Prog. Theor. Exp. Phys. **2018**, 103B04 (22 pages)
DOI: 10.1093/ptep/pty105

Magnetic monopoles in pure $SU(2)$ Yang–Mills theory with a gauge-invariant mass

Shogo Nishino[1,*], Ryutaro Matsudo[2], Matthias Warschinke[1], and Kei-Ichi Kondo[1,2]

[1] *Department of Physics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan*
[2] *Department of Physics, Graduate School of Science and Engineering, Chiba University, Chiba 263-8522, Japan*
[*] E-mail: shogo.nishino@chiba-u.jp

Received June 1, 2018; Revised August 16, 2018; Accepted August 31, 2018; Published October 31, 2018

In this paper, we show the existence of magnetic monopoles in the pure $SU(2)$ Yang–Mills theory when a gauge-invariant mass term is introduced. This result follows from the recent proposal for obtaining gauge field configurations in the Yang–Mills theory from the solutions of the field equations in the "complementary" gauge–scalar model. The gauge-invariant mass term is obtained through a change of variables and a gauge-independent description of the Brout–Englert–Higgs mechanism, which relies neither on the spontaneous breaking of gauge symmetry nor on the assumptions of the nonvanishing vacuum expectation value of the scalar field. We solve under the static and spherically symmetric ansatz the field equations of the $SU(2)$ Yang–Mills theory coupled to a single adjoint scalar field whose radial degree of freedom is eliminated. We show that the solution can be identified with the gauge field configuration of a magnetic monopole with a minimum magnetic charge in the massive Yang–Mills theory. Moreover, we compare the magnetic monopole of the massive Yang–Mills theory obtained in this way with the Wu–Yang magnetic monopole in the pure Yang–Mills theory and the 't Hooft–Polyakov magnetic monopole in the Georgi–Glashow gauge–scalar model.

Subject Index B03

1. Introduction

In high-energy physics, quark confinement is a long-standing problem to be solved in the framework of quantum chromodynamics (QCD). The dual superconductivity picture [1–3] for the QCD vacuum is known as one of the most promising scenarios for quark confinement. For this hypothesis to be realized, however, condensations of some magnetic objects are indispensable. However, the relevant magnetic objects are not included in the action of QCD. Therefore, we begin our arguments by showing the existence of magnetic monopoles in the Yang–Mills theory even in the absence of the usual scalar field. Such magnetic monopoles in the pure Yang–Mills theory, which we call *Yang–Mills magnetic monopoles*, should be compared with the 't Hooft–Polyakov magnetic monopole [4–7] in the Georgi–Glashow model, which includes the scalar field in the action from the beginning; see, e.g., Refs. [8–12] for reviews of magnetic monopoles. The Yang–Mills magnetic monopoles are expected to be obtained as topological defects or topological solitons afterwards, since they are not included in the original QCD action.

Funded by SCOAP[3]

Nevertheless, we know [10–12] that the pure Yang–Mills theory with no scalar fields has the topological soliton only in 4D Euclidean space. Indeed, such topological solitons are known as instantons and anti-instantons, in agreement with the nontrivial homotopy group $\pi_3(S^3) = \mathbb{Z}$. This statement is no longer true once we introduce the mass term to the Yang–Mills theory, which we call the *massive Yang–Mills theory* hereafter. In the massive Yang–Mills theory, it is shown that there exists the other topological soliton suggested from the nontrivial homotopy group $\pi_2(S^2) = \mathbb{Z}$. This is nothing but a magnetic monopole. It is reasonable to consider the massive Yang–Mills theory, in light of the conjecture that the quantum Yang–Mills theory has a mass gap, even if the classical Yang–Mills theory is a conformal theory with no mass scale [13]. Indeed, recent investigations arrived at a consensus that gluons behave as massive particles in the low-energy (momentum) region, which is called the *decoupling solution* [14–21]. In view of these, it is worth investigating the existence of magnetic monopole configurations in the massive Yang–Mills theory. However, a naive mass term for the gluon field breaks the gauge symmetry.

Recently, it has been shown that the gauge-invariant mass term of the Yang–Mills field can be introduced by combining the change of variables and a *gauge-independent description of the Brout–Englert–Higgs (BEH) mechanism* [22], which neither relies on the spontaneous breaking of gauge symmetry nor on the assumptions of the nonvanishing vacuum expectation value of the scalar field. The gauge-independent BEH mechanism [22] with a single adjoint scalar field $\boldsymbol{\phi}(x)$ leads to the separation of the gauge field $\mathscr{A}(x)$ into the *massive mode* $\mathscr{W}(x)$ and the *residual mode* $\mathscr{R}(x)$ in the gauge–scalar model,

$$\mathscr{A}(x) = \mathscr{W}(x) + \mathscr{R}(x), \tag{1}$$

where $\mathscr{R}(x)$ transforms in the same way as the original gauge field $\mathscr{A}(x)$ and $\mathscr{W}(x)$ transforms in the adjoint way under the gauge transformation $\mathscr{W}(x) \to U(x)\mathscr{W}(x)U^{-1}(x)$. Here $\mathscr{W}(x)$ and $\mathscr{W}(x)$ are written in terms of $\mathscr{A}(x)$ and $\boldsymbol{\phi}(x)$ [22]. For preceding works related to the gauge-invariant mass term for the non-Abelian gauge field, see Ref. [23] and references therein.

This fact provides a natural understanding of the gauge field decomposition in the Yang–Mills theory called the *Cho–Duan–Ge–Faddeev–Niemi–Shabanov (CDGFNS) decomposition* (Refs. [24–36] and Y. M. Cho, unpublished preprint) and the subsequent reformulations of the Yang–Mills theory using the new field variables [37–39]; see, e.g., Ref. [40] for a review. In the CDGFNS decomposition, indeed, the gauge field $\mathscr{A}(x)$ is decomposed into two pieces:

$$\mathscr{A}(x) = \mathscr{X}(x) + \mathscr{V}(x), \tag{2}$$

where $\mathscr{V}(x)$ is called the *restricted* (or *residual*) *field*, which transforms in the same way as the original gauge field $\mathscr{A}(x)$, and $\mathscr{X}(x)$ is called the *remaining* (or *coset*) *field*, which transforms in the adjoint way under the gauge transformation $\mathscr{X}(x) \to U(x)\mathscr{X}(x)U^{-1}(x)$. The key ingredient in the CDGFNS decomposition is the so-called *color direction field* $\boldsymbol{n}(x)$, which transforms in the adjoint way under the local gauge transformation $\boldsymbol{n}(x) \to U(x)\boldsymbol{n}(x)U^{-1}(x)$. However, introducing the color field is a difficult part in understanding the CDGFNS decomposition.

According to the gauge-independent BEH mechanism, the color field $\boldsymbol{n}(x)$ in the reformulated Yang–Mills theory follows from the normalized adjoint scalar field $\hat{\boldsymbol{\phi}}(x)$ in the "complementary" gauge–scalar model[1]:

$$\hat{\boldsymbol{\phi}}(x) \to \boldsymbol{n}(x). \tag{3}$$

This way of introducing the color field will facilitate understanding of the role of the color field itself. The remaining field $\mathscr{X}(x)$ is identified with the massive mode $\mathscr{W}(x)$, and the restricted field $\mathscr{V}(x)$ with the residual mode $\mathscr{R}(x)$. Consequently, a gauge-invariant mass term $M_{\mathscr{X}}^2 \mathrm{tr}(\mathscr{X}_\mu \mathscr{X}^\mu)$ in the reformulated Yang–Mills theory follows according to the gauge-independent BEH mechanism from the kinetic term of the gauge–scalar model:

$$(\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}) \cdot (\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}) = M_{\mathscr{W}}^2 \mathrm{tr}(\mathscr{W}_\mu \mathscr{W}^\mu) = M_{\mathscr{X}}^2 \mathrm{tr}(\mathscr{X}_\mu \mathscr{X}^\mu). \tag{4}$$

Consequently, we can introduce a gauge-invariant mass term $M_{\mathscr{X}}^2 \mathrm{tr}(\mathscr{X}_\mu \mathscr{X}^\mu)$ in the pure Yang–Mills theory.

To obtain the pure Yang–Mills theory from the complementary gauge–scalar model, we must solve an issue. The naively extended Yang–Mills theory written in terms of the field variables $(\mathscr{A}, \boldsymbol{n})$ has extra degrees of freedom originating from the color field $\boldsymbol{n}(x)$ if we wish to obtain the gauge theory that is equipollent to the original Yang–Mills theory. For this purpose, we impose an additional condition to relate the gauge field $\mathscr{A}(x)$ and the color field $\boldsymbol{n}(x)$ in such a way that the color field is given as a functional in terms of the gauge field: $\boldsymbol{n} = \boldsymbol{n}[\mathscr{A}]$. This condition is called the *reduction condition*. By using the resulting color field, we can define the magnetic charge in a gauge-invariant way.

Such color field configurations satisfying the reduction condition are obtained from the field equations of the complementary gauge–scalar model, since it is shown [22] that the simultaneous solutions of the coupled field equations in the "complementary" gauge–scalar model automatically satisfy the reduction condition. Thus, we can construct gauge-invariant magnetic monopoles in the massive Yang–Mills theory using the color field obtained in this way.

We show in this paper that magnetic monopoles do exist in the pure $SU(2)$ Yang–Mills theory with a gauge-invariant mass term. In fact, we solve under the static and spherically symmetric ansatz the field equations of the $SU(2)$ gauge–scalar model with an adjoint scalar field whose radial degree of freedom is eliminated to be identified with the color direction field in the pure Yang–Mills theory. Then we obtain a gauge field configuration for a magnetic monopole with a minimum magnetic charge in the massive $SU(2)$ Yang–Mills theory. In particular, we compare the magnetic monopole obtained in this way in the massive Yang–Mills theory with the Wu–Yang magnetic monopole [44] in the pure Yang–Mills theory and the 't Hooft–Polyakov magnetic monopole in the Georgi–Glashow gauge–scalar model.

It should be remarked that, within the framework of the reformulated Yang–Mills theory, the configurations of the color field $\boldsymbol{n}[\mathscr{A}]$ have been obtained by solving the reduction condition for a given configuration of the gauge field $\mathscr{A}(x)$, e.g., instantons and merons in Refs. [45–47]. We now revisit this problem from the opposite direction such that the gauge field configurations are obtained for a given configuration of the color field or the normalized scalar field.

[1] The "complementarity" originates from the confinement–Higgs complementarity in the gauge–scalar model that says that there is no phase transition between the two phases, confinement and Higgs, which are analytically connected in the phase diagram [41–43]. See Ref. [23] for the precise definition and more details on "complementarity".

