



Entanglement entropy at higher orders for the states of $a = 3 \theta = 1$ Lifshitz theory

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Abstract

We evaluate the entanglement entropy of strips for boosted D3-black-branes compactified along the light-cone coordinate. The bulk theory describes 3-dimensional $a = 3 \theta = 1$ Lifshitz theory on the boundary. The area of small strips is evaluated perturbatively up to second order, where the leading term has a logarithmic dependence on strip width l , whereas entropy of the excitations is found to be proportional to l^4 . The entanglement temperature falls off as $1/l^3$ on expected lines. The size of the subsystem has to be bigger than typical ‘Lifshitz scale’ in the theory. At second order, the redefinition of temperature (or strip width) is required so as to meaningfully describe the entropy corrections in the form of a first law of entanglement thermodynamics.

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1. Introduction

The AdS/CFT correspondence [1] has remained a central idea for holographic studies in string theory. The holography relates conformal field theory living on the boundary of anti-de Sitter spacetime with the gravity theory within the bulk. Along these lines finding the entanglement entropy of strongly coupled quantum systems at criticality has also been a focus of several studies [2,4]. In these calculations entanglement entropy can be obtained [2,3] by estimating the area of

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codimension two surfaces embedded inside the bulk geometry. The boundary of such extremal surfaces coincides with the boundary of the subsystem in the CFT. Recently it has been observed that the excitations in the CFT follow entanglement laws similar to the black hole thermodynamic laws [5–7]; see also [9], [12], [10], [13]. It is understood now that the entanglement entropy (S_E) and the energy of small excitations (E) in AdS spacetime obey a definite relation

$$\Delta E = T_E \Delta S_E + \mathcal{V} \Delta \mathcal{P} + \mu_E \Delta N$$

This equation is described as the first law of entanglement thermodynamics. The charge contributions can simply arise for a boosted black-brane vacua [7], where the charged excitations could be either Kaluza–Klein (KK) momentum modes along a compactified brane direction or the dual winding modes of a string.

The backgrounds of our interest here are the nonrelativistic Lifshitz spacetimes. We would like to holographically study these solutions and check if similar entanglement law could be written for them. Typically a Lifshitz like geometry [14] has a line element

$$ds^2 = -\frac{dt^2}{z^{2a}} + \frac{dx_1^2 + \cdots + dx_D^2}{z^2} + \frac{dz^2}{z^2}$$

as subspace where time and space scale asymmetrically ($z \rightarrow \lambda z$, $t \rightarrow \lambda^a t$, $x_i \rightarrow \lambda x_i$). The Lorentz symmetry is explicitly broken. The parameter a is called the dynamical exponent of time and D is total spatial dimensions. Whereas a hyperscaling violating Lifshitz (hvLif) geometry

$$ds^2 = z^\theta \left(-\frac{dt^2}{z^{2a}} + \frac{dx_1^2 + \cdots + dx_D^2}{z^2} + \frac{dz^2}{z^2} \right)$$

has an overall conformal factor [4]. So an overall scaling $g_{\mu\nu} \rightarrow \lambda^\theta g_{\mu\nu}$ of the metric is involved. As an unique example in ten dimensions, especially $Lif_4^{a=2} \times S^1 \times S^5$ vacua are recently constructed in [15], as solutions of 10-dimensional massive type IIA supergravity theory [16]. These are understood to describe strongly coupled Lifshitz $a = 2$ theory in three spacetime dimensions at the fixed point. In these bulk solutions ‘massive’ strings are tied up with D2–D8 parallel brane system that exhibit scaling symmetry. These vacua can be related via massive/generalized T-duality [17] to D3–D7 axion ‘flux’ vacua [18]. In the ordinary type IIA/B string theory and M-theory, the boosted black brane solutions compactified along a lightcone direction, can also give rise to Lifshitz solutions [19,21]. The latter class of brane solutions all have conformal scaling (or hyperscaling) properties as listed in [21], see [22] for discussion of $\theta = 1$ case. There are other instances also in gauged supergravities where Lifshitz vacua can be obtained, see for example [25].

In this work we shall only study boosted black D3-brane system in lightcone coordinates, with one lightcone coordinate compactified on a circle [28]. These compactified bulk solutions describe a thermal state of 3-dimensional $Lif^{a=3}$ theory with hyperscaling parameter $\theta = 1$. The corresponding ground state (at zero temperature) is described by ten-dimensional solutions discussed in [19]. These zero temperature solutions were obtained by performing a double scaling limit in which horizon size is taken to vanishing value associated with an infinite boost. The fact that there is hyperscaling violation in these Lifshitz solutions (when explicit lightcone compactification is performed) was pointed out subsequently in the work [22]. Actually these hvLif theories fall in a special category where $\theta = D - 1$ ($D = 2$ is number of spatial dimensions). Precisely for these hvLif states the entanglement entropy has logarithmic violation [4,23,24]. We shall be encountering some of these results in our work as we progress. Both the solutions (finite

