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Finite volume mass gap and free energy of the SU(N) × SU(N) chiral sigma model

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We compute the free energy in the presence of a chemical potential coupled to a conserved charge in the effective SU(N) × SU(N) scalar field theory to third order for asymmetric volumes in general d dimensions, using dimensional regularization. We also compute the mass gap in a finite box with periodic boundary conditions. The results can be used in $d = 4$ together with measurements of the observables in small volumes through simulations of quantum chromodynamics with N-massless quarks, to estimate combinations of low-energy constants of chiral perturbation theory.

Subject Index B30, B36

1. Introduction

Chiral perturbation theory (χPT) [1,2] is an effective theory describing the low-energy dynamics of the lowest-lying pseudoscalar mesons. The parameters of the theory are couplings appearing in the effective chiral Lagrangian, the pion decay constant F_π ($= F$ in the chiral limit) and other low-energy constants (LECs). These parameters can be determined by phenomenology, or by lattice simulations of quantum chromodynamics (QCD). For a detailed summary of various determinations of the LECs the reader is referred to the FLAG review [3].

For $N_{\mathrm{f}} = 2$ the relevant χPT has SU(2) × SU(2) $\simeq$ O(4) symmetry. As a consequence in the past many theoretical χPT computations, in particular those pertaining to finite volume, have been performed for the slightly simpler model with O(n) symmetry. One special environment is the so-called δ-regime first discussed by Leutwyler [4] where the system is in a periodic spatial box of sides L_s and $m_\pi L_s$ is small (i.e., small or zero quark mass) whereas $F_\pi L_s$ is large. In 2009 Hasenfratz [5] computed the mass gap in the delta-regime to third-order χPT with the hope that a comparison with a precise lattice measurement of the low-lying stable masses in this regime may be used to determine some combination of the LECs.

In a previous paper [6] we computed the change in the free energy due to a chemical potential coupled to a conserved charge in the non-linear O(n) sigma model with two regularizations, lattice regularization (with standard action) and dimensional regularization (DR) in a general d-dimensional asymmetric volume with periodic boundary conditions (pbc) in all directions. This freedom allowed us to establish for $d = 4$ two independent relations among the four-derivative couplings appearing in the effective Lagrangians and in turn this allows conversion of results for physical quantities computed by the lattice regularization to those involving scales introduced in DR.

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In particular we could convert the computation of the mass gap in a periodic box by Niedermayer and Weiermann [7] using lattice regularization to a result involving parameters of the dimensionally regularized effective theory, and we verified this result by a direct computation [6] (which disagrees slightly with the previous computation [5]).

Although $N_{\rm f} = 2$ is the phenomenologically most relevant case due to the low mass of the physical pions, χPT with $N_{\rm f} > 2$ can also have useful applications [3]. With this in mind in this paper we extend the computations to the case of $\mathrm{SU}(N) \times \mathrm{SU}(N)$. After recollecting the structure of the effective Lagrangian in the next section we compute the free energy in Sect. 3 and the mass gap in a finite periodic box in Sect. 4.

In this paper we do not analyze explicit chiral symmetry breaking. In QCD the effect of including a small quark mass on the finite volume spectrum has been computed for $N_{\rm f} = 2$ to leading order in Ref. [4], and to next-to-leading order by Weingart [8,9]. Furthermore Matzelle and Tiburzi [10] have studied the effect of small symmetry breaking in the quantum mechanical (QM) rotator picture ($N_{\rm f} = 2$), and extended the results to small non-zero temperatures. In a recent related paper [11] we have computed the isospin susceptibility in the effective $\mathrm{O}(n)$ scalar field theory, to third-order χPT in the delta-regime using the QM rotator picture including an explicit symmetry-breaking term, and showed consistency with standard χPT computations.

2. The effective Lagrangian

The dynamical fields are matrices $U(x) \in \mathrm{SU}(N)$. In the chiral limit the action is invariant under global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ transformations of the fields

$$U(x) \to g_L U(x) g_R^\dagger. \tag{1}$$

In this limit the leading-order effective Lagrangian is given by [1]:

$$\mathcal{L}_1 = \frac{1}{4g_0^2} \mathrm{tr}\left(\partial_\mu U^\dagger \partial_\mu U\right). \tag{2}$$

For $N \geq 4$ there are four linearly independent (omitting total derivatives and terms containing the equation of motion) four-derivative terms in the effective Lagrangian [1]:

$$\mathcal{L}_2 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \mathcal{L}_2^{(i)}, \tag{3}$$

with

$$\mathcal{L}_2^{(0)} = \mathrm{tr}\left(\partial_\mu U^\dagger \partial_\nu U \partial_\mu U^\dagger \partial_\nu U\right), \tag{4}$$

$$\mathcal{L}_2^{(1)} = \mathrm{tr}^2\left(\partial_\mu U^\dagger \partial_\mu U\right), \tag{5}$$

$$\mathcal{L}_2^{(2)} = \mathrm{tr}\left(\partial_\mu U^\dagger \partial_\nu U\right) \mathrm{tr}\left(\partial_\mu U^\dagger \partial_\nu U\right), \tag{6}$$

$$\mathcal{L}_2^{(3)} = \mathrm{tr}\left(\partial_\mu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\nu U\right). \tag{7}$$

The four-derivative couplings in Eq. (3) are related to the standard ones [1] as $G_4^{(i)} = -4L_i$. Since we work here in Euclidean space-time, our couplings differ in sign[1]. Note also the absence of the four-derivative term $\mathrm{tr}(\Box U^\dagger \Box U)$ in the above list; as explained in Ref. [12], this term can be eliminated by redefinition of the field U. The argument is reproduced for completeness in Appendix C.

For $N < 4$ these four operators are not all independent. One has [12]

$$\mathcal{L}_2^{(0)} = -\frac{1}{2}\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}, \qquad \mathcal{L}_2^{(3)} = \frac{1}{2}\mathcal{L}_2^{(1)} \qquad (N=2), \tag{8}$$

$$\mathcal{L}_2^{(0)} = \frac{1}{2}\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} - 2\mathcal{L}_2^{(3)} \qquad (N=3). \tag{9}$$

A proof of Eq. (9) is given in Appendix B. Accordingly, in Eq. (3) one can restrict the summation to $i = 1, 2$ for $N = 2$ and to $i = 1, 2, 3$ for $N = 3$.

From these relations it follows that the results obtained for general N should at $N = 2$ be invariant under the transformation

$$\begin{aligned} &G_4^{(0)} \to G_4^{(0)} + \alpha_1, \quad G_4^{(1)} \to G_4^{(1)} + \alpha_2, \\ &G_4^{(2)} \to G_4^{(2)} - \alpha_1, \quad G_4^{(3)} \to G_4^{(3)} + \alpha_1 - 2\alpha_2 \quad (N=2), \end{aligned} \tag{10}$$

and at $N = 3$ under

$$\begin{aligned} &G_4^{(0)} \to G_4^{(0)} + 2\alpha, \quad G_4^{(1)} \to G_4^{(1)} - \alpha, \\ &G_4^{(2)} \to G_4^{(2)} - 2\alpha, \quad G_4^{(3)} \to G_4^{(3)} + 4\alpha \quad (N=3). \end{aligned} \tag{11}$$

The $\mathrm{SU}(N)\times\mathrm{SU}(N)$ model for $N = 2$ flavors is equivalent to the O(4) non-linear sigma model [13] (with fields S_i, $i = 0, \ldots, 3$, and $S^2 = 1$) where

$$\overline{\mathcal{L}}_1 = \frac{1}{2g_0^2}\left(\partial_\mu S \cdot \partial_\mu S\right), \tag{12}$$

and

$$\overline{\mathcal{L}}_2 = \sum_{i=2,3} \frac{g_4^{(i)}}{4} \overline{\mathcal{L}}_2^{(i)} \tag{13}$$

with

$$\overline{\mathcal{L}}_2^{(2)} = \left(\partial_\mu S \cdot \partial_\mu S\right)^2, \tag{14}$$

$$\overline{\mathcal{L}}_2^{(3)} = \left(\partial_\mu S \cdot \partial_\nu S\right)\left(\partial_\mu S \cdot \partial_\nu S\right). \tag{15}$$

Writing

$$U(x) = S_0(x) + i\sum_{a=1}^{3} \sigma^a S_a(x), \qquad S^2(x) = 1, \tag{16}$$

[1] To avoid confusion with the box size L_μ we shall only use the renormalized couplings L_i^{r} in the final results.

where σ^a are the Pauli matrices, one obtains

$$\overline{\mathcal{L}}_2^{(2)} = \frac{1}{4}\mathcal{L}_2^{(1)}, \qquad \overline{\mathcal{L}}_2^{(3)} = \frac{1}{4}\mathcal{L}_2^{(2)}. \tag{17}$$

This leads to the identification [1]

$$g_4^{(2)} = 4G_4^{(1)}, \qquad g_4^{(3)} = 4G_4^{(2)}. \tag{18}$$

These and the relations (10), (11) can serve as checks on the final results.

2.1. *Perturbative expansion*

Here we work in a continuum volume $V = L_t \times L_s^{d_s}$, $d_s = d - 1$. In this section we impose periodic boundary conditions (pbc) on the dynamical variables in all directions. We dimensionally regularize by adding q extra compact dimensions of size $\widehat{L}$ (also with pbc) and analytically continue the resulting loop formulae to $q = -2\epsilon$. We define $D = d + q, V_D = V\widehat{L}^q$, and the aspect ratios $\ell = L_t/L_s, \hat{\ell} \equiv \widehat{L}/L_s$[2].

For the perturbative expansion we parameterize U with scalar fields $\xi_a(x)$[3]:

$$U(x) = u\overline{U}(x), \ \ \overline{U}(x) = \exp\left(ig_0\xi(x)\right), \tag{19}$$

where u is a constant matrix and

$$\xi = \sum_{a=1}^{N_1} \lambda^a \xi_a, \tag{20}$$

where the Hermitian λ-matrices are defined and some of their properties noted in Appendix A. Further,

$$N_1 \equiv N^2 - 1, \tag{21}$$

and the fields ξ satisfy the constraints

$$\int_x \xi_a(x) = 0, \quad \forall a. \tag{22}$$

The effective action $A_{2,\mathrm{eff}}$ for the ξ fields including only terms from $\mathcal{L}_1$ in Eq. (2), the measure and zero modes, has the perturbative expansion

$$A_{2,\mathrm{eff}} = A_{2,0} + g_0^2 A_{2,1} + g_0^4 A_{2,2} + \mathcal{O}\left(g_0^6\right), \tag{23}$$

where

$$A_{2,0} = \frac{1}{2}\int_x \partial_\mu \xi_a(x)\partial_\mu \xi_a(x), \tag{24}$$

$$A_{2,1} = A_{2,1}^{(a)} + A_{2,1}^{(b)}, \tag{25}$$

[2] It is advantageous to treat these extra dimensions with a different size, since an extra check of the calculation is provided by the requirement that physical quantities are independent of this choice.

