



Non-perturbative spinning black holes in dynamical Chern–Simons gravity



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ABSTRACT

Spinning black holes in dynamical Einstein–Chern–Simons gravity are constructed by directly solving the field equations, without resorting to any perturbative expansion. This model is obtained by adding to the Einstein–Hilbert action a particular higher-curvature correction: the Pontryagin density, linearly coupled to a scalar field. The spinning black holes are stationary, axi-symmetric, asymptotically flat generalisations of the Kerr solution of Einstein’s gravity, but they possess a non-trivial (odd-parity) scalar field. They are regular on and outside the horizon and satisfy a generalized Smarr relation. We discuss the deviations from Kerr at the level of the spin and mass distribution, the horizon angular velocity, the ergo-region and some basic properties of geodesic motion. For sufficiently small values of the Chern–Simons coupling our results match those previously obtained using a perturbative approach.

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1. Introduction

Amongst all possible four dimensional extensions of General Relativity (GR) containing higher curvature corrections [1], the cases of Gauss–Bonnet (GB) and Chern–Simons (CS) models are of special interest. Both models rely on coupling topological terms to matter fields – in the simplest case, a real scalar field –, making the (otherwise) topological terms dynamical and contributing to the field equations. These extra terms, quadratic in the curvature, lead to new effects in the strong-field regime, manifesting themselves most naturally in the BH solutions of these models.

There are, however, important differences between the (scalar-) GB and CS extensions of GR. Firstly, whereas the GB term affects the properties of solutions already in the static sector, the CS term leads to different results only in the presence of a parity-odd source, such as rotation. Thus, any static solution of GR is also a solution of dynamical Einstein–Chern–Simons (ECS) gravity [2]. Secondly, while the fully nonlinear generalization of the astrophysically relevant Kerr metric in Einstein–Gauss–Bonnet (EGB) gravity has been constructed [3,4], so far only partial (perturbative) results

have been known in the ECS case,¹ see e.g. [6–14]. The absence of a fully nonlinear version of the Kerr metric in ECS gravity is presumably due to the complexity of the problem, since the equations of motion contain third order derivatives [15]; by contrast, they remain second order in EGB gravity, since the GB invariant is an Euler density, and Euler densities are the building blocks of the most general metric gravitational theory with second order field equations, i.e. Lovelock gravity [17]. Still, ECS gravity remains physically interesting, both as an example of parity violating gravity and due to its motivation in quantum gravity approaches, such as string theory [16] and loop quantum gravity [18,19] – see [20] for a review.

The perturbative construction of BH solutions in ECS gravity, as in [6–13], has the advantage of yielding closed form expressions, leading to insights on the trends introduced by the CS coupling. It is clear, however, that a number of important features, occurring in the fast spinning and/or large coupling regimes, cannot be captured by this perturbative approach. The main purpose of this work is to present a general framework which allows a non-perturbative approach to constructing the modified Kerr solution in ECS model,

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¹ The only non-perturbative results are those in [5] for the (more academic) Taub-NUT solution.

together with some numerical results illustrating non-perturbative solutions, including fast rotating BHs.

This paper is organised as follows. In Section 2 we present the model, including the equations of motion, boundary conditions and relevant physical quantities. Some details on the numerical approach are also presented. The results are presented in Section 3, where we comment both on the properties of illustrative solutions and on an overview of the solution's properties as one deviates from the GR limit. Final remarks are presented in Section 4. We use units with $c = 1$.

2. The model

2.1. Action, equations of motion and ansatz

A general ECS gravity model is described by the action

$$S = \int d^4x \sqrt{-g} \times \left[\frac{R}{16\pi G} + \frac{\alpha}{4} f(\phi) {}^*R R - \frac{1}{2} g^{ab} (\nabla_a \phi) (\nabla_b \phi) - V(\phi) \right], \quad (1)$$

where ϕ is a real scalar field with a potential $V(\phi)$, $f(\phi)$ is a coupling function and α a dimensionful constant, an input parameter of the theory. As usual, g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci scalar. The quantity ${}^*R R$ is the Pontryagin density, also referred to as the Chern–Simons scalar, defined via²

$${}^*R R = {}^*R^a{}_b{}^{cd} R^b{}_{acd}, \quad \text{with } {}^*R^a{}_b{}^{cd} \equiv \frac{1}{2} \epsilon^{cdef} R^a{}_{bef}, \quad (3)$$

where ϵ^{cdef} is the 4-dimensional Levi-Civita tensor.