This paper is organized as follows. In Sect. 2, we review how to obtain the gauge-invariant massive Yang–Mills theory (Yang–Mills theory with a gauge-invariant mass term) by starting from the "complementary" $SU(2)$ gauge–adjoint scalar model with a fixed radial degree of freedom. In Sect. 3, we summarize the essentials for the 't Hooft–Polyakov magnetic monopole. In Sect. 4, we show by using the scaling argument due to Derrick [48] that there can exist magnetic monopoles in the massive Yang–Mills theory. In Sect. 5, we obtain the magnetic monopole solution with a minimum magnetic charge under the static and spherically symmetric ansatz. In Sect. 6, we discuss the short-distance and long-distance behavior of the gauge field and the chromo-magnetic field, in comparison with the 't Hooft–Polyakov magnetic monopole. For the gauge field, we also perform a decomposition based on the reformulation to investigate how the respective decomposed field behaves in the short-distance and long-distance regions. In the final section, we discuss how the magnetic monopoles obtained in the massive Yang–Mills theory are responsible for quark confinement from the viewpoint of the dual superconductivity and are consistent with the existence of a mass gap. In Appendix A, we explain the method used for numerically solving the monopole equation.

2. The massive Yang–Mills theory "complementary" to the gauge–adjoint scalar model

In this section, we review the procedure [22] for obtaining the massive $SU(2)$ Yang–Mills theory from the "complementary" $SU(2)$ gauge–adjoint scalar model described by the Lagrangian density

$$\mathscr{L} = -\frac{1}{4}\mathscr{F}^A_{\mu\nu}\mathscr{F}^{\mu\nu A} + \frac{1}{2}\left(\mathscr{D}^{AB}_\mu[\mathscr{A}]\phi^B\right)\left(\mathscr{D}^{\mu AC}[\mathscr{A}]\phi^C\right) + u\left(\phi^A\phi^A - v^2\right), \tag{5}$$

with

$$\mathscr{F}^A_{\mu\nu}(x) = \partial_\mu\mathscr{A}^A_\nu(x) - \partial_\nu\mathscr{A}^A_\mu(x) - g\epsilon^{ABC}\mathscr{A}^B_\mu(x)\mathscr{A}^C_\nu(x), \tag{6}$$

$$\mathscr{D}^{AB}_\mu[\mathscr{A}]\phi^B(x) = \partial_\mu\phi^A(x) - g\epsilon^{ABC}\mathscr{A}^B_\mu(x)\phi^C(x), \tag{7}$$

where $u = u(x)$ is the Lagrange multiplier field to incorporate the radially fixed constraint

$$\phi^A(x)\phi^A(x) = v^2 \quad (A = 1,2,3,\ v > 0). \tag{8}$$

In what follows, we introduce, respectively, the inner and exterior products for the Lie-algebra-valued fields by

$$\mathcal{P}\cdot\mathcal{Q} := \mathcal{P}^A\mathcal{Q}^A, \quad \mathcal{P}\times\mathcal{Q} := \epsilon^{ABC}T_A\mathcal{P}^B\mathcal{Q}^C, \tag{9}$$

with the generator of the Lie algebra T_A.

To begin with, we construct a composite vector boson field $\mathscr{X}_\mu(x)$ from $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$ as

$$g\mathscr{X}_\mu(x) := \hat{\boldsymbol{\phi}}(x) \times \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x), \tag{10}$$

which can be considered as identifying the normalized scalar field

$$\hat{\boldsymbol{\phi}}(x) := \frac{1}{v}\boldsymbol{\phi}(x), \tag{11}$$

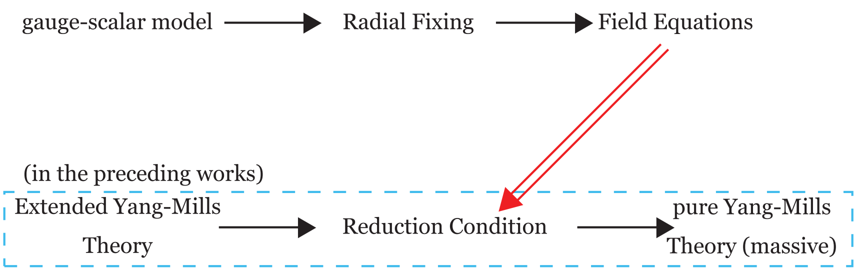

Fig. 1. The outline to obtain the massive Yang–Mills theory from the "complementary" gauge–scalar model. The double-lined arrow stands for our approach in this paper. The dotted box shows the approach in Refs. [39,40].

and the color field $\boldsymbol{n}(x)$ in Refs. [39,40] (the dotted blue box in Fig. 1). Notice that $\mathscr{X}_\mu(x)$ transforms in the adjoint way under the gauge transformation $U(x) \in G = SU(2)$:

$$\begin{aligned} g\mathscr{X}_\mu(x) \to g\mathscr{X}'_\mu(x) =& \hat{\boldsymbol{\phi}}'(x) \times \mathscr{D}_\mu[\mathscr{A}']\hat{\boldsymbol{\phi}}'(x) = U(x)\hat{\boldsymbol{\phi}}(x)U^\dagger(x) \times U(x)\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)U^\dagger(x) \\ =& U(x)\hat{\boldsymbol{\phi}}(x) \times \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)U^\dagger(x) = U(x)\mathscr{X}_\mu(x)U^\dagger(x). \end{aligned} \tag{12}$$

Moreover, the kinetic term of the scalar field is identical to the mass term of the vector field $\mathscr{X}_\mu(x)$:

$$\frac{1}{2}\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} = \frac{1}{2}M_{\mathscr{X}}^2\mathscr{X}_\mu \cdot \mathscr{X}^\mu, \quad M_{\mathscr{X}} := gv, \tag{13}$$

as long as the radial degree of freedom of the scalar field is fixed [22]. It is clear that by observing Eq. (12) the obtained mass term of $\mathscr{X}_\mu(x)$ is gauge invariant. Therefore, $\mathscr{X}_\mu(x)$ can become massive without breaking the original gauge symmetry. This gives a gauge-independent definition of the massive modes of the gauge field in the operator level. It should be emphasized that we do not choose a specific vacuum of $\boldsymbol{\phi}(x)$ and hence no spontaneous symmetry breaking occurs.

By using the definition of the massive vector field $\mathscr{X}_\mu(x)$, the original gauge field $\mathscr{A}_\mu(x)$ is separated into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x), \tag{14}$$

where the field $\mathscr{V}_\mu(x)$ can be written in terms of $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$:

$$g\mathscr{V}_\mu(x) = g\mathscr{A}_\mu(x) - g\mathscr{X}_\mu(x) = gc_\mu(x)\hat{\boldsymbol{\phi}}(x) - \hat{\boldsymbol{\phi}}(x) \times \partial_\mu\hat{\boldsymbol{\phi}}(x), \quad c_\mu(x) = \mathscr{A}_\mu(x) \cdot \hat{\boldsymbol{\phi}}(x). \tag{15}$$

Then, we regard a set of field variables $\{c_\mu(x), \mathscr{X}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ as being obtained from $\{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ based on a change of variables:

$$\{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\} \to \{c_\mu(x), \mathscr{X}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}, \tag{16}$$

and identify $c_\mu(x)$, $\mathscr{X}_\mu(x)$, and $\hat{\boldsymbol{\phi}}(x)$ with the fundamental field variables for describing the massive Yang–Mills theory anew, which means that we should perform the quantization with respect to the variables $\{c_\mu(x), \mathscr{X}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ appearing in the path-integral measure.

In the gauge–scalar model, $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$ are independent; however, the Yang–Mills theory should be described by $\mathscr{A}_\mu(x)$ alone. Hence the scalar field $\boldsymbol{\phi}(x)$ must be supplied by the gauge field $\mathscr{A}_\mu(x)$ due to the strong interactions, or, in other words, $\boldsymbol{\phi}(x)$ should be given as a functional of the gauge field $\mathscr{A}_\mu(x)$.

Moreover, notice that the degrees of freedom of the original gauge field $\mathscr{A}_\mu^A(x)$ in pure $SU(2)$ Yang–Mills theory in D-dimensional space-time are $[\mathscr{A}_\mu^A(x)] = 3 \times D = 3D$. Here, we have omitted the infinite degrees of freedom of the space-time points. On the other hand, the new field variables have $[c_\mu(x)] = D$, $[\hat{\boldsymbol{\phi}}(x)] = 2$, $[\mathscr{X}_\mu^A(x)] = 2 \times D = 2D$, respectively[2]. We can therefore observe that the theory with the new field variables has two extra degrees of freedom if we wish to obtain the (pure) Yang–Mills theory from the "complementary" gauge–scalar model. These are eliminated by imposing the two constraints that we call the *reduction condition*. We choose, e.g.,

$$\boldsymbol{\chi}(x) := \hat{\boldsymbol{\phi}}(x) \times \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x) = 0. \tag{17}$$

The reduction condition indeed eliminates the two extra degrees of freedom introduced by the radially fixed scalar field into the Yang–Mills theory, since

$$\boldsymbol{\chi}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0. \tag{18}$$

Following the Faddeev–Popov procedure, we insert unity into the functional integral to incorporate the reduction condition:

$$1 = \int \mathcal{D}\boldsymbol{\chi}^\theta \, \delta\left(\boldsymbol{\chi}^\theta\right) = \int \mathcal{D}\boldsymbol{\theta} \, \delta\left(\boldsymbol{\chi}^\theta\right) \Delta^{\text{red}}, \tag{19}$$

where $\boldsymbol{\chi}^\theta := \boldsymbol{\chi}[\mathscr{A}, \boldsymbol{\phi}^\theta]$ is the reduction condition written in terms of $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}^\theta$, which is the local rotation of $\boldsymbol{\phi}(x)$ by $\boldsymbol{\theta} = \boldsymbol{\theta}(x) = \theta^A(x)T_A$, and $\Delta^{\text{red}} := \det\left(\frac{\delta\boldsymbol{\chi}^\theta}{\delta\boldsymbol{\theta}}\right)$ denotes the Faddeev–Popov determinant associated with the reduction condition $\boldsymbol{\chi} = 0$. Then, we obtain

$$\begin{aligned} Z &= \int \mathcal{D}\hat{\boldsymbol{\phi}}\mathcal{D}\mathscr{A} \, \delta(\boldsymbol{\chi}) \, \Delta^{\text{red}} \exp\left\{iS_{\text{YM}}[\mathscr{A}] + iS_{\text{kin}}[\mathscr{A}, \boldsymbol{\phi}]\right\} \\ &= \int \mathcal{D}\hat{\boldsymbol{\phi}}\mathcal{D}c\mathcal{D}\mathscr{X} \, J\delta(\widetilde{\boldsymbol{\chi}}) \, \widetilde{\Delta}^{\text{red}} \exp\left\{iS_{\text{YM}}[\mathscr{V} + \mathscr{X}] + iS_m[\mathscr{X}]\right\}. \end{aligned} \tag{20}$$

The Jacobian J associated with the change of variables is equal to one: $J = 1$ [40]. Therefore, we obtain the massive Yang–Mills theory that keeps the original gauge symmetry:

$$\mathscr{L}_{\text{mYM}} = -\frac{1}{4}\mathscr{F}_{\mu\nu}[\mathscr{V} + \mathscr{X}] \cdot \mathscr{F}^{\mu\nu}[\mathscr{V} + \mathscr{X}] + \frac{1}{2}M_{\mathscr{X}}^2 \mathscr{X}_\mu \cdot \mathscr{X}^\mu, \quad M_{\mathscr{X}} := gv > 0. \tag{21}$$

The obtained massive Yang–Mills theory indeed has the same degrees of freedom as the usual Yang–Mills theory because the massive vector boson $\mathscr{X}_\mu(x)$ is constructed by combining the original gauge field $\mathscr{A}_\mu(x)$ and the normalized scalar field $\hat{\boldsymbol{\phi}}(x)$ where $\hat{\boldsymbol{\phi}}(x)$ is now a (complicated) functional of $\mathscr{A}_\mu(x)$ through the reduction condition (17).

