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Letter

Family unification in special grand unification

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We discuss family unification in grand unified theory (GUT) based on an $SU(19)$ GUT gauge group broken to its subgroups, including a special subgroup. In the $SU(19)$ GUT on the six-dimensional (6D) orbifold space $M^4 \times T^2/\mathbb{Z}_2$, three generations of the 4D Standard Model Weyl fermions can be embedded into 6D bulk Weyl fermions in an $SU(19)$ second-rank anti-symmetric tensor representation. 6D and 4D gauge anomalies can be canceled out by considering proper matter content without 4D exotic chiral fermions at low energies.

Subject Index B40, B41, B42, B43

1. *Introduction* The existence of three chiral generations of quarks and leptons is one of the most mysterious facts in particle physics. In addition, their hierarchical mass structures generated by the Higgs mechanism via their corresponding Yukawa couplings strongly suggest the existence of a hidden structure of nature. There have been many attempts to understand the origin of chiral generations and/or their hierarchical mass structures by considering, e.g., so-called horizontal symmetry (or family symmetry) in four-dimensional (4D) theories [1–7], geometrical structures in higher-dimensional theories [8–10], and string theories [11–13].

As is well known, quarks and leptons for each generation in the Standard Model (SM) can be unified into one multiplet (or two multiplets) in grand unified theories (GUTs) [14]. There are many GUTs in the 4D framework [14–19] and higher-dimensional space frameworks [20–32] (for a review, see Refs. [33,34]).

There have been some attempts to unify GUT and family groups into a larger GUT group in 4D and higher-dimensional theories [35–42]. However, such attempts are based on GUT groups and their limited subgroups, so-called *regular subgroups*, e.g., $E_8 \supset E_7 \supset E_6 \supset SO(10) \supset SU(5) \supset G_{\rm SM}$ $(:= SU(3)_C \times SU(2)_L \times U(1)_Y)$. There are other subgroups called *special subgroups* (or *non-regular subgroups*), e.g., $SO(248) \supset E_8$, $USp(56) \supset E_7$, $SU(27) \supset E_6$, and $SU(16) \supset SO(10)$. (For Lie groups and their subgroups, see, e.g., Refs. [33,34,43–49].)

Recently, new-type GUTs called *special GUTs* based on GUT groups $SO(32)$ and $SU(16)$ and their special subgroup $SO(10)$ have been proposed in Refs. [50,51]. The main results of $SO(32)$ and $SU(16)$ special GUTs are summarized as follows. In an $SU(16)$ special GUT based on its GUT group $SU(16)$ broken to its special subgroup $SO(10)$, a 4D $SU(16)$ **16** Weyl fermion can be identified with one generation of quarks and leptons; 4D $SU(16)$ gauge anomaly cancellation does not work in the 4D framework, while it works in the 6D framework without any exotic chiral fermions once

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we take into account $SU(16)$ symmetry breaking effects [50]. Almost the same results are obtained in an $SO(32)$ special GUT [51].

In a special GUT framework, family unification can be considered by using GUT groups and their "regular-type" and "product-type" subgroups; an example of the former is $SU(19) \supset SU(16) \times SU(3) \times U(1)$; an example of the latter is $SU(48) \supset SU(16) \times SU(3)$, where $SU(16)$ contains an ordinary GUT gauge group $SO(10)$ and $SU(3)$ is a family gauge group. (Their branching rules of $SU(19) \supset SU(16) \times SU(3) \times U(1)$, $SU(48) \supset SU(16) \times SU(3)$, etc. can be calculated, e.g., by using the projection matrix method shown in Refs. [34,45,46].)

First, for a regular-type case, an example of GUT gauge groups and their subgroup pair is $SU(19) \supset SU(16) \times SU(3) \times U(1) \supset SO(10) \times SU(3) \times U(1)$. This is a simple extension of $SU(16)$. The branching rule of $SU(19) \supset SU(16) \times SU(3) \times U(1) \supset SO(10) \times SU(3) \times U(1)$ for the $SU(19)$ defining representation is

$$\mathbf{19} = (\mathbf{16}, \mathbf{1})(3) \oplus (\mathbf{1}, \mathbf{3})(-16). \tag{1.1}$$

In this case, e.g., an $SU(19)$ second-rank anti-symmetric tensor representation contains three generations of quarks and leptons. Its branching rule is given by

$$\mathbf{171} = (\mathbf{16}, \mathbf{3})(-13) \oplus (\mathbf{120}, \mathbf{1})(6) \oplus (\mathbf{1}, \overline{\mathbf{3}})(-32), \tag{1.2}$$

where an $SU(16)$ **120** representation is complex while an $SO(10)$ **120** representation is real. A 4D Weyl fermion in an $SO(10)$ **120** representation is vectorlike, so when we take into account symmetry breaking effects for $SU(19)$ to $SO(10)$, only three 4D $SO(10)$ **16** Weyl fermions remain chiral. Also, an $SU(19)$ second-rank symmetric tensor representation **190** contains an $SU(16) \times SU(3)$ $(\mathbf{16}, \mathbf{3})$ representation. Note that the $SU(19)$ **190** contains unwilling $SU(16) \times SU(3)$ complex representations. The $SU(19)$ adjoint representation **360** contains not only an $SU(16) \times SU(3)$ $(\mathbf{16}, \overline{\mathbf{3}})$ but also its conjugate representation $(\overline{\mathbf{16}}, \mathbf{3})$. (In Ref. [52], R. M. Fonseca has also pointed out that a 4D Weyl fermion in an $SU(19)$ **171** representation contains the SM fermions plus vectorlike particles only, which was found by using the Susyno program [48].)