The gravitational equations derived from (1) can be written in a GR-like form:

$$E_{ab} \equiv G_{ab} - 8\pi G T_{ab} = 0, \quad (4)$$

where G_{ab} is the Einstein tensor and the *effective* energy momentum tensor

$$T_{ab} \equiv T_{ab}^{(\phi)} - 2\alpha C_{ab}, \quad (5)$$

is a combination of the scalar field energy-momentum tensor, $T_{ab}^{(\phi)}$,

$$T_{ab}^{(\phi)} \equiv (\nabla_a \phi) (\nabla_b \phi) - g_{ab} \left[\frac{1}{2} (\nabla_c \phi) (\nabla^c \phi) + V(\phi) \right], \quad (6)$$

and a contribution from the CS term,

$$C^{ab} = (\nabla_c f(\phi)) \epsilon^{cde(a} \nabla_e R^b) + (\nabla_c \nabla_d f(\phi)) {}^*R^{d(ab)c}. \quad (7)$$

The scalar field equation derived from (1) is a modified Klein-Gordon equation,

$$\nabla^2 \phi = \frac{dV}{d\phi} - \frac{\alpha}{4} \frac{df(\phi)}{d\phi} {}^*R R. \quad (8)$$

To make contact with previous literature, in this work we shall report results for a massless, non-self-interacting scalar field, and a linear coupling function

$$V(\phi) = 0, \quad \text{and} \quad f(\phi) = \phi. \quad (9)$$

Some brief comments on more general cases will be made in Section 4.

To obtain stationary and axi-symmetric BH spacetimes, possessing two commuting Killing vector fields, ξ and η , we use a coordinate system adapted to these symmetries. Then: $\xi = \partial_t$, $\eta = \partial_\varphi$. Such spacetimes are usually described by a Lewis–Papapetrou-type ansatz [31], which contains four unknown functions. We shall use here a metric ansatz originally introduced in [21,22], which factorizes the asymptotics

$$ds^2 = -e^{2F_0(r,\theta)} N dt^2 + e^{2F_1(r,\theta)} \left(\frac{dr^2}{N} + r^2 d\theta^2 \right) + e^{2F_2(r,\theta)} r^2 \sin^2 \theta (d\varphi - W(r,\theta) dt)^2, \quad (10)$$

where $N \equiv 1 - r_H/r$, and r_H is a constant. The scalar field ϕ depends on the r, θ coordinates only:

$$\phi \equiv \phi(r, \theta). \quad (11)$$

2.2. Boundary conditions and physical quantities

We are interested in asymptotically flat solutions. This implies the following boundary conditions³

$$F_i|_{r=\infty} = W|_{r=\infty} = \phi|_{r=\infty} = 0, \quad (12)$$

where $i = 0, 1, 2$. Since the scalar field is massless, one can construct an approximate solution of the field equations compatible with these asymptotics as a power series in $1/r$. The leading order terms of such an expansion are:

$$\begin{aligned} F_0(r, \theta) &= \frac{c_t}{r} + \dots, & F_1(r, \theta) &= -\frac{c_t}{r} + \dots, \\ F_2(r, \theta) &= -\frac{c_t}{r} + \dots, \\ W(r, \theta) &= \frac{c_\varphi}{r^3} + \dots, & \phi(r, \theta) &= \frac{q \cos \theta}{r^2} + \dots, \end{aligned} \quad (13)$$

where c_t , c_φ and q are constants fixed by the numerics; q corresponds to the dipole moment of the scalar field, which has no monopole term.

On the symmetry axis, *i.e.* at $\theta = 0, \pi$, axi-symmetry and regularity require that

$$\partial_\theta F_i|_{\theta=0,\pi} = \partial_\theta W|_{\theta=0,\pi} = \partial_\theta \phi|_{\theta=0,\pi} = 0. \quad (14)$$

Again, an approximate expansion of the solution compatible with these conditions can be constructed; at, say, $\theta = 0$ one finds

$$\mathcal{F}_a(r, \theta) = \mathcal{F}_{a0}(r) + \theta^2 \mathcal{F}_{a2}(r) + \mathcal{O}(\theta^4), \quad (15)$$

where $\mathcal{F}_a = \{F_0, F_1, F_2, W; \phi\}$, and the essential data, which is fixed by the numerics, is encoded in the functions $\mathcal{F}_{a0} = \{F_{i0}, W_0, \phi_0\}$. Absence of conical singularities requires, moreover, $F_1|_{\theta=0,\pi} = F_2|_{\theta=0,\pi}$.

For the considered coupling function (9), the problem possesses a well defined parity: the metric functions are invariant under a reflection along the equatorial plane $\theta = \pi/2$, while the scalar field changes sign, $\phi \rightarrow -\phi$. This symmetry is used to integrate the field equations for $0 \leq \theta \leq \pi/2$ only.

² Note that the Pontryagin term is topological and can be expressed as a total divergence [20]

$$\nabla_a K^a = \frac{1}{2} {}^*R R, \quad \text{with } K^a \equiv \epsilon^{abcd} \Gamma_{bm}^n \left(\partial_c \Gamma_{dn}^m + \frac{2}{3} \Gamma_{cl}^m \Gamma_{dn}^l \right). \quad (2)$$

³ Setting $\phi|_{r=\infty} = 0$ removes the shift symmetry $\phi \rightarrow \phi + \text{const.}$ of the ECS model (1) with (9).

For the metric ansatz (10), the event horizon is located at a surface with constant radial variable, $r = r_H > 0$. By introducing a new radial coordinate $x \equiv \sqrt{r^2 - r_H^2}$, the horizon boundary conditions and numerical treatment of the problem are simplified. These boundary conditions are

$$\partial_x F_i|_{x=0} = \partial_x \phi|_{x=0} = 0, \quad W|_{x=0} = \Omega_H, \quad (16)$$

where Ω_H is the horizon angular velocity, such that the Killing vector $\chi = \xi + \Omega_H \eta$ is orthogonal and null on the horizon. These conditions are consistent with the near horizon solution⁴:

$$\mathcal{F}_a(r, \theta) = \mathcal{F}_{a0}(\theta) + x^2 \mathcal{F}_{a2}(\theta) + \mathcal{O}(x^4), \quad (17)$$

where the essential functions are \mathcal{F}_{i0} (also $F_0|_{r_H} = F_1|_{r_H}$).