It should be remarked that the solutions of the field equations of the gauge–scalar model satisfy the reduction condition automatically. (But the converse is not true.) The field equations besides Eq. (8) are obtained as

$$\mathscr{D}^\mu[\mathscr{A}]\mathscr{F}_{\mu\nu} + g\boldsymbol{\phi} \times \mathscr{D}_\nu[\mathscr{A}]\boldsymbol{\phi} = 0, \tag{22}$$

$$\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} - 2u\boldsymbol{\phi} = 0. \tag{23}$$

[2] The massive vector field $\mathscr{X}_\mu(x)$ obeys the condition

$$\mathscr{X}_\mu(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0.$$

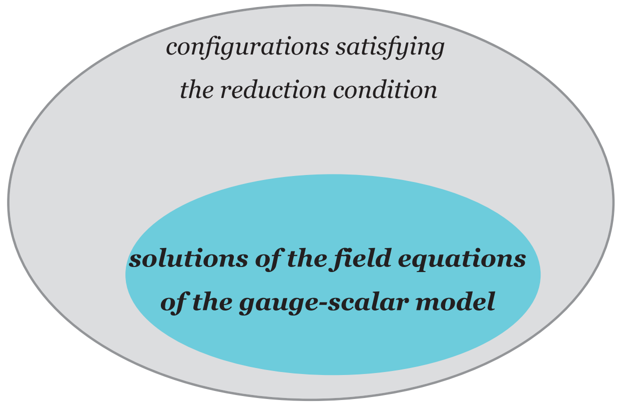

Fig. 2. The relation between the solutions of the field equations of the gauge–scalar model and the reduction condition.

To eliminate the Lagrange multiplier field u in Eq. (23) we take the inner product of Eq. (23) and $\boldsymbol{\phi}(x)$ and use Eq. (8) to obtain

$$u = \frac{1}{2v^2}\boldsymbol{\phi} \cdot \left(\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}\right) = \frac{1}{2}\hat{\boldsymbol{\phi}} \cdot \left(\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}\right). \tag{24}$$

The field equations (22) and (23) are rewritten in terms of $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$:

$$\mathscr{D}^\mu[\mathscr{A}]\mathscr{F}_{\mu\nu} + gv^2\hat{\boldsymbol{\phi}} \times \mathscr{D}_\nu[\mathscr{A}]\hat{\boldsymbol{\phi}} = 0, \tag{25}$$

$$\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}} - \left(\hat{\boldsymbol{\phi}} \cdot \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}\right)\hat{\boldsymbol{\phi}} = 0. \tag{26}$$

By applying the covariant derivative $\mathscr{D}^\nu[\mathscr{A}]$ to Eq. (25), the reduction condition is naturally induced:

$$0 = -\mathscr{D}^\nu[\mathscr{A}]\mathscr{D}^\mu[\mathscr{A}]\mathscr{F}_{\mu\nu} = gv^2\hat{\boldsymbol{\phi}} \times \mathscr{D}^\nu[\mathscr{A}]\mathscr{D}_\nu[\mathscr{A}]\hat{\boldsymbol{\phi}} = gv^2\boldsymbol{\chi}. \tag{27}$$

Moreover, by taking the exterior product of Eq. (26) and $\hat{\boldsymbol{\phi}}(x)$, the reduction condition is induced again:

$$\begin{aligned} 0 &= \hat{\boldsymbol{\phi}} \times \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}} - \left(\hat{\boldsymbol{\phi}} \cdot \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}\right)\left(\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}}\right) \\ &= \hat{\boldsymbol{\phi}} \times \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}} = \boldsymbol{\chi}. \end{aligned} \tag{28}$$

Hence, the simultaneous solutions of the coupled field equations (25) and (26) automatically satisfy the reduction condition (17). Figure 2 shows the relation between the solutions of the field equations of the gauge–adjoint scalar model and the reduction condition. From this relation, we find that the solutions of the coupled field equations of the gauge–scalar model (25) and (26) can play the very important role of the configurations satisfying the reduction condition (17) in a massive Yang–Mills theory through the path integral (20).

3. The Georgi–Glashow model and the 't Hooft–Polyakov monopole: radially variable case

In this section, we shall summarize the essence of the Georgi–Glashow model. We introduce the Georgi–Glashow model by the Lagrangian density:

$$\mathscr{L}_{\rm GG} = -\frac{1}{4}\mathscr{F}_{\mu\nu} \cdot \mathscr{F}^{\mu\nu} + \frac{1}{2}\left(\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}\right) \cdot \left(\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}\right) - \frac{\lambda^2 g^2}{4}\left(\boldsymbol{\phi} \cdot \boldsymbol{\phi} - v^2\right)^2. \tag{29}$$

We take the standard static and spherically symmetric ansatz for the 't Hooft–Polyakov monopole with a unit magnetic charge [4–7]:

$$g\mathscr{A}_0^A(x) = 0, \quad g\mathscr{A}_j^A(x) = \epsilon^{jAk}\frac{x^k}{r}\frac{1-f(r)}{r} \quad \phi^A(x) = v\frac{x^A}{r}h(r), \tag{30}$$

where r is the 3D radius $r := \sqrt{x^2+y^2+z^2}$. The profile functions $f(r)$ and $h(r)$ are unknown functions to be determined by solving the field equations.

In the spherically symmetric case, we can rewrite the Lagrangian $L_{\rm GG}$,

$$L_{\rm GG} = \int d^3x\, \mathscr{L}_{\rm GG} = \int_0^\infty dr\, \widetilde{\mathscr{L}}_{\rm GG}, \quad \widetilde{\mathscr{L}}_{\rm GG} := 4\pi r^2 \mathscr{L}_{\rm GG}, \tag{31}$$

in terms of the redefined Lagrangian density $\widetilde{\mathscr{L}}_{\rm GG}$:

$$\widetilde{\mathscr{L}}_{\rm GG} = \frac{4\pi}{g^2}\left[-f'^2(r) - \frac{(f^2(r)-1)^2}{2r^2} - \frac{1}{2}g^2v^2r^2h'^2(r) - g^2v^2f^2(r)h^2(r) - \frac{\lambda^2}{4}g^4v^4r^2\left(h^2(r)-1\right)^2\right]. \tag{32}$$

By rescaling

$$r \to \rho := gvr, \tag{33}$$

with ρ now being dimensionless, $\widetilde{\mathscr{L}}_{\rm GG}$ is cast into

$$\widetilde{\mathscr{L}}_{\rm GG} = 4\pi v^2\left[-f'^2(\rho) - \frac{(f^2(\rho)-1)^2}{2\rho^2} - \frac{1}{2}\rho^2h'^2(\rho) - f^2(\rho)h^2(\rho) - \frac{\lambda^2}{4}\rho^2\left(h^2(\rho)-1\right)^2\right]. \tag{34}$$

The field equations are obtained as

$$f''(\rho) = \frac{f^3(\rho) - f(\rho)}{\rho^2} + f(\rho)h^2(\rho), \tag{35}$$

$$\left(\rho^2h'(\rho)\right)' = 2f^2(\rho)h(\rho) + \lambda^2\rho^2\left(h^3(\rho) - h(\rho)\right). \tag{36}$$

We assume asymptotic behavior for small ρ as the power series in ρ:

$$f(\rho) = 1 + \sum_{n=0}^{\infty} F_n\rho^n, \quad h(\rho) = \sum_{n=0}^{\infty} H_n\rho^n. \tag{37}$$

By substituting these series expansions into the field equations (35) and (36), we can determine the coefficients as

$$f(\rho) = 1 - F_2\rho^2 + \frac{3F_2^2 + H_1^2}{10}\rho^4 - \frac{14F_2^3 + 12F_2H_1^2 + \lambda H_1^2}{140}\rho^6 + \cdots, \tag{38}$$

$$h(\rho) = H_1\rho - \frac{\lambda H_1 + 4F_2H_1}{10}\rho^3 + \frac{48F_2^2H_1 + 8\lambda F_2H_1 + \lambda^2H_1 + 2(5\lambda+2)H_1^3}{280}\rho^5 + \cdots. \tag{39}$$

4. Scaling argument of the massive Yang–Mills theory and the Georgi–Glashow model

In this section we examine the existence of the static and stable configuration in the massive Yang–Mills theory. For this purpose, we follow the scaling argument due to Derrick [48]. In this section only, we consider an arbitrary spatial dimension d.

In the gauge–adjoint scalar model with a radial-fixing constraint, the static energy E can be written after eliminating the Lagrange multiplier field $u(x)$ as

$$E = \int d^d x \left[\frac{1}{4} \mathscr{F}_{jk} \cdot \mathscr{F}_{jk} + \frac{v^2}{2} \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right) \cdot \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right) + \frac{v^2}{2} \left(1 - \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} \right) \hat{\boldsymbol{\phi}} \cdot \mathscr{D}_j[\mathscr{A}]\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right]. \quad (40)$$

By rescaling the spatial variable $\boldsymbol{x}$ as $\boldsymbol{x} \to \mu \boldsymbol{x}$, the fields are transformed as $\Phi(\boldsymbol{x}) \to \Phi^{(\mu)}(\boldsymbol{x})$ in general: For the scalar and the vector field,

$$\hat{\boldsymbol{\phi}}^{(\mu)}(\boldsymbol{x}) = \hat{\boldsymbol{\phi}}(\mu \boldsymbol{x}), \quad \mathscr{A}_j^{(\mu)}(\boldsymbol{x}) = \mu \mathscr{A}_j(\mu \boldsymbol{x}), \quad (41)$$

which yields

$$\left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right)^{(\mu)}(\boldsymbol{x}) = \mu \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right)(\mu \boldsymbol{x}), \quad \mathscr{F}_{jk}^{(\mu)}(\boldsymbol{x}) = \mu^2 \mathscr{F}_{jk}(\mu \boldsymbol{x}). \quad (42)$$

Then the scaled energy $E(\mu, d)$ obeys

$$E(\mu, d) = \mu^{4-d} E_4 + \mu^{2-d} E_2, \quad (43)$$

where

$$E_4 := \int d^d x \, \frac{1}{4} \mathscr{F}_{jk} \cdot \mathscr{F}_{jk}, \quad (44)$$

$$E_2 := \int d^d x \left[\frac{v^2}{2} \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right) \cdot \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right) + \frac{v^2}{2} \left(1 - \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} \right) \hat{\boldsymbol{\phi}} \cdot \mathscr{D}_j[\mathscr{A}]\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}} \right]. \quad (45)$$

For the massive Yang–Mills theory (21), the scaled energy $E(\mu, d)$ obeys the same equation as Eq. (43) with the replacement:

$$E_4 := \int d^d x \, \frac{1}{4} \mathscr{F}_{jk} \cdot \mathscr{F}_{jk}, \quad E_2 := \int d^d x \, \frac{1}{2} M_{\mathscr{X}}^2 \mathscr{X}_j \cdot \mathscr{X}_j. \quad (46)$$

We find that $E(\mu, d)$ has a stationary point in $2 < d < 4$ spatial dimensions:

$$\frac{dE(\mu, d)}{d\mu} = 0 \quad \text{at} \quad \mu = \sqrt{\frac{(d-2)E_2}{(4-d)E_4}}, \quad (47)$$

implying that there can exist a stable configuration with a finite energy that differs from the vacuum configuration. It should be noticed that such a stable configuration can exist *only* in $d = 3$. Therefore, we can obtain the static topological soliton in the $(3 + 1)$-dimensional massive Yang–Mills theory. The explicit construction is given in the next section.