temperature and the zero temperature one) allow us to embed codimension-2 strip like surfaces (at constant lightcone time) inside the bulk. We evaluate the area of small strips using perturbative method up to second order, by using the procedure introduced in [7]. We find that for small strips (but bigger than some critical size) the leading term in the entanglement entropy has indeed logarithmic dependence on strip width l . Whereas the subleading term which accounts for the entropy of excitations goes as l^4 . The entanglement temperature falls off as $\frac{1}{l^3}$. These results are on expected lines. Quite importantly, the (KK) charge contribution in the first law is present at the first order itself, unlike in the relativistic cases studied in [7] where the contribution of the charges appears only at the second order. At the second order, once again we find that the first law relation requires the entanglement temperature (and strip width) to be suitably corrected or renormalized.

The paper is presented as follows. In section-2 we write down the $Lif^{a=3, \theta=1}$ hyperscaling solutions of our interest both with black hole excitations and the zero temperature counterpart. In section-3 we evaluate the entanglement entropy at first order and present the form of first law. In section-4 we obtain second order corrections and rewrite the new form of first law and determine the corrections to associated thermodynamic quantities. We find that the strip width (so also subsystem volume) has to be renormalized and redefined. The final summary is presented in section-5.

2. Entanglement entropy for $a = 3 \theta = 1$ Lifshitz system

It has been known that the boosted $AdS_5 \times S^5$ black hole background compactified along a light-cone coordinate can describe excitations of $hvLif^{a=3}$ system [19,21]. These black hole solutions were first explored for their non-relativistic properties in [28]. These type IIB string vacua can be written as

$$ds^2 = L^2 \left(-\frac{z_l^4 f}{z^6} (dx^+)^2 + \frac{z^2}{4z_l^4} (dx^- - \omega)^2 + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{f z^2} \right) + L^2 d\Omega_3^2, \tag{1}$$

supported by constant dilaton and a self-dual 5-form field strength. The function f is

$$f(z) = 1 - \frac{z^4}{z_0^4}, \tag{2}$$

where $z = z_0$ is the black hole horizon. It will be assumed that x^- is compactified on a circle of radius r^- . The fiber 1-form is given by

$$\omega = \frac{z_l^4}{z^4} \left(2 - \frac{z^4}{z_0^4} \right) dx^+ \tag{3}$$

The radius of curvature L is taken very large in string units ($\alpha' = 1$) so that the stringy excitations are suppressed. The parameter z_l is an intermediate (free) UV scale, rather we shall suitably call it as ‘Lifshitz scale’ in the theory. We take a wide parameter range such that $z_0 \gg z_l$. This is so because we wish to study small excitations only. (Also let us note that at any stage the Lifshitz scale z_l can be related to z_0 through the boost of lightcone coordinates, i.e. one can write $z_l^2 = z_0^2/\lambda$, with $\lambda \geq 1$ being the lightcone boost parameter.)

Further we shall take the $a = 3 \theta = 1$ Lifshitz solutions [19] as the ground state. Let us explain it here. Recalling [19], one can take simultaneous double limits $\lambda \rightarrow \infty$, $z_0 \rightarrow \infty$, while keeping the ratio $\frac{\lambda}{z_0^2} = \frac{1}{z_l^2}$ (say) fixed. These limits take us to (hyperscaling) Lifshitz $a = 3$ vacua, namely

$$\begin{aligned}
 ds_{Lif}^2 &= L^2 \left(-\frac{z_l^4(dx^+)^2}{z^6} + \frac{z^2}{4z_l^4}(dx^- - \frac{2z_l^4}{z^4}dx^+)^2 + \frac{dx_1^2 + dx_2^2 + dz^2}{z^2} + d\Omega_5^2 \right) \\
 &\equiv L^2 \left(-\frac{dx^+dx^-}{z^2} + \frac{z^2}{4z_l^4}(dx^-)^2 + \frac{dx_1^2 + dx_2^2 + dz^2}{z^2} + d\Omega_5^2 \right) \tag{4}
 \end{aligned}$$

This zero temperature background is characterized by the scale z_l , which also defines the charge (number) density of the states of this system at zero temperature [19]. It only suggests that $a = 3$ hyperscaling Lifshitz ground state system exists for any given z_l . The z_l is treated as Lifshitz (intermediate) scale in the black hole solution (1), when we switch on the temperature. That is we are interested in the excitations around the hyperscaling Lifshitz vacua (4).