The ADM mass M and the total angular momentum J of the solutions are read off from the asymptotics of the metric functions,

$$g_{tt} = -1 + \frac{2GM}{r} + \dots, \quad g_{\varphi t} = -\frac{2GJ}{r} \sin^2 \theta + \dots \quad (18)$$

As usual (see, e.g., [23]), M and J can be split into the horizon contribution, respectively M_H and J_H , – computed as a Komar integral on the horizon – and the “matter” contribution, respectively, M_ψ and J_ψ , in this case composed by the scalar field and CS parts. The latter are computed as volume integrals of the appropriate energy-momentum tensor components:

$$M = M_H + M_\psi, \quad \text{with } M_\psi = -2 \int_{\Sigma} dS_a \left(T_b^a \xi^b - \frac{1}{2} T \xi^a \right), \quad (19)$$

$$J = J_H + J_\psi, \quad \text{with } J_\psi = \int_{\Sigma} dS_a \left(T_b^a \eta^b - \frac{1}{2} T \eta^a \right), \quad (20)$$

where Σ is a spacelike surface, bounded by the sphere at infinity S_∞^2 and the horizon \mathcal{H} . In the above relations, M_ψ and J_ψ encode the contribution of the *effective* “matter” distribution to the total mass and angular momentum. For Kerr BHs, $M = M_H$ and $J = J_H$; this is not so for ECS BHs. Moreover, since $T_t^{(\phi)} - \frac{1}{2} T^{(\phi)} = T_\varphi^{t(\phi)} = 0$, only the CS part of the *effective* energy-momentum tensor (5) contributes to the energy and angular momentum “matter” densities, which are determined by the C_t^t and C_φ^t components, respectively (since $C_a^a = 0$).

The BH horizon introduces a temperature T_H and horizon area A_H ,

$$T_H = \frac{1}{4\pi r_H} e^{(F_0 - F_1)|_{r_H}}, \quad A_H = 2\pi r_H^2 \int_0^\pi d\theta \sin \theta e^{(F_1 + F_2)|_{r_H}}. \quad (21)$$

Then a straightforward computation shows that the following Smarr relation holds in ECS theory:

$$M + \mathcal{U} = 2(\Omega_H J + T_H S), \quad (22)$$

where

$$\mathcal{U} = 2\alpha \int_{\Sigma} dS_a \xi^a (\nabla \phi)^2, \quad (23)$$

⁴ A similar near horizon expansion for the Kerr solution in a generic EGB model implies the existence of a critical set of limiting configurations. That is, the second order term ϕ_2 solves a quadratic equation; the vanishing of this equation's discriminant selects solutions that form a part of the boundary of the domain of existence. No such restriction is found for the ECS model.

and S is the BH entropy in the ECS model, which is the sum of two different contributions [32]

$$S = S_E + S_{CS}, \quad \text{with } S_E \equiv \frac{A_H}{4G}, \quad S_{CS} \equiv \pi \alpha \int_{\mathcal{H}} \phi^* R^{abcd} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \hat{\epsilon}, \quad (24)$$

$\hat{\epsilon}_{ab}$ being the binormal of the horizon, which is normalized such that $\hat{\epsilon}_{ab} \hat{\epsilon}^{ab} = -2$. One can also show that the following relation holds

$$T_H S_{CS} = -\alpha \int \hat{\epsilon} \phi K^r. \quad (25)$$

2.3. Scaling and dimensionless quantities

The dependence on Newton's constant G disappears from the field equations under the rescaling

$$\phi \rightarrow \phi / \sqrt{8\pi G}, \quad \alpha \rightarrow \alpha / \sqrt{8\pi G}. \quad (26)$$

This rescaling makes the scalar field dimensionless, whereas α remains a dimensionful constant. The field equations still possess the scaling symmetry

$$r \rightarrow \lambda r, \quad \alpha \rightarrow \lambda^2 \alpha, \quad (27)$$

where λ is a positive constant, under which global quantities transform as $M \rightarrow \lambda M$, $J \rightarrow \lambda^2 J$. In the following we shall work with dimensionless quantities which are invariant under (27):

$$\xi \equiv \frac{\alpha \sqrt{8\pi G}}{M^2}, \quad j \equiv \frac{J}{M^2}, \quad w_H \equiv \Omega_H M. \quad (28)$$

2.4. The numerical scheme

Within our approach, the ECS equations reduce to a system of five coupled non-linear partial differential equations⁵ for the functions \mathcal{F}_i , together with two constraints. The set of five equations consists of the Klein-Gordon equation (8) together with suitable combinations of the ECS equations (4) $\{E_r^r + E_\theta^\theta = 0; E_\varphi^\varphi = 0; E_t^t = 0; E_\varphi^t = 0\}$.

