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Schwinger mechanism with stochastic quantization

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ABSTRACT

We prescribe a formulation of the particle production with real-time Stochastic Quantization. To construct the retarded and the time-ordered propagators we decompose the stochastic variables into positive- and negative-energy parts. In this way we demonstrate how to derive a standard formula for the Schwinger mechanism under time-dependent electric fields. We discuss a mapping to the Schwinger–Keldysh formalism and a relation to the classical statistical simulation.

1. Introduction

Direct simulations of the quantum field theory formulated on discretized space–time, that is, lattice simulations have proved to be a powerful numerical tool to reveal non-perturbative aspects of the theory. It is, however, not always guaranteed that one can dig meaningful information out from the lattice calculations. Because the numerical algorithm relies on the importance sampling, the method ceases to work as soon as the integrand becomes negative (or complex in general). In gauge theories the most notorious example to hinder the lattice numerical approach is the "sign problem" associated with finite density of fundamental fermions [1,2] (for reviews; see Ref. [3]). The sign problem is activated also when the theory has a Chern–Simons term that is necessary to access the θ-vacuum structure [4–7].

In addition to these Euclidean examples one cannot avoid encountering the sign problem if one attacks the real-time problem in Minkowskian space–time. The complex phase originates from the path-integral weight, e^{iS}. The real-time simulation is one of the most challenging topics in modern quantum field theories; the transport coefficients of a fluid, the particle emission rate in strongly correlated systems, and so on, are needed in various physics circumstances. One can still utilize the conventional lattice technique as long as the analytical continuation from Euclidean space–time is a legitimate procedure [8–11]. The applicability of such approach is, however, limited to static (or steady) phenomena or linear-response perturbation at best. Full quantum simulations would demand an alternative quantization machinery in different directions from the importance sampling. For a promising candidate, in this work, we will advocate the Stochastic Quantization [12,13] (for reviews, see Ref. [14]) and take a concrete example of real-time physics problem.

One of the most important and most ubiquitous phenomena that call for real-time quantization is the problem of the particle production from the *vacuum*. In the quantum field theory, in fact, the vacuum is not empty but is full of quanta, and some of them could *tunnel* the potential barrier out from the vacuum. Celebrated examples of such tunneling phenomena include the Schwinger mechanism that refers to the vacuum-insulation breakdown under external electric fields [15,16] (for a review, see Ref. [17]), and the Hawking radiation that refers to the spontaneous radiation process from black holes, namely, the particle production under external gravitational fields [18,19].

In this work we shall focus specifically on a theoretical reformulation of the Schwinger mechanism on the basis of the Stochastic Quantization. For attempts in different directions the readers can consult the literature [20–22]. Because the Stochastic Quantization is a functional description in terms of classical fields, we must first establish a prescription to derive various kinds of propagators which are written most conveniently with creation/annihilation operators. In Refs. [23–25] it has been shown that the inclusive spectrum is to be expressed in the following manner:

$$\frac{dN}{d^3\boldsymbol{p}} = \frac{1}{(2\pi)^3 2\mathcal{E}_{\rm out}(\boldsymbol{p})} \lim_{t=t'\to\infty} \big[\partial_{t'} + i\mathcal{E}_{\rm out}(\boldsymbol{p})\big] \times \big[\partial_t - i\mathcal{E}_{\rm out}(\boldsymbol{p})\big] \big\langle \hat{\rho}_{\rm in} \hat{\phi}^\dagger(t',\boldsymbol{p}) \hat{\phi}(t,\boldsymbol{p})\big\rangle. \tag{1}$$

The initial density matrix is assumed to be $\hat{\rho}_{\rm in} = |0_{\rm in}\rangle\langle 0_{\rm in}|$ throughout this work. The finite-temperature extension is rather straightforward [25]. We note that this two-point function (called the Wightman function) is nothing but $D_F(t,\boldsymbol{p};t',-\boldsymbol{p}) - D_R(t,\boldsymbol{p};$

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$t', -\boldsymbol{p})$ where $D_F(t, \boldsymbol{p}; t', \boldsymbol{p}')$ and $D_R(t, \boldsymbol{p}; t', \boldsymbol{p}')$, respectively, represent the time-ordered and the retarded propagators. In the present work we limit ourselves to the simplest case of complex scalar field theory (i.e., scalar QED) under an external electric field, which is easily translated to spinor matter.