The entanglement entropy has also been studied for these BH systems in the work [26]. It was pointed out there that due to the boost there is an asymmetry in the entanglement along various directions of the boundary theory. This asymmetry should show up in the entanglement entropy calculations and the first law as well. Up to first order in perturbative expansion (for small subsystem) it has been explicitly shown that it indeed is the case [8]. The observation was that the entanglement asymmetry is entirely due to pressure asymmetry in the theory. But in the current paper we shall be studying only those strip systems which lie in the transverse to the boost direction, it is because the solutions are compactified along the boosted lightcone coordinate, so that we can view it as a hyperscaling Lifshitz theory. The entanglement entropy along the boost direction can also be studied at higher orders, but this will require some careful considerations, because the constant time slices will not be existing. Technically one has to resort to a covariant slice analysis [3]. Up to first order the calculations are all easy and there are no such hurdles. But beyond first order we have to only consider covariant approach such as [3]. We leave it for a separate investigation.

2.1. Small strip systems

The entanglement entropy for a subsystem on the boundary of the background (1) can be studied by using Ryu–Takayanagi proposal [2]. Here x_1 and x_2 are two flat directions along the brane, while spatial lightcone coordinate x^- is compactified. We choose a strip along x_1 direction with an interval $-l/2 \leq x_1 \leq l/2$. We wish to embed co-dimension two strip (a constant x^+ surface) inside the bulk geometry. The two straight boundaries of the extremal strip surface coincide with the two ends of the interval $\Delta x_1 = l$. The size of other coordinates are taken as; $x^- \simeq x^- + 2\pi r^-, 0 \leq x_2 \leq l_2$, where l_2 is taken very large, $l_2 \gg l$.

Following Ryu–Takayanagi prescription the entanglement entropy of a strip subsystem is given in terms of the geometrical area of co-dimension two surface (with light-cone time x^+ taken constant everywhere on the surface). We thus get

$$\begin{aligned}
 S_E &\equiv \frac{\mathcal{A}_{Strip}}{4G_5} \\
 &= \frac{L^3 \pi r_- l_2}{2G_5 z_l^2} \int_{\epsilon}^{z^*} dz \frac{1}{z\sqrt{f}} \sqrt{1 + f(\partial_z x^1)^2} \tag{5}
 \end{aligned}$$

where G_5 is 5-dimensional Newton’s constant. Here $\epsilon \sim 0$, is the cut-off scale in UV. (We need to pay special attention to $z = z_l$ scale in the theory. The bulk geometry (1) is not well defined beyond $z = z_l$, as the size of the x^- circle becomes sub-stringy in $z < z_l$ near boundary region.

A way to overcome this problem is that beyond $z = z_l$ one can switch over to T-dual type-IIA background, where the circle size instead will increase. Doing this however does not affect the entropy functional given in (5). Hence so far as the area functional is concerned it appears immune to $z = z_l$. Nevertheless z_l is an important scale in the Lifshitz theory and we can add suitable counter terms as we shall discuss next.) The z_* is the turning point of the strip. Next the area functional is extremized through the equation of motion

$$\frac{dx_1}{dz} = \frac{z}{z_*\sqrt{f}} \frac{1}{\sqrt{1 - (\frac{z}{z_*})^2}} \tag{6}$$

It implies that the boundary value $x^1(0) = l/2$ is given by the following integral relation

$$\frac{l}{2} = \int_0^{z_*} dz \frac{z}{z_*\sqrt{f}} \frac{1}{\sqrt{1 - (\frac{z}{z_*})^2}} \tag{7}$$

which relates the width l with the turning point z_* . The turning-point of the strip lies at the mid-point $x^1(z_*) = 0$ of the boundary interval due to symmetry. The entropy for extremal strip system can now be described as

$$S_E = \frac{L^3\pi r^{-l_2}}{2G_5z_l^2} \int_{\epsilon}^{z_*} \frac{dz}{z} \frac{1}{\sqrt{f}} \frac{1}{\sqrt{1 - (\frac{z}{z_*})^2}} \tag{8}$$

The Lifshitz scale z_l is an important fixed parameter in these vacua, but it only appears as a constant multiplier outside the integrand.

2.2. A hierarchy of scales and perturbative expansion

When strip width l is small the turning point generically lies in the proximity of asymptotic region. Therefore one can safely assume $z_* \ll z_0$. However, our main focus here will be on those subsystems (or critical surfaces) for which following hierarchy of scales is obeyed

$$z_l < z_* \ll z_0 . \tag{9}$$

This will specially require us to take z_l (UV) and z_0 (IR) to be widely separated scales. This $[z_0, z_l]$ interval is known as the ‘Lifshitz window’ region in [21,20]. A large Lifshitz window is desirable here for perturbative expansion to work out properly, as we are seeking to evaluate the entanglement entropy (8) by expanding it around a zero temperature vacua (4) (i.e. treating $a = 3\theta = 1$ Lifshitz vacua [19] as the ground state). Under these conditions we can estimate area entropy perturbatively by expanding the integrand around its central value.