The resulting equations, however, contain third order derivatives of the metric functions F_0 , F_2 and W . To obtain a standard form of the problem, we introduce a set of ‘auxiliary’ functions S_i , Q_i , with

$$S_1 = F_{0,r}, \quad S_2 = F_{2,r}, \quad S_3 = W_{,r} \quad \text{and} \\ Q_1 = F_{0,\theta}, \quad Q_2 = F_{2,\theta}, \quad Q_3 = W_{,\theta}. \quad (29)$$

These ‘auxiliary’ functions satisfy the following boundary conditions

$$S_i|_{r=\infty} = Q_i|_{r=\infty} = 0, \quad \partial_\theta S_i|_{\theta=0,\pi/2} = \partial_\theta Q_i|_{\theta=0,\pi/2} = 0, \\ \partial_x S_i|_{x=0} = \partial_x Q_i|_{x=0} = 0, \quad (30)$$

which are compatible with the approximate expression of the solutions given above.

⁵ These equations are long, each of them containing several hundred independent terms.

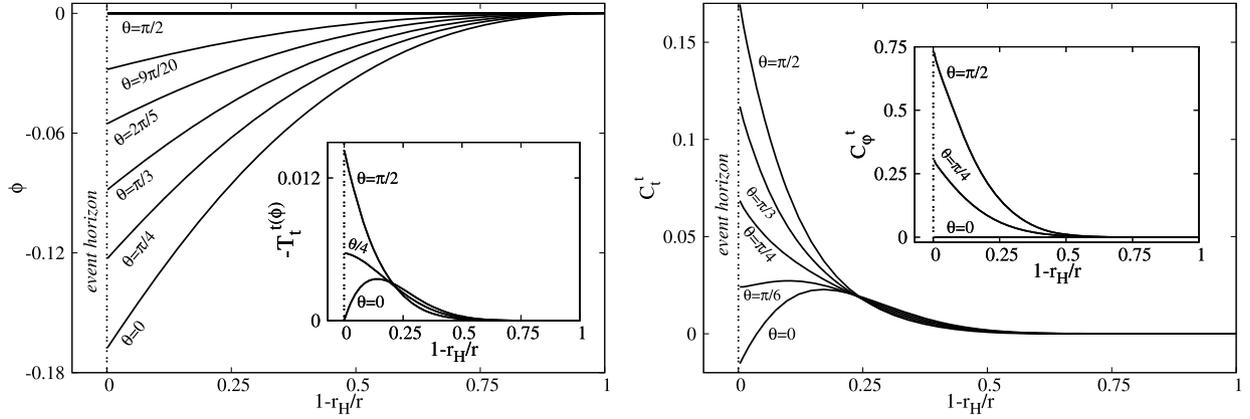


Fig. 1. (Left panel) Radial variation of the scalar field ϕ and of the $T_t^{(\phi)}$ -component of its energy–momentum tensor for several values of θ . (Right panel) Same for the C_t^t and C_t^θ -components of the *effective* energy–momentum tensor associated with the energy and angular momentum densities. The ECS BH in this plot has the input parameters $r_H = 1$, $\Omega_H = 0.2$ and $\alpha = 0.3$, while $\xi = 0.951$ and $j = 0.493$.

The equations $E_\theta^r = 0$ and $E_r^r - E_\theta^\theta = 0$ yield two constraints which are monitored in numerics.⁶ Typically they are satisfied at the level of the overall numerical accuracy.

Our numerical treatment can be summarized as follows: (i) we use the radial variable x introduced above; (ii) this coordinate is compactified, $\bar{x} \equiv x/(1+x)$. This transformation maps the semi infinite x -domain $[0, \infty)$ to the finite \bar{x} -domain $[0, 1]$; (iii) the equations for \mathcal{F} are discretized on some given grid in \bar{x} and θ . Various grid choices have been considered, but most of the results have been obtained for an equidistant grid with 150×30 points. The grid covers the integration region $0 \leq \bar{x} \leq 1$ and $0 \leq \theta \leq \pi/2$; (iv) all numerical calculations are performed with a professional package [24], which uses a Newton–Raphson method. We remark that for all solutions obtained we have monitored the Ricci and Kretschmann scalars, and, at the level of the numerical accuracy, we have not observed any sign of a singular behaviour. As a further test, we have verified that our results for small ξ and j are in good agreement with the perturbative results in [10].

In this scheme, there are three input parameters: **i**) the event horizon velocity Ω_H ; **ii**) the event horizon radius r_H in the metric ansatz (10); **iii**) the coupling constant α . The first two parameters are geometric quantities, while the third one characterizes the theory.

In our approach we start with an Einstein gravity Kerr solution with given r_H, Ω_H as initial guess⁷ for an ECS BH with a small value of α . Then we increase the value of α slowly. The iterations converge, and, in principle, repeating the procedure we obtain in this way solutions for increasingly higher values of α . Around one thousand different solutions were constructed in this way, covering a part of the domain of existence of ECS BHs.