2. Stochastic quantization

The key idea of the Stochastic Quantization is that one can quantize field theories using a classical equation of motion with one artificial axis (i.e., quantum or Suzuki–Trotter axis [26]) denoted here by θ and with stochastic variables $\eta(x, \theta)$. We thus need to solve a *complex Langevin* equation, which turns out to be accompanied by i in Minkowskian space–time. Let us take a quick flash at the way to retrieve free propagators. As a matter of fact, a functional formulation usually comes along with the time-ordered propagator, whereas in the real-time problems we often need the retarded and advanced propagators as well. It is crucial, therefore, to establish the correct description of them within the Stochastic Quantization (without going back to the operator formalism). For a free scalar field theory the classical equation of motion reads,

$$\frac{\partial \phi_p(t,\theta)}{\partial \theta} = \mathrm{i}\big[-\partial_t^2 - \mathcal{E}^2(\boldsymbol{p})\big]\phi_p(t,\theta) + \eta_p(t,\theta) \tag{2}$$

with $\mathcal{E}(\boldsymbol{p}) \equiv \sqrt{\boldsymbol{p}^2 + m^2}$. Here, we took the Fourier transform with respect to spatial coordinates. For our purpose to cope with a time-dependent but spatially homogeneous background field, it is convenient to keep t not changed to the frequency.

In the complex scalar field theory of our interest, we need to introduce another independent field $\bar{\phi}(t, \theta)$ and associated stochastic variable $\bar{\eta}_p(t, \theta)$. In this partially Fourier transformed representation we should define the average over the stochastic variables as follows:

$$\begin{aligned}&\big\langle \eta_p(t,\theta)\bar{\eta}_{p'}\big(t',\theta'\big)\big\rangle_\eta = 2\delta\big(t-t'\big)(2\pi)^3\delta^{(3)}\big(\boldsymbol{p}+\boldsymbol{p}'\big)\delta\big(\theta-\theta'\big),\\ &\big\langle \eta_p(t,\theta)\eta_{p'}\big(t',\theta'\big)\big\rangle_\eta = \big\langle \bar{\eta}_p(t,\theta)\bar{\eta}_{p'}\big(t',\theta'\big)\big\rangle_\eta = 0.\end{aligned} \tag{3}$$

When we solve Eq. (2), the most useful boundary condition is $\phi_p(t, 0) = 0$. We could have taken a non-zero value, but then we should supplement a proper subtraction in the end. We can easily find a formal solution of the complex Langevin equation given explicitly as

$$\phi_p(t,\theta) = \int_0^\theta d\theta' e^{\mathrm{i}[-\partial_t^2 - \mathcal{E}^2(\boldsymbol{p}) + \mathrm{i}\epsilon](\theta-\theta')}\eta_p\big(t,\theta'\big). \tag{4}$$

We inserted $\mathrm{i}\epsilon$ to guarantee the convergence in the $\theta \to \infty$ limit, which corresponds to the $\mathrm{i}\epsilon$ prescription to derive the time-ordered propagator.

After taking the average we can simplify the expression of the two-point function to reach the following form:

$$\begin{aligned}&\big\langle \phi_p(t,\theta)\bar{\phi}_{p'}\big(t',\theta\big)\big\rangle_\eta\\ &\quad = \frac{\mathrm{i}}{-\partial_t^2 - \mathcal{E}^2(\boldsymbol{p}) + \mathrm{i}\epsilon}\big[1 - e^{2\mathrm{i}(-\partial_t^2 - \mathcal{E}_{\boldsymbol{p}}^2 + \mathrm{i}\epsilon)\theta}\big]\\ &\quad\quad \times (2\pi)^3\delta^{(3)}\big(\boldsymbol{p}+\boldsymbol{p}'\big)\delta\big(t-t'\big).\end{aligned} \tag{5}$$

When we take the $\theta \to \infty$ limit, the exponential oscillatory term drops off, and the resultant expression is reduced to the standard form of the time-ordered propagator, i.e., $D_F(t, \boldsymbol{p}; t', \boldsymbol{p}')$.