Next, for a product-type case, an example of GUT gauge groups and their subgroup pair is $SU(48) \supset SU(16) \times SU(3) \supset SO(10) \times SU(3)$. The branching rule for the $SU(48)$ defining representation is

$$\mathbf{48} = (\mathbf{16}, \mathbf{3}). \tag{1.3}$$

The 4D Weyl fermion in the $SU(48)$ defining representation can be identified with three chiral generations of quarks and leptons. To the best of my knowledge, there is no way to construct an $SU(48)$ gauge theory that contains only three chiral generations without any gauge anomalies, at least for a 4D, 5D, or 6D framework. We will not discuss this possibility in this letter.

In this letter we discuss a 6D $SU(19)$ special GUT on $M^4 \times T^2/\mathbb{Z}_2$, whose GUT group includes an $SO(10)$ GUT group and an $SU(3)_F$ family group. The main purpose of this paper is to show that three generations of the 4D SM Weyl fermions can be embedded into 6D bulk Weyl fermions in an $SU(19)$ second-rank anti-symmetric tensor representation. 6D and 4D gauge anomalies can be canceled out by considering proper matter content without 4D exotic chiral fermions at low energies.

This letter is organized as follows. In Sect. 2, before we discuss a special GUT based on an $SU(19)$ gauge group, we discuss basic properties of $SU(19)$ and its subgroups mainly by using the technique in Ref. [34]. In Sect. 3 we construct a 6D $SU(19)$ special GUT on $M^4 \times T^2/\mathbb{Z}_2$. Section 4 is devoted to a summary and discussion.

$$\begin{array}{lll} SU(19) & \underset{\text{BCs}}{\longrightarrow} & SU(16)\times SU(3)\times U(1) \\ & \underset{\langle\Phi_{\mathbf{10830}}\rangle\neq 0}{\longrightarrow} & SO(10)\times SU(3) \\ & \underset{\langle\Phi_{\mathbf{19}}\rangle\text{'s}\neq 0}{\longrightarrow} & SU(5) \\ & \underset{\langle\Phi_{\mathbf{360}}\rangle\neq 0}{\longrightarrow} & SU(3)\times SU(2)\times U(1)=G_{\text{SM}}. \end{array}$$

Fig. 1. A symmetry breaking pattern of $SU(19)$ to G_{SM}. BCs stands for an orbifold boundary condition. $\Phi_{\mathbf{x}}$ represents a scalar field in a representation $\mathbf{x}$ of $SU(19)$. We assume that the appropriate component of each $\Phi_{\mathbf{x}}$ develops its non-vanishing VEV $\langle\Phi_{\mathbf{x}}\rangle\neq 0$.

2. *Basics for $SU(19)$ and its subgroups* First, we check how to embed three generations of the SM Weyl fermions into 4D Weyl fermion in an $SU(19)$ second-rank anti-symmetric tensor representation **171**. For regular and special embeddings $SU(19)\supset SU(16)\times SU(3)_F\times U(1)_Z\supset SO(10)\times SU(3)_F\times U(1)_Z$, an $SU(19)$ second-rank anti-symmetric tensor representation **171** is decomposed into an $SU(16)$ second-rank anti-symmetric tensor representation **120**, $SU(3)_F$ defining representations $\overline{\mathbf{3}}$, and $SU(16)\times SU(3)_F$ bi-fundamental representations $(\mathbf{16},\mathbf{3})$ given in Eq. (1.2). Further, as is well known, the $SO(10)$ spinor representation **16** is decomposed into $G_{\text{SM}}\times U(1)_X=SU(3)_C\times SU(2)_L\times U(1)_Y\times U(1)_X$ representations:

$$\begin{aligned}\mathbf{16}=&(\mathbf{3},\mathbf{2})(-1)(1)\oplus(\overline{\mathbf{3}},\mathbf{1})(-2)(-3)\oplus(\overline{\mathbf{3}},\mathbf{1})(4)(1)\\ &\oplus(\mathbf{1},\mathbf{2})(3)(-3)\oplus(\mathbf{1},\mathbf{1})(-6)(1)\oplus(\mathbf{1},\mathbf{1})(0)(5).\end{aligned}\tag{2.1}$$

That is, three generations of the SM Weyl fermions are embedded into a 4D $SU(19)$ **171** Weyl fermion. In addition, an $SU(16)$ complex representation **120** is identified with an $SO(10)$ real representation **120**. A 4D $SU(16)$ **120** Weyl fermion is chiral, while a 4D $SO(10)$ **120** Weyl fermion is vectorlike. When $SU(19)$ is broken to $SO(10)$, $SU(3)_F\times U(1)_Z$ is broken to nothing. A 4D Weyl fermion in the $SO(10)\times SU(3)$ $(\mathbf{1},\overline{\mathbf{3}})$ is chiral, while three 4D $SO(10)$ **1** Weyl fermions are vectorlike. Thus, once $SU(19)$ is broken to $SO(10)$, a 4D $SU(19)$ **191** Weyl fermion is decomposed into three 4D $SO(16)$ **16** Weyl fermions and vectorlike fermions.

Next, we consider a symmetry breaking pattern from $SU(19)$ to G_{SM}. One way of achieving this is to use orbifold symmetry breaking boundary conditions (BCs) and several GUT-breaking Higgses. One example is to choose orbifold BCs breaking $SU(19)$ to $SU(16)\times SU(3)_F\times U(1)$ and to introduce $SU(19)$ **10830**, **360**, **19** scalar fields, where we assume their proper components acquire non-vanishing VEVs (see Fig. 1). First, the following orbifold BC for the $SU(19)$ defining representation **19** breaks $SU(19)$ to $SU(16)\times SU(3)_F\times U(1)_Z$:

$$P_{\mathbf{19}}=\text{diag}\,(I_{16},-I_3)\,,\tag{2.2}$$

where $P_{\mathbf{19}}$ stands for a projection matrix defined in Eq. (3.5). The non-vanishing VEV of the $SU(19)$ **10830** scalar field is responsible for breaking $SU(19)\supset SU(16)\times SU(3)_F\times U(1)_Z$ to $SO(10)\times SU(3)_F$, where its branching rule of $SU(19)\supset SU(16)\times SU(3)_F\times U(1)_Z$ is given by

$$\begin{aligned}\mathbf{10830}=&(\mathbf{5440},\mathbf{1})(12)\oplus(\mathbf{1360},\mathbf{3})(-7)\oplus(\mathbf{136},\overline{\mathbf{6}})(-26)\\ &\oplus(\mathbf{120},\overline{\mathbf{3}})(-26)\oplus(\mathbf{16},\mathbf{8})(-45)\oplus(\mathbf{1},\mathbf{6})(-64).\end{aligned}\tag{2.3}$$