[3] For SU(2) the identification to the O(4) fields S_a, $a = 1, 2, 3$, is $S_a = g_0\pi_a = \xi_a \sin(g_0|\xi|)/|\xi|$, where $|\xi| = \sqrt{\sum_b \xi_b\xi_b}$.

$$A_{2,1}^{(a)} = \frac{N}{3V_D}\int_x \sum_a \xi_a(x)\xi_a(x), \tag{26}$$

$$A_{2,1}^{(b)} = \frac{1}{48}\int_x \mathrm{tr}\left(\left[\xi(x), \partial_\mu\xi(x)\right]^2\right), \tag{27}$$

and

$$A_{2,2} = A_{2,2}^{(a)} + A_{2,2}^{(b)} + A_{2,2}^{(c)}, \tag{28}$$

$$A_{2,2}^{(a)} = \frac{1}{1440V_D}\int_x \sum_a \mathrm{tr}\left(\lambda^a\left[\xi(x),\left[\xi(x),\left[\xi(x),\left[\xi(x),\lambda^a\right]\right]\right]\right]\right), \tag{29}$$

$$A_{2,2}^{(b)} = \frac{1}{1152V_D^2}\int_{xy}\sum_{a,b}\mathrm{tr}\left(\lambda^b\left[\xi(x),\left[\xi(x),\lambda^a\right]\right]\right)\mathrm{tr}\left(\lambda^a\left[\xi(y),\left[\xi(y),\lambda^b\right]\right]\right), \tag{30}$$

$$A_{2,2}^{(c)} = \frac{1}{1440}\int_x \mathrm{tr}\left(\left[\xi(x),\left[\xi(x),\partial_\mu\xi(x)\right]\right]^2\right), \tag{31}$$

where the terms $A_{2,1}^{(a)}, A_{2,2}^{(a)}, A_{2,2}^{(b)}$ come from the zero-mode action derived in Appendix D.

The total effective action, including also the four-derivative terms, has then a perturbative expansion of the form

$$\mathcal{A} = \sum_{r=0}\mathcal{A}_r g_0^{2r}, \tag{32}$$

with

$$\mathcal{A}_r = A_{2,r} + \sum_{i=0}^{3}\frac{G_4^{(i)}}{4}A_{4,r}^{(i)}. \tag{33}$$

Note that

$$A_{4,0}^{(i)} = 0 = A_{4,1}^{(i)}, \ \ \forall i, \tag{34}$$

and

$$A_{4,2}^{(0)} = \int_x \mathrm{tr}\left(\partial_\mu\xi(x)\partial_\nu\xi(x)\partial_\mu\xi(x)\partial_\nu\xi(x)\right), \tag{35}$$

$$A_{4,2}^{(1)} = \int_x \mathrm{tr}^2\left(\partial_\mu\xi(x)\partial_\mu\xi(x)\right), \tag{36}$$

$$A_{4,2}^{(2)} = \int_x \mathrm{tr}\left(\partial_\mu\xi(x)\partial_\nu\xi(x)\right)\mathrm{tr}\left(\partial_\mu\xi(x)\partial_\nu\xi(x)\right), \tag{37}$$

$$A_{4,2}^{(3)} = \int_x \mathrm{tr}\left(\partial_\mu\xi(x)\partial_\mu\xi(x)\partial_\nu\xi(x)\partial_\nu\xi(x)\right). \tag{38}$$

The free two-point function is given by

$$\langle\xi_a(x)\xi_b(y)\rangle_0 = \delta_{ab}G(x-y), \tag{39}$$

with propagator

$$G(x) = \frac{1}{V_D} \sum_p{}' \frac{e^{ipx}}{p^2}, \tag{40}$$

where the sum is over momenta $p_\mu = 2\pi n_\mu / L_\mu$, $n_\mu \in \mathbb{Z}$ and the prime on the sum means that $p = 0$ is omitted.

3. The chemical potential

The chemical potential h is introduced by the substitution

$$\partial_0 \to \partial_0 + h \left[\frac{\lambda^3}{2}, \cdot \right]. \tag{41}$$

This gives an additional h-dependent part $\mathcal{A}_h$ to the total action of the form

$$\mathcal{A}_h = A_{2h} + \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} A_{4h}^{(i)}. \tag{42}$$

Further, writing

$$A_{2h} = ihB_2 + h^2 C_2 + \cdots, \tag{43}$$

$$A_{4h}^{(i)} = ihB_4^{(i)} + h^2 C_4^{(i)} + \cdots, \tag{44}$$

we have

$$B_2 = -\frac{i}{4g_0^2} \int_x \mathrm{tr} \left(\lambda^3 \left[U(x), \partial_0 U^\dagger(x) \right] \right), \tag{45}$$

$$C_2 = \frac{1}{16g_0^2} \int_x \mathrm{tr} \left(\left[\lambda^3, U(x) \right] \left[\lambda^3, U^\dagger(x) \right] \right). \tag{46}$$

The four-derivative operators $B_4^{(i)}, C_4^{(i)}$ are given in Appendix F.

The h-dependent part of the free energy f_h is defined as

$$e^{-Vf_h} = \langle e^{-\mathcal{A}_h} \rangle_{\mathcal{A}} = 1 - \langle \mathcal{A}_h \rangle_{\mathcal{A}} + \frac{1}{2} \langle \mathcal{A}_h^2 \rangle_{\mathcal{A}} + \cdots, \tag{47}$$

giving up to the order h^2

$$Vf_h = \langle \mathcal{A}_h \rangle_{\mathcal{A}} - \frac{1}{2} \langle \mathcal{A}_h^2 \rangle_{\mathcal{A}} + \frac{1}{2} \langle \mathcal{A}_h \rangle_{\mathcal{A}}^2 + \cdots. \tag{48}$$

Note that for an observable X

$$\langle X \rangle_{\mathcal{A}} = \langle X \rangle_0 - g_0^2 \langle X \mathcal{A}_1 \rangle_0^c - g_0^4 \langle X \mathcal{A}_2 \rangle_0^c + \frac{1}{2} g_0^4 \langle X \mathcal{A}_1^2 \rangle_0^c + \cdots. \tag{49}$$

Now

$$\langle B_2 \rangle_{\mathcal{A}} = 0 = \langle B_4^{(i)} \rangle_{\mathcal{A}} \quad \forall i, \tag{50}$$

so we have for the susceptibility

$$\chi \equiv -2 \lim_{h\to 0} \left(f_h/h^2\right) = -2 \sum_{s=1}^{5} F_s, \tag{51}$$

with

$$F_1 = \frac{1}{V_D} \langle C_2 \rangle_{\mathcal{A}}, \tag{52}$$

$$F_2 = \frac{1}{2} \frac{1}{V_D} \langle B_2^2 \rangle_{\mathcal{A}}, \tag{53}$$

$$F_3 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \frac{1}{V_D} \langle C_4^{(i)} \rangle_{\mathcal{A}}, \tag{54}$$

$$F_4 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \frac{1}{V_D} \langle B_2 B_4^{(i)} \rangle_{\mathcal{A}}, \tag{55}$$

$$F_5 = \frac{1}{2} \sum_{i,j=0}^{3} \frac{G_4^{(i)}}{4} \frac{G_4^{(j)}}{4} \frac{1}{V_D} \langle B_4^{(i)} B_4^{(j)} \rangle_{\mathcal{A}}. \tag{56}$$

Averaging over the zero modes, denoting $\overline{U}(x) = e^{ig_0 \lambda \xi(x)}$,

$$\begin{aligned}\frac{1}{V_D} \int du\, C_2 &= \frac{1}{8g_0^2} \int_x du \,\mathrm{tr}\left(\lambda^3 u \overline{U}(x) \lambda^3 \overline{U}^\dagger(x) u^\dagger - \left(\lambda^3\right)^2\right) \\ &= -\frac{1}{4g_0^2},\end{aligned} \tag{57}$$

where we have used Eq. (E3). So

$$F_1 = -\frac{1}{4g_0^2}. \tag{58}$$

Next

$$\frac{1}{V_D} \int du\, B_2^2 = \frac{1}{g_0^4} W, \tag{59}$$

with W given by

$$W = -\frac{1}{16V_D} \int_{xy} \int du \,\mathrm{tr}\left(\lambda^3 \left[u\overline{U}(x), \partial_0 \overline{U}^\dagger(x) u^\dagger\right]\right) \mathrm{tr}\left(\lambda^3 \left[u\overline{U}(y), \partial_0 \overline{U}^\dagger(y) u^\dagger\right]\right). \tag{60}$$

For the averages we have

$$\langle W \rangle_{\mathcal{A}} = \frac{1}{N_1} \langle [W] \rangle_{\mathcal{A}}, \tag{61}$$

where $[W]$ is obtained from W by replacing $\lambda^3_{ij}\lambda^3_{kl}$ with $\sum_a \lambda^a_{ij}\lambda^a_{kl}$. Using completeness in the form (A.12) we get

$$[W] = -\frac{1}{8V_D} \int_{xy} \int du \,\mathrm{tr}\left(\left[u\overline{U}(x), \partial_0 \overline{U}^\dagger(x) u^\dagger\right] \left[u\overline{U}(y), \partial_0 \overline{U}^\dagger(y) u^\dagger\right]\right)$$

$$= \frac{1}{8V_D}\int_{xy}\int du\, \mathrm{tr}\, (J_-(x)J_-(y) + J_+(x)J_+(y)$$
$$+uJ_-(x)u^\dagger J_+(y) + J_+(x)uJ_-(y)u^\dagger\Big), \tag{62}$$

where

$$J_+(x) \equiv i\partial_0\overline{U}^\dagger(x)\overline{U}(x),\ \ J_-(x) \equiv i\partial_0\overline{U}(x)\overline{U}^\dagger(x). \tag{63}$$

Note that $J_\pm$ are Hermitian $J_\pm^\dagger(x) = J_\pm(x)$ and traceless

$$\mathrm{tr}(J_\pm(x)) = 0, \tag{64}$$

and have a perturbative expansion[4]

$$J_\pm(x) = \pm g_0\left(\frac{\exp(\mathrm{Ad}[\pm ig_0\xi(x)]) - 1}{\mathrm{Ad}[\pm ig_0\xi(x)]}\right)\partial_0\xi(x) \tag{65}$$

$$= \pm g_0\sum_{r=1}^{\infty}\frac{1}{r!}\,(\mathrm{Ad}[\pm ig_0\xi(x)])^{r-1}\,\partial_0\xi(x)$$
$$= \pm g_0\partial_0\xi(x) + i\frac{g_0^2}{2}\,[\xi(x),\partial_0\xi(x)] \mp \frac{g_0^3}{6}\,[\xi(x),[\xi(x),\partial_0\xi(x)]]$$
$$- i\frac{g_0^4}{24}\,[\xi(x),[\xi(x),[\xi(x),\partial_0\xi(x)]]] + \mathcal{O}\left(g_0^5\right). \tag{66}$$