It is a non-trivial question how to construct other types of the propagators. Since the creation and annihilation operators correspond to the negative- and the positive-energy parts of the field operator, it is then quite natural to decompose the stochastic variable as $\eta_p(t, \theta) = \eta_p^+(t, \theta) + \eta_p^-(t, \theta)$ where

$$\eta_p^\pm(t,\theta) \equiv \int_0^\infty \frac{d\omega}{2\pi}\tilde{\eta}_p(\pm\omega,\theta)e^{\mp \mathrm{i}\omega t}. \tag{6}$$

Here $\tilde{\eta}_p(\omega, \theta)$ represents the Fourier transform of $\eta_p(t, \theta)$. We also do the same for $\bar{\eta}_p(t, \theta)$ and then $\delta(t - t')$ in Eq. (3) is replaced with $2\pi\delta(\omega + \omega')$ in the two-point function of $\tilde{\eta}_p(\omega, \theta)$ and $\tilde{\bar{\eta}}_p(\omega', \theta')$. Accordingly we can introduce variants of Eq. (4), namely:

$$\phi_p^\pm(t,\theta) \equiv \int_0^\theta d\theta' e^{\mathrm{i}[-\partial_t^2 - \mathcal{E}^2(\boldsymbol{p}) + \mathrm{i}\epsilon](\theta-\theta')}\eta_p^\pm\big(t,\theta'\big). \tag{7}$$

It is an important ingredient in our formulation to define:

$$\psi_p^\pm(t,\theta) \equiv \int_0^\theta d\theta' e^{-\mathrm{i}[-\partial_t^2 - \mathcal{E}^2(\boldsymbol{p}) - \mathrm{i}\epsilon](\theta-\theta')}\eta_p^\pm\big(t,\theta'\big), \tag{8}$$

which solves a slightly deformed equation of motion with the sign of i flipped in Eq. (2), in other words, the equation of motion derived from the sign-flipped action. As we discuss later, thus, $\psi_p^\pm(t, \theta)$ can be interpreted as the field along the backward time path.

The time-ordered propagator involves only the components with $\phi_p^\pm(t, \theta)$ and our main proposition here is to utilize $\psi_p^\pm(t, \theta)$ as an additional building block of other types of the propagators:

$$\begin{aligned}&D_R\big(t,\boldsymbol{p};t',\boldsymbol{p}'\big)\\ &\quad = \lim_{\theta\to\infty}\big\langle \phi_p^+(t,\theta)\bar{\phi}_{p'}^-\big(t',\theta\big) - \psi_p^-(t,\theta)\bar{\psi}_{p'}^+\big(t',\theta\big)\big\rangle_\eta.\end{aligned} \tag{9}$$

We can also write the advanced propagator down in the same way by means of an appropriate combination of $\phi_p^\pm(t, \theta)$ and $\psi_p^\pm(t, \theta)$. In view of Eq. (1), therefore, we can identify an expression directly relevant to the particle production as

$$\begin{aligned}&D_F\big(t,\boldsymbol{p};t',\boldsymbol{p}'\big) - D_R\big(t,\boldsymbol{p};t',\boldsymbol{p}'\big)\\ &\quad = \lim_{\theta\to\infty}\big\langle \phi_p^-(t,\theta)\bar{\phi}_{p'}^+\big(t',\theta\big) + \psi_p^-(t,\theta)\bar{\psi}_{p'}^+\big(t',\theta\big)\big\rangle_\eta.\end{aligned} \tag{10}$$

We emphasize that, though our prescription may look ad-hoc at first glance, this is a unique choice so that the convergence factor $\mathrm{i}\epsilon$ has a right sign in the propagator as $p_0^2 - \mathcal{E}^2(\boldsymbol{p}) \pm \mathrm{sgn}(p_0)\mathrm{i}\epsilon$, after taking the Fourier transform from t to p_0.