An $SU(16)$ **5440** contains a singlet under its $SO(10)$ special subgroup. Its non-vanishing VEV can break $SU(16)$ to its special subgroup $SO(10)$ [50,51], where their $SO(10)$ decompositions are given in Ref. [34] by

$$\mathbf{5440} = \mathbf{4125} \oplus \overline{\mathbf{1050}} \oplus \mathbf{210} \oplus \mathbf{54} \oplus \mathbf{1}. \tag{2.4}$$

The VEV of an $SU(19)$ **19** scalar breaks ($SU(19) \supset$)$SO(10) \times SU(3)_F \times U(1)_Z$ to $SU(5) \times SU(3)_F$ or ($SU(19) \supset$)$SO(10) \times SU(3)_F \times U(1)_Z$ to $SO(10) \times SU(2)_F$, where its breaking rule is given in Eq. (1.1). If the three VEVs of the proper components of $SU(19)$ **19** scalars can break ($SU(19) \supset$)$SO(10) \times SU(3)_F \times U(1)_Z$ to $SU(5)$, the VEV of the $SU(19)$ **360** scalar further reduces ($SU(19) \supset SO(10) \times SU(3)_F \supset SO(10)) \supset SU(5)$ to G_{SM}, where its branching rule of $SU(19) \supset SU(16) \times SU(3)_F \times U(1)_Z$ is given by

$$\mathbf{360} = (\mathbf{255}, \mathbf{1})(0) \oplus (\mathbf{1}, \mathbf{8})(0) \oplus (\mathbf{1}, \mathbf{1})(0) \oplus (\mathbf{16}, \overline{\mathbf{3}})(19) \oplus (\overline{\mathbf{16}}, \mathbf{3})(-19), \tag{2.5}$$

where the $SU(16)$ **255** is decomposed into $SO(10)$ ($\mathbf{210} \oplus \mathbf{45}$), and the $SO(10)$ **45** is decomposed into $SU(5)$ ($\mathbf{24} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}$). (For further information, see, e.g., Ref. [34].)

3. *$SU(19)$ special grand unification* As in Refs. [50,51], we consider an $SU(19)$ special GUT on 6D orbifold spacetime $M^4 \times T^2/\mathbb{Z}_2$ with the Randall–Sundrum (RS) type metric [31,32,53] given by

$$ds^2 = e^{-2\sigma(y)}(\eta_{\mu\nu}dx^\mu dx^\nu + dv^2) + dy^2, \tag{3.1}$$

where y is the coordinate of RS warped space, v is the coordinate of S^1, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, $\sigma(y) = \sigma(-y) = \sigma(y + 2\pi R_5)$, $\sigma(y) = k|y|$ for $|y| \leq \pi R_5$, and $v \sim v + 2\pi R_6$. There are four fixed points on $T^2/\mathbb{Z}_2$ at $(y_0, v_0) = (0, 0)$, $(y_1, v_1) = (\pi R_5, 0)$, $(y_2, v_2) = (0, \pi R_6)$, and $(y_3, v_3) = (\pi R_5, \pi R_6)$. For each fixed point, the $\mathbb{Z}_2$ parity reflection is described by

$$P_j : (x_\mu, y_j + y, v_j + v) \to (x_\mu, y_j - y, v_j - v), \tag{3.2}$$

where $j = 0, 1, 2, 3$, and $P_3 = P_1P_0P_2 = P_2P_0P_1$. The fifth and sixth dimensional translations $U_5 : (x_\mu, y, v) \to (x_\mu, y + 2\pi R_5, v)$ and $U_6 : (x_\mu, y, v) \to (x_\mu, y, v + 2\pi R_6)$ satisfy $U_5 = P_1P_0$ and $U_6 = P_2P_0$, respectively.

We consider the matter content in the $SU(19)$ special GUT that consists of a 6D $SU(19)$ bulk gauge boson A_M: 6D $SU(19)$ **171** positive Weyl fermions with the orbifold BCs $(\eta_0, \eta_1, \eta_2, \eta_3) = (+, +, -, -)$ and $(\eta_0, \eta_1, \eta_2, \eta_3) = (+, +, +, +)$ $\Psi_{\mathbf{171}+}$ and $\Psi'_{\mathbf{171}+}$, and 6D negative Weyl fermions with $(-, +, -, +)$ and $(-, +, +, -)$ $\Psi_{\mathbf{171}-}$ and $\Psi'_{\mathbf{171}-}$, where η_j ($j = 0, 1, 2, 3$) stands for parity assignment for each 6D fermion; 5D $SU(19)$, **10830**, **360** and **19** brane scalar bosons at $y = 0$ $\Phi_{\mathbf{10830}}$, $\Phi_{\mathbf{360}}$, $\Phi_{\mathbf{19}}$, $\Phi'^{(\alpha)}_{\mathbf{19}}$ ($\alpha = 1, 2$); a 4D $SU(16) \times SU(3)_F \times U(1)_Z$ $(\overline{\mathbf{120}}, \mathbf{1})(0)$ Weyl fermion, four 4D $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{120}, \mathbf{1})(6)$ Weyl fermions, 64 4D $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{1}, \overline{\mathbf{3}})(13)$ Weyl fermions, four 4D $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{1}, \overline{\mathbf{3}})(-32)$ Weyl fermions at the fixed point $(y_0, v_0) = (0, 0)$ $\psi_{\overline{\mathbf{120}}}$, $\psi^{(a)}_{\mathbf{120}}$ ($a = 1, 2, 3, 4$), $\psi^{(b)}_{\overline{\mathbf{3}}}$ ($b = 1, 2, \ldots, 64$), $\psi'^{(c)}_{\overline{\mathbf{3}}}$ ($c = 1, 2, 3, 4$). The matter content of the $SU(19)$ special GUT is summarized in Table 1. Note that the 5D brane scalars are responsible for achieving the appropriate symmetry and the 4D brane fermions are necessary to realize 4D gauge anomaly cancellation. Only their conditions do not uniquely determine the matter content, so one may choose another matter content; e.g., one may introduce 6D bulk scalars instead of 5D brane scalars. We will see the roles of the bulk and brane fields in the following.