In contrast to the EGB case [4], no clear upper bound seems to exist on the value of α , or equivalently on the dimensionless parameter ξ . For any initial Kerr configuration, however, the numerical accuracy decreases with increasing α , the convergence of the numerical iteration becoming slower and requiring a very large number of iterations (or even being lost). As such, we could not scan the full domain of existence of the solutions. Very likely, this is a numerical problem only; we suspect that a better approach

could show that, for a given j (or w_H) no upper bound exists for the parameter ξ .

3. Numerical results

3.1. General properties

From the results of the numerical integration we have observed that a Kerr BH solution for any j allows a generalization in the ECS model. As expected, the deviation from the GR solution increases with the value of the coupling constant $\xi \sim \alpha/M^2$. Since we were not able to identify the existence of an upper bound for ξ , neither at the numerical nor at the analytical level, for all curves displayed in this work, the end points correspond to configurations where the numerical results stopped being reliable, rather than some fundamental obstruction.⁸

Let us first comment on features of an illustrative solution. Unlike Kerr, ECS spinning BHs have a non-trivial scalar field profile outside the horizon. A typical ϕ profile is shown in Fig. 1 (main left panel). The northern hemisphere scalar field is negative and a monotonically increasing function of the radial coordinate for each $\theta \neq 0$. At the equator it vanishes (as required) and in the southern hemisphere it changes sign (not shown). This scalar field profile is distinct from those observed in other cases of asymptotically flat spinning BHs with scalar hair, namely those in EGB gravity and Kerr BHs with synchronised scalar hair in GR; in both these cases a typical scalar field profile is not monotonic – see Figure 2 in [4] and Figure 16 in [4]. A discussion of the energy/angular momentum distribution of the scalar field for the ECS BHs is delicate, since the most meaningful energy/angular momentum densities, *i.e.* those entering eqs. (19) and (20) vanish. Still, some intuition may, perhaps, be gained from the inset of Fig. 1 (left panel) where we exhibit only the time–time component of the scalar field energy momentum tensor. Such energy–density is asymptotically vanishing; since it also vanishes at the BH poles, it attains a maximum along the symmetry axis at some distance from the horizon. The right panel of Fig. 1 shows the analogous quantity for the CS term contribution to the effective energy momentum tensor (main panel), where a similar behaviour is observed. The corresponding angular momentum density is shown in the inset.

⁶ One can easily verify that the remaining equations vanish identically, $E_r^\phi = E_r^t = E_\theta^\phi = E_\theta^t = 0$, the circularity condition being satisfied. As such, the employed Ansatz is consistent.

⁷ In practice, the scaling symmetry (27) is used to fix the value of the event horizon radius.

⁸ The typical numerical error for the solutions displayed in this work is estimated of the order of 10^{-3} or lower, except close to the end points of the curves, where it increases to around a few percent.

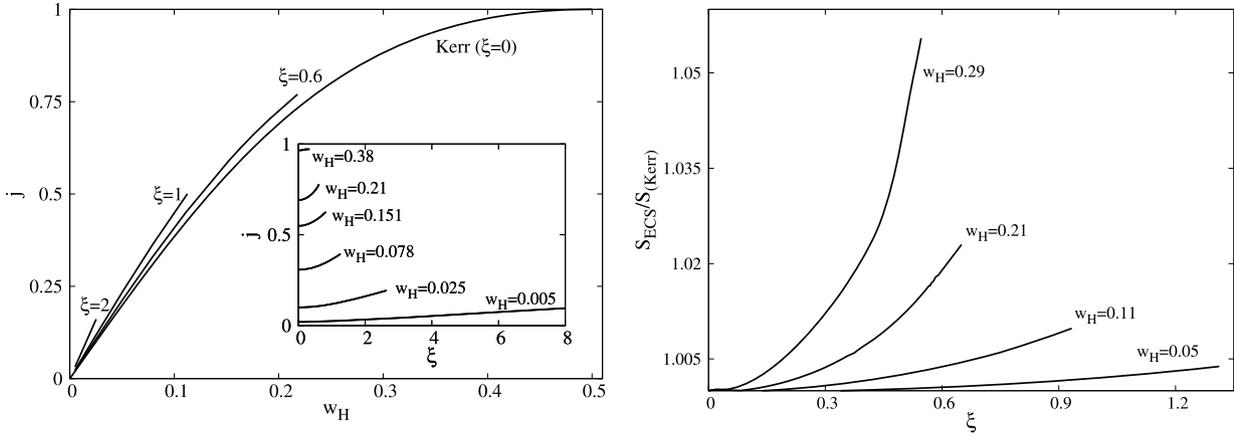


Fig. 2. (Left panel) The $j(w_H)$ -diagram of ECS BHs, for several values of the dimensionless coupling constant ξ . The inset shows the $j(\xi)$ curves for different values of w_H . (Right panel) The ratio between the BH entropy in ECS model and the corresponding Einstein gravity solutions is shown for several values of w_H .