3. Time-dependent background field

From now on we shall turn the time-dependent potential on, denoted by $\mathcal{V}_p(t)$, which yields a complex Langevin equation,

$$\frac{\partial \phi_p^\pm(t,\theta)}{\partial \theta} = \mathrm{i}\big[-\partial_t^2 + \mathcal{V}_p(t)\big]\phi_p^\pm(t,\theta) + \eta_p^\pm(t,\theta) \tag{11}$$

and a similar one for $\psi_p^\pm(t, \theta)$ with i in the right-hand side changed to $-\mathrm{i}$. We assume a time-dependent but spatially homogeneous electric field $\boldsymbol{E}(t)$ and thus $\mathcal{V}_p(t)$ is given explicitly as

$$\mathcal{V}_p(t) = -m^2 - \big[\boldsymbol{p} - e\boldsymbol{A}(t)\big]^2 \tag{12}$$

with $\boldsymbol{E}(t) = -\partial_t \boldsymbol{A}(t)$. As long as $\mathcal{V}_p(t)$ does not involve momentum transfer, the spatial derivatives are diagonalized in this partially Fourier transformed representation. In the in- and the out-states the interaction falls off, so that the asymptotic states have $\mathcal{V}_p(t \sim$

$t_I) = -\mathcal{E}_{in}^2(\boldsymbol{p})$ and $\mathcal{V}_p(t \sim t_F) = -\mathcal{E}_{out}^2(\boldsymbol{p})$. Let us demonstrate how our formulas (1) and (10) work for the estimate of the produced particle number.

We can easily solve (11) for general $\mathcal{V}_p(t)$ to find the explicit form of the solution as

$$\phi_p^{\pm}(t,\theta) = \int_0^{\theta} d\theta' e^{i[-\partial_t^2 + \mathcal{V}_p(t) + i\epsilon](\theta-\theta')} \eta_p^{\pm}(t,\theta') \tag{13}$$

and we can solve for $\psi_p^{\pm}(t,\theta)$ as well. We now get ready to compute $D_R(t,\boldsymbol{p};t',\boldsymbol{p}')$ according to our prescription.

The final answer should not depend on how we treat the η-average as long as $\eta_p^{\pm}(t,\theta)$'s are generated consistently as the Gaussian noise (3). Instead of taking the Gaussian average, we can simplify the calculation by means of $\eta_p^{\pm}(t,\theta)$ decomposed with a complete set of the solutions of the following equation of motion:

$$\left[-\partial_t^2 + \mathcal{V}_p(t)\right]\chi_\omega^{\pm}(t) = \left[\omega^2 - \mathcal{E}_{in}^2(\boldsymbol{p})\right]\chi_\omega^{\pm}(t), \tag{14}$$

where in the right-hand side, ω [or $\omega^2 - \mathcal{E}_{in}^2(\boldsymbol{p})$] is an eigenvalue to label the complete set, and the superscript $\pm$ corresponds to the boundary condition,

$$\chi_\omega^{\pm}(t \to t_I) \to e^{\mp i\omega t}, \tag{15}$$

which is chosen for convenience to meet the boundary condition of Eq. (6) at $t = t_I$. Here, let us consider the electric field along x^3 and take $\boldsymbol{A}(t) = (0,0,A^3(t))$. We note that $\chi_\omega^{\pm}(t)$ correspond to the positive and negative energy solutions of the classical equation of motion in Ref. [27] and thus the Bogoliubov coefficients of $\chi_\omega^{\pm}(t)$ yield the produced particle spectrum [27,28].