This result should be compared with the pure (massless) Yang–Mills theory and the $SU(2)$ Georgi–Glashow model: The scaled energies for these theories are respectively given by

$$E_{\rm YM}(\mu, d) = \mu^{4-d} E_4, \quad (48)$$

$$E_{\rm GG}(\mu, d) = \mu^{4-d} E_4 + \mu^{2-d} E_2 + \mu^{-d} E_0, \quad (49)$$

where E_0 in $E_{\rm GG}(\mu, d)$ comes from the potential term:

$$E_0 := \int d^d x \, \frac{\lambda^2 g^2}{4} \left(\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) - v^2 \right)^2. \quad (50)$$

Notice that for the pure (massless) Yang–Mills theory only the term of the gauge field exists and hence there is no stationary point under the scaling, which implies the non-existence of the static and stable soliton solutions in the $(3+1)$-dimensional massless Yang–Mills theory. For the Georgi–Glashow model, the scaling argument does not prohibit the existence of the static soliton solution; i.e., indeed, there exists the 't Hooft–Polyakov magnetic monopole for $d = 3$.

5. The magnetic monopole solution in the massive Yang–Mills theory

Because of the constraint (8) the normalized scalar field $\hat{\boldsymbol{\phi}}(x)$ takes the value in the target space of the 2D sphere S^2. Then, by regarding $\hat{\boldsymbol{\phi}}(x)$ as the map

$$\hat{\boldsymbol{\phi}}(x) : S^2_{\text{phys}} \to S^2_{\text{target}}, \tag{51}$$

there could exist topological soliton solutions related to the nontrivial homotopy group $\pi_2(S^2) = \mathbb{Z}$.

We adopt the same ansatz as the 't Hooft–Polyakov monopole:

$$g\mathscr{A}_0^A(x) = 0, \quad g\mathscr{A}_j^A(x) = \epsilon^{jAk}\frac{x^k}{r}\frac{1-f(r)}{r}, \tag{52}$$

and[3]

$$\phi^A(x) = v\frac{x^A}{r}h(r) \quad \Longleftrightarrow \quad \hat{\phi}^A(x) = \frac{x^A}{r}h(r). \tag{53}$$

The profile functions $f(r)$ and $h(r)$ are unknown functions to be determined by solving the field equations.

We redefine the Lagrangian density $\mathscr{L}$ by $\widetilde{\mathscr{L}} = 4\pi r^2\mathscr{L}$:

$$\widetilde{\mathscr{L}} = \frac{4\pi}{g^2}\left[-f'^2(r) - \frac{(f^2(r)-1)^2}{2r^2} - \frac{1}{2}g^2v^2r^2h'^2(r) - g^2v^2f^2(r)h^2(r) + ug^2v^2r^2\left(h^2(r)-1\right)\right], \tag{54}$$

where the prime denotes a derivative with respect to r.

The equations for the profile functions $f(r)$ and $h(r)$ are obtained as

$$f''(r) = \frac{f^3(r) - f(r)}{r^2} + g^2v^2h^2(r)f(r), \tag{55}$$

$$\left(r^2h'(r)\right)' = 2f^2(r)h(r) - 2ur^2h(r), \tag{56}$$

$$h^2(r) - 1 = 0. \tag{57}$$

Equation (57) comes from the constraint and can be solved:

$$h(r) = \pm 1. \tag{58}$$

[3] It should be noted that in this setup the reduction condition (17) is automatically satisfied due to its tensor structure (without knowing the profile functions):

$$\begin{aligned}\boldsymbol{\chi} &= \hat{\boldsymbol{\phi}} \times \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}} = \epsilon^{ABC}T_A\hat{\phi}^B\left(-\mathscr{D}_j[\mathscr{A}]\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}}\right)^C \\ &= -\epsilon^{ABC}T_A\frac{x^B}{r}\frac{x^C}{r}h(r)\left[\frac{d^2h(r)}{dr^2} + \frac{2}{r}\frac{dh(r)}{dr} - \frac{2}{r^2}h(r)f^2(r)\right] = 0, \quad T_A = \frac{\sigma_A}{2}.\end{aligned}$$

By substituting Eq. (58) into the other equations, we have

$$f''(r) = \frac{f^3(r) - f(r)}{r^2} + g^2 v^2 f(r), \tag{59}$$

$$0 = f^2(r) - ur^2. \tag{60}$$

Thus we can determine the Lagrange multiplier field $u = u(r)$ by

$$u(r) = \frac{f^2(r)}{r^2}, \tag{61}$$

once the remaining equation (59), which we call the monopole equation, is solved. By rescaling $r \to \rho := M_{\mathscr{X}} r$, the monopole equation reads

$$f''(\rho) = \frac{f^3(\rho) - f(\rho)}{\rho^2} + f(\rho). \tag{62}$$

First, we examine the asymptotic behavior of $f(r)$. The static energy E is given by

$$\begin{aligned} E &= \frac{4\pi}{g^2} \int_0^\infty dr \left[f'^2(r) + \frac{(f^2(r) - 1)^2}{2r^2} + g^2 v^2 f^2(r) \right] \\ &= \frac{4\pi M_{\mathscr{X}}}{g^2} \int_0^\infty d\rho \left[f'^2(\rho) + \frac{(f^2(\rho) - 1)^2}{2\rho^2} + f^2(\rho) \right], \end{aligned} \tag{63}$$

where in the second equality we have rescaled $r \to \rho$. One can find the boundary conditions for $f(\rho)$ by requiring the energy E to be finite:

$$f(\rho) \xrightarrow{\rho \to 0} \pm 1 + \mathcal{O}(\rho^{1/2}), \quad f(\rho) \xrightarrow{\rho \to \infty} 0 + \mathcal{O}(\rho^{-1}). \tag{64}$$

For small ρ, we further require

$$f(\rho) \xrightarrow{\rho \to 0} +1 + \mathcal{O}(\rho^\alpha),\ \alpha > 1, \tag{65}$$

so that the gauge field $\mathscr{A}_j^A(x)$ becomes non-singular at the origin.

Here, one finds that $f(\rho) \equiv 0$ is a solution of the monopole equation (62). This is nothing but the Wu–Yang magnetic monopole. However, this solution yielding $\mathscr{X}_j(x) = 0$ does not satisfy the boundary condition (64) for $\rho \approx 0$, which leads to infinite energy $E = \infty$. Conversely, the solution $f(\rho) \neq 0$ means $\mathscr{X}_\mu(x) \neq 0$, which yields a finite energy $E < \infty$.

In order to obtain the asymptotic behavior of $f(\rho)$ for small ρ, let us define $f(\rho) = 1 + g(\rho)$ with $|g(\rho)| \ll 1$ and linearize the monopole equation (62):

$$\rho^2 g''(\rho) - 2g(\rho) - \rho^2 g(\rho) = \rho^2. \tag{66}$$

The linear differential equation (66) for $g(\rho)$ has the following general solution:

$$\begin{aligned} g(\rho) = &C_1 \left(\cosh\rho - \frac{\sinh\rho}{\rho} \right) + C_2 \left(\frac{\cosh\rho}{\rho} - \sinh\rho \right) \\ &- 1 + \left(\cosh\rho - \frac{\sinh\rho}{\rho} \right) \operatorname{Chi}\rho + \left(\frac{\cosh\rho}{\rho} - \sinh\rho \right) \operatorname{Shi}\rho, \end{aligned} \tag{67}$$

where we have introduced the hyperbolic cosine and sine integral Chi x and Shi x respectively, defined with the Euler constant γ by

$$\mathrm{Chi}\, x := \gamma + \log x + \int_0^x dt\, \frac{\cosh t - 1}{t}, \quad \mathrm{Shi}\, x := \int_0^x dt\, \frac{\sinh t}{t}. \tag{68}$$

Here the first two terms of Eq. (67) correspond to the general solution consisting of two independent special solutions ($\cosh\rho - \frac{\sinh\rho}{\rho}$) and ($\frac{\cosh\rho}{\rho} - \sinh\rho$) of the homogeneous equation obtained by eliminating the inhomogeneous term ρ^2 of Eq. (66), and the remaining terms represent a special solution of the inhomogeneous equation (66).

Under the boundary conditions $g(0) = 0$ and $g'(0) = 0$, we can determine only one coefficient $C_2 = 0$:

$$g(\rho) = C_1\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right) - 1 + \left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\mathrm{Chi}\,\rho + \left(\frac{\cosh\rho}{\rho} - \sinh\rho\right)\mathrm{Shi}\,\rho. \tag{69}$$

The Taylor expansion of the solution (69) around the origin $\rho = 0$ has the form

$$g(\rho) = \widetilde{C}\rho^2 + \frac{1}{3}\rho^2\log\rho + \mathcal{O}(\rho^4), \quad \widetilde{C} := \frac{1}{9}(-4 + 3\gamma + 3C_1). \tag{70}$$

Thus, under the boundary conditions $f(0) = 1$ and $f'(0) = 0$, we can set the asymptotic form of $f(\rho)$ around the origin:

$$f(\rho) = 1 + \widetilde{C}\rho^2 + \frac{1}{3}\rho^2\log\rho + \cdots \quad (\rho \approx 0), \tag{71}$$

where $\widetilde{C}$ is arbitrary at this stage. Notice that the extra logarithmic behavior of $f(\rho)$ appears. This is due to the radially fixed constraint $h(0) = 1$, and the singularity at the origin should be cured by $f(\rho)$ itself. In the case of the 't Hooft–Polyakov monopoles, on the other hand, the field equations are satisfied by the power series in ρ, Eqs. (38) and (39), without the logarithmic terms for $0 \le \lambda < \infty$.