Note that for pbc

$$\int_x J_\pm(x) = 0. \tag{67}$$

Using Eq. (E3) we get simply

$$[W] = \frac{1}{8V_D}\int_{xy}\mathrm{tr}\,(J_-(x)J_-(y) + J_+(x)J_+(y)). \tag{68}$$

This has a perturbative expansion

$$[W] = g_0^4 W_2 + g_0^6 W_3 + \cdots \tag{69}$$

with

$$W_2 = -\frac{1}{16V_D}\int_{xy}\mathrm{tr}\,([\xi(x),\partial_0\xi(x)]\,[\xi(y),\partial_0\xi(y)])$$
$$= \frac{1}{2V_D}f_{abe}f_{cde}\int_{xy}\xi_a(x)\partial_0\xi_b(x)\xi_c(y)\partial_0\xi_d(y), \tag{70}$$

and

$$W_3 = W_3^{(1)} + W_3^{(2)}, \tag{71}$$

[4] For a, b in the Lie algebra $\mathrm{Ad}(a)b = [a, b]$.

with

$$W_3^{(1)} = \frac{1}{144V_D} \int_{xy} \mathrm{tr}\,([\xi(x),[\xi(x),\partial_0\xi(x)]]\,[\xi(y),[\xi(y),\partial_0\xi(y)]]), \tag{72}$$

$$W_3^{(2)} = \frac{1}{96V_D} \int_{xy} \mathrm{tr}\,([\xi(x),\partial_0\xi(x)]\,[\xi(y),[\xi(y),[\xi(y),\partial_0\xi(y)]]]). \tag{73}$$

Expanding Eq. (53) in a perturbative series

$$F_2 = \sum_{r=0}^{\infty} F_{2,r} g_0^{2r}, \tag{74}$$

we have at leading order

$$F_{2,0} = \frac{1}{2N_1}\langle W_2\rangle_0. \tag{75}$$

Now

$$\begin{aligned}\langle W_2\rangle_0 &= \frac{1}{8}Z_1 \int_x [\partial_0 G(x)]^2 \\ &= \frac{1}{8}Z_1 \bar{I}_{21}.\end{aligned} \tag{76}$$

Here Z_1 is a group factor defined in Eq. (A.16) in Appendix A where other such factors $Z_i, i = 2,\ldots,8$ appearing below are also defined and evaluated. Further, the dimensionally regularized sums $\bar{I}_{nm}$ are formally defined by

$$\bar{I}_{nm} = \frac{1}{V_D}\sum_p{}' \frac{\left(p_0^2\right)^m}{\left(p^2\right)^n}. \tag{77}$$

So we have at leading order

$$F_{2,0} = \frac{N}{2}\bar{I}_{21}. \tag{78}$$

At next order

$$F_{2,1} = \frac{1}{2N_1}\left[\langle W_3\rangle_0 - \langle W_2 A_{2,1}\rangle_0^c\right]. \tag{79}$$

First[5]

$$\langle W_3^{(1)}\rangle_0 = \frac{1}{96}(Z_2+Z_3)\overline{W} = \frac{1}{2}N^2 N_1 \overline{W}, \tag{80}$$

where

$$\overline{W} = -\int_x G(x)^2 \partial_0^2 G(x). \tag{81}$$

[5] We used $\int_y \partial_0^y \left[G(x-y)^2 \partial_0^x G(x-y)\right] = 0$.

This two-loop function, the “massless sunset diagram”, is calculated in detail in Ref. [14].

Secondly

$$\langle W_3^{(2)}\rangle_0 = \frac{1}{48} Z_5 G(0) \int_x [\partial_0 G(x)]^2 \tag{82}$$

$$= -\frac{5}{3} N^2 N_1 \bar{I}_{10} \bar{I}_{21}. \tag{83}$$

Next

$$\begin{aligned}\langle W_2 A_{2,1}^{(a)}\rangle_0^c &= -\frac{N}{48 V_D^2} \int_{xyu} \langle \mathrm{tr}\left([\xi(x), \partial_0 \xi(x)]\,[\xi(y), \partial_0 \xi(y)]\right) \xi_a(u)\xi_a(u)\rangle_0^c \\ &= \frac{4N^2 N_1}{3 V_D^2} \int_{xyu} G(x-u) G(y-u) \partial_0^x \partial_0^y G(x-y) \\ &= \frac{4}{3} N^2 N_1 \frac{1}{V_D} \bar{I}_{31}.\end{aligned} \tag{84}$$

Furthermore

$$\begin{aligned}\langle W_2 A_{2,1}^{(b)}\rangle_0^c &= -\frac{1}{768 V_D} \int_{xyu} \langle \mathrm{tr}\left([\xi(x), \partial_0 \xi(x)]\,[\xi(y), \partial_0 \xi(y)]\right) \mathrm{tr}\left(\left[\xi(u), \partial_\mu \xi(u)\right]^2\right)\rangle_0^c \\ &= w_2^{(a)} + w_2^{(b)} + w_2^{(c)},\end{aligned} \tag{85}$$

with

$$\begin{aligned}w_2^{(a)} &= \frac{N}{96 V_D} G(0) \int_{xyu} \langle \mathrm{tr}\left([\xi(x), \partial_0 \xi(x)]\,[\xi(y), \partial_0 \xi(y)]\,\partial_\mu \xi^a(u) \partial_\mu \xi^a(u)\right)\rangle_0^c \\ &= -\frac{N}{12 V_D} G(0) Z_1 \int_{xyu} \partial_\mu^u G(x-u) \partial_\mu^u G(y-u) \partial_0^x \partial_0^y G(x-y) \\ &= -\frac{2}{3} N^2 N_1 \bar{I}_{10} \bar{I}_{21}.\end{aligned} \tag{86}$$

$$\begin{aligned}w_2^{(b)} &= \frac{N}{96 V_D} \Box G(0) \int_{xyu} \langle \mathrm{tr}\left([\xi(x), \partial_0 \xi(x)]\,[\xi(y), \partial_0 \xi(y)]\,\xi^a(u) \xi^a(u)\right)\rangle_0^c \\ &= \frac{2}{3} N^2 N_1 \frac{1}{V_D} \bar{I}_{31}.\end{aligned} \tag{87}$$

Note that $\bar{I}_{00} = -\Box G(0) = -1/V_D$ since with dimensional regularization one sets $\delta(0) = 0$. Finally

$$\begin{aligned}w_2^{(c)} &= -\frac{1}{96 V_D} (Z_6 + Z_7) \int_{xyu} G(x-u) \partial_0^{x2} G(x-u) G(y-u) \partial_0^{y2} G(y-u) \\ &= -N^2 N_1 \bar{I}_{21}^2.\end{aligned} \tag{88}$$

3.1. Contribution from the four-derivative terms

For the averages we have

$$\left\langle C_4^{(i)} \right\rangle_{\mathcal{A}} = \frac{1}{N_1} \left\langle \left[C_4^{(i)} \right] \right\rangle_{\mathcal{A}}, \tag{89}$$

where $\left[C_4^{(i)}\right]$ is obtained in Appendix F from $C_4^{(i)}$ by replacing $\lambda^3_{ij}\lambda^3_{kl}$ with $\sum_a \lambda^a_{ij}\lambda^a_{kl}$ and averaging over the constant modes. From these expressions we obtain

$$\begin{aligned} F_{3,1} = & -\frac{G_4^{(0)}}{N}\left\{\frac{1}{V_D} + 2N_1\bar{I}_{11}\right\} + G_4^{(1)}\left\{\frac{N_1}{V_D} - 2\bar{I}_{11}\right\} \\ & + G_4^{(2)}\left\{\frac{1}{V_D} - N^2\bar{I}_{11}\right\} + \frac{G_4^{(3)}}{N}\left\{\frac{N_1}{V_D} - (N^2-2)\bar{I}_{11}\right\}. \end{aligned} \tag{90}$$

One can check that $F_{3,1} = 0$ for $N = 3$ when one sets $G_4^{(0)} = 2, G_4^{(1)} = -1, G_4^{(2)} = -2, G_4^{(3)} = 4$, as required by Eq. (9).

Finally

$$F_{4,1} = F_{5,1} = 0. \tag{91}$$

3.2. Summary

Collecting the results together, the expansion of the susceptibility with DR is given by

$$\chi = \frac{1}{2g_0^2}\left(1 + g_0^2 R_1 + g_0^4 R_2 + \cdots\right), \tag{92}$$

with

$$R_1 = -2N\bar{I}_{21}, \tag{93}$$

and

$$R_2 = R_2^{(a)} + R_2^{(b)}, \tag{94}$$

with

$$R_2^{(a)} = N^2\left\{-\overline{W} + 2\bar{I}_{21}\left[\bar{I}_{10} - \bar{I}_{21}\right] + \frac{4}{V_D}\bar{I}_{31}\right\}, \tag{95}$$

$$\begin{aligned} R_2^{(b)} = & -4\left[-\frac{G_4^{(0)}}{N}\left\{\frac{1}{V_D} + 2N_1\bar{I}_{11}\right\} + G_4^{(1)}\left\{\frac{N_1}{V_D} - 2\bar{I}_{11}\right\}\right. \\ & \left. + G_4^{(2)}\left\{\frac{1}{V_D} - N^2\bar{I}_{11}\right\} + \frac{G_4^{(3)}}{N}\left\{\frac{N_1}{V_D} - (N^2-2)\bar{I}_{11}\right\}\right] \end{aligned} \tag{96}$$

$$\begin{aligned} = & -\frac{4}{N}\left[-G_4^{(0)} + NN_1G_4^{(1)} + NG_4^{(2)} + N_1G_4^{(3)}\right]\frac{1}{V_D} \\ & + \frac{4}{N}\left[2N_1G_4^{(0)} + 2NG_4^{(1)} + N^3G_4^{(2)} + (N^2-2)G_4^{(3)}\right]\bar{I}_{11}. \end{aligned} \tag{97}$$

For $N = 2, 3$ the relations (10), (11) are satisfied.