Because $\mathcal{V}_p(t)$ is real, $\chi_{-\omega}^{\mp}(t) = \chi_\omega^{\pm}(t)$ follows. We can deform the definition of positive- and negative-energy parts at $t = t_I$ using this complete set:

$$\eta_p^{\pm}(t,\theta) \equiv \int_0^{\infty} \frac{d\omega}{2\pi} \tilde{\eta}_p(\pm\omega,\theta)\chi_\omega^{\pm}(t), \tag{16}$$

which coincides with Eq. (6) in the in-state at $t = t_I$. We would emphasize again that this parametrization is just for practical convenience and we could have kept using the definition of Eq. (6) to come up to the same answer; the difference is whether we should cope with the complicated t-dependent evolution operator in the exponential as seen in Eq. (13) or make it t-independent with the complicated wave-function $\chi_p^{\pm}(t)$ (which is reminiscent of a transition between the Schrödinger and the Heisenberg pictures in quantum mechanics).

With help of eigenfunctions of Eq. (14) we can readily derive the following form of the retarded propagator,

$$D_R(t,\boldsymbol{p};t',\boldsymbol{p}') = (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i\chi_\omega^{+}(t)\chi_\omega^{-}(t')}{\omega^2 - \mathcal{E}_{in}^2(\boldsymbol{p}) + \mathrm{sgn}(\omega)i\epsilon}. \tag{17}$$

For the particle production problem we need to calculate $D_F - D_R$ which reads:

$$\begin{aligned} &D_F(t,\boldsymbol{p};t',\boldsymbol{p}') - D_R(t,\boldsymbol{p};t',\boldsymbol{p}') \\ &= (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}') \int_{-\infty}^{0} \frac{d\omega}{2\pi}(-i) \\ &\quad \times 2\pi\delta(\omega^2 - \mathcal{E}_{in}(\boldsymbol{p})^2) \cdot i\chi_\omega^{+}(t)\chi_\omega^{-}(t') \\ &= (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}') \frac{\chi_{-\mathcal{E}_{in}(\boldsymbol{p})}^{+}(t)\chi_{-\mathcal{E}_{in}(\boldsymbol{p})}^{-}(t')}{2\mathcal{E}_{in}(\boldsymbol{p})}. \end{aligned} \tag{18}$$

We note that the delta function picks up an eigenvalue of $\omega = -E_{in}(\boldsymbol{p})$ only that makes the right-hand side of Eq. (14) vanishing! Therefore, $\chi_{-\mathcal{E}_{in}(\boldsymbol{p})}^{\pm}(t)$ satisfies the classical equation of motion in the ordinary field theory.

With the initial condition (15) the solution of the equation of motion should behave like $\chi_{-\mathcal{E}_{in}(\boldsymbol{p})}^{-}(t) = e^{-i\mathcal{E}_{in}(\boldsymbol{p})t}$ near the in-state at $t = t_I$ and we can parametrize:

$$\chi_{-\mathcal{E}_{in}(\boldsymbol{p})}^{-}(t) = \sqrt{\frac{\mathcal{E}_{in}(\boldsymbol{p})}{\mathcal{E}_{out}(\boldsymbol{p})}}\left[\alpha_{\boldsymbol{p}} e^{-i\mathcal{E}_{out}(\boldsymbol{p})t} + \beta_{\boldsymbol{p}}^{*} e^{i\mathcal{E}_{out}(\boldsymbol{p})t}\right], \tag{19}$$

near the out-state at $t = t_F$. From these asymptotic forms it is easy to find the following expression near the out-state as

$$\begin{aligned} &D_F(t,\boldsymbol{p};t',\boldsymbol{p}') - D_R(t,\boldsymbol{p};t',\boldsymbol{p}') \\ &= (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}')\frac{1}{2\mathcal{E}_{out}(\boldsymbol{p})}\left\{|\alpha_{\boldsymbol{p}}|^2 e^{i\mathcal{E}_{out}(\boldsymbol{p})(t-t')} \right. \\ &\quad \left. + |\beta_{\boldsymbol{p}}|^2 e^{-i\mathcal{E}_{out}(\boldsymbol{p})(t-t')} + 2\mathrm{Re}\left[\alpha_{\boldsymbol{p}}\beta_{\boldsymbol{p}} e^{-i\mathcal{E}_{out}(\boldsymbol{p})(t+t')}\right]\right\}, \end{aligned} \tag{20}$$

which recovers the results in Ref. [25] and leads to the well-known formula of the produced particle spectrum [27,28]:

$$\frac{dN}{d^3\boldsymbol{p}} = \delta^{(3)}(0)|\beta_{\boldsymbol{p}}|^2. \tag{21}$$