We first proceed to obtain the perturbative expansion of the l -integral (7) up to first order, assuming $\frac{z_*^4}{z_0^4} \ll 1$,

$$\begin{aligned} l &= 2z_* \int_0^1 d\xi \frac{\xi}{\sqrt{1 - \xi^2}} \left(1 + \frac{z_*^4}{2z_0^4} \xi^4 + \dots\right) \\ &= 2z_* b_0 + \frac{z_*^5}{z_0^4} b_1 + \dots \end{aligned} \tag{10}$$

where for simplicity we introduced $\xi \equiv \frac{z}{z_*}$ and $R \equiv (1 - \xi^2)$.¹ The ellipses stand for second and higher order terms which we neglect in this section. By inverting the above series we get a turning point expansion

$$z_* = \bar{z}_* \left(1 - \frac{\bar{z}_*^4}{z_0^4} \frac{b_1}{2b_0} \right) + \dots \tag{11}$$

where $\bar{z}_* \equiv \frac{l}{2b_0}$ is the turning point for pure hvLif ground state (4) (i.e. in the absence of excitations or black holes). This relationship is an important first step before we proceed to the area calculation.

Next we consider the area of the strip (8). We evaluate the integral quantity (which is independent of z_l)

$$A \equiv \frac{2}{z_l^2} \int_{\epsilon}^{z_*} \frac{dz}{z} \frac{1}{\sqrt{f} \sqrt{1 - (\frac{z}{z_*})^2}} \tag{12}$$

by expanding the integrand perturbatively as

$$A \equiv \frac{2}{z_l^2} \left(\int_{\epsilon/z_*}^1 \frac{d\xi}{\xi} \frac{1}{\sqrt{R}} + \frac{z_*^4}{2z_0^4} \int_{\epsilon/z_*}^1 d\xi \frac{\xi^3}{\sqrt{R}} + \dots \right). \tag{13}$$

The contribution of first term is singular when $\epsilon \rightarrow 0$ (near the boundary). Also as mentioned before, going beyond $z = z_l$, the x^- circle in (1) becomes sub-stringy, so near boundary region $z < z_l$ needs to be carefully considered. We thus note that, in the corresponding dual geometry the size of T-dual x^- circle will anyway expand for $z < z_l$. While the functional form of integral in eq. (12) remains unchanged under this duality. Thus there appears to be no pathological problem in the near boundary region $0 < z < z_l$. Nevertheless, to be on the safe side we subtract the following contribution (as a counter term)

$$A_{CT} = \frac{2}{z_l^2} \int_{\epsilon}^{z_l} \frac{dz}{z} = \frac{2}{z_l^2} \ln \frac{z_l}{\epsilon} \tag{14}$$

from the area integral A given above. This precisely amounts to subtracting the contribution of two disconnected (no turning point) strips hanging between $z = z_l$ and the $z = \epsilon$ inside the hvLif geometry (4). Note that A_{CT} has no dependence on z_0 , which is a parameter controlling the excitations. So it is totally a harmless subtraction from point of view of the excitations (our goal is to know the entropy of the excitations and it will not be affected). So we extract the finite area contribution as

¹ The value of expansion coefficients b_0, b_1, b_2 can be evaluated, $b_0 = \int_0^1 d\xi \frac{\xi}{\sqrt{R}} = \frac{1}{2} B(1, 1/2) = 1$, $b_1 = \int_0^1 d\xi \frac{\xi^5}{\sqrt{R}} = \frac{1}{2} B(3, 1/2) = \frac{8}{15}$, $b_2 = \int_0^1 d\xi \frac{\xi^9}{\sqrt{R}} = \frac{1}{2} B(5, 1/2)$, where $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ are the Beta-functions.

$$\begin{aligned}
 A_{finite} &= A - A_{CT} = \frac{2}{z_l^2} \left(\int_{\epsilon/z_*}^1 \frac{d\xi}{\xi} \frac{1}{\sqrt{R}} + \frac{z_*^4}{2z_0^4} \int_{\epsilon/z_*}^1 d\xi \frac{\xi^3}{\sqrt{R}} + \dots \right) - \frac{2}{z_l^2} \int_{\epsilon/z_*}^{z_l/z_*} \frac{d\xi}{\xi} \\
 &= \frac{2}{z_l^2} \ln \frac{2z_*}{z_l} + \frac{1}{z_l^2} \frac{z_*^4}{z_0^4} \int_0^1 d\xi \frac{\xi^3}{\sqrt{R}} + \dots
 \end{aligned}
 \tag{15}$$

and the limit $\epsilon \rightarrow 0$ is understood to have been implemented in the second equality. In the next step by substituting the expansion of z_* in the eq. (15), we get up to first order