Table 1. The matter content in the $SU(19)$ special GUT on $M^4 \times T^2/\mathbb{Z}_2$. The representations of $SU(19)$ and 6D, 5D, 4D Lorentz group, the orbifold BCs of 6D bulk fields and 5D brane fields, and the spacetime location of 5D and 4D fields are shown. Orbifold BCs stand for parity assignment $\begin{pmatrix} \eta_2 & \eta_3 \\ \eta_0 & \eta_1 \end{pmatrix}$ for 6D fields and $\begin{pmatrix} \eta_2 \\ \eta_0 \end{pmatrix}$ for 5D fields. The orbifold BCs of the 6D $SU(19)$ gauge field A_M are given in Eqs. (3.3) and (3.4). $\alpha = 1,2$; $a = 1,2,3,4$; $b = 1,2,\ldots,64$; $c = 1,2,3,4$.

6D bulk field	A_M	$\Psi_{\mathbf{171}+}$	$\Psi'_{\mathbf{171}+}$	$\Psi_{\mathbf{171}-}$	$\Psi'_{\mathbf{171}-}$
$SU(19)$	**360**	**171**	**171**	**171**	**171**
$SO(5,1)$	**6**	$\mathbf{4}_+$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{4}_-$
Orbifold BC		$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$

5D brane field	$\Phi_{\mathbf{10830}}$	$\Phi_{\mathbf{360}}$	$\Phi_{\mathbf{19}}$	$\Phi'^{(\alpha)}_{\mathbf{19}}$
$SU(19)$	**10830**	**360**	**19**	**19**
$SO(4,1)$	**1**	**1**	**1**	**1**
Orbifold BC	$\begin{pmatrix} - \\ - \end{pmatrix}$	$\begin{pmatrix} + \\ + \end{pmatrix}$	$\begin{pmatrix} + \\ + \end{pmatrix}$	$\begin{pmatrix} + \\ - \end{pmatrix}$
Spacetime	$y=0$	$y=0$	$y=0$	$y=0$

4D field	$\psi_{\overline{\mathbf{120}}}$	$\psi^{(a)}_{\mathbf{120}}$	$\psi^{(b)}_{\overline{\mathbf{3}}}$	$\psi'^{(c)}_{\overline{\mathbf{3}}}$
$SU(16)\times SU(3)_F$	$(\overline{\mathbf{120}},\mathbf{1})$	$(\mathbf{120},\mathbf{1})$	$(\mathbf{1},\overline{\mathbf{3}})$	$(\mathbf{1},\overline{\mathbf{3}})$
$U(1)_Z$	0	6	13	−32
$SL(2,\mathbb{C})$	(1/2, 0)	(1/2, 0)	(1/2, 0)	(1/2, 0)
Spacetime (y,v)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

First, a 6D $SU(19)$ bulk gauge boson A_M is decomposed into a 4D gauge field A_μ and fifth- and sixth-dimensional gauge fields A_y and A_v. The orbifold BCs of the 6D $SU(19)$ gauge field are given by

$$\begin{pmatrix} A_\mu \\ A_y \\ A_v \end{pmatrix}(x, y_j - y, v_j - v) = P_{j\mathbf{19}} \begin{pmatrix} A_\mu \\ -A_y \\ -A_v \end{pmatrix}(x, y_j + y, v_j + v) P_{j\mathbf{19}}^{-1}, \tag{3.3}$$

where $P_{j\mathbf{19}}$ is a projection matrix satisfying $(P_{j\mathbf{19}})^2 = I_{19}$. We consider the orbifold BCs P_2 and P_3 preserving $SU(19)$ symmetry, while the orbifold BCs P_0 and P_1 reduce $SU(19)$ to its regular subgroup $SU(16) \times SU(3)_F \times U(1)_Z$. We take $P_{j\mathbf{19}}$ as

$$P_{j\mathbf{19}} = \begin{cases} I_{19} & \text{for } j = 2,3, \\ \mathrm{diag}\,(I_{16}, -I_3) & \text{for } j = 0,1. \end{cases} \tag{3.4}$$

In this case, the 4D $SU(19)$ **360** gauge field A_μ have Neumann BCs at the fixed points (y_2, v_2) and (y_3, v_3), while the fifth- and sixth-dimensional gauge fields A_y and A_v have Dirichlet BCs because of the negative sign in Eq. (3.3). On the other hand, since $SU(19)$ symmetry is broken to $SU(16) \times SU(3)_F \times U(1)_Z$ at the fixed points (y_0, v_0) and (y_1, v_1), by using the branching rules of the $SU(19)$ adjoint representation **360** given in Eq. (2.5) as well as Eqs. (3.3) and (3.4), the $SU(16) \times SU(3)_F \times U(1)_Z$ $((\mathbf{255},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{8})(0) \oplus (\mathbf{1},\mathbf{1})(0))$ and $\big((\mathbf{16},\overline{\mathbf{3}})(19) \oplus (\overline{\mathbf{16}},\mathbf{3})(-19)\big)$