3.3. *Renormalization of the free energy in $d = 4$*

We first recall some results obtained in Ref. [14] for the behavior of the functions as $q \to 0$:

$$\bar{I}_{10} = -\beta_1(\ell)L_s^{-2} + \mathcal{O}(q), \tag{98}$$

$$\bar{I}_{11} = \frac{1}{L_s^4}\left\{\frac{1}{2}(1 - q\ln L_s)\left[\gamma_1(\ell) - \frac{1}{2}\right] + q\mathcal{W}_1(\ell,\hat{\ell})\right\} + \mathcal{O}\left(q^2\right), \tag{99}$$

$$\bar{I}_{21} = \frac{1}{8\pi L_s^2}(\gamma_2(\ell) - 1) + \mathcal{O}(q), \tag{100}$$

$$\bar{I}_{31} = -\frac{1}{32\pi^2}\left[\frac{1}{q} - \ln L_s - \frac{1}{2}\gamma_3(\ell)\right] + \mathcal{O}(q), \tag{101}$$

where the shape functions $\beta_1(\ell)$, $\gamma_i(\ell)$ and $\overline{\mathcal{W}}(\ell)$ are given in Ref. [6], and for the two-loop function

$$\overline{W} = \frac{1}{16\pi^2 L_s^4}\left\{\left[\frac{1}{q} - 2\ln L_s\right]\mathcal{W}_0(\ell) + \frac{1}{3\ell}\ln(\hat{\ell}) - \frac{10}{3}\mathcal{W}_1(\ell,\hat{\ell}) + \overline{\mathcal{W}}(\ell)\right\} + \mathcal{O}(q), \tag{102}$$

with the non-singular shape function [6]:

$$\mathcal{W}_0(\ell) = \frac{5}{3}\left(\frac{1}{2} - \gamma_1(\ell)\right) - \frac{1}{3\ell}. \tag{103}$$

The shape function $\mathcal{W}_1(\ell,\hat{\ell})$ occurring in Eqs. (99) and (102) is not needed here (see below).

Below we switch to the conventional couplings $L_i = -G_4^{(i)}/4$ and express the bare couplings through the renormalized ones by

$$L_i = L_i^{\mathrm{r}} + \frac{v_i}{16\pi^2}\mu^{D-4}\left(\frac{1}{D-4} + \overline{C}\right), \tag{104}$$

where

$$\overline{C} = \log\bar{c} = -\frac{1}{2}(\ln(4\pi) - \gamma_E + 1) = -1.476\,904\,292. \tag{105}$$

By convention [1] the renormalized couplings are taken at the scale $\mu = M_\pi$, where M_π is the mass of the charged pion.

Requiring cancellation of the $\propto 1/(D-4)$ terms in R_2 one obtains two relations:

$$\begin{aligned} &Nv_2 - v_0 + (N^2 - 1)(Nv_1 + v_3) = \frac{5}{48}N^3, \\ &2Nv_0 - Nv_3 + (N^2 - 2)(v_2 - 2v_1) = 0. \end{aligned} \tag{106}$$

Due to these relations the terms $\ln(\hat{\ell})$ and $\mathcal{W}_1(\ell,\hat{\ell})$, depending on auxiliary, unphysical box size, also cancel. The relations (106) are satisfied by the coefficients v_i, which were calculated in an elegant way by Gasser and Leutwyler [1][6]:

$$v_0 = N/48, \quad v_1 = 1/16, \quad v_2 = 1/8, \quad v_3 = N/24. \tag{107}$$

Finally one has

$$L_s^2 \chi = \frac{1}{2} F^2 L_s^2 \left(1 + \frac{1}{F^2 L_s^2} (L_s^2 R_1) + \frac{1}{F^4 L_s^4} (L_s^4 R_2) + \mathcal{O}\left((F L_s)^{-6} \right) \right) \tag{108}$$

where

$$\begin{aligned} L_s^2 R_1 &= -\frac{N}{4\pi} (\gamma_2 - 1), \\ L_s^4 R_2 &= -\frac{N^2}{32\pi^2} \left[(\gamma_2 - 1)^2 + 8\pi (\gamma_2 - 1)\beta_1 + 2\overline{\mathcal{W}} - \frac{2}{\ell}\gamma_3 \right] \\ &\quad + \frac{5N^2}{48\pi^2} \left[\frac{1}{\ell} - \gamma_1 + \frac{1}{2} \right] \log(\bar{c} L_s M_\pi) \\ &\quad - \frac{8}{N} \left[2(N^2-1) L_0^{\mathrm{r}} + 2N L_1^{\mathrm{r}} + N^3 L_2^{\mathrm{r}} + (N^2-2) L_3^{\mathrm{r}} \right] \left(\gamma_1 - \frac{1}{2} \right) \\ &\quad + \frac{16}{N} \left[-L_0^{\mathrm{r}} + N(N^2-1) L_1^{\mathrm{r}} + N L_2^{\mathrm{r}} + (N^2-1) L_3^{\mathrm{r}} \right] \frac{1}{\ell} \qquad (N \geq 4). \end{aligned} \tag{109}$$

For $N = 3$ one should here omit the term proportional to L_0^{r}. Similarly, for $N = 2$ one should omit L_0^{r} and L_3^{r}. In addition, to use the conventional notation (stemming from the O(4) formulation), one should make the replacement $L_1^{\mathrm{r}} \to l_1^{\mathrm{r}}/4$, $L_2^{\mathrm{r}} \to l_2^{\mathrm{r}}/4$. This result is also invariant under the transformations corresponding to Eqs. (10) and (11).

For the O(n) case one has [6]

$$\begin{aligned} L^4 R_2^{\mathrm{O}(n)} &= -\frac{n-2}{16\pi^2} \left[(\gamma_2 - 1)^2 + 8\pi\,(\gamma_2 - 1)\,\beta_1 + 2\overline{\mathcal{W}} - \frac{(n-2)}{\ell}\gamma_3 \right] \\ &\quad + \frac{n-2}{24\pi^2} \left[\frac{3n-7}{\ell} - 5\left(\gamma_1 - \frac{1}{2} \right) \right] \log(\bar{c} L M_\pi) \\ &\quad - 2\left(2 l_1^{\mathrm{r}} + n l_2^{\mathrm{r}} \right) \left(\gamma_1 - \frac{1}{2} \right) + 4\left((n-1) l_1^{\mathrm{r}} + l_2^{\mathrm{r}} \right) \frac{1}{\ell}. \end{aligned} \tag{110}$$

Our result (109) for $N = 2$ flavors agrees with this taken at $n = 4$.

4. Computation of the mass gap on a periodic strip

In this section we will compute the mass gap of the 4D chiral SU(N) × SU(N) model on a periodic strip. We will follow the method first used in Ref. [15] and later in Ref. [7]. In the latter reference the computation was done using lattice regularization. Here we will employ dimensional regularization as we did in Ref. [6]. The dynamical fields $U(x)$ are now defined in a volume

$$\Lambda = \left\{ x; x_0 \in [-T, T], x_\mu \in [0, L], \text{for } \mu = 1, \ldots, d-1, x_\mu \in [0, \widehat{L}], \text{for } \mu = d, \ldots, D-1 \right\}, \tag{111}$$

[6] The coefficients in Ref. [1] are written out explicitly only for $N = 3$ but the previous steps are done for general N. The $N = 3$ coefficients Γ_i in Ref. [1] are given by $\Gamma_1 = v_1 + v_0/2 = 3/32$, $\Gamma_2 = v_2 + v_0 = 3/16$, and $\Gamma_3 = v_3 - 2v_0 = 0$.

with periodic boundary conditions in the $D-1$ "spatial" directions, and free boundary conditions in the time direction.

Here we will only give a brief description of the computation since it closely follows that for the O(n) model [6]. We first compute the two-point function

$$\begin{aligned} C_0(x) &= \lim_{T\to\infty} \frac{1}{N} \left\langle \mathrm{tr}\left(U^\dagger(x)U(0)\right)\right\rangle \\ &\propto e^{-(\mathcal{E}_1-\mathcal{E}_0)|x_0|}, \quad (|x_0|\to\infty). \end{aligned} \tag{112}$$

It follows that the mass gap

$$\mathcal{E}_1 - \mathcal{E}_0 = -\lim_{x_0\to\infty} \frac{\partial}{\partial x_0} \ln C(x_0). \tag{113}$$

Since $C_0(x)$ has a perturbative expansion of the form

$$C_0(x) = 1 + \sum_{\nu=1}^{\infty} g_0^{2\nu} C_0^{(\nu)}(x), \tag{114}$$

Eq. (113) yields the power series

$$\mathcal{E}_1 - \mathcal{E}_0 = \frac{1}{2\overline{V}_D} \sum_{\nu=1}^{\infty} g_0^{2\nu} \triangle^{(\nu)}, \tag{115}$$

where $\overline{V}_D \equiv L^{d-1}\widehat{L}^{D-d}$. If for $x_0 \to \infty$

$$C_0^{(\nu)}(x) \sim \sum_{r=0}^{\nu} \bar{c}_r^{(\nu)} \left(\frac{x_0}{2\overline{V}_D}\right)^r + \text{exponentially damped} \tag{116}$$

then

$$\triangle^{(1)} = -\bar{c}_1^{(1)}, \tag{117}$$

$$\triangle^{(2)} = -\bar{c}_1^{(2)} + \bar{c}_1^{(1)}\bar{c}_0^{(1)} = -\bar{c}_1^{(2)} - \triangle^{(1)}\bar{c}_0^{(1)}, \tag{118}$$

$$\begin{aligned} \triangle^{(3)} &= -\bar{c}_1^{(3)} + \bar{c}_1^{(2)}\bar{c}_0^{(1)} + \bar{c}_0^{(2)}\bar{c}_1^{(1)} - \bar{c}_1^{(1)}\bar{c}_0^{(1)2} \\ &= -\bar{c}_1^{(3)} - \triangle^{(2)}\bar{c}_0^{(1)} - \triangle^{(1)}\bar{c}_0^{(2)}. \end{aligned} \tag{119}$$

It thus suffices to compute the coefficients $c_i^{(r)}$ with $i = 0, 1$[7].

The fields $U(x)$ are parameterized as in Eq. (19) but now the $\xi(x)$ field satisfies the Neumann boundary conditions [15]

$$\partial_0\xi(x) = 0 \quad \text{for } x_0 = \pm T, \tag{120}$$

and periodic boundary conditions in the spatial directions.