$$\begin{aligned}
 A_{finite} &= \frac{2}{z_l^2} \ln \frac{2\bar{z}_*}{z_l} - \frac{2}{z_l^2} \frac{\bar{z}_*^4}{z_0^4} \frac{b_1}{2} + \frac{2}{z_l^2} \frac{\bar{z}_*^4}{z_0^4} \frac{a_1}{2} \\
 &= \frac{2}{z_l^2} \ln \frac{l}{z_l} + \frac{1}{z_l^2} \frac{\bar{z}_*^4}{z_0^4} (a_1 - b_1) \\
 &= A_0 + \frac{1}{z_l^2} \frac{\bar{z}_*^4}{z_0^4} \frac{a_1}{5}
 \end{aligned}
 \tag{16}$$

where a_1, b_1, \dots are finite coefficients.² The leading finite term is simply given by

$$A_0 = \frac{1}{z_l^2} \ln \frac{l^2}{z_l^2}
 \tag{17}$$

Thus the entanglement entropy for strip can be written as

$$S_E = S_{(0)} + \frac{L^3 \pi r_{-l_2}}{4G_5} \frac{a_1}{5z_l^2} \frac{\bar{z}_*^4}{z_0^4}
 \tag{18}$$

where the leading term is

$$S_{(0)} = \frac{L^3 \pi r_{-l_2}}{4G_5} \frac{1}{z_l^2} \ln \frac{l^2}{z_l^2}
 \tag{19}$$

It is clear that $S_{(0)} > 0$ only when $l > z_l$ and that is why the hierarchy of the scales (9) was adopted. It also does not look like an AdS_5 ground state entropy which instead goes as $-\frac{1}{l^2}$ [2]. Therefore the logarithmic dependence on l ought to be recognized as a contribution of $a = 3, \theta = 1$ Lifshitz ground state [4,23]. This also happens because we have chosen to study $x^+ = constant$ strip subsystems. Had we chosen to evaluate entanglement entropy for usual $x^0 = constant$ (fixed Lorentzian time) strips, we instead would get the leading contribution precisely that for AdS_5 ; see [7] for a second order perturbative calculation in relativistic theory.

The leading logarithmic term depends on z_l (UV scale) and the width $l (> z_l)$, and not on z_0 (the scale describing the excitations). But both of these quantities are fixed for a given subsystem. Thus $S_{(0)}$ is essentially a fixed quantity and it cannot be viewed as part of the excitations. Subtracting the leading term leaves us with the net *vacuum-subtracted* entropy of the excitations around Lifshitz theory as

² The expansion coefficients are $a_1 = \int_0^1 d\xi \frac{\xi^3}{\sqrt{1-\xi^2}} = \frac{1}{2} B(2, 1/2) = \frac{2}{3}, a_2 = \int_0^1 d\xi \frac{\xi^7}{\sqrt{1-\xi^2}} = \frac{1}{2} B(4, 1/2) = \frac{6}{7} b_1$.

$$\Delta S_E^{(1)} = \frac{L^3 \pi r_- l_2}{4G_5} \frac{a_1}{5z_l^2} \frac{l^4}{(2z_0)^4} + \text{higher order corrections} \tag{20}$$

This result is true up to first order in the ratio $\frac{z_*^4}{z_0^4}$. At higher order there will be further corrections on the right hand side to add. It can be immediately observed that the entanglement entropy of excitations is proportional to l^4 and depends on z_0 also, the parameter describing excitations. In contrast, for Lorentz covariant AdS_{d+1} ground state the entropy of excitations rather increases quadratically as l^2 [5].

3. The entanglement first law

The boundary theory is a 3-dimensional Lifshitz theory, since the lightcone direction, namely x^- , is compactified. The excitation energy and the pressure can be obtained by expanding the geometry (1) in Fefferman–Graham coordinates near the boundary [27]. The energy density of the excitations is given by [28]

$$\mathcal{E} = \frac{L^3 r_-}{16G_5} \frac{1}{z_0^4}, \tag{21}$$

whereas the charge (number) density is

$$\rho = \frac{N}{\text{volume}} = \frac{L^3 r_-^2}{8G_5} \frac{1}{z_l^4} \tag{22}$$

The charge density in the Lifshitz theory at zero temperature is usually very large whereas other quantities can be vanishingly small [19]. It is obvious here too as $\rho \propto \frac{1}{z_l^4}$ and given our hierarchy of the scales $z_l < z_* \ll z_0$. The ‘entanglement’ chemical potential, obtained by measuring the value of KK field ω_+ at the turning point, is

$$\begin{aligned} \mu_E &= \frac{1}{r_-} \left(2 \frac{z_l^4}{z_*^4} - \frac{z_l^4}{z_0^4} \right) = \frac{1}{r_-} \left(\frac{2z_l^4}{z_*^4} + \left(\frac{4b_1}{b_0} - 1 \right) \frac{z_l^4}{z_0^4} \right) \\ &= \mu_E^{Lif} + \frac{1}{r_-} \left(\frac{4b_1}{b_0} - 1 \right) \frac{z_l^4}{z_0^4} \end{aligned} \tag{23}$$

where to obtain second equality the turning point expansion (11) has been used. The leading term $\mu_E^{Lif} = \frac{1}{r_-} \frac{2z_l^4}{z_*^4}$ is chemical potential corresponding to the hvLif ground state (4). The sub-leading term is however of *universal* nature, because it is independent of l . Thus the net change in chemical potential due to excitations is

$$\Delta \mu_E^{(1)} = \mu_E - \mu_E^{Lif} \simeq \frac{1}{r_-} \left(\frac{4b_1}{b_0} - 1 \right) \frac{z_l^4}{z_0^4} \tag{24}$$