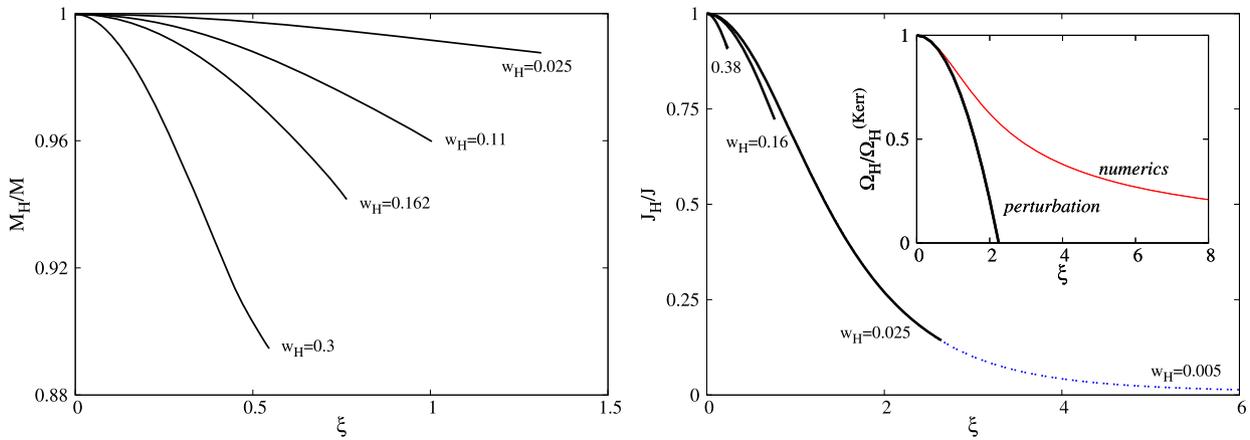


Fig. 3. Ratio M_H/M (left panel) and J_H/J (right panel) as a function of ξ for several values of w_H . (Inset) Ratio $\Omega_H/\Omega_H^{(Kerr)}$ as a function of ξ for BHs with the same M, J .

We now turn our attention to trends in the space of solutions. In Fig. 2 (left panel) we show the dimensionless spin j as a function of the dimensionless horizon angular velocity w_H for several values of ξ . One observes that the GR pattern is shared by the ECS solutions: j and w_H are positively correlated, but increasing ξ the same w_H requires an increasingly larger j . Such increase of j with ξ , for fixed w_H is detailed in the inset (for Kerr BHs $j = 4w_H/(1 + 4w_H^2)$). This confirms that ECS spinning BHs require more (dimensionless) angular momentum to support the same (dimensionless) angular velocity as Kerr BHs. This property is likely related to the fact that not all angular momentum is stored inside the horizon for ECS BHs. We emphasize that we have constructed ECS BHs with $0 < j < 1$, although the near-extremal configurations⁹ were found for small values of ξ only. All ECS spinning BHs constructed so far obey the Kerr bounds for dimensionless spin, $j < 1$, and dimensionless angular velocity, $w_H < 1/2$.

In Fig. 2 (right panel) we show the ratio between the BH entropy in ECS model (24), and that of corresponding Kerr solutions with the same (M, J) , as a function of the dimensionless coupling constant ξ , for several values of dimensionless horizon angular velocity w_H . We note that, as expected [32], the contribution of the CS term to the total BH entropy is always positive. Also, it turns

out that for the same global charges, the entropy of the solutions is maximized by the ECS BHs.

To gain some insight on how the mass and angular momentum of ECS BHs are distributed in and outside the horizon, in Fig. 3 we display the ratios M_H/M and J_H/J . One can see that most of the mass is stored inside the horizon, the ratio M_H/M decreasing with both w_H and ξ . By contrast, for small w_H and sufficiently large ξ , most of the angular momentum can be stored *outside* the horizon. From the data sample analysed, however, we found no evidence of a counterrotating horizon, $J_H/J < 0$, as suggested by perturbative studies.

A somewhat unexpected result concerns the ratio $\Omega_H/\Omega_H^{(Kerr)}$ for ECS and GR BHs with the same mass and angular momentum – Fig. 3 (right panel inset). *A priori*, the value of this ratio will depend both on ξ and j ; however, our results show that the j -dependence is weak for all solutions constructed so far; consequently, the same curve (red thin line in the inset of Fig. 3) provides a good fit for all data, regardless of j . The same panel also shows the perturbation theory result [10] $\Omega_H/\Omega_H^{(Kerr)} = 1 - \frac{709}{3584}\xi^2$; one can observe it provides a good approximation up to ξ of order one.

We have also verified that the ECS BHs are (generically) algebraically general (Petrov type I). Also, since the metric functions are always smooth and finite outside the horizon, the Lorentzian signature of the metric is preserved there. Moreover, in all dataset analysed we observed the absence of closed causal curves.

⁹ Only non-extremal solutions are reported here. Extremal ECS BHs should, however, also exist [32]. In contrast to the EGBs case [4], the contribution of the CS term is compatible with a non-singular extremal horizon.

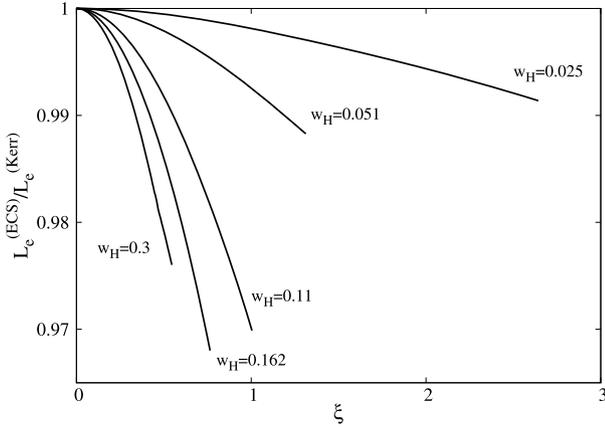


Fig. 4. Ratio between the ECS ergocircle and the corresponding value for a Kerr BH with the same (M, J) as a function of ξ for several values of w_H .