For large ρ, we adopt the asymptotic form:

$$f(\rho) = e^{-\rho}\sum_{n=0}^{\infty} D_n\rho^{-n} \quad (\rho \approx \infty). \tag{72}$$

In a similar way to the above, we can determine the coefficients D_n as

$$f(\rho) = D_0 e^{-\rho}\left(1 - \frac{1}{2\rho} + \frac{3}{8\rho^2} - \cdots\right), \tag{73}$$

where the overall factor D_0 is arbitrary at this stage. The monopole equation (62) can be solved in a numerical way; see Appendix A for details. The coefficients $\widetilde{C}$ and D_0 can be determined in a numerical way as well.

Figure 3 shows the obtained solution $f(\rho)$ of the monopole equation (62) and a corresponding scalar profile function $h(\rho)$ as a function of ρ, which should be compared with the usual 't Hooft–Polyakov monopole solution. The 't Hooft–Polyakov monopole solution with a large coupling $\lambda \gg 1$ approaches the Yang–Mills magnetic monopole except for in the neighborhood of the origin $\rho \approx 0$: In the 't Hooft–Polyakov monopole, the scalar profile function $h_{\mathrm{tHP}}(\rho)$ starts from zero, $h_{\mathrm{tHP}}(0) = 0$, even in the limit $\lambda \to \infty$, while the scalar profile function of the Yang–Mills magnetic monopole has a constant value $h(0) = 1$ due to the constraint (57).

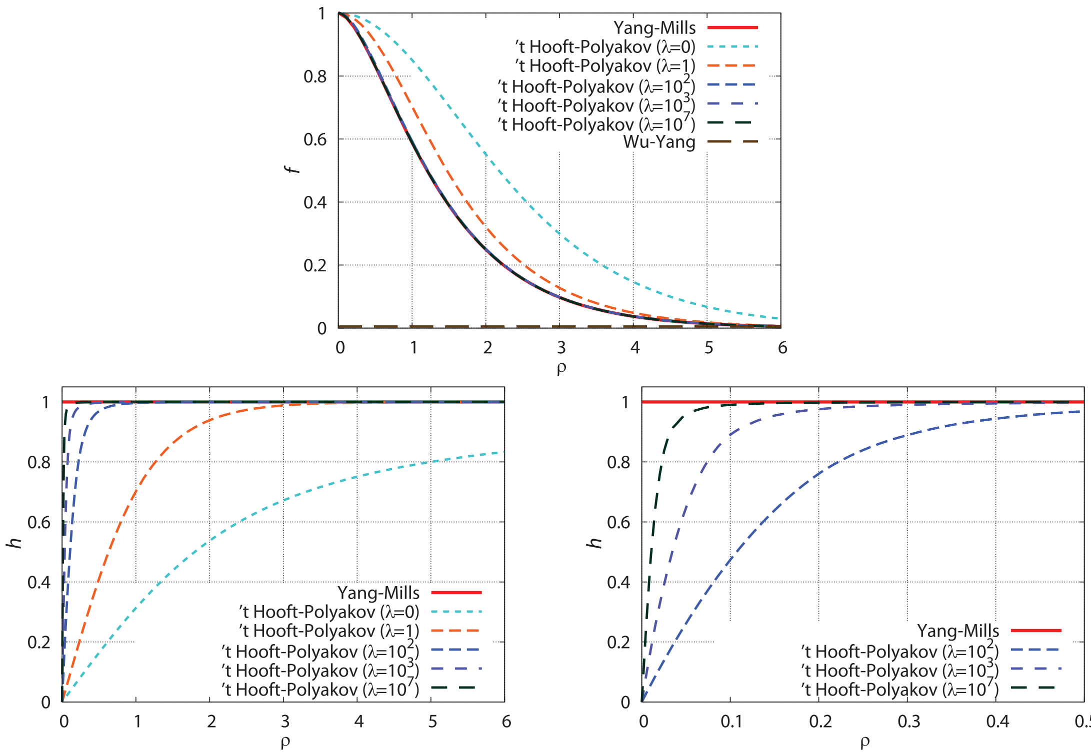

Fig. 3. (Top) The solution f of the Yang–Mills monopole equation (62) as a function of $\rho = M_{\mathscr{X}} r$ to be compared with the 't Hooft–Polyakov monopole solutions (for $\lambda = 0, 1, 10^2, 10^3$, and 10^7) and the Wu–Yang magnetic monopole with $f \equiv 0$. (Bottom left) The corresponding solution h for the scalar field as a function of ρ. The radially fixed constraint $h(\rho) \equiv 1$ holds even at the origin $\rho = 0$ in the Yang–Mills monopole, while the naive $\lambda \to \infty$ limit of the 't Hooft–Polyakov monopole approaches the limit value, $h_{\text{tHP}}(\rho) \to 1$, for $\rho > 0$. (Bottom right) An enlarged figure for $h(\rho)$ around the origin $\rho \approx 0$: the Yang–Mills monopole $h \equiv 1$ is to be compared with the 't Hooft–Polyakov monopole with $h_{\text{tHP}}(0) = 0$ in the case of $\lambda = 10^2, 10^3$, and 10^7.

From this numerical solution, we can calculate the static energy or the rest mass of a magnetic monopole E as

$$E = \frac{4\pi M_{\mathscr{X}}}{g^2} \int_0^\infty d\rho \left[f'^2(\rho) + \frac{(f^2(\rho) - 1)^2}{2\rho^2} + f^2(\rho) \right] \simeq \frac{4\pi M_{\mathscr{X}}}{g^2} \times 1.782\,06. \tag{74}$$

This result also shows that the obtained solution $f(\rho)$ is different from the Bogomol'nyi–Prasad–Sommerfield (BPS) monopole [49,50]: By definition, the energy in the BPS limit is given by

$$E = \frac{4\pi v}{g} = \frac{4\pi M_{\mathscr{X}}}{g^2}, \quad M_{\mathscr{X}} = gv. \tag{75}$$

We define the energy density $e(\rho)$ by

$$E = \int d^3x\, \mathscr{H}(r) = \int_0^\infty d\rho\, 4\pi\rho^2 \frac{M_{\mathscr{X}}}{g^2} \mathscr{H}(\rho) = \frac{4\pi M_{\mathscr{X}}}{g^2} \int_0^\infty d\rho\, e(\rho), \tag{76}$$

where $\mathscr{H}(r)$ is the Hamiltonian density. The energy density $e(\rho)$ can be written as

$$e(\rho) = f'^2(\rho) + \frac{(f^2(\rho) - 1)^2}{2\rho^2} + \frac{1}{2}\rho^2 h'^2(\rho) + f^2(\rho)h^2(\rho) + \rho^2 V(h^2). \tag{77}$$

Figure 4 is a plot of the energy density $e(\rho)$ as a function of ρ obtained from the solution $f(\rho)$, which should also be compared with the case of the 't Hooft–Polyakov monopoles. One can find

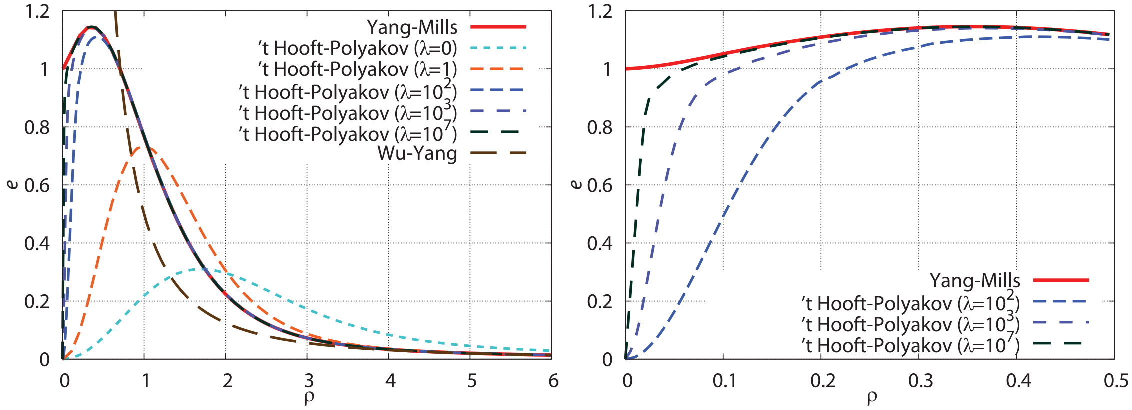

Fig. 4. (Left panel) The energy density e of the Yang–Mills monopole as a function of $\rho = M_{\mathscr{X}} r$ to be compared with the 't Hooft–Polyakov monopoles (for $\lambda = 0, 1, 10^2, 10^3$, and 10^7) and the Wu–Yang magnetic monopole (diverging at the origin). (Right panel) An enlarged figure for e around the origin $\rho \approx 0$: the Yang–Mills solution is to be compared with the 't Hooft–Polyakov monopole with $\lambda = 10^2, 10^3$, and 10^7.

that the energy density of the Yang–Mills monopole is very different from the 't Hooft–Polyakov solution at the origin even in the limit of $\lambda \to \infty$ even though they have the same energy value. This is caused by the radially fixed condition: In the 't Hooft–Polyakov case, $e(0) = 0$ originates from $h(0) = 0$, which persists even in the limit $\lambda \to \infty$, while in our case, $h(0) = 1$ with no potential term $V(h^2) = 0$, the contribution from the fourth term in Eq. (77) for $e(\rho)$ survives at the origin due to $f(0) = 1$.

Based on Eq. (74), we estimate the static mass of the Yang–Mills monopole as

$$E = 0.93 \pm 0.04\,\text{GeV}, \tag{78}$$

where we have used the value for the off-diagonal gluon mass $M_{\mathscr{X}} = 1.2\,\text{GeV}$ obtained by previous studies on a lattice [51] and the typical value of the running coupling constant $\alpha_s(p) := g^2(p)/4\pi \approx 2.3 \pm 0.1$ at $p \simeq M_{\mathscr{X}} \approx 1.2\,\text{GeV}$ obtained in Refs. [52,53]. This result should be compared with the $SU(5)$ grand unified theory (GUT) monopole [54] and the $SU(2) \times U(1)$ electroweak monopoles (Cho–Maison monopoles) [55,56]. For the GUT monopole, the monopole mass exists around 10^{14}–10^{15} TeV for the $SU(5)$ GUT scale 10^{13} TeV. The mass of the electroweak monopole is estimated as 4–7 TeV, which is much heavier than the mass of the W boson $m_W \approx 80\,\text{GeV}$ because of the smallness of the running coupling constant $\alpha_s(p) \approx 0.12$ at the weak scale, $p \simeq m_Z \approx 91\,\text{GeV}$.