components of the 4D gauge field A_μ have Neumann and Dirichlet BCs at the fixed points (y_0, v_0) and (y_1, v_1), respectively; the $SU(16) \times SU(3)_F \times U(1)_Z$ $((\mathbf{255},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{8})(0) \oplus (\mathbf{1},\mathbf{1})(0))$ and $\big((\mathbf{16},\overline{\mathbf{3}})(19) \oplus (\overline{\mathbf{16}},\mathbf{3})(-19)\big)$ components of the fifth- and sixth-dimensional gauge fields A_y and A_v have Dirichlet and Neumann BCs, respectively. Thus, since the $SU(16) \times SU(3)_F \times U(1)_Z$ $((\mathbf{255},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{8})(0) \oplus (\mathbf{1},\mathbf{1})(0))$ components of the 4D gauge field A_μ have four Neumann BCs at the four fixed points (y_j, v_j) $(j = 0, 1, 2, 3)$, they have zero modes corresponding to 4D $SU(16)$, $SU(3)_F$, and $U(1)_Z$ gauge fields; since the other components of A_μ and any component of A_y and A_v have four Dirichlet BCs or two Neumann and two Dirichlet BCs at the four fixed points, they do not have zero modes. The orbifold BCs reduce $SU(19)$ to $SU(16) \times SU(3)_F \times U(1)_Z$.

To achieve the SM gauge symmetry $G_{\rm SM}$ at low energies, we consider the symmetry breaking sector via spontaneous symmetry breaking. We introduce 5D $SU(19)$ **10830**, **360**, and **19** brane scalar fields, $\Phi_{\mathbf{10830}}$, $\Phi_{\mathbf{360}}$, $\Phi_{\mathbf{19}}$, and $\Phi_{\mathbf{19}}^{\prime(\alpha)}$ $(\alpha = 1, 2)$ on the 5D brane $(y = 0)$. Their orbifold BCs are given by

$$\Phi_{\mathbf{x}}^{(\prime)}(x, v_\ell - v) = \eta_{\ell\mathbf{x}}^{(\prime)} P_{\ell\mathbf{x}} \Phi_{\mathbf{x}}^{(\prime)}(x, v_\ell + v), \tag{3.5}$$

where $\ell = 0, 2$, $\mathbf{x}$ stands for **10830**, **360**, and **19**, $\eta_{\ell\mathbf{x}}$ is a positive or negative sign, and $P_{\ell\mathbf{x}}$ is a projection matrix. We take $\eta_{\ell\mathbf{10830}} = -\eta_{\ell\mathbf{360}} = -\eta_{\ell\mathbf{19}} = \eta'_{0\mathbf{19}} = -\eta'_{2\mathbf{19}} = -1$. The branching rules of $SU(19) \supset SU(16) \times SU(3)_F \times U(1)_Z$ for **10830**, **360**, and **19** are given in Eqs. (2.3), (2.5), and (1.1), respectively. For $\Phi_{\mathbf{10830}}$, the $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{5440},\mathbf{1})(12) \oplus (\mathbf{120},\overline{\mathbf{3}})(-26) \oplus (\mathbf{1},\mathbf{6})(-64)$ components have zero modes; for $\Phi_{\mathbf{360}}$, the $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{255},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{8})(0) \oplus (\mathbf{1},\mathbf{1})(0)$ components have zero modes; for $\Phi_{\mathbf{19}}$, the $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{16},\mathbf{1})(3)$ components have zero modes; and for $\Phi_{\mathbf{19}}^{\prime(\alpha)}$, the $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{1},\mathbf{3})(-16)$ components have zero modes. We assume that a scalar field $\Phi_{\mathbf{10830}}$ is responsible for breaking $(SU(19) \supset)SU(16) \times SU(3)_F \times U(1)_Z$ to $SO(10) \times SU(3)_F$; two scalar fields $\Phi_{\mathbf{19}}^{\prime(\alpha)}$ are responsible for breaking $(SU(19) \supset SU(16) \times SU(3)_F \times U(1)_Z) \supset SO(10) \times SU(3)_F$ to $SO(10)$; the non-vanishing VEV of the scalar field $\Phi_{\mathbf{19}}$ breaks $(SU(19) \supset)SO(10)$ to $SU(5)$; the non-vanishing VEV of $\Phi_{\mathbf{360}}$ breaks $(SU(19) \supset)SU(5)$ to $G_{\rm SM}$.

The SM Weyl fermions are identified with zero modes of a 6D $SU(19)$ **171** Weyl bulk fermion. The orbifold BCs of 6D $SU(19)$ **171** positive or negative Weyl bulk fermions can be written by

$$\Psi_{\mathbf{171}\pm}^{(\prime)}(x, y_j - y, v_j - v) = \eta_{j\mathbf{171}\pm}^{(\prime)} \overline{\gamma} P_{j\mathbf{171}} \Psi_{\mathbf{171}\pm}^{(\prime)}(x, y_j + y, v_j + v), \tag{3.6}$$

where the $\pm$ subscript of Ψ stands for 6D chirality, $\eta_{j\mathbf{171}\pm}^{(\prime)}$ is a positive or negative sign, $\prod_{j=0}^{3} \eta_{j\mathbf{171}\pm}^{(\prime)} = 1$, the 6D gamma matrices γ^a $(a = 1, 2, \dots, 7)$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ $(\eta^{ab} = {\rm diag}(-I_1, I_5))$, $\overline{\gamma} := -i\gamma^5\gamma^6 = \gamma_{\rm 6D}^7 \gamma_{\rm 4D}^5$, $\gamma_{\rm 4D}^5 = I_2 \otimes \sigma^3 \otimes I_2$, $\gamma_{\rm 6D}^7 = I_4 \otimes \sigma^3$, and $P_{j\mathbf{171}\pm}^{(\prime)}$ is a projection matrix. (The same notation is used in Refs. [31,32].) In our notation, a 6D Dirac fermion $\Psi_{\rm D}^{\rm 6D}$ and 6D positive and negative Weyl fermions $\Psi_\pm^{\rm 6D} := P_\pm^{\rm 6D} \Psi_{\rm D}^{\rm 6D}$ $(P_\pm^{\rm 6D} := (1 \pm \gamma_{\rm 6D}^7)/2)$ can be expressed by using 4D left- and right-handed Weyl fermions $\psi_{L/R\pm}^{\rm 4D} (= P_{L/R}^{\rm 4D} \psi_{\rm D}^{\rm 4D})$ $(P_{L/R}^{\rm 4D} := (1 \pm \gamma_{\rm 4D}^5)/2)$, where the subscripts L/R and $\pm$ stand for 4D and 6D chiralities, respectively:

$$\Psi_{\rm D}^{\rm 6D} := \begin{pmatrix} \psi_{R+}^{\rm 4D} \\ \psi_{L+}^{\rm 4D} \\ \psi_{R-}^{\rm 4D} \\ \psi_{L-}^{\rm 4D} \end{pmatrix}, \quad \Psi_+^{\rm 6D} = P_+^{\rm 6D} \Psi_{\rm D}^{\rm 6D} = \begin{pmatrix} \psi_{R+}^{\rm 4D} \\ \psi_{L+}^{\rm 4D} \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_-^{\rm 6D} = P_-^{\rm 6D} \Psi_{\rm D}^{\rm 6D} = \begin{pmatrix} 0 \\ 0 \\ \psi_{R-}^{\rm 4D} \\ \psi_{L-}^{\rm 4D} \end{pmatrix}. \tag{3.7}$$

Note that to see the relation between $P_{j\mathbf{171}}$ and $P_{j\mathbf{19}}$, we express the orbifold BCs of $\Psi^{(\prime)}_{\mathbf{171}\pm}$ by using a 19×19 matrix form, the same as that of the gauge field in Eq. (3.3). We can write the $(\alpha\beta)$ component of $\Psi^{(\prime)}_{\mathbf{171}\pm}$ as $[\Psi^{(\prime)}_{\mathbf{171}\pm}]_{\alpha\beta} = \sum_{a<b}\psi^{(\prime)ab}_{\mathbf{171}\pm}[M_{\mathbf{171}ab}]_{\alpha\beta}$ $(a,b,\alpha,\beta = 1,2,\ldots,19)$, where the $\psi^{(\prime)ab}_{\mathbf{171}\pm}$s are the component fields of $\Psi^{(\prime)}_{\mathbf{171}\pm}$ expanded by $M_{\mathbf{171}ab}$s, and $[M_{\mathbf{171}ab}]_{\alpha\beta} = (1/\sqrt{2})\left(\delta_{a\alpha}\delta_{b\beta} - \delta_{a\beta}\delta_{b\alpha}\right)$. In this notation, the orbifold BCs of the 6D $SU(19)$ anti-symmetric tensor fermion $\Psi^{(\prime)}_{\mathbf{171}\pm}$ can be expressed by using the projection matrix $P_{j\mathbf{19}}$ given in Eq. (3.4) instead of $P_{j\mathbf{171}}$:

$$[\Psi^{(\prime)}_{\mathbf{171}\pm}(x,y_j-y,v_j-v)]_{\alpha\beta} = \eta^{(\prime)}_{j\mathbf{171}\pm}\overline{\gamma}[P_{j\mathbf{19}}]_{\alpha\kappa}[\Psi^{(\prime)}_{\mathbf{171}\pm}(x,y_j+y,v_j+v)]_{\kappa\lambda}[P_{j\mathbf{19}}]_{\lambda\beta}, \tag{3.8}$$

where $\alpha,\beta,\kappa,\lambda = 1,2,\ldots,19$, and $[P_{j\mathbf{19}}]_{\alpha\beta}$ denotes the $(\alpha\beta)$ element of the projection matrix $P_{j\mathbf{19}}$ given in Eq. (3.4). (The projection matrix of any $SU(19)$ tensor product representation can be expressed by the projection matrix of the $SU(19)$ defining representation $P_{j\mathbf{19}}$.)

Here, we check zero modes of, e.g., a 6D $SU(19)$ **171** positive Weyl fermion with orbifold BCs $(\eta_0,\eta_1,\eta_2,\eta_3) = (+,+,-,-)$ $\Psi_{\mathbf{171}+}$. At fixed points (y_2,v_2) and (y_3,v_3), the 4D $SU(19)$ **171** left-handed Weyl fermion components have Neumann BCs, while the 4D $SU(19)$ **171** right-handed Weyl fermion components have Dirichlet BCs. At fixed points (y_0,v_0) and (y_1,v_1), the 4D $SU(16)\times SU(3)_F\times U(1)_Z$ $(\mathbf{16},\mathbf{3})(-13)$ and $(\mathbf{120},\mathbf{1})(6)\oplus(\mathbf{1},\overline{\mathbf{3}})(-32)$ left-handed Weyl fermion components have Neumann and Dirichlet BCs, respectively, while the 4D $SU(16)\times SU(3)_F\times U(1)_Z$ $(\mathbf{16},\mathbf{3})(-13)$ and $(\mathbf{120},\mathbf{1})(6)\oplus(\mathbf{1},\overline{\mathbf{3}})(-32)$ right-handed Weyl fermion components have Dirichlet and Neumann BCs, respectively. In this case, only the 4D $SU(16)\times SU(3)_F\times U(1)_Z$ $(\mathbf{16},\mathbf{3})(-13)$ left-handed Weyl fermion has zero modes. Also, for $\Psi'_{\mathbf{171}+}$, the 4D $SU(16)\times SU(3)_F\times U(1)_Z$ $(\mathbf{120},\mathbf{1})(6)\oplus(\mathbf{1},\overline{\mathbf{3}})(-32)$ right-handed Weyl fermion has zero modes. They are vectorlike once we take into account the symmetry breaking effects of $SU(16)\times SU(3)_F\times U(1)_Z$ to $SO(10)$. Also, for $\Psi_{\mathbf{171}-}$ and $\Psi'_{\mathbf{171}-}$, there is no zero mode. The parity assignments of $\Psi^{(\prime)}_{\mathbf{171}\pm}$ are summarized in Table 2.