It is remarkable that, using the quantities defined so far, from (20) we can construct the following first law-like relation

$$\Delta S_E^{(1)} = \frac{1}{T_E} (\Delta E + \frac{1}{2} N \Delta \mu_E^{(1)}) \tag{25}$$

where the net charge contained in the subsystem is simply $N = \rho l_2 l$ and energy of excitations $\Delta E = l_2 l \mathcal{E}$. The entanglement temperature is given by

$$T_E = \frac{2^6 z_l^2}{\pi} \frac{1}{l^3}. \tag{26}$$

Importantly the temperature is inversely proportional to the cubic power of the strip width l . This conveys the fact that the dynamical exponent of time for the Lifshitz theory is indeed *three*, and it collaborates with early work [19].

We add some remarks here. In the first law (25) the charge and chemical potential contribution is present at the first order itself, unlike in the relativistic case where no charge appears at the first order. In the relativistic case the charges appeared only at the second order in perturbation, see [7]. The reason for this major difference may be the fact that the charge density is very high in the Lifshitz theory, i.e.

$$\frac{\rho}{\rho_c} = \frac{z_0^4}{z_l^4} \gg 1,$$

where $\rho_c = \frac{L^3 r_-^2}{8G_5 z_0^2} \frac{1}{z_0}$ is some critical (reference) charge density.

The von Neumann entanglement entropy $S_E = -\text{Tr}_A \sigma_A \ln \sigma_A$ of a quantum subsystem A requires the knowledge of a reduced density matrix (obtained by tracing out the states over the complimentary system),

$$\sigma_A = \frac{e^{-H_A}}{Z} \tag{27}$$

where partition function $Z = \text{Tr}_A e^{-H_A}$. The H_A is the reduced Hamiltonian describing the subsystem. In this approach, at first order we expect that the modular (entanglement) Hamiltonian H_E of the subsystem to be related as, [11,12],

$$\Delta S_E^{(1)} = \frac{1}{T_E} (\Delta E + \frac{1}{2} N \Delta \mu_E) = \langle \Delta H_E^{(1)} \rangle. \tag{28}$$

A variational form of first law The small fluctuations of bulk parameters (z_0, z_l) determine the variations of the thermodynamic quantities of boundary nonrelativistic theory. In the present $h\nu \text{Lif}^{a=3}$ case we are interested in the study of the ensembles with fixed KK charges which can only be done by keeping z_l fixed, so we will only allow z_0 to have a spread. The small variation of chemical potential becomes (at first order)

$$\delta \mu_E = (4b_1 - 1) \frac{z_l^4}{r_-} \delta \left(\frac{1}{z_0^4} \right) \tag{29}$$

as given $b_0 = 1$. One can see that the product $\rho \cdot \delta \mu_E$ is of the same order as $\delta \mathcal{E}$ and thus it will eventually be related to it. Hence the states of the system describe a canonical ensemble and therefore knowing the fluctuations of a single quantity, such as $\delta \mathcal{E}$, is sufficient to describe the state of the system. Under these restrictions (since $\delta S_{(0)} = 0$ as z_l is fixed) we find from eq. (18) the variational form of first law is

$$\delta S_E = \frac{1}{T_E} \delta E' \tag{30}$$

where new energy $E' \equiv E + \frac{1}{2} \mu_E N$ has been defined so that the entanglement temperature is the same as (26). For a comparison with the black hole first law, we wish to recall the thermal first law [28] for boosted BH background, which for fixed charge density ($z_l = \text{fixed}$), gets reduced to

$$\delta S_{th} = \frac{1}{T_{th}} \delta E \tag{31}$$

where

$$T_{th} = \frac{z_l^2}{\pi z_0^3} \tag{32}$$

is thermal temperature. It is worthwhile to note that not only the z_l dependence in entanglement temperature is exactly the same as that in thermal temperature but the dynamical exponent of time also comes out as 3. Usually for smaller subsystems the entanglement temperature is higher than compared to the thermal one (if any). It is appropriate to compare the two in the present Lifshitz case. The ratio comes out to be

$$\frac{T_{th}}{T_E} = \frac{1}{8} \left(\frac{l}{2z_0} \right)^3 \ll 1 \tag{33}$$

Since $l \ll 2z_0$, there will exist a big hierarchy in two temperature scales where the degree is determined by the dynamical exponent of time. Though the ratio remains independent of z_l (or charge density of Lifshitz states), but it crucially depends on the value of dynamical exponent, which obviously enhances this hierarchy.