The Yang–Mills monopole mass, 0.93 GeV, obtained in this paper corresponds to the heaviest one in the family of 't Hooft–Polyakov monopoles in the Georgi–Glashow model, since the energy (76) is monotonically increasing in the coupling constant λ, while the lightest mass, 0.52 GeV, occurs if the coupling λ vanishes, namely, in the BPS limit. It should be noted that the Yang–Mills monopole mass $E \approx 0.93\,\text{GeV}$ and the off-diagonal gluon mass $M_{\mathscr{X}} \approx 1.2\,\text{GeV}$ are of the same order as the typical scale of the strong interactions: $\mathcal{O}(1)$. In view of these, the existence of the Yang–Mills monopole with a reasonable mass is a remarkable step for quark confinement to be realized due to condensation of the relevant Yang–Mills monopoles according to the dual superconductor picture, although we need more serious investigations to conclude whether or not the interactions among monopoles are indeed sufficient for realizing monopole condensations, as examined in the 3D case by Polyakov [57].

6. Behavior of the gauge and chromo-magnetic fields

6.1. The gauge field

We shall separate the gauge field $\mathscr{A}_\mu(x)$ into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x), \tag{79}$$

where

$$g\mathscr{X}_\mu = \hat{\boldsymbol{\phi}} \times \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}, \quad g\mathscr{V}_\mu = g\mathscr{A}_\mu - g\mathscr{X}_\mu = g(\mathscr{A}_\mu \cdot \hat{\boldsymbol{\phi}})\hat{\boldsymbol{\phi}} + \partial_\mu\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}}. \tag{80}$$

In the present ansatz, by using the normalized scalar field $\hat{\boldsymbol{\phi}}(x)$ with $h(r) = +1$ and the Pauli matrices $T_A = \frac{1}{2}\sigma_A$,

$$\hat{\boldsymbol{\phi}}(x) = \frac{x^A}{r}\frac{\sigma_A}{2}, \tag{81}$$

they are explicitly written as

$$g\mathscr{V}_j(x) = \frac{\epsilon^{jAk}x^k}{r^2}\frac{\sigma_A}{2}, \quad g\mathscr{X}_j(x) = -\frac{\epsilon^{jAk}x^k}{r^2}\frac{\sigma_A}{2}f(r), \tag{82}$$

and their time components vanish: $\mathscr{V}_0(x) = 0$, $\mathscr{X}_0(x) = 0$.

In what follows, we adopt the polar coordinate system (r, θ, φ) for the spatial coordinates:

$$g\mathscr{A}_r(x) = 0, \quad g\mathscr{A}_\theta(x) = A(r)T_\theta, \quad g\mathscr{A}_\varphi(x) = A(r)T_\varphi, \tag{83}$$

$$g\mathscr{V}_r(x) = 0, \quad g\mathscr{V}_\theta(x) = V(r)T_\theta, \quad g\mathscr{V}_\varphi(x) = V(r)T_\varphi, \tag{84}$$

$$g\mathscr{X}_r(x) = 0, \quad g\mathscr{X}_\theta(x) = X(r)T_\theta, \quad g\mathscr{X}_\varphi(x) = X(r)T_\varphi, \tag{85}$$

where we have defined

$$T_\theta = \frac{1}{2}\begin{pmatrix} 0 & ie^{-i\varphi} \\ -ie^{i\varphi} & 0 \end{pmatrix}, \quad T_\varphi = \frac{1}{2}\begin{pmatrix} -\sin\theta & \cos\theta e^{-i\varphi} \\ \cos\theta e^{i\varphi} & \sin\theta \end{pmatrix}, \tag{86}$$

and

$$A(r) = \frac{1 - f(M_{\mathscr{X}}r)}{r}, \quad V(r) = \frac{1}{r}, \quad X(r) = -\frac{f(M_{\mathscr{X}}r)}{r}. \tag{87}$$

Figure 5 is a plot of the fields A, V, and X as functions of $\rho = M_{\mathscr{X}}r$, which shows that the original gauge field $\mathscr{A}(x)$ is indeed regular at the origin:

$$A(r) = M_{\mathscr{X}}^2\left[-\widetilde{C}r - \frac{1}{3}r\log(M_{\mathscr{X}}r) + \mathcal{O}(r^3)\right], \tag{88}$$

as is expected. On the other hand, the fields $\mathscr{V}(x)$ and $\mathscr{X}(x)$ diverge at the origin $r = 0$.

We perform a singular gauge transformation, which makes $\hat{\boldsymbol{\phi}}(x)$ diagonal: $\hat{\boldsymbol{\phi}}_\infty = \frac{1}{2}\sigma_3$:

$$\hat{\boldsymbol{\phi}}(x) = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \to \hat{\boldsymbol{\phi}}'(x) = U(x)\hat{\boldsymbol{\phi}}(x)U^{-1}(x) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: \hat{\boldsymbol{\phi}}_\infty, \tag{89}$$

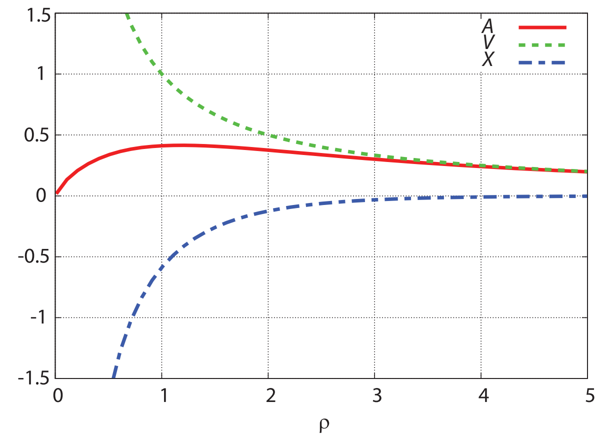

Fig. 5. The behavior of A, V, and X as functions of $\rho = M_{\mathscr{X}} r$. Here $A(x) = V(x) + X(x)$, where $V(x)$ agrees with the Wu–Yang monopole and $X(x)$ corresponds to the massive mode.

or equivalently $\hat{\phi}'^A(x) = \delta^{A3}$. Such a gauge transformation can be done by using the following $SU(2)$ matrix $U(x)$:

$$U(x) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\varphi} \\ -\sin\frac{\theta}{2} e^{i\varphi} & \cos\frac{\theta}{2} \end{pmatrix} \in SU(2). \tag{90}$$

As mentioned before, $\mathscr{X}_\mu(x)$ is transformed in an adjoint way:

$$\mathscr{X}_\mu(x) \to \mathscr{X}'_\mu(x) = U(x)\mathscr{X}_\mu(x)U^{-1}(x), \tag{91}$$

while, as a consequence, $\mathscr{V}_\mu(x)$ has the same gauge transformation property as the original gauge field $\mathscr{A}_\mu(x)$:

$$\mathscr{V}_\mu(x) \to \mathscr{V}'_\mu = U(x)\left(\mathscr{V}_\mu(x) + \frac{i}{g}\partial_\mu\right)U^{-1}(x). \tag{92}$$

Thus, $\mathscr{V}(x)$ and $\mathscr{X}(x)$ are transformed by $U(x)$ as

$$g\mathscr{V}'_r(x) = 0, \quad g\mathscr{V}'_\theta(x) = 0, \quad g\mathscr{V}'_\varphi(x) = -\frac{1}{r}\frac{1-\cos\theta}{\sin\theta}T_3, \tag{93}$$

$$g\mathscr{X}'_r(x) = 0, \quad g\mathscr{X}'_\theta(x) = -\frac{f(M_{\mathscr{X}}r)}{r}T_-, \quad g\mathscr{X}'_\varphi(x) = -\frac{f(M_{\mathscr{X}}r)}{r}T_+, \tag{94}$$

where we have defined

$$T_+ := T_1\cos\varphi + T_2\sin\varphi, \quad T_- := T_1\sin\varphi - T_2\cos\varphi, \quad T_A = \frac{\sigma_A}{2}. \tag{95}$$

One can find that the field $\mathscr{V}(x)$ is nothing but the Wu–Yang potential [44], which has singularities of the Dirac string type [58] located on the negative part of the z-axis. Moreover, we find that by recalling $f(M_{\mathscr{X}}r) \propto \exp(-M_{\mathscr{X}}r)$ at $r \approx \infty$ the field $\mathscr{X}(x)$ indeed falls off exponentially, and hence we can identify $\mathscr{X}(x)$ with the massive (or high-energy) mode.

For the Yang–Mills magnetic monopole obtained in the massive Yang–Mills theory, we do not need to introduce artificial regularization by hand to remedy the short-distance (or ultraviolet) singularity and instability of the Wu–Yang magnetic monopole in the pure massless Yang–Mills theory as worked out in Refs. [59–61]. The regularized solution of the Yang–Mills field equation was obtained so that the Wu–Yang solution for $r > r_0$ and another solution for $r < r_0$ are connected at $r = r_0$ to make the energy finite; see pp. 503–504 and Appendix B of Ref. [61]. The Yang–Mills magnetic monopole $\mathscr{A}(x)$ obtained in this paper approaches the Wu–Yang type $\mathscr{V}(x)$ for large r, while for small r it approaches the regular form and the energy becomes finite. This is attributed to the behavior of the massive mode $\mathscr{X}(x)$. For large r, $\mathscr{X}(x)$ falls off quickly to guarantee $\mathscr{A}(x) \simeq \mathscr{V}(x)$, while for small r, $\mathscr{X}(x)$ also becomes singular but with a signature opposite to $\mathscr{V}(x)$ to cancel the singularity of $\mathscr{V}(x)$, leading to a finite Yang–Mills field, $\mathscr{A}(x) = \mathscr{V}(x) + \mathscr{X}(x) \simeq 0$ near $x = 0$.

6.2. *The chromo-magnetic field*

We examine the magnetic charge q_m obtained by the chromo-magnetic field $\mathscr{B}_j^A(x)$:

$$g\mathscr{B}_j^A(x) = \frac{1}{2}\epsilon_{jkl} g\mathscr{F}_{kl}^A(x) = \frac{x^A x^j}{r^4}\left(1 - f^2(M_{\mathscr{X}} r)\right) - \left(\frac{\delta^{Aj}}{r} - \frac{x^A x^j}{r^3}\right)\frac{df(M_{\mathscr{X}} r)}{dr}. \tag{96}$$

The magnetic charge q_m and its density $\rho_m(r)$ are defined by

$$q_m = \int d^3x\, \mathscr{B}_j^A \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}}\right)^A = \frac{4\pi}{g}\int_0^\infty dr\, \rho_m(r), \quad \rho_m(r) := r^2 \mathscr{B}_j^A \left(\mathscr{D}_j[\mathscr{A}]\hat{\boldsymbol{\phi}}\right)^A. \tag{97}$$

The magnetic charge density $\rho_m(r)$ can be written in terms of the profile functions $f(M_{\mathscr{X}} r)$ (and $h(M_{\mathscr{X}} r)$):

$$\rho_m(r) = \frac{d}{dr}\left[h(M_{\mathscr{X}} r)\left(1 - f^2(M_{\mathscr{X}} r)\right)\right]. \tag{98}$$

From the definition of q_m, this chromo-magnetic field $\mathscr{B}_j^A(x)$ indeed has a nontrivial magnetic charge q_m:

$$\begin{aligned} q_m &:= \frac{4\pi}{g}\int_0^\infty dr\, \rho_m(r) = \frac{4\pi}{g}\int_0^\infty dr\, \frac{d}{dr}\left[h(M_{\mathscr{X}} r)\left(1 - f^2(M_{\mathscr{X}} r)\right)\right] \\ &= \frac{4\pi}{g}\left[h(M_{\mathscr{X}} r)\left(1 - f^2(M_{\mathscr{X}} r)\right)\right]\Bigg|_{r=0}^{r=\infty} = \frac{4\pi}{g}. \end{aligned} \tag{99}$$

In the top panel of Fig. 6 we give the magnetic charge density $\rho_m(r)$, which is also compared with the ’t Hooft–Polyakov magnetic monopole. We observe that the Yang–Mills magnetic monopole is more localized in the vicinity of the origin than any ’t Hooft–Polyakov magnetic monopole, and is the same size as the $\lambda = \infty$ ’t Hooft–Polyakov monopole.