The corresponding free two-point function is given by

$$\langle \xi_a(x)\xi_b(y)\rangle_0 = \delta_{ab} G(x,y), \tag{121}$$

[7] Computation of higher coefficients $c_i^{(r)}$ $i > 1$ can serve as useful checks since these are fixed by the requirement of exponentiation.

with

$$G(x,y) = \frac{1}{\overline{V}_D}\left(\frac{x_0^2+y_0^2}{4T} - \frac{1}{2}|x_0-y_0| + \frac{T}{6}\right) + \sum_{m=-\infty}^{\infty}\Big\{R\left(x_0-y_0+4mT, \mathbf{x}-\mathbf{y}\right) + R\left(x_0+y_0+2(2m+1)T, \mathbf{x}-\mathbf{y}\right)\Big\}, \tag{122}$$

where

$$R(z) = \frac{1}{2\overline{V}_D}\sum_{\mathbf{p}\neq 0}\frac{1}{\omega_{\mathbf{p}}}e^{-\omega_{\mathbf{p}}|z_0|}e^{i\mathbf{p}\mathbf{z}}, \tag{123}$$

where the sum goes over $p_\mu = \frac{2\pi\nu_\mu}{L_\mu},\ \mu = 1,\ldots,D-1$ with $\nu_\mu \in \mathbb{Z}$, and

$$\omega_{\mathbf{p}} = |\mathbf{p}|. \tag{124}$$

Expanding

$$\frac{1}{N}\mathrm{tr}\left(U^\dagger(x)U(0)\right) = 1 + \sum_{\nu=1}^{\infty} g_0^{2\nu}\theta_\nu(x) + \sum_{\nu=1}^{\infty} g_0^{2\nu+1}\rho_\nu(x), \tag{125}$$

the operators ρ_ν are not of interest to us here since their expectation values with operators even in ξ are zero, and the two-point correlation function has a perturbative expansion of the form

$$\left\langle\frac{1}{N}\mathrm{tr}\left(U^\dagger(x)U(0)\right)\right\rangle = 1 + \sum_{\nu=1}^{\infty} g_0^{2\nu}\omega_\nu(x), \tag{126}$$

with

$$\omega_1(x) = \langle\theta_1(x)\rangle_0, \tag{127}$$

$$\omega_2(x) = \langle\theta_2(x)\rangle_0 - \langle\theta_1(x)\mathcal{A}_1\rangle_0^{\mathrm{c}}, \tag{128}$$

$$\omega_3(x) = \langle\theta_3(x)\rangle_0 - \langle\theta_2(x)\mathcal{A}_1\rangle_0^{\mathrm{c}} - \langle\theta_1(x)\mathcal{A}_2\rangle_0^{\mathrm{c}} + \frac{1}{2}\langle\theta_1(x)\mathcal{A}_1^2\rangle_0^{\mathrm{c}}, \tag{129}$$

where $\langle\cdots\rangle^{\mathrm{c}}$ denote connected parts.

The interaction terms in the total action have the same form as in the previous section apart from the integration range, which is now Λ, and the volume factors V_D in the expressions for $A_{2,1}^{(a)}, A_{2,2}^{(a)}, A_{2,2}^{(b)}$ should be replaced by $|\Lambda| = 2T\overline{V}_D$.

The computation now proceeds as in Ref. [6], and here we only give the final results. In lowest order

$$\bar{c}_1^{(1)} = -\frac{2N_1}{N}, \tag{130}$$

$$\bar{c}_0^{(1)} = -\left(\frac{2N_1}{N}\right)R(0), \tag{131}$$

where $R(0)$ is dimensionally regularized [6]. So the leading-order contribution to the mass gap, first computed by Leutwyler [4], is given by

$$\triangle^{(1)} = \frac{2N_1}{N}. \tag{132}$$

In the next order

$$\bar{c}_1^{(2)} = \frac{2N_1(N_1-1)}{N^2} R(0), \tag{133}$$

$$\bar{c}_0^{(2)} = \frac{N_1(N_1-1)}{N^2} R(0)^2, \tag{134}$$

yielding

$$\triangle^{(2)} = 2N_1 R(0). \tag{135}$$

Finally at third order we obtain

$$\bar{c}_1^{(3)} = \frac{N_1}{N^3}\left[N^4 + 2N^2 - 4\right] R(0)^2 - 2N_1 N\left(W + \frac{3}{8}Y\right) + \bar{c}_1^{(3;13)}, \tag{136}$$

where

$$W = -\int_{-\infty}^{\infty} dz_0 \int_{\mathbf{z}} R(z)^2 \partial_0^2 R(z), \tag{137}$$

and

$$Y = \frac{1}{\overline{V}_D^2} \sum_{\mathbf{p}\neq 0} \frac{1}{\mathbf{p}^2}. \tag{138}$$

The term $\bar{c}_1^{(3;13)}$ appearing in Eq. (136) is the contribution to the correlator from the four-derivative terms:

$$\bar{c}_1^{(3;13)} = -\frac{8N_1}{N^2}\left[2(N^2-1)G_4^{(0)} + 2NG_4^{(1)} + N^3 G_4^{(2)} + (N^2-2)G_4^{(3)}\right]\ddot{R}(0). \tag{139}$$

The third-order energy shift is given by:

$$\begin{aligned}\triangle^{(3)} &= N_1 N\left[2W + \frac{3}{4}Y + R(0)^2\right] \\ &\quad + \frac{8N_1}{N^2}\left[2(N^2-1)G_4^{(0)} + 2NG_4^{(1)} + N^3 G_4^{(2)} + (N^2-2)G_4^{(3)}\right]\ddot{R}(0).\end{aligned} \tag{140}$$

Defining the moment of inertia Θ through[8]

$$m_1 = \frac{N_1}{N\Theta}, \tag{141}$$

then

$$\frac{\Theta}{\overline{V}_D} = \frac{1}{g_0^2}\left[1 + \Theta_1 g_0^2 + \Theta_2 g_0^4 + \cdots\right], \tag{142}$$

[8] For $N = 2$ this is consistent with the standard definition of Θ for O(4).

with

$$\Theta_1 = -NR(0), \tag{143}$$

$$\Theta_2 = -\frac{N}{2N_1}\triangle^{(3)} + N^2R(0)^2 \tag{144}$$

$$= -N^2\left(W + \frac{3}{8}Y - \frac{1}{2}R(0)^2\right) - \frac{4}{N}\left[2(N^2-1)G_4^{(0)} + 2NG_4^{(1)} + N^3G_4^{(2)} + (N^2-2)G_4^{(3)}\right]\ddot{R}(0). \tag{145}$$

For O(n) we had Ref. [6] for $d = 4$:

$$m_1 = \frac{(n-1)}{2\overline{\Theta}}, \tag{146}$$

with

$$\frac{\overline{\Theta}}{F^2L^3} = 1 + \overline{\Theta}_1(FL)^{-2} + \overline{\Theta}_2(FL)^{-4} + \cdots \tag{147}$$

and

$$\overline{\Theta}_1 = -(n-2)L^2R(0), \tag{148}$$

$$\overline{\Theta}_2 = (n-2)L^4\left[-2W + R(0)^2 - \frac{3}{4}Y\right] + 4\,(2l_1 + nl_2)\,\ddot{R}(0). \tag{149}$$

We can check (for $d = 4$) using Eq. (18) (and recalling $l_1 = -g_4^{(2)}/4, l_2 = -g_4^{(3)}/4$) and setting $F^2 = 1/g_0^2$ that

$$L^2\,[\Theta_1]_{N=2} = \left[\overline{\Theta}_1\right]_{n=4}, \tag{150}$$

$$L^4\,[\Theta_2]_{N=2} = \left[\overline{\Theta}_2\right]_{n=4}. \tag{151}$$

4.1. *Renormalization of the mass gap in* $d = 4$

The mass gap does not lead to a new renormalization condition beyond Eq. (106) required by the free energy considered in this paper. As discussed in Ref. [11], the reason for this is that they are closely related: knowing Θ determines the mass spectrum of the Hamiltonian states and these determine the free energy.

In Ref. [14] we find

$$W = \frac{5}{24\pi^2}\ddot{R}(0)\left[\frac{1}{D-4} - \ln L\right] + \frac{c_w}{L^4}, \tag{152}$$

with

$$c_w = 0.098\,682\,9798. \tag{153}$$

Further,

$$-L^2R(0) = -L\bar{I}_{10}^{(3)} = \beta_1^{(3)} = 0.225\,784\,9594, \tag{154}$$

and $L^4\ddot{R}(0)$ can be expressed through the known shape coefficients as[9]

$$-L^4\ddot{R}(0) \equiv \rho = 8\pi^2\beta_2^{(3)}(1) = \frac{1}{2}\alpha_2^{(3)}(1) + \frac{3}{4} = \lim_{\ell\to\infty}\left[\frac{1}{2}\left(\gamma_1^{(4)}(\ell) - \frac{1}{2}\right) + \frac{1}{\ell}\right]$$
$$= 0.837\,536\,9107. \quad (155)$$

Also

$$Y = L^{-3}\overline{I}_{10}^{(3)} = L^{-3}\left[\overline{G}_1\right]_{\mathrm{HL}}^{(d=3)} = -\beta_1^{(3)}L^{-4}. \quad (156)$$

After introducing the renormalized couplings (104) one obtains

$$\begin{aligned} L^2\Theta_1 &= N\beta_1^{(3)}, \\ L^4\Theta_2 &= \frac{1}{2}N^2\left[\beta_1^{(3)}\left(\beta_1^{(3)} + \frac{3}{4}\right) - 2c_w\right] - \frac{5N^2\rho}{24\pi^2}\log(\overline{c}LM) \\ &\quad - \frac{16\rho}{N}\left[2(N^2-1)L_0^{\mathrm{r}} + 2NL_1^{\mathrm{r}} + N^3L_2^{\mathrm{r}} + (N^2-2)L_3^{\mathrm{r}}\right] \qquad (N \geq 4). \end{aligned} \quad (157)$$

Note that the combination of the renormalized couplings is the same as one of the combinations appearing in Eq. (97). Again for $N = 3$ one should omit L_0^{r}, while for $N = 2$ the couplings L_0^{r} and L_3^{r} should be omitted, and L_1^{r} and L_2^{r} should be replaced by $4l_1^{\mathrm{r}}$ and $4l_2^{\mathrm{r}}$ respectively (see Appendix F).