3.2. Other features

All ECS BHs have an ergoregion, defined as the domain in which $\xi = \partial_t$ is positive (exterior to the horizon). This region is bounded by the event horizon and by the surface where

$$g_{tt} = -e^{2F_0}N + W^2e^{2F_2}r^2 \sin^2\theta = 0. \quad (31)$$

For the Kerr spacetime, this surface has a spherical topology and touches the horizon at the poles. As discussed in [25], the ergoregion can be more complicated for BHs with scalar hair, with the possible existence of an additional $S^1 \times S^1$ ergo-surface (ergo-torus). We have found that this is not the case for ECS BHs, where all solutions are Kerr-like and possess a single topologically S^2 ergo-surface.

The effect of the CS term on the “size” of the ergoregion is illustrated in Fig. 4, by using the measure L_e , the proper length of the ergocircle along the equatorial plane:

$$L_e^{(ECS)} = r_e^2 e^{F_2(r_e, \pi/2)}, \quad \text{with } g_{tt}(r_e, \pi/2) = 0. \quad (32)$$

We have denoted as $L_e^{(Kerr)}$ the corresponding value for a Kerr BH with the same mass and angular momentum. One can see that the generic effect of the CS term is to reduce the size of the ergoregion as compared to the GR case. Although increasing ξ one observes a higher value of j for the same ω_H , cf. Fig. 2, the angular momentum is stored both in and outside the horizon, thus making the ergoregion grow less than for a comparable Kerr BH. A similar situation has been observed for Kerr BHs with synchronised hair, which has the physical impact of making superradiant instabilities weaker for the hairy BHs than for comparable Kerr BHs [25–27].

Let us also briefly look at some basic features of geodesic motion. It is known that the Kerr spacetime supports unstable photon orbits with a fixed radial coordinate [28,29]. A subset of the latter are restricted to the equatorial plane $\theta = \pi/2$, and comprises two independent circular photon orbits with opposite rotation directions, usually dubbed as *light rings*.

For a stationary spacetime, the light ring positions can be obtained by analysing the following condition in the equatorial plane [30]

$$\partial_r h_{\pm} = 0, \quad \text{with } h_{\pm} = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{tt}}. \quad (33)$$

Each sign \pm leads to one of the two light rings. As shown in Fig. 5 (left panel), the light ring qualitative structure is still the same as

in Kerr and in the dataset analysed the differences with respect to comparable Kerr BHs are small, at percent level. We thus predict that the BH shadows are going to be very similar to those of Kerr BHs in this region of the solution space. We have denoted as $R_{\pm}^{(ECS)}$ the proper length of the light ring orbit,

$$R_{\pm}^{(ECS)} \equiv r_{\pm} e^{F_2(r_{\pm}, \pi/2)}, \quad (34)$$

where r_{\pm} are the solutions of Eq. (33).

We have also studied the angular frequency at the ISCO for a large set of ECS BHs. The geodesic motion is studied along the equatorial plane, $\theta = \pi/2$; then the Lagrangian ruling the motion of a timelike test particle (the only case studied here) is

$$2\mathcal{L} = e^{2F_1} \dot{r}^2 + e^{2F_2} r^2 (\dot{\varphi} - W\dot{t})^2 - e^{2F_0} N \dot{t}^2 = -1. \quad (35)$$

Note that F_i, W depend only on r for equatorial motions; also a dot denotes a derivative w.r.t. proper time. Stationarity and axisymmetry imply the existence of the first integrals

$$\begin{aligned} e^{2F_2} r^2 (\dot{\varphi} - W\dot{t}) &\equiv L, \\ (e^{2F_0} N - e^{2F_2} r^2 W^2) \dot{t} + e^{2F_2} r^2 W \dot{\varphi} &\equiv E, \end{aligned} \quad (36)$$

where E and L are the specific energy and angular momentum of the test particle. Then the orbital angular velocity is expressed as

$$\Omega_c = \frac{\dot{\varphi}}{\dot{t}} = W - \frac{e^{2F_0-2F_2} L N}{r^2 (LW - E)}. \quad (37)$$

The equation governing the variation of the radial coordinate r for an orbit on the equatorial plane is

$$\dot{r}^2 = V(r) = e^{-2F_1} N \left(-1 - e^{-2F_2} \frac{L^2}{r^2} + \frac{e^{-2F_0} (E - LW)^2}{N} \right). \quad (38)$$

The requirement for a circular orbit at $r = r_c$ is $V(r_c) = V'(r_c) = 0$ which results in two algebraic equations for E, L which are solved analytically, possessing two distinct pairs of solutions (E_+, L_+) and (E_-, L_-) , corresponding to co-rotating and counter-rotating trajectories.