4. The entropy at second order

We wish to evaluate the area of the lightcone strip up to next higher order and evaluate corresponding entropy corrections. The higher order results provide us with better precision and improved estimate of the entanglement entropy since exact analytical calculations cannot be done. First the expansion of the turning point has to be obtained. We expand the integrand in eq. (7) up to second order in $\frac{z_*^4}{z_0^4} \ll 1$, which gives us the series

$$\begin{aligned} l &= 2z_* \int_0^1 d\xi \frac{\xi}{\sqrt{1-\xi^2}} \left(1 + \frac{z_*^4}{2z_0^4} \xi^4 + \frac{3z_*^8}{8z_0^8} \xi^8 + \text{higher orders} \right) \\ &= 2z_* \left(1 + \frac{z_*^4}{z_0^4} \frac{b_1}{2} + \frac{z_*^8}{z_0^8} \frac{3b_2}{8} \right) + \dots \end{aligned} \tag{34}$$

where the ellipses stand for third and higher order terms. By inverting the above expansion one can obtain

$$z_* = \bar{z}_* \left(1 + \frac{\bar{z}_*^4}{z_0^4} \frac{b_1}{2} + \frac{\bar{z}_*^8}{z_0^8} \left(\frac{3}{8} b_2 - b_1^2 \right) \right)^{-1} \tag{35}$$

where $\bar{z}_* = l/2$ is the turning point value for the ground state. The A expansion up to second order is, keeping the counter term same as in the previous section,

$$\begin{aligned} A - A_{CT} &= \frac{2}{z_l^2} \left(\int_{\epsilon/z_*}^1 \frac{d\xi}{\xi} \frac{1}{\sqrt{R}} + \frac{z_*^4}{2z_0^4} \int_{\epsilon/z_*}^1 d\xi \frac{\xi^3}{\sqrt{R}} + \frac{3z_*^8}{8z_0^8} \int_{\epsilon/z_*}^1 d\xi \frac{\xi^7}{\sqrt{R}} + \dots \right) - \frac{2}{z_l^2} \int_{\epsilon/z_*}^{z_l/z_*} \frac{d\xi}{\xi} \\ &= \frac{2}{z_l^2} \left(\ln \frac{2z_*}{z_l} + \frac{z_*^4}{2z_0^4} \int_0^1 d\xi \frac{\xi^3}{\sqrt{R}} + \frac{3z_*^8}{8z_0^8} \int_0^1 d\xi \frac{\xi^7}{\sqrt{R}} + \dots \right) \end{aligned}$$

$$= \frac{2}{z_l^2} \ln \frac{2z_*}{z_l} + \frac{1}{z_l^2} \frac{z_*^4}{z_0^4} a_1 + \frac{3}{4z_l^2} \frac{z_*^8}{z_0^8} a_2 \tag{36}$$

where $\epsilon \rightarrow 0$ limit has been implemented. The coefficients a_1, a_2 are defined earlier. Substituting the z_* -expansion (35) in the A expansion (36), we finally get

$$A_{finite} = A_0 + A_1 + A_2 \tag{37}$$

where A_0 and A_1 are the leading order and first order terms, respectively. These are the same as obtained in the previous section. The new contribution at second order is

$$A_2 \equiv \frac{1}{4z_l^2} (9b_1^2 - 8a_1b_1 - 3(b_2 - a_2)) \frac{z_*^8}{z_0^8} \tag{38}$$

With this the entanglement entropy calculated up to second order becomes

$$S_E = S_{(0)} + \frac{L^3 \pi r_- l_2}{4G_5} (A_1 + A_2). \tag{39}$$

So overall change up to second order is

$$\Delta S_E^{(2)} = S_E - S_{(0)} = \frac{L^3 \pi r_- l_2}{4G_5} \cdot \frac{a_1 Q}{5z_l^2} \cdot \frac{l^4}{2^4 z_0^4} \tag{40}$$

where Q factor is defined as

$$Q = (1 - \frac{26}{105} \frac{z_*^4}{z_0^4}) < 1 \tag{41}$$

which involves first order term only. It is always smaller than unity. This in fact implies that overall entanglement entropy after the inclusion of second order corrections has indeed decreased,

$$\Delta S_E^{(2)} < \Delta S_E^{(1)}. \tag{42}$$

This is a common observation in many CFTs including the relativistic ones. This calculation ends our perturbative results up to second order. In the next step we would like to see if the second order corrections can be absorbed in the redefinitions of various entanglement quantities like T_E and μ_E .