5. Final remarks

5.1. Sensitivity to the four-derivative couplings L_i^{r} for $d = 4$

Here we make a few remarks concerning the sensitivity of the observables on the four-derivative couplings L_i^{r}. The sensitivity of the isospin susceptibility at $\ell = 1$ (hypercubic lattice) is obtained from Eq. (109) (observing that $\gamma^{(4)}(1) = 0$):

$$\frac{\delta_L\chi}{\chi} = \frac{1}{F^4L_s^4}\left(136\delta L_1^{\mathrm{r}} + 52\delta L_2^{\mathrm{r}} + 52\delta L_3^{\mathrm{r}}\right) \quad (N = 3,\ \ell = 1). \quad (158)$$

For a long cubic tube, $\ell \gg 1$, the sensitivity of the susceptibility and of the mass gap on L_i^{r} are

$$\begin{aligned} \frac{\delta_L\chi}{\chi} = \frac{\delta_L m_1}{m_1} &= -\frac{16}{3F^4L_s^4}\rho\left(6\delta L_1^{\mathrm{r}} + 27\delta L_2^{\mathrm{r}} + 7\delta L_3^{\mathrm{r}}\right), \\ &= \frac{1}{F^4L_s^4}\left(-26.8\delta L_1^{\mathrm{r}} - 120.6\delta L_2^{\mathrm{r}} - 31.3\delta L_3^{\mathrm{r}}\right) \quad (N = 3,\ \ell \gg 1). \end{aligned} \quad (159)$$

Note that all coefficients change sign as ℓ varies from 1 to ∞; this feature can be used to select optimal values of ℓ for certain purposes, e.g., to reduce the influence on the uncertainty of the L_i^{r} on determination of F.

[9] The expression in the square brackets above converges exponentially fast for $\ell \to \infty$; at $\ell = 4$ it already agrees to 9 digits with the limiting value.

5.2. Large-ℓ behavior

Comparing Θ_2 with R_2 in Eq. (109), using the large-ℓ behavior of the shape coefficients from Ref. [11] one finds that

$$\lim_{\ell\to\infty}\left(R_1+\frac{N}{6}\ell\right)=\Theta_1, \tag{160}$$

$$\lim_{\ell\to\infty} R_2=\Theta_2. \tag{161}$$

For the susceptibility calculated in χPT for the long cylinder geometry this gives a remarkably simple result:

$$L_s^2\chi=\frac{\Theta}{2L_s}-\frac{N}{12}\frac{L_t}{L_s}+\mathcal{O}\left(\frac{1}{F^4L_s^4}\right). \tag{162}$$

In the O(n) model for $\ell\to\infty$ one obtains $L_s^4R_2=\mathrm{const}(n-2)(n-4)\ell^2+\mathcal{O}(1)$, in contrast to the SU($N$) $\times$ SU(N) model. It is interesting to observe that in the cases of $n=2$ and $n=4$ the manifold S_{n-1} on which the system is moving is a group manifold, U(1) and SU(2) with symmetries U(1)$\times$U(1) and SU(2), correspondingly, while for general O(n) the expansion parameter for large ℓ is $\ell/(F^2L_s^2)$; in these special cases the expansion parameter seems to be $1/(F^2L_s^2)$ (see Eq. (3.6) of Ref. [11]).

Equation (162) is obtained assuming $L_s\ll L_t\ll F^2L_s^3$. This is a high-temperature expansion for the spatially constant modes and at the same time a low-temperature expansion for the $\mathbf{p}\neq 0$ modes. The leading term, $\Theta/2L_s$ is the classical result. The second one is the leading quantum correction; it appears both for O(n) and for SU(N) $\times$ SU(N), and does not depend on the dynamics. Note that $L_s^2\chi\propto\langle T_3^2\rangle=\langle C_2\rangle/(N^2-1)$ where C_2 is the quadratic Casimir invariant; hence in the more natural choice $(N^2-1)L_s^2\chi$ the curvature of the SU(N) manifold, $N(N^2-1)/12$, appears.

A specific feature of the SU(N) $\times$ SU(N) case is that in the χPT result (162) the $\propto L_s/\Theta\sim 1/(F^2L_s^2)$ term is absent, i.e., the LECs to NNL order are hidden in the first term alone. Related to this observation, there is strong evidence that in the SU(N) $\times$ SU(N) rotator approximation (describing the contribution of the spatially constant modes) there are no power-like corrections to the first two terms in Eq. (162) for general N (cf. F. Niedermayer and P. Weisz, manuscript in preparation). For the SU(2) $\times$ SU(2) $\simeq$ O(4) case this can be shown analytically; writing

$$\log(z_0(u))=\log(\sqrt{\pi}/4)-\frac{3}{2}\log u+u+\phi(u) \tag{163}$$

from Eq. (A.38) of Ref. [11] it follows that the correction term $\phi(u)$ decreases faster than any power of u. In fact it is extremely small already at $u=0.1$; one has $\phi(0.1)=-5.4\times10^{-41}$. For $N=3,4$, and 5 this was shown numerically (F. Niedermayer and P. Weisz, manuscript in preparation).

A derivation of the susceptibility from an SU(N) $\times$ SU(N) rotator (for general N) will be presented in a separate paper (F. Niedermayer and P. Weisz, manuscript in preparation). Suffice it here to say that in this scenario we have numerically shown the absence of power-like corrections for $N=3,4$, and 5.

The absence of a $\sim 1/\Theta^2$ term in the SU(3) $\times$ SU(3) rotator approximation does, however, not necessarily mean that a term $\mathcal{O}\left(F^{-4}L_s^{-4}\right)$ cannot be present in Eq. (162), since the simple rotator model requires modifications in order to match χPT at higher orders.

In Eq. (160) the limit $\ell \to \infty$ is reached exponentially fast, while in Eq. (161) apart from the exponentially small corrections there are $\propto 1/\ell$ corrections as well. This gives for the deviation of the susceptibility $\chi_{\rm rot}$ calculated for the *standard* rotator[10] from the χPT result χ in the NNL order

$$F^4 L_s^4 \frac{\chi - \chi_{\rm rot}}{\chi} = \frac{16}{N\ell}\left[(2N^2-3)(L_0^{\rm r}+L_3^{\rm r}) + N(N^2+1)(L_1^{\rm r}+L_2^{\rm r})\right] + \frac{5N^2}{16\pi^2\ell}\left[\log\left(\overline{c}L_s M_\pi\right) + \frac{1}{2}\alpha_0^{(3)}(1) - \frac{1}{3}\right] + \cdots . \tag{164}$$

The omitted terms at this order vanish exponentially as $\ell \to \infty$. The $1/\ell$ term given above should come from the distortion of the rotator spectrum in the region of energies $E \ll 1/L_s$, far below the threshold for the $\mathbf{p} \neq 0$ modes. In other words, the true rotator Hamiltonian differs from that of the standard rotator in higher order. A similar situation was found in Ref. [6] for the case of the O(n) model. The corresponding correction for the SU(N) $\times$ SU(N) case is discussed in a forthcoming paper (F. Niedermayer and P. Weisz, manuscript in preparation).

Finally we remark that the case of two dimensions $d = 2$ will be treated in more detail in the forthcoming paper referred to above.

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Appendix A. SU(N) Gell-Mann matrices

The $N \times N$ Gell-Mann Hermitian λ-matrices satisfy

$$\mathrm{tr}\lambda^a = 0, \tag{A.1}$$

$$\mathrm{tr}\left(\lambda^a\lambda^b\right) = 2\delta_{ab}, \tag{A.2}$$

$$\lambda^a\lambda^b = \frac{2}{N}\delta_{ab} + \left(d_{abc} + if_{abc}\right)\lambda^c, \tag{A.3}$$

where f_{abc} is totally anti-symmetric and d_{abc} is totally symmetric and

$$\sum_a d_{aac} = 0. \tag{A.4}$$

Note the identities

$$f_{abc}f_{cde} + f_{dbc}f_{ace} + f_{ebc}f_{adc} = 0, \tag{A.5}$$

$$f_{abc}d_{cde} + f_{dbc}d_{ace} + f_{ebc}d_{adc} = 0, \tag{A.6}$$

$$f_{abc}f_{dec} = \frac{2}{N}\left(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}\right) + d_{adc}d_{bec} - d_{aec}d_{bdc}, \tag{A.7}$$

and

$$f_{abc}f_{dbc} = N\delta_{ad}, \tag{A.8}$$

$$d_{abc}d_{dbc} = \frac{(N^2-4)}{N}\delta_{ad}. \tag{A.9}$$

[10] With the standard Hamiltonian proportional to the quadratic Casimir invariant C_2.

Completeness reads

$$\sum_a \lambda^a_{ij}\lambda^a_{kl} = 2\delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}. \tag{A.10}$$

From this we immediately get

$$\sum_a \lambda^a\lambda^a = \frac{2N_1}{N}\mathbf{1}, \qquad \sum_a \lambda^a\lambda^b\lambda^a = -\frac{2}{N}\lambda^b, \tag{A.11}$$

and

$$\sum_a \mathrm{tr}\left(\lambda^a A\right)\mathrm{tr}\left(\lambda^a B\right) = 2\mathrm{tr}(AB) - \frac{2}{N}\mathrm{tr}(A)\mathrm{tr}(B), \tag{A.12}$$

$$\sum_a \mathrm{tr}\left(\lambda^a A\lambda^a B\right) = 2\mathrm{tr}(A)\mathrm{tr}(B) - \frac{2}{N}\mathrm{tr}(AB). \tag{A.13}$$

For $N = 2$, $\lambda^a = \sigma^a$, the Pauli matrices. Also for an SU(2) matrix

$$U = \exp\left(i\sum_{a=1}^{3} v_a\sigma^a\right) = \cos(\sqrt{v^2}) + i\frac{\sin(\sqrt{v^2})}{\sqrt{v^2}}\sum_{a=1}^{3} v_a\sigma^a. \tag{A.14}$$

Note that for $N = 3$ we have the extra identity [16]

$$d_{abc}d_{cde} + d_{dbc}d_{ace} + d_{ebc}d_{adc} = \frac{1}{3}\left(\delta_{ab}\delta_{de} + \delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd}\right). \tag{A.15}$$