The solutions for E, L are then replaced in the expression of $V''(r_c)$, requiring $V''(r_c) \leq 0$ for stability. For the configurations studied so far we have noticed a (qualitative) analogy with the Kerr BH. First, circular geodesic motion is only possible for radii larger than a minimum value, $r_c > r_{min}$, a constraint imposed by requiring the energy E to be real. Then for $r_{min} < r_c < r_{ISCO}$ only unstable circular orbits can exist, i.e. with $V''(r_c) > 0$. For $r_c > r_{ISCO}$, circular orbits are stable.

In Fig. 5 (right panel) we exhibit the angular frequency at the ISCO for co-rotating and counter-rotating geodesics, where the value of Ω_c is normalized with respect to that of a Kerr BH with equal mass and angular momentum. Only small deviations from GR were found so far, no larger than a few percent, similarly to the case of the light rings described above. More significant differences are likely to occur for larger values of ξ .

4. Further remarks

The main purpose of this work was to provide a concrete approach to constructing the nonperturbative spinning BHs in dynamical ECS gravity, together with a preliminary discussion of their (basic) physical properties. These configurations can be viewed as the counterparts of Kerr solutions in the presence of a CS term in the gravitational action. Our results here show that such BHs qualitatively share some basic properties of the GR BHs.

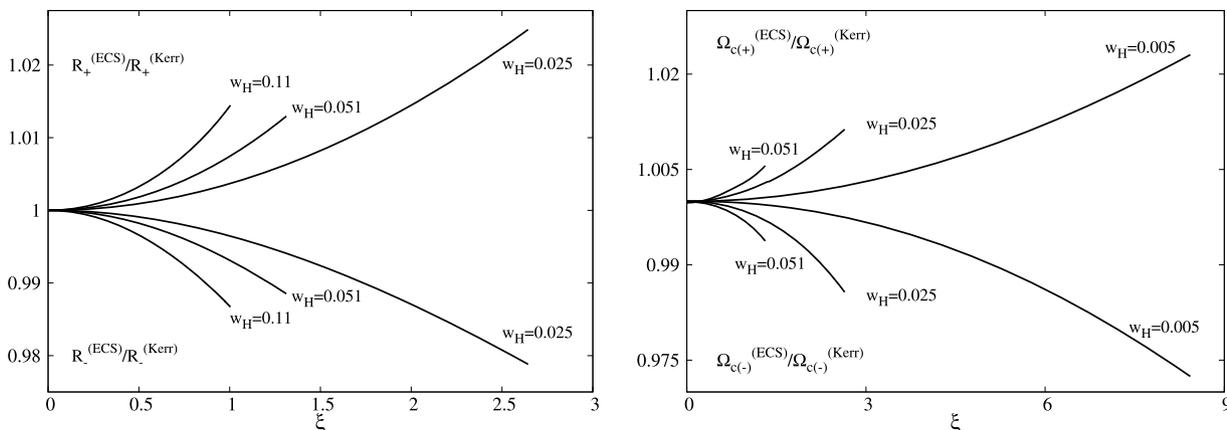


Fig. 5. (Left panel) Ratio between the ECS light ring proper radius and the corresponding value for a Kerr BH with the same (M, J) , as a function of the coupling constant ξ for several values of w_H . (Right panel) The same for the ratio between the angular frequency at the ISCO.

The research initiated here can be furthered in many possible directions. An important issue is to thoroughly scan the parameter space of solutions and clarify its boundaries and possible limiting configurations. The stability of the ECS BHs is another important point, particularly in view of the expected Ostrogradsky instabilities associated to the third order equations of motion. If and how these instabilities reflect on the solutions described herein is an important question, although, due to the complexity of the field equations, any result in this direction will be highly challenging task. In this context, let us remark that, as discussed above, these spinning BHs have an ergoregion. Thus, similar to the Kerr case, they should be afflicted by superradiant instabilities in the presence of (massive) bosonic fields [33]. Yet another interesting direction would be to further explore the astrophysical signatures of these BHs. An obvious task here will be to study the geodesics in a more systematic way and to compute, e.g., the shadows, or the X-ray spectrum in the presence of an accretion disk, contrasting the results with those for the Kerr solution. Phenomenological studies of these features for EGB BHs have revealed only small quantitative differences occur, with respect to Kerr BHs [34,35].

It would also be interesting to extend the solutions in this work to models beyond the simple choices in (9) for $V(\phi)$ and $f(\phi)$. Working with the same linear scalar coupling to the CS term, we have constructed families of solutions for a massive scalar field, $V(\phi) = \frac{1}{2}\mu^2\phi^2$. This leads to a more complicated landscape, since one more length scale is present. Our preliminary numerical results, however, suggest that a qualitative similar picture to that found in the massless case.

The situation can be different for other choices of the coupling, e.g. $f(\phi) = \phi^2$, leading to striking new features. For example, with this coupling, the phenomenon of spontaneous scalarisation recently discussed in the context of GB scalar–tensor models [36–38] (see also [39]) should also occur: the Kerr BH is also a solution of the fully non-linear model (with $\phi = 0$), but it may get, in some regions of the parameter space, spontaneously scalarised into a non-GR ECS spinning BH.

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