Here, we check the contribution to 6D bulk and 4D brane anomalies from the above 6D Weyl fermion sets. The fermion set does not contribute to 6D $SU(19)$ gauge anomalies because of the same number of 6D $SU(19)$ **171** positive and negative Weyl fermions. We need to check 4D gauge anomaly cancellation at four fixed points (y_j,v_j) $(j = 0,1,2,3)$ by using the 4D anomaly coefficients listed in Ref. [34]. From Table 2, at three fixed points (y_j,v_j) $(j = 1,2,3)$ there are two 4D left- and right-handed Weyl fermions in $(\mathbf{16},\mathbf{3})(-13)$, $(\mathbf{120},\mathbf{1})(6)$, and $(\mathbf{1},\overline{\mathbf{3}})(-32)$ of $SU(16)\times SU(3)_F\times U(1)$ from the 6D $SU(19)$ **171** positive and negative Weyl fermions $\Psi_{\mathbf{171}+}$, $\Psi'_{\mathbf{171}+}$, $\Psi_{\mathbf{171}-}$, and $\Psi'_{\mathbf{171}-}$. The vectorlike matter sets do not produce any 4D gauge anomalies. At the other fixed point (y_0,v_0), there can be 4D pure $SU(16)$, pure $SU(3)_F$, pure $U(1)_Z$, mixed $SU(16)-SU(16)-U(1)_Z$, mixed $SU(3)_F-SU(3)_F-U(1)_Z$, and mixed grav. − grav. − $U(1)_Z$ anomalies. In fact, the 6D $SU(19)$ **171** positive and negative Weyl fermions generate 4D pure $SU(16)$, pure $SU(3)_F$, pure $U(1)_Z$, mixed $SU(16)-SU(16)-U(1)_Z$, mixed $SU(3)_F-SU(3)_F-U(1)_Z$, and mixed grav. − grav. − $U(1)_Z$ anomalies. We focus on how to cancel the 4D anomalies at the fixed point (y_0,v_0) below.

To achieve 4D gauge anomaly cancellation at the fixed point (y_0,v_0), we need to introduce 4D Weyl fermions in appropriate representations of $SU(16)\times SU(3)_F\times U(1)$. First, we consider the pure $SU(16)$ gauge anomaly cancellation. The 4D $SU(16)$ gauge anomaly of 12 4D $SU(16)$ **16** left-handed Weyl fermions and four 4D $SU(16)$ **120** right-handed Weyl fermions is canceled out by the anomaly of three 4D $SU(16)$ **120** left-handed Weyl fermions. From Table 1, there are one 4D $SU(16)$ $\overline{\mathbf{120}}$ left-handed Weyl fermion $\psi_{\overline{\mathbf{120}}}$ and four 4D $SU(16)$ **120** left-handed Weyl fermions $\psi^{(a)}_{\mathbf{120}}$ $(a = 1,2,3,4)$ in the model. The $SU(16)$ gauge anomaly of $\psi_{\overline{\mathbf{120}}}$ is canceled out by one of, e.g., $\psi^{(1)}_{\mathbf{120}}$. The other three 4D $SU(16)$ **120** left-handed Weyl fermions $\psi^{(a)}_{\mathbf{120}}$ $(a = 2,3,4)$ contribute

Table 2. The parity assignments $\begin{pmatrix} \eta^{(\prime)}_{2\mathbf{171}\pm}\overline{\gamma}P^{(\prime)}_{2\mathbf{171}\pm} & \eta^{(\prime)}_{3\mathbf{171}\pm}\overline{\gamma}P^{(\prime)}_{3\mathbf{171}\pm} \\ \eta^{(\prime)}_{0\mathbf{171}\pm}\overline{\gamma}P^{(\prime)}_{0\mathbf{171}\pm} & \eta^{(\prime)}_{1\mathbf{171}\pm}\overline{\gamma}P^{(\prime)}_{1\mathbf{171}\pm} \end{pmatrix}$ of the 4D $SU(16)\times SU(3)\times U(1)$ left- and right-handed Weyl fermion components of the 6D $SU(19)$ **171** positive and negative Weyl fermions $\Psi_{\mathbf{171}+}$, $\Psi'_{\mathbf{171}+}$, $\Psi_{\mathbf{171}-}$, and $\Psi'_{\mathbf{171}-}$ given in Eq. (3.6).

	$\Psi_{\mathbf{171}+}$		$\Psi'_{\mathbf{171}+}$	
$SU(16)\times SU(3)\times U(1)$	Left	Right	Left	Right
$(\mathbf{16},\mathbf{3})(-13)$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$	$\begin{pmatrix} - & - \\ - & - \end{pmatrix}$	$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$
$(\mathbf{120},\mathbf{1})(6)$	$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{pmatrix} - & - \\ - & - \end{pmatrix}$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$
$(\mathbf{1},\overline{\mathbf{3}})(-32)$	$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{pmatrix} - & - \\ - & - \end{pmatrix}$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$

	$\Psi_{\mathbf{171}-}$		$\Psi'_{\mathbf{171}-}$	
$SU(16)\times SU(3)\times U(1)$	Left	Right	Left	Right
$(\mathbf{16},\mathbf{3})(-13)$	$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$	$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$
$(\mathbf{120},\mathbf{1})(6)$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$	$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$
$(\mathbf{1},\overline{\mathbf{3}})(-32)$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$	$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$

the $SU(16)$ gauge anomaly on the fixed point (y_0, v_0). Thus, the 4D brane fermions $\psi_{\overline{\mathbf{120}}}$ and $\psi^{(a)}_{\mathbf{120}}$ $(a = 1, 2, 3, 4)$ cancel the 4D $SU(16)$ gauge anomaly from the bulk fermions. Second, for the pure $SU(3)_F$ gauge anomaly, the 4D $SU(3)$ gauge anomaly of 64 4D $SU(3)$ **3** left-handed Weyl fermions and four 4D $SU(3)$ $\overline{\mathbf{3}}$ right-handed Weyl fermions is canceled out by the anomaly of 68 4D $SU(3)$ $\overline{\mathbf{3}}$ left-handed Weyl fermions. Third, the 4D pure $U(1)_Z$, mixed $SU(16) - SU(16) - U(1)_Z$, mixed $SU(3)_F - SU(3)_F - U(1)_Z$, and mixed grav. − grav. − $U(1)_Z$ anomalies are canceled out if the matter content is vectorlike from the view of the $U(1)_Z$ gauge theory. The matter content shown in Table 1 satisfies all the above requirements, so any 6D and 4D gauge anomalies at the fixed points are canceled out.