Appendix A.1. Group factors appearing in the perturbative computation

$$Z_1 \equiv -\sum_{a,b}\mathrm{tr}\left(\left[\lambda^a,\lambda^b\right]\left[\lambda^a,\lambda^b\right]\right) = 8NN_1, \tag{A.16}$$

$$\begin{aligned} Z_2 &\equiv \mathrm{tr}\left(\left[\lambda^a,\left[\lambda^b,\lambda^c\right]\right]\left[\lambda^a,\left[\lambda^b,\lambda^c\right]\right]\right) \\ &= 32 f_{bce}f_{aeg}f_{bcd}f_{adg} = 32N^2N_1, \end{aligned} \tag{A.17}$$

$$\begin{aligned} Z_3 &\equiv \mathrm{tr}\left(\left[\lambda^a,\left[\lambda^b,\lambda^c\right]\right]\left[\lambda^b,\left[\lambda^a,\lambda^c\right]\right]\right) \\ &= 32 f_{bce}f_{aeg}f_{bad}f_{cdg} = 16N^2N_1, \end{aligned} \tag{A.18}$$

$$Z_4 \equiv \mathrm{tr}\left(\left[\lambda^a,\left[\lambda^a,\lambda^b\right]\right]\left[\lambda^c,\left[\lambda^c,\lambda^b\right]\right]\right) = 32N^2N_1. \tag{A.19}$$

$$\begin{aligned} Z_5 \equiv \mathrm{tr}\Big(\left[\lambda^a,\lambda^b\right]\Big(&\left[\lambda^c,\left[\lambda^c,\left[\lambda^a,\lambda^b\right]\right]\right] + \left[\lambda^c,\left[\lambda^a,\left[\lambda^c,\lambda^b\right]\right]\right] \\ &+ \left[\lambda^a,\left[\lambda^c,\left[\lambda^c,\lambda^b\right]\right]\right]\Big)\Big) = -Z_2 - Z_3 - Z_4 = -80N^2N_1. \end{aligned} \tag{A.20}$$

$$\begin{aligned} Z_6 &\equiv \mathrm{tr}\left(\left[\lambda^a,\lambda^b\right]\left[\lambda^c,\lambda^d\right]\right)\mathrm{tr}\left(\left[\lambda^a,\lambda^b\right]\left[\lambda^c,\lambda^d\right]\right) \\ &= 64 f_{abe}f_{cde}f_{abg}f_{cdg} = 64N^2N_1, \end{aligned} \tag{A.21}$$

$$Z_7 \equiv \mathrm{tr}\left(\left[\lambda^a,\lambda^b\right]\left[\lambda^c,\lambda^d\right]\right)\mathrm{tr}\left(\left[\lambda^a,\lambda^c\right]\left[\lambda^b,\lambda^d\right]\right)$$

$$= 64 f_{abe} f_{cde} f_{acg} f_{bdg} = 32 N^2 N_1, \tag{A.22}$$

$$Z_8 \equiv \sum_{a,b,c} \mathrm{tr}\left(\lambda^a \lambda^b \lambda^c \lambda^a \lambda^b \lambda^c\right) = \frac{8N_1(N^2+1)}{N^2}. \tag{A.23}$$

Appendix B. Proof of Eq. (9)

We start from the trivial identity

$$\mathrm{tr}(ABAB) = \frac{1}{2}\mathrm{tr}(\{A,B\}^2) - \mathrm{tr}(A^2B^2). \tag{B1}$$

Let A, B be traceless SU(N) matrices $A = A_a\lambda^a, B = B_a\lambda^a$; we have

$$\mathrm{tr}(A^2B^2) = A^a A^b B^c B^d \left[\frac{4}{N}\delta_{ab}\delta_{cd} + 2d_{abe}d_{cde}\right], \tag{B2}$$

$$\mathrm{tr}(ABAB) = -\mathrm{tr}(A^2B^2) + 2A^a B^b A^c B^d \left[\frac{4}{N}\delta_{ab}\delta_{cd} + 2d_{abe}d_{cde}\right]. \tag{B3}$$

So

$$\mathrm{tr}(ABAB) + 2\mathrm{tr}(A^2B^2) = 2\left[2A^a B^b A^c B^d + A^a A^b B^c B^d\right]\left[\frac{2}{N}\delta_{ab}\delta_{cd} + d_{abe}d_{cde}\right] \tag{B4}$$

$$= 2A^a A^b B^c B^d \left[d_{abe}d_{cde} + d_{ace}d_{bde} + d_{ade}d_{bce} + \frac{2}{N}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)\right]. \tag{B5}$$

For SU(3) we have using Eq. (A.15)

$$\mathrm{tr}(ABAB) + 2\mathrm{tr}(A^2B^2) = 2A^a A^b B^c B^d \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \tag{B6}$$

$$= \frac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}(B^2) + \mathrm{tr}^2(AB), \quad N = 3. \tag{B7}$$

Now consider $U(x)$ slowly varying in x. Define $U(x) = U(0)V(x)$; then close to $x = 0$, $V(x)$ is close to the identity matrix

$$V(x) = 1 + ix_\mu A_\mu + \cdots \tag{B8}$$

where A_μ are traceless Hermitian matrices. Then the $\mathcal{L}_i$ at $x = 0$ can be computed by replacing $\partial_\mu U$ with $iU(0)A_\mu$. The factors involving $U(0)$ cancel and then Eq. (9) follows using Eq. (B6).

Appendix C. Redundancy of $\mathrm{tr}(\Box U^\dagger \Box U)$

For the SU(2) case the four-derivative operator

$$\mathcal{L}_2^{(X)} = \mathrm{tr}\left(\Box U^\dagger \Box U\right), \tag{C1}$$

corresponding to $(\Box S \cdot \Box S)$ in O(4), turns out to be redundant: it can be eliminated by changing the integration variable $U(x)$ in the path integral. Below we show that this remains true for general SU(N).

Consider the change of variables

$$U \to U e^{\alpha F} = U\left(1 + \alpha F + \mathcal{O}\left(\alpha^2\right)\right) \tag{C2}$$

where F is a traceless anti-Hermitian matrix.

By choosing

$$F = \frac{1}{2}\left(U^\dagger \Box U - \Box U^\dagger U\right), \tag{C3}$$

one has

$$U \to U + \alpha\, UF = U + \frac{\alpha}{2}\left(\Box U - U \Box U^\dagger U\right) + \mathcal{O}\left(\alpha^2\right). \tag{C4}$$

For the SU(2) case this corresponds to the change of variables

$$S \to S + \alpha\,[\Box S - S(S \cdot \Box S)] + \mathcal{O}\left(\alpha^2\right), \tag{C5}$$

which is the transformation used to show the redundancy of the operator $\mathrm{tr}(\Box S\, \Box S)$.

We have still to show that F is indeed traceless. One has

$$\mathrm{tr} F = \frac{1}{2}\mathrm{tr}\left(U^\dagger \Box U - \Box U^\dagger U\right) = \mathrm{Im}\,\mathrm{tr}\left(U^\dagger \Box U\right). \tag{C6}$$

Further, we can write

$$\mathrm{Im}\,\mathrm{tr}\left(U^\dagger(x) \Box U(x)\right)\Big|_{x=0} = \mathrm{Im}\,\mathrm{tr}\left[\Box\left(U^\dagger(0) U(x)\right)\right]\Big|_{x=0}. \tag{C7}$$

One has

$$W(x) \equiv U^\dagger(0) U(x) = \exp\left(i x_\mu A_\mu + \frac{1}{2} i x_\mu x_\nu B_{\mu\nu} + \mathcal{O}\left(x^3\right)\right), \tag{C8}$$

where A_μ and $B_{\mu\nu}$ are traceless Hermitian matrices. From this it follows that

$$\mathrm{Im}\,\Box W(x)|_{x=0} = \mathrm{Im}\,\mathrm{tr}\left(-A_\mu A_\mu + i B_{\mu\mu}\right) = 0. \tag{C9}$$

Therefore we can conclude that the operator $\mathrm{tr}(\Box U^\dagger\, \Box U)$ can be transformed away by a field redefinition.

A similar discussion to that presented above has been given by Leutwyler in Eq. (11.6) and the following paragraph of Ref. [12].

Appendix D. Faddeev–Popov trick for the zero modes

Consider the SU(N) partition function formally given by

$$Z = \int \left[\prod_x dU(x)\right] e^{-A(U)}. \tag{D1}$$

We parameterize U as in Eq. (19). The integral over the constant u factors out for this consideration. The action and measure are invariant under global SU(N) transformations

$$U(x) \to V U(x), \tag{D2}$$

which induces a change

$$\xi_a(x) \to \xi_a^V(x). \tag{D3}$$

Define à la Faddeev–Popov $\Phi[\xi]$ through the integral

$$1 = \Phi[\xi] \int dV \prod_{a}^{N^2-1} \delta\left(\int_x \xi_a^V(x)\right). \tag{D4}$$

Now the action, measure, and also $\Phi[\xi]$ are invariant under SU(N) transformations, so inserting 1 in the form of the rhs of Eq. (D4) in the partition function we obtain

$$Z = \int dV \int \prod_x \left[\frac{d\xi(x)}{M[\xi(x)]}\right] e^{-A(\xi)} \Phi[\xi] \prod_{a=1}^{N^2-1} \delta\left(\int_x \xi_a(x)\right). \tag{D5}$$

The group volume is an irrelevant factor. Also, for DR we set $M[\xi(x)] = 1,\ \forall x$.

Now we only need $\Phi[\xi]$ near the surface $\int_x \xi(x) = 0$ and we can consider an infinitesimal transformation

$$V = 1 + i\alpha_a \lambda^a + \mathcal{O}\left(\alpha^2\right). \tag{D6}$$

This induces a change

$$g_0 \xi_a^V(x)\lambda^a = g_0 \xi_a(x)\lambda^a + \alpha_a t^a(x) + \mathcal{O}\left(\alpha^2\right), \tag{D7}$$

with $t(x)$ obtained by solving (the argument x understood)

$$(1 + i\alpha_a\lambda^a + \mathcal{O}\left(\alpha^2\right))e^{ig_0\xi_b\lambda^b} = e^{ig_0\xi_a\lambda^a + i\alpha_a t^a} + \mathcal{O}\left(\alpha^2\right), \tag{D8}$$

thereby yielding

$$\begin{aligned} t^a &= \lambda^a + g_0 \frac{i}{2}\left[\lambda^a, \xi\right] - g_0^2 \frac{1}{12}\left[\xi, \left[\xi, \lambda^a\right]\right] \\ &\quad - g_0^4 \frac{1}{720}\left[\xi, \left[\xi, \left[\xi, \left[\xi, \lambda^a\right]\right]\right]\right] + \cdots && \text{(D9)} \\ &= T_{ab}\lambda^b, && \text{(D10)} \end{aligned}$$

with

$$\begin{aligned} T_{ab} &= \frac{1}{2}\mathrm{tr}\left(\lambda^b t^a\right) && \text{(D11)} \\ &= \delta_{ab} + g_0 f_{abc}\xi_c - g_0^2 \frac{1}{24}\mathrm{tr}\left(\lambda^b\left[\xi, \left[\xi, \lambda^a\right]\right]\right) \\ &\quad - g_0^4 \frac{1}{1440}\mathrm{tr}\left(\lambda^b\left[\xi, \left[\xi, \left[\xi, \left[\xi, \lambda^a\right]\right]\right]\right]\right) + \cdots . && \text{(D12)} \end{aligned}$$