4.1. Renormalization of thermodynamic observables

As we have seen that entanglement entropy of the strip system gets corrected at higher orders in perturbative calculation. It is reasonable to expect that other thermodynamic variables also receive similar corrections at higher orders. We already saw that the chemical potential μ_E indeed gets corrected. Using second order turning point expansion (35), one can determine

$$\begin{aligned} \mu_E &= \frac{1}{r_-} (2 \frac{z_l^4}{z_*^4} - \frac{z_l^4}{z_0^4}) \\ &\simeq \frac{1}{r_-} \left(\frac{2z_l^4}{z_*^4} + \frac{z_l^4}{z_0^4} \left[\left(\frac{4b_1}{b_0} - 1 \right) + \frac{z_*^4}{z_0^4} (3b_2 - 5b_1^2) \right] \right) \\ &= \mu_E^{Lif} + \frac{1}{r_-} (4b_1 \mathcal{Z} - 1) \frac{z_l^4}{z_0^4} \end{aligned} \tag{43}$$

where $\mathcal{Z} = 1 + \frac{z_*^4 (3b_2 - 5b_1^2)}{z_0^4 4b_1}$. Thus the net difference in chemical potential due to excitations

$$\Delta\mu_E^{(2)} = \frac{1}{r_-} (4b_1 \mathcal{Z} - 1) \frac{z_l^4}{z_0^4} \tag{44}$$

has second order terms also. It can be seen that entropy expression (40) can be reexpressed as a first law

$$\Delta S_E^{(2)} = \frac{1}{T_E} (\Delta E_R + \frac{1}{2} N_R \Delta \mu_E^{(2)}) \tag{45}$$

The corresponding entanglement temperature is given by

$$T_E = \frac{64z_l^2}{\pi(l_R)^3} \simeq \frac{64z_l^2}{\pi l^3} \frac{1}{(1 - \frac{1}{35} \frac{l^4}{(2z_0)^4})} \tag{46}$$

which involves cubic power of renormalized length. Similarly the ‘renormalized’ energy and charge within subsystem are also given in terms of l_R

$$\Delta E_R = l_2 l_R \mathcal{E}, \quad N_R = l_2 l_R \rho. \tag{47}$$

In the above ‘renormalized’ width of the strip is defined as

$$l_R \equiv l \left(\frac{Q}{\mathcal{Z}}\right)^{\frac{1}{4}} \simeq l \tilde{Q} \tag{48}$$

where $\tilde{Q} = 1 - \frac{1}{105} \frac{z_*^4}{z_0^4}$. The new subsystem width l_R includes terms only up to first order $\frac{z_*^4}{z_0^4}$. The result also suggests $l_R < l$, which is consistent with the fact that at the second order overall entanglement entropy decreases.

In summary, in this universal approach all thermodynamic (extensive) quantities describing subsystem are assumed to be dependent on the renormalized width through the volume factor. It is also hypothesized that in terms of the corrected quantities the entanglement first law should hold true at each order, like our approach in [7]. We also speculate that at the second order a new modular Hamiltonian can be inferred as

$$\Delta S_E^{(2)} = \langle \Delta H_E^{(2)} \rangle. \tag{49}$$

5. Summary

We calculated the entanglement entropy of a strip like subsystem on the boundary of 10-dimensional boosted black 3-brane solutions. These solutions when compactified along a lightcone coordinate describe excitations of 3-dimensional $a = 3 \theta = 1$ hyperscaling Lifshitz theory, at fixed charge (momentum) density. The theory has a natural scale z_l , determined by the charge density. The area of the strip geometry for constant ‘lightcone time’ is evaluated perturbatively up to second order. The ‘finite’ contribution of the $a = 3 \theta = 1$ Lifshitz ground state is found to be

$$S_{(0)} = \frac{L^3 \pi r_- l_2}{4G_5 z_l^2} \ln \frac{l^2}{z_l^2}$$

where an allowed range is $l > 2z_l$. Due to the z_l dependence, this entropy is qualitatively different from the 2D CFT entropy which behaves as $\sim \ln(l/\epsilon)$ (ϵ being UV cutoff) [2].

The entanglement entropy of the excitations is however found to be proportional to l^4 , whereas the entanglement temperature falls off as $\frac{1}{l^3}$. These results are essentially along expected lines, indicating that the dynamical exponent of time for the hyperscaling Lifshitz background is three. Notably these results are distinct when compared with the relativistic counterpart where the entanglement entropy of excitations instead grows as l^2 , and temperature goes as $\frac{1}{l}$, at first order [5]. A renormalization of entanglement width is proposed at second order when we try to write down first law of thermodynamics

$$\Delta S_E^{(2)} = \frac{1}{T_E} (\Delta E_R + \frac{1}{2} N_R \Delta \mu_E^{(2)})$$

This conclusion falls along the lines with the hypothesis invoked in an earlier work [7] for the relativistic case. The results in our paper can be generalized to study higher dimensional Lifshitz theories with varied dynamical exponents, for example the cases listed in [21], which follow from lightcone compactification of boosted black Dp -branes and black Mp -branes.

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