pi^2}{4374} + \frac{1009 \pi^4}{1620} + \frac{221 \pi^6}{5103} \right] \\
+ C_A C_F n_f T_F \left[ \frac{14828 \zeta_3}{81} - \frac{64 \pi^2 \zeta_5}{3} + \frac{32 \zeta_5}{3} - \frac{2942843}{6561} \right] \\
+ \frac{136648 \pi^2}{2187} - \frac{418 \pi^4}{405} \right] \\
+ C_F^2 n_f T_F \left[ \frac{22432 \zeta_3}{243} - \frac{272 \pi^2 \zeta_5}{27} + \frac{160 \pi^2}{3} - \frac{261587}{486} \right] \\
+ \frac{4853 \pi^2}{54} - \frac{5876 \pi^4}{1215} \right] \\
+ C_F n_f^2 T_F \left[ \frac{1504 \zeta_3}{243} - \frac{249806}{6561} - \frac{1864 \pi^2}{243} + \frac{8 \pi^4}{45} \right].
\]

It is often convenient to work with the Laplace transform

\[
J_q(\nu, \mu) = \int_0^\infty ds e^{-\nu s} J_q(s),
\]

because the convolutions of Eq. (2) type turn in to simple products in Laplace space. The Laplace space equivalents to our Eqs. (14) and (15) can be read off from Ref. [22]. The new three-loop constant related to Eq. (16) in their notation is

\[
c_s^3 = 25.06777873 C_A^3 + 32.81169125 C_A C_F^2 \\
- 0.7795843561 C_F^3 C_A - 31.65196210 C_A C_F n_f T_F \\
- 61.78995095 C_A^2 n_f T_F + 28.49157341 C_F n_f^2 T_F^2,
\]

where for brevity we have evaluated the exact analytical result to ten valid digits for each color factor. The constant \( c_s^3 \) equals the position space coefficient \( j_3 \) affecting the \( \alpha_s \) determinations in Refs. [23,24,26], where until now \( j_3 = 0 \pm 3000 \) has been assumed. Evaluating Eq. (18) for \( N_c = 3 \), \( T_F = 1/2 \), and \( n_f = 5 \) we have \( j_3 = -128.6512525 \).

In Ref. [68] the \( N^0 \)LO nonlogarithmic constant of the (normalized) thrust cumulant cross section in the singular limit was obtained from a fit to fixed-order data produced by the Monte Carlo program EERAD3 [69], albeit with large numerical errors. With our new three-loop jet function constant in Eq. (16) and the known three-loop hard function [23] at hand we can use this result to extract a rough estimate for the unknown thrust \((q \bar{q} \text{ channel})\) soft function constant at three loops. In Laplace (position) space and adopting the notation of Ref. [22] \( c_s^3 \) and Ref. [23] \( s_3 \) we find \( (N_c = 3, T_F = 1/2, n_f = 5) \)

\[
c_s^3 = 2 s_3 + 691 = -19988 \pm 14400 \text{ (stat)} \pm 4000 \text{ (syst)}.\]
Summary.—In this Letter we have presented our calculation of the quark jet function $J_q(s)$ at three loops. The main result is the three-loop contribution to the $\delta(s)$ coefficient and given in Eq. (16). All other terms at this order can be derived from RG consistency conditions in terms of previous results, see Eq. (15). The new contribution is a necessary ingredient to many $N^3LL'$ resummed processes with final state jets. It has, e.g., a direct impact on existing $\alpha_s$ determinations from $e^+e^-$ event shapes. Our calculation also represents the first step toward possible applications of the $N$-jettiness IR slicing (or subtraction) method at $N^3$LO.

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[7] Neglecting potential factorization breaking effects due to Glauber modes [8–11] in hadron collisions, which are, however, absent up to three loops in fixed-order perturbation theory [12].
[28] For thrust, C parameter, and heavy jet mass at $N^3LL'$ also the respective soft function correction is currently missing. In the primed counting scheme the $N^3$LO boundary terms of the functions in the factorization formula are included for $N^3LL'$ accuracy. For details and advantages of this scheme see, e.g., Ref. [29].
[49] Bubble-type subintegrals can of course be conveniently integrated out in $d$ dimensions beforehand.
Due to Casimir scaling $\Gamma_n = C_A/C_F \Gamma_n^{\text{QCD}}$ for $n = 0, 1, 2$. Beyond three loops Casimir scaling is violated, cf. Refs. \cite{64-66}.