4. *Summary and discussion* In this letter we have pointed out that in a special GUT framework, family unification may be achieved by using GUT groups and their "regular-type" and "product-type" subgroups, such as $SU(19) \supset SU(16) \times SU(3) \times U(1)$ and $SU(48) \supset SU(16) \times SU(3)$, respectively.

For a "regular-type" subgroup $SU(19) \supset SU(16)\times SU(3)\times U(1)$, we have constructed an $SU(19)$ special GUT by using a special breaking $SU(16)$ to $SO(10)$. In this framework, the zero modes of a 6D $SU(19)$ **171** Weyl fermion can be identified with three generations of quarks and leptons; the 6D $SU(19)$ gauge anomaly on the bulk and the 4D $SU(19)$ or $SU(16)\times SU(3)\times U(1)$ gauge anomalies at each fixed point can be canceled out; as in the $SU(16)$ special GUT [50], exotic chiral fermions do not exist due to a special feature of the $SU(16)$ complex representation $\overline{\mathbf{120}}$ once we take into account the symmetry breaking of $SU(19)$ to $SO(10)$.

To cancel 4D pure $SU(16)$, pure $SU(3)_F$, pure $U(1)_Z$, and mixed anomalies on a fixed point, we introduced a lot of 4D Weyl fermions. For the mixed anomalies, one may rely on the Green–Schwarz anomaly cancellation mechanism [54] for the 4D version [55,56]. It may be achieved by introducing a pseudo-scalar field that transforms non-linearly under the anomalous $U(1)_Z$ symmetry. In this case, the number of 4D Weyl fermions can be drastically reduced.

We comment on the SM fermion masses in the $SU(19)$ special GUT. Since three generations of the SM fermions are unified into a 6D $SU(19)$ **171** Weyl fermion, the masses of all quarks and leptons are degenerate without $SU(19)(\supset SU(16) \times SU(3)_F \supset SO(10) \times SU(3)_F)$ breaking effects. We assumed that since the non-vanishing VEVs of 5D brane scalars $\Phi'^{(\alpha)}_{\mathbf{19}}$ break the $SU(3)_F$ symmetry, there is no reason to expect the unified masses of first, second, and third generations of up-type and down-type quarks, charged leptons, and neutrinos, respectively; in addition, since the non-vanishing VEVs of 5D brane scalars $\Phi_{\mathbf{10830}}$, $\Phi_{\mathbf{19}}$, and $\Phi_{\mathbf{360}}$ break $SU(16)(\supset SO(10))$ to G_{SM}, there is no reason to expect the degenerate mass of quarks and leptons for each generation. As discussed in, e.g., Refs. [31,32], on the UV brane $y = 0$, we can introduce $SU(19)$-invariant brane interaction terms among the 6D bulk fermions and the 5D brane scalars because $SU(19)$ tensor products, e.g., $\mathbf{171} \otimes \mathbf{171} \otimes \overline{\mathbf{10830}}$, $\mathbf{171} \otimes \overline{\mathbf{171}} \otimes \mathbf{360}$, $\mathbf{171} \otimes \overline{\mathbf{171}} \otimes \mathbf{19} \otimes \overline{\mathbf{19}}$, etc. contain a singlet. The $SU(16) \times SU(3)_F \times U(1)_Z$ $(\mathbf{16}, \mathbf{3})(-13)$ and $(\mathbf{120}, \mathbf{1})(6)$ components of the $SU(19)$ **171** bulk fermions can be mixed via the VEVs of the 5D brane scalars once their corresponding brane interaction terms or effective brane mass terms are generated, where $SO(10)(\subset SU(19))$ **120** contains $(\mathbf{3}, \mathbf{2})(-1)$, $(\overline{\mathbf{3}}, \mathbf{1})(4)$, $(\overline{\mathbf{3}}, \mathbf{1})(-2)$, $(\mathbf{1}, \mathbf{2})(3)$, and $(\mathbf{1}, \mathbf{1})(0)$ of G_{SM}. We expect that the effective mass terms divide the degenerate mass for each generation into up-type quark, down-type quark, charged lepton, and neutrino masses, where some VEVs or coupling constants must be hierarchical to realize mass hierarchies for up-type and down-type quarks and charged leptons. To realize tiny neutrino masses, it seems to be better to introduce 5D symplectic Majorana fermions [57] on the UV brane $y = 0$. $SU(19)$ brane interaction terms among the 6D bulk fermions, the 5D brane scalars, and the 5D symplectic Majorana fermions lead to tiny neutrino masses via a seesaw mechanism as discussed in Refs. [31,58]. The above brane interaction terms are essential to realize not only quark and lepton masses but also their mixing matrices, i.e., the Cabibbo–Kobayashi–Maskawa [59,60] and Maki–Nakagawa–Sakata [61] matrices. Since we can introduce a lot of 5D brane interaction terms in the $SU(19)$ special GUT, the model seems to realize the SM fermion masses, but seems to give us no prediction about quark and lepton masses and mixings. We will leave the detailed analysis for future studies.

We have discussed how to embed three chiral generations of quarks and leptons in a triplet (a finite-dimensional representation) of a non-Abelian compact group $SU(3)_F$. Another direction for unifying generations may be considered by using non-Abelian non-compact groups (e.g., $SU(1, 1)$) and their infinite-dimensional representation [5,62–68].

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