In order to investigate the behavior of the chromo-magnetic field $\mathscr{B}_j^A(x)$ around $r \approx 0$, we turn to the polar coordinate representation:

$$\begin{aligned} g\mathscr{B}_r(x) &= \frac{1 - f^2(M_{\mathscr{X}} r)}{r^2}\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}, \\ g\mathscr{B}_\theta(x) &= -\frac{1}{r}\frac{df(M_{\mathscr{X}} r)}{dr}\frac{1}{2}\begin{pmatrix} \sin\theta & -\cos\theta e^{-i\varphi} \\ -\cos\theta e^{i\varphi} & -\sin\theta \end{pmatrix}, \\ g\mathscr{B}_\varphi(x) &= -\frac{1}{r}\frac{df(M_{\mathscr{X}} r)}{dr}\frac{1}{2}\begin{pmatrix} 0 & ie^{-i\varphi} \\ -ie^{i\varphi} & 0 \end{pmatrix}. \end{aligned} \tag{100}$$

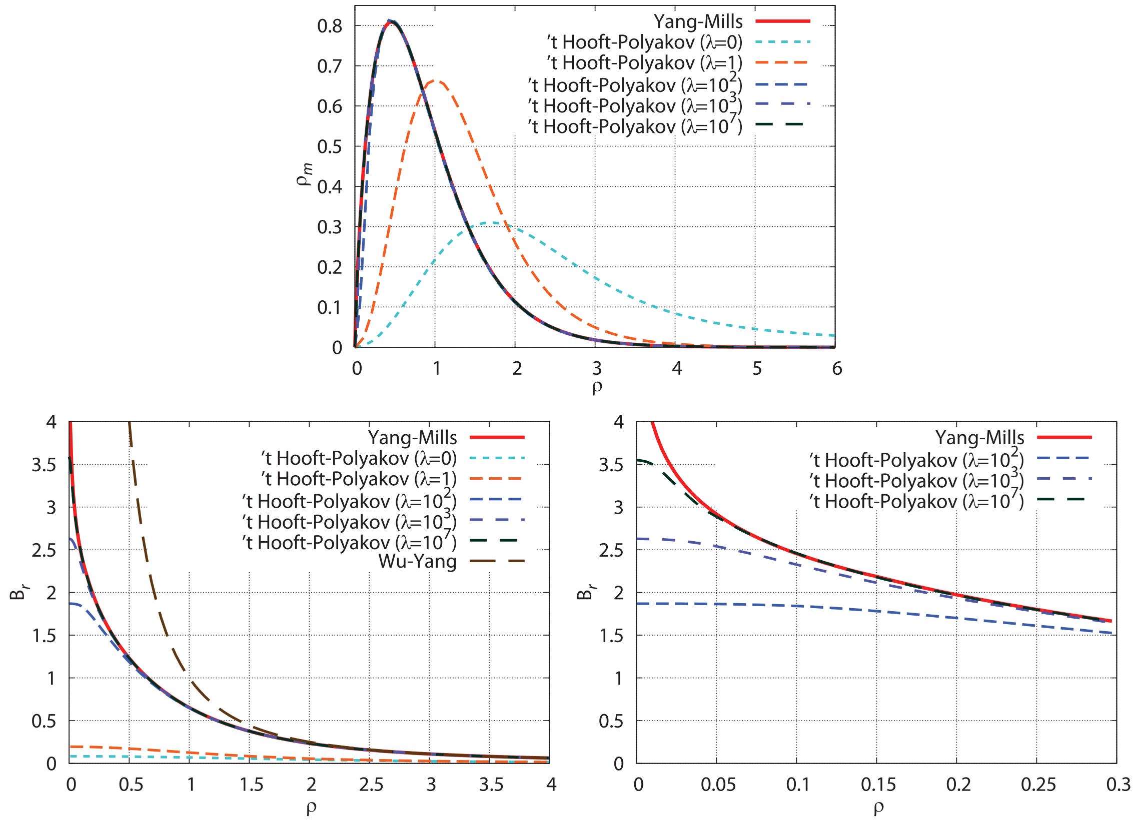

Fig. 6. The short-distance behavior of (top) the magnetic charge density ρ_m, (bottom left) the gauge-invariant chromo-magnetic field $B_r := \mathscr{B}'_r \cdot \hat{\boldsymbol{\phi}}_\infty$ as functions of $\rho = M_{\mathscr{X}} r$. (Bottom right) An enlarged figure around the origin $\rho \approx 0$ of the chromo-magnetic field of the Yang–Mills monopole to be compared with the 't Hooft–Polyakov monopole at large values of λ; e.g., $\lambda = 10^2, 10^3$, and 10^7. For the Wu–Yang monopole, the magnetic charge density is proportional to the delta function: $\rho_m \propto \delta(x)$. Here, the magnetic field of the 't Hooft–Polyakov monopole is finite at the origin, while the magnetic field of the Yang–Mills monopole is divergent logarithmically at the origin.

Then, $\mathscr{B}(x)$ is transformed by $U(x)$ in Eq. (90), $\mathscr{B}(x) \to \mathscr{B}'(x) = U(x)\mathscr{B}(x)U^{-1}(x)$:

$$g\mathscr{B}'_r(x) = \frac{1 - f^2(M_{\mathscr{X}} r)}{r^2} T_3, \quad g\mathscr{B}'_\theta(x) = -\frac{1}{r}\frac{df(M_{\mathscr{X}} r)}{dr} T_+, \quad g\mathscr{B}'_\varphi(x) = \frac{1}{r}\frac{df(M_{\mathscr{X}} r)}{dr} T_-. \tag{101}$$

For $\mathscr{B}(x)$ to be gauge invariant, we take the inner product $\mathscr{B}' \cdot \hat{\boldsymbol{\phi}}_\infty$:

$$g\mathscr{B}'_r(x) \cdot \hat{\boldsymbol{\phi}}_\infty = \frac{1 - f^2(M_{\mathscr{X}} r)}{r^2}, \quad g\mathscr{B}'_\theta(x) \cdot \hat{\boldsymbol{\phi}}_\infty = g\mathscr{B}'_\varphi(x) \cdot \hat{\boldsymbol{\phi}}_\infty = 0. \tag{102}$$

We find that in the radially fixed case of Yang–Mills theory, the chromo-magnetic field diverges at the origin due to its logarithmic behavior:

$$g\mathscr{B}'_r(x) \cdot \hat{\boldsymbol{\phi}}_\infty = \frac{1 - f^2(M_{\mathscr{X}} r)}{r^2} = M_{\mathscr{X}}^2 \left[-\frac{2}{3} \log(M_{\mathscr{X}} r) + (\text{finite terms}) \right]. \tag{103}$$

See the panels in the second line of Fig. 6. This magnetic field $\mathscr{B}'_r(x) \cdot \hat{\boldsymbol{\phi}}_\infty$ should be compared with that of the 't Hooft–Polyakov monopole, which has a finite value even at the origin.

It should be noticed that we have included the volume element $4\pi r^2$ in the definition of the energy and magnetic charge densities because we started from the Lagrangian including $4\pi r^2$. By excluding the factor r^2 from the energy and magnetic charge densities, in the radially fixed case, they diverge at the origin $r = 0$ due to the logarithmic term $\log(M_{\mathscr{X}} r)$. This divergence, however, is not essential for the calculation of the physical quantities, since, for instance, in order to evaluate the magnetic charge q_m we need the volume element $4\pi r^2$, which makes the divergence disappear.

7. Conclusion and discussion

In this paper we have constructed the magnetic monopole *configuration* in the $SU(2)$ Yang–Mills theory even in the absence of the scalar field by incorporating a gauge-invariant mass term. Such a gauge-invariant mass term is obtained through a gauge-independent description of the BEH mechanism proposed in Ref. [22]. The procedure for obtaining the relevant magnetic monopole is guided by the "complementarity" between the $SU(2)$ gauge–adjoint scalar model with the radial-fixing constraint and the massive $SU(2)$ Yang–Mills theory [22]. In fact, we have obtained the static and spherically symmetric magnetic monopole configuration in the $SU(2)$ massive Yang–Mills theory by solving the field equations of the "complementary" $SU(2)$ gauge–adjoint scalar model with a radially fixed scalar field. We have found that the static energy or the rest mass of the obtained Yang–Mills magnetic monopole is finite and proportional to the mass $M_{\mathscr{X}}$ of the massive components $\mathscr{X}$ of the Yang–Mills gauge field $\mathscr{A}$.

In the long-distance region, we observed that the Yang–Mills magnetic monopole configuration $\mathscr{A}$ reduces to the restricted field $\mathscr{V}$, which agrees with the Wu–Yang magnetic monopole as a consequence of the suppression of the massive modes $\mathscr{X}$ in the long-distance region. This feature is similar to the usual 't Hooft–Polyakov monopoles. In the short-distance region, on the other hand, the Wu–Yang magnetic monopole becomes singular, while the 't Hooft–Polyakov monopole remains non-singular even at the origin. In the Yang–Mills magnetic monopole, we found that the massive components $\mathscr{X}$ play the very important role of canceling the singularity of $\mathscr{V}$ in the short-distance region such that the original gauge field $\mathscr{A}$ remains non-singular at the origin. This regularity of the Yang–Mills magnetic monopole is guaranteed by the logarithmic behavior of the gauge field itself without the aid of the scalar field, which vanishes at the origin as seen in 't Hooft–Polyakov monopoles. This behavior renders the energy of the Yang–Mills magnetic monopole finite even if the magnitude of the scalar field is fixed. It should be remarked that the chromo-magnetic field $\mathscr{B}$ is divergent at the origin due to the logarithmic behavior of the solution $f(\rho)$, which is, however, unessential for obtaining finite physical quantities such as energy, magnetic charge density, and magnetic flux.

By using the Yang–Mills magnetic monopole found in this paper, we can show quark confinement in the 3D Yang–Mills theory in the same way as the 3D Georgi–Glashow model shown by Polyakov [57] without introducing the artificially regularized Yang–Mills magnetic monopole [59–61] for avoiding the short-distance singularity and instability of the Wu–Yang magnetic monopole.