So

$$\begin{aligned} \Phi[\xi]^{-1} &= \int \prod_a \left[d\alpha_a \delta(\alpha_b \overline{T}_{ab}[\xi])\right] && \text{(D13)} \\ &= \left(\det \overline{T}[\xi]\right)^{-1}, && \text{(D14)} \end{aligned}$$

with (setting $\int_x \xi_a(x) = 0$)

$$\overline{T}_{ab}[\xi] = \delta_{ab} - g_0^2 \overline{T}_{ab}^{(1)} - g_0^4 \overline{T}_{ab}^{(2)} + \cdots \tag{D15}$$

where

$$\overline{T}^{(1)}_{ab} = \frac{1}{24V_D}\int_x \mathrm{tr}\left(\lambda^b\left[\xi(x),\left[\xi(x),\lambda^a\right]\right]\right), \tag{D16}$$

$$\overline{T}^{(2)}_{ab} = \frac{1}{1440V_D}\int_x \mathrm{tr}\left(\lambda^b\left[\xi(x),\left[\xi(x),\left[\xi(x),\left[\xi(x),\lambda^a\right]\right]\right]\right]\right). \tag{D17}$$

The zero-mode action is then given by

$$A_{\mathrm{zero}} = -\ln\Phi[\xi] \tag{D18}$$

$$= -\mathrm{tr}\ln\left(\overline{T}[\xi]\right) \tag{D19}$$

$$= -(N^2-1)\ln V_D + g_0^2\mathrm{tr}\overline{T}^{(1)} + g_0^4\mathrm{tr}\left\{\overline{T}^{(2)} + \frac{1}{2}\overline{T}^{(1)2}\right\} + \cdots. \tag{D20}$$

We have in particular

$$\mathrm{tr}\overline{T}^{(1)} = \frac{N}{3V_D}\int_x\sum_a \xi_a(x)^2. \tag{D21}$$

Appendix E. Some integrals over SU(N) matrices

We give integrals with the Haar measure over SU(N) matrices u (see, e.g., Ref. [17]):

$$\int du = 1. \tag{E1}$$

$$\int du\, u_{ij}(u^\dagger)_{kl} = \frac{1}{N}\delta_{il}\delta_{jk}. \tag{E2}$$

It follows that

$$\int du\,\mathrm{tr}(uAu^\dagger B) = \frac{1}{N}\mathrm{tr}(A)\mathrm{tr}(B), \tag{E3}$$

$$\int du\,\mathrm{tr}(uA)\mathrm{tr}(u^\dagger B) = \frac{1}{N}\mathrm{tr}(AB). \tag{E4}$$

$$\int du\, u_{i_1j_1}\cdots u_{i_Nj_N} = \frac{1}{N!}\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}, \tag{E5}$$

where $\epsilon_{i_1\cdots i_N}$ is the totally anti-symmetric tensor with $\epsilon_{1\ldots N} = 1$.

Note that for $N = 2$ and an SU(2) matrix V

$$\epsilon_{ik}\epsilon_{jl}V_{kl} = V^*_{ij} = \left(V^\dagger\right)_{ji} \tag{E6}$$

(the conjugate of the fundamental representation is equivalent to the fundamental representation for SU(2)). So

$$\int du\,\mathrm{tr}(uA)\mathrm{tr}(uB) = \frac{1}{2}\mathrm{tr}(AB^\dagger), \quad N = 2. \tag{E7}$$

Appendix F. Expressions involving the four-derivative terms

The four-derivative terms $B_4^{(i)}$, $C_4^{(i)}$ appearing in Eq. (44) are given by:

$$B_4^{(0)} = -i \int_x \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_\mu U(x) \partial_0 U^\dagger(x) \partial_\mu U(x)\right) + \text{h.c.} \tag{F1}$$

$$\begin{aligned} C_4^{(0)} = \frac{1}{4}\int_x \mathrm{tr}\Big\{ &\partial_\mu U^\dagger(x)\left[\lambda^3, U(x)\right] \partial_\mu U^\dagger(x)\left[\lambda^3, U(x)\right] \\ &+ \partial_0 U(x) \partial_0 U^\dagger(x)\left[\lambda^3, U(x)\right]\left[\lambda^3, U^\dagger(x)\right] \\ &+ \partial_0 U^\dagger(x) \partial_0 U(x)\left[\lambda^3, U^\dagger(x)\right]\left[\lambda^3, U(x)\right]\Big\} + \text{h.c.} \end{aligned} \tag{F2}$$

$$B_4^{(1)} = -i \int_x \mathrm{tr}\left(\partial_0 U^\dagger(x)\left[\lambda^3, U(x)\right]\right) \mathrm{tr}\left(\partial_\mu U^\dagger(x) \partial_\mu U(x)\right) + \text{h.c.}, \tag{F3}$$

$$\begin{aligned} C_4^{(1)} = \frac{1}{4}\int_x \Big\{ &\mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right]\left[\lambda^3, U(x)\right]\right) \mathrm{tr}\left(\partial_\mu U^\dagger(x) \partial_\mu U(x)\right) \\ &+ \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x)\right) \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x)\right) \\ &+ \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x)\right) \mathrm{tr}\left(\partial_0 U^\dagger(x)\left[\lambda^3, U(x)\right]\right)\Big\} + \text{h.c.} \end{aligned} \tag{F4}$$

$$B_4^{(2)} = -i \int_x \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_\mu U(x)\right) \mathrm{tr}\left(\partial_0 U^\dagger(x) \partial_\mu U(x)\right) + \text{h.c.}, \tag{F5}$$

$$\begin{aligned} C_4^{(2)} = \frac{1}{4}\int_x \Big\{ &\mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right]\left[\lambda^3, U(x)\right]\right) \mathrm{tr}\left(\partial_0 U^\dagger(x) \partial_0 U(x)\right) \\ &+ \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_\mu U(x)\right) \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_\mu U(x)\right) \\ &+ \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x)\right) \mathrm{tr}\left(\partial_0 U^\dagger(x)\left[\lambda^3, U(x)\right]\right)\Big\} + \text{h.c.} \end{aligned} \tag{F6}$$

$$B_4^{(3)} = -i \int_x \mathrm{tr}\left(\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x) \partial_\mu U^\dagger(x) \partial_\mu U(x)\right) + \text{h.c.}, \tag{F7}$$

$$\begin{aligned} C_4^{(3)} = \frac{1}{4}\int_x \mathrm{tr}\Big\{ &\left[\lambda^3, U^\dagger(x)\right]\left[\lambda^3, U(x)\right] \partial_\mu U^\dagger(x) \partial_\mu U(x) \\ &+ \left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x)\left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x) \\ &+ \left[\lambda^3, U^\dagger(x)\right] \partial_0 U(x) \partial_0 U^\dagger(x)\left[\lambda^3, U(x)\right]\Big\} + \text{h.c.} \end{aligned} \tag{F8}$$

Using Eqs. (A.12), (A.13), (E3), and (E4) the average $\left[C_4^{(0)}\right]$ in Eq. (89) is given by

$$\begin{aligned} \left[C_4^{(0)}\right] &= \frac{1}{4}\int_x \int du\, \mathrm{tr}\Big\{\partial_\mu \overline{U}^\dagger(x) u^\dagger\left[\lambda^a, u\overline{U}(x)\right] \partial_\mu \overline{U}^\dagger(x) u^\dagger\left[\lambda^a, u\overline{U}(x)\right] \\ &+ u\partial_0 \overline{U}(x) \partial_0 \overline{U}^\dagger(x) u^\dagger\left[\lambda^a, u\overline{U}(x)\right]\left[\lambda^a, \overline{U}^\dagger(x) u^\dagger\right] \\ &+ \partial_0 \overline{U}^\dagger(x) \partial_0 \overline{U}(x)\left[\lambda^a, \overline{U}^\dagger(x) u^\dagger\right]\left[\lambda^a, u\overline{U}(x)\right]\Big\} + \text{h.c.} \end{aligned} \tag{F9}$$

$$= \frac{1}{2}\int_x \int du\, \Big\{2\mathrm{tr}\left(u\partial_\mu \overline{U}(x)\right) \mathrm{tr}\left(u^\dagger \partial_\mu \overline{U}^\dagger(x)\right)$$

$$+\,\mathrm{tr}\left(u\overline{U}(x)\right)\left(\mathrm{tr}\left(u^\dagger\overline{U}^\dagger(x)\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)+\mathrm{tr}\left(u^\dagger\partial_0\overline{U}^\dagger(x)\partial_0\overline{U}(x)\overline{U}^\dagger(x)\right)\right)$$
$$+\,\mathrm{tr}\left(u^\dagger\overline{U}^\dagger(x)\right)\left(\mathrm{tr}\left(u\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\overline{U}(x)\right)+\mathrm{tr}\left(u\overline{U}(x)\partial_0\overline{U}^\dagger(x)\partial_0\overline{U}(x)\right)\right)$$
$$-4N\,\mathrm{tr}\left(\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)\Big\}+\mathrm{h.c.} \tag{F10}$$
$$=\frac{2}{N}\int_x\left\{\mathrm{tr}\left(\partial_\mu\overline{U}(x)\partial_\mu\overline{U}^\dagger(x)\right)-2N_1\,\mathrm{tr}\left(\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)\right\}. \tag{F11}$$

Similarly for $\left[C_4^{(i)}\right]$, $i=1,2,3$, one obtains

$$\left[C_4^{(1)}\right]=-2\int_x\left\{N_1\,\mathrm{tr}\left(\partial_\mu\overline{U}(x)\partial_\mu\overline{U}^\dagger(x)\right)+2\mathrm{tr}\left(\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)\right\}, \tag{F12}$$
$$\left[C_4^{(2)}\right]=-2\int_x\left\{N^2\mathrm{tr}\left(\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)+\mathrm{tr}\left(\partial_\mu\overline{U}(x)\partial_\mu\overline{U}^\dagger(x)\right)\right\}, \tag{F13}$$
$$\left[C_4^{(3)}\right]=-\frac{2}{N}\int_x\left\{(N^2-2)\mathrm{tr}\left(\partial_0\overline{U}(x)\partial_0\overline{U}^\dagger(x)\right)+N_1\,\mathrm{tr}\left(\partial_\mu\overline{U}(x)\partial_\mu\overline{U}^\dagger(x)\right)\right\}. \tag{F14}$$

Appendix G. Some relations for the O(4) couplings

We give some relations between different conventions for the O(4) couplings to connect with those used in Ref. [6]:

$$l_i=l_i^{\mathrm{r}}+\frac{w_i}{16\pi^2}\left(\frac{1}{D-4}+\log(\overline{c}M)\right) \tag{G1}$$

where $w_1=n/2-5/3$, $w_2=2/3$ and choosing the scale $\mu=M$, the mass of the charged pion.

Further,

$$l_i=\frac{w_i}{16\pi^2}\left(\frac{1}{D-4}+\log(\overline{c}\Lambda_i)\right). \tag{G2}$$

From here

$$l_i^{\mathrm{r}}=\frac{w_i}{16\pi^2}\log\left(\Lambda_i/M\right)=\frac{w_i}{32\pi^2}\bar{l}_i \tag{G3}$$

since $\bar{l}_i=\log\left(\Lambda_i^2/M^2\right)$.

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