Acknowledgements

The authors would like to express their sincere thanks to the referee for giving valuable comments to revise the manuscript. S.N. would like to thank Nakamura Sekizen-kai for a scholarship. R.M. was supported by a Grant-in-Aid for Japan Society for the Promotion of Science (JSPS) Research Fellow Grant Number 17J04780. M.W. was supported by the Ministry of Education, Culture, Sports, Science and Technology, Japan (MEXT scholarship). K.-I.K. was supported by a Grant-in-Aid for Scientific Research, JSPS KAKENHI Grant Number (C) No. 15K05042.

Funding

Open Access funding: SCOAP[3].

Appendix. A numerical treatment

We have used the integral equation method [62] to solve the monopole equation. We can rewrite the monopole equation (62) as

$$\mathcal{L}_\rho f(\rho) = f^3(\rho) - f(\rho) + \rho^2 f(\rho) + 2\rho f'(\rho) - (\rho^2 + \nu(\nu+1))f(\rho), \tag{A.1}$$

where we have introduced the differential operator $\mathcal{L}_\rho$ defined by

$$\mathcal{L}_\rho := \rho^2 \frac{d^2}{d\rho^2} + 2\rho \frac{d}{d\rho} - (\rho^2 + \nu(\nu+1)). \tag{A.2}$$

By using the Green's function $G(\rho, s)$ for the differential operator $\mathcal{L}_\rho$, i.e., $\mathcal{L}_\rho G(\rho, s) = -\delta(\rho - s)$, which is given using the modified spherical Bessel functions as

$$G(\rho, s) = \begin{cases} k_\nu(\rho) i_\nu(s) & (s < \rho) \\ k_\nu(s) i_\nu(\rho) & (\rho > s) \end{cases}, \tag{A.3}$$

Eq. (62) can be rewritten into the integral equation

$$f(\rho) = \int_0^\infty ds\; G(\rho, s) \mathfrak{F}(f(s), f'(s), s), \tag{A.4}$$

where $\mathfrak{F}$ stands for the inhomogeneous terms

$$\mathfrak{F}(f(s), f'(s), s) = f^3(s) - f(s) + s^2 f(s) + 2s f'(s) - (s^2 + \nu(\nu+1))f(s). \tag{A.5}$$

Here we have chosen the modified spherical Bessel functions to accelerate the convergence of the numerical calculations.

According to Ref. [62], we solve the integral equation (A.4) by numerical iterations:

$$f^{(n)}(\rho) = f^{(0)}(\rho) + \int_0^\infty ds\; G(\rho, s) \left\{ \mathfrak{F}(f(s), f'(s), s) + \mathcal{L}_s f^{(0)}(s) \right\}, \tag{A.6}$$

where $f^{(0)}(\rho)$ is a trial function for numerical calculations and $f^{(n)}(\rho)$ is the appropriate solution obtained at n iterations. We have chosen $f^{(0)}(\rho)$ to satisfy the boundary conditions of the original $f(\rho)$:

$$f^{(0)}(\rho) = \operatorname{sech} \rho. \tag{A.7}$$

After $n = 30$ iterations, we find that $f^{(n)}(\rho)$ converges to a reliable result. For the 't Hooft–Polyakov monopole in the Georgi–Glashow model, we can perform the same procedure as that given in Ref. [62]. The result of our numerical calculations is consistent with the 't Hooft–Polyakov monopole [63–65] for the infinite coupling $\lambda = \infty$ in the Georgi–Glashow model.

References

[1] Y. Nambu, Phys. Rev. D **10**, 4262 (1974).
[2] G. 't Hooft, in *High Energy Physics*, ed. A. Zichichi (Editorice Compositori, Bologna, 1975).
[3] S. Mandelstam, Phys. Rep. **23**, 245 (1976).
[4] G. 't Hooft, Nucl. Phy. B **79**, 276 (1974).
[5] A. M. Polyakov, Sov. Phys. JETP **41**, 988 (1975).
[6] A. M. Polyakov, Phys. Lett. B **59**, 82 (1975).
[7] A. M. Polyakov, JETP Lett. **20**, 194 (1974).
[8] Ya. Shnir, *Magnetic Monopoles* (Springer, Berlin, 2005).
[9] P. Goddard and D. I. Olive, Rep. Prog. Phys. **41**, 1357 (1978).
[10] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, UK, 2004).
[11] E. J. Weinberg, *Classical Solutions in Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2012).
[12] R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1987).
[13] A. Jaffe and E. Witten, *Yang–Mills existence and mass gap: Millennium Prize problem* (Clay Mathematics Institute) (available at: http://www.claymath.org/, date last accessed June 1, 2018).
[14] Ph. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Péne, and J. Rodrìguez-Quintero, J. High Energy Phys. **0806**, 012 (2008).
[15] Ph. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Péne, and J. Rodrìguez-Quintero, J. High Energy Phys. **0806**, 099 (2008) [arXiv:0803.2161 [hep-ph]] [Search INSPIRE] .
[16] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **78**, 025010 (2008) [arXiv:0802.1870 [hep-ph]] [Search INSPIRE].
[17] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, PoS **LAT2007**, 290 (2007) [arXiv:0710.1968 [hep-lat]] [Search INSPIRE].
[18] A. Cucchieri and T. Mendes, PoS **LAT2007**, 297 (2007) [arXiv:0710.0412 [hep-lat]] [Search INSPIRE].
[19] A. Sternbeck, L. von Smekal, D. B. Leinweber, and A. G. Williams, PoS **LAT2007**, 340 (2007) [arXiv:0710.1982 [hep-lat]] [Search INSPIRE].
[20] C. S. Fischer, A. Maas, and J. M. Pawlowski, Ann. Phys. **324**, 2408 (2009) [arXiv:0810.1987 [hep-ph]] [Search INSPIRE].
[21] J. Braun, H. Gies, and J. M. Pawlowski, Phys. Lett. B **684**, 262 (2010) [arXiv:0708.2413 [hep-th]] [Search INSPIRE].
[22] K.-I. Kondo, Phys. Lett. B **762**, 219 (2016) [arXiv:1606.06194 [hep-th]] [Search INSPIRE].
[23] K.-I. Kondo, Eur. Phys. J. C **78**, 577 (2018) [arXiv:1804.03279 [hep-th]] [Search INSPIRE].
[24] Y. S. Duan and M. L. Ge, Sci. Sin. **11**, 1072 (1979).
[25] Y. M. Cho, Phys. Rev. D **21**, 1080 (1980).
[26] Y. M. Cho, Phys. Rev. D **23**, 2415 (1981).
[27] Y. M. Cho, Phys. Rev. Lett. **44**, 1115 (1980).
[28] Y. M. Cho, and D. G. Pak, Phys. Rev. D **65**, 074027 (2002) [arXiv:hep-th/0301013] [Search INSPIRE].
[29] W. S. Bae, Y. M. Cho, and S. W. Kimm, Phys. Rev. D **65**, 025005 (2002).
[30] L. Faddeev and A. J. Niemi, Phys. Rev. Lett. **82**, 1624 (1999).
[31] L. Faddeev and A. J. Niemi, Phys. Lett. B **449**, 214 (1999).
[32] L. Faddeev and A. J. Niemi, Phys. Rev. B **464**, 90 (1999).
[33] T. A. Bolokhov and L. D. Faddeev, Theor. Math. Phys. **139**, 679 (2004).
[34] L. Faddeev and A. J. Niemi, Nucl. Phys. B **776**, 38 (2007).
[35] S. V. Shabanov, Phys. Lett. B **458**, 322 (1999).
[36] S. V. Shabanov, Phys. Lett. B **463**, 263 (1999).
[37] K.-I. Kondo, T. Murakami, and T. Shinohara, Eur. Phys. J. C **42**, 475 (2005).
[38] K.-I. Kondo, T. Shinohara, and T. Murakami, Prog. Theor. Phys. **120**, 1 (2008).
[39] K.-I. Kondo, T. Murakami, and T. Shinohara, Prog. Theor. Phys. **115**, 201 (2006).
[40] K.-I. Kondo, S. Kato, A. Shibata, and T. Shinohara, Phys. Rep. **579**, 1 (2015) [arXiv:1409.1599 [hep-th]] [Search INSPIRE].
[41] K. Osterwalder and E. Seiler, Ann. Phys. **110**, 440 (1978).
[42] E. Seiler, Lect. Notes Phys. **159**, 1 (1982).
[43] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979).

[44] T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).
[45] K.-I. Kondo, N. Fukui, A. Shibata, and T. Shinohara, Phys. Rev. D **78**, 065033 (2008) [arXiv:0806.3913 [hep-th]] [Search INSPIRE].
[46] N. Fukui, K.-I. Kondo, A. Shibata, and T. Shinohara, Phys. Rev. D **82**, 045015 (2010) [arXiv:1005.3157 [hep-th]] [Search INSPIRE].
[47] N. Fukui, K.-I. Kondo, A. Shibata, and T. Shinohara, Phys. Rev. D **86**, 065020 (2012) [arXiv:1205.4972 [hep-th]] [Search INSPIRE].
[48] G. H. Derrick, J. Math. Phys. **5**, 1252 (2004).
[49] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976).
[50] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
[51] A. Shibata, S. Kato, K.-I. Kondo, T. Murakami, T. Shinohara, and S. Ito, Phys. Lett. B **653**, 101 (2007) [arXiv:0706.2529 [hep-lat]] [Search INSPIRE].
[52] J. C. R. Bloch, Few Body Syst. **33**, 111 (2003).
[53] J. C. R. Bloch, A. Cucchieri, K. Langfeld, and T. Mendes, Nucl. Phys. B **687**, 76 (2004) [arXiv:hep-lat/0312036] [Search INSPIRE].
[54] C. P. Dokos and T. N. Tomaras, Phys. Rev. D **21**, 2940 (1980).
[55] Y. M. Cho and D. Maison, Phys. Lett. B **391**, 360 (1997) [arXiv:hep-th/9601028] [Search INSPIRE].
[56] K. Kimm, J. H. Yoon, and Y. M. Cho, Eur. Phys. J. C **75**, 67 (2015) [arXiv:1305.1699 [hep-ph]] [Search INSPIRE].
[57] A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
[58] P. A. M. Dirac, Proc. R. Soc. Lond. A **133**, 60 (1931).
[59] S. R. Das and S. R. Wadia, Phys. Rev. D **53**, 5856 (1996).
[60] T. Yoneya, Phys. Rev. D **16**, 2567 (1977).
[61] T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B **129**, 493 (1977).
[62] F. A. Bais and J. R. Primack, Phys. Rev. D **13**, 819 (1976).
[63] P. Vinciarelli, CERN-TH-2246 (1976).
[64] W. Troost and P. Vinciarelli, CERN-TH-2195 (1976).
[65] T. W. Kirkman and C. K. Zachos, Phys. Rev. D **24**, 999 (1981).