We make a remark that Eq. (14) provides us with a basis of the so-called over-the-barrier scattering picture for the Schwinger mechanism [27,28] (see also Refs. [29–31] which can be understood in this picture).

4. Relation to other formalisms

Now that we have reached the final expression of the particle production, let us deepen a physical insight from the point of view of both formal and numerical aspects.

As we already mentioned, $\psi_p^{\pm}(t,\theta)$ plays a similar role to the field along the backward time path that appears in the Schwinger–Keldysh or closed-time path (CTP) formalism [32,33]. In fact we can find a mapping to two-point functions in the canonical quantization, that is:

$$\lim_{\theta\to\infty}\langle\phi_p^{+}(t,\theta)\bar{\phi}_{p'}^{-}(t',\theta)\rangle_\eta = \langle\Theta(t-t')\hat{\phi}_p(t,\theta)\hat{\phi}_{p'}^{\dagger}(t',\theta)\rangle, \tag{22}$$

$$\lim_{\theta\to\infty}\langle\phi_p^{-}(t,\theta)\bar{\phi}_{p'}^{+}(t',\theta)\rangle_\eta = \langle\Theta(t'-t)\hat{\phi}_{p'}^{\dagger}(t',\theta)\hat{\phi}_p(t,\theta)\rangle, \tag{23}$$

$$\lim_{\theta\to\infty}\langle\psi_p^{+}(t,\theta)\bar{\psi}_{p'}^{-}(t',\theta)\rangle_\eta = \langle\Theta(t'-t)\hat{\phi}_p(t,\theta)\hat{\phi}_{p'}^{\dagger}(t',\theta)\rangle, \tag{24}$$

$$\lim_{\theta\to\infty}\langle\psi_p^{-}(t,\theta)\bar{\psi}_{p'}^{+}(t',\theta)\rangle_\eta = \langle\Theta(t-t')\hat{\phi}_{p'}^{\dagger}(t',\theta)\hat{\phi}_p(t,\theta)\rangle \tag{25}$$

with $\Theta(t)$ being the Heaviside step function. We use the hat to indicate the quantum operator. The Schwinger–Keldysh formalism consists of 2×2 matrix propagators which we can construct from the above two-point functions as

$$\begin{aligned} &D_{++}(t,\boldsymbol{p};t',\boldsymbol{p}') \\ &\equiv \langle T[\hat{\phi}_p(t,\theta)\hat{\phi}_{p'}^{\dagger}(t',\theta)]\rangle \\ &= \lim_{\theta\to\infty}\langle\phi_p^{+}(t,\theta)\bar{\phi}_{p'}^{-}(t',\theta) + \phi_p^{-}(t,\theta)\bar{\phi}_{p'}^{+}(t',\theta)\rangle_\eta, \end{aligned} \tag{26}$$

$$\begin{aligned} &D_{--}(t,\boldsymbol{p};t',\boldsymbol{p}') \\ &\equiv \langle\tilde{T}[\hat{\phi}_p(t,\theta)\hat{\phi}_{p'}^{\dagger}(t',\theta)]\rangle \\ &= \lim_{\theta\to\infty}\langle\psi_p^{+}(t,\theta)\bar{\psi}_{p'}^{-}(t',\theta) + \psi_p^{-}(t,\theta)\bar{\psi}_{p'}^{+}(t',\theta)\rangle_\eta, \end{aligned} \tag{27}$$

$D_{+-}(t,\boldsymbol{p};t',\boldsymbol{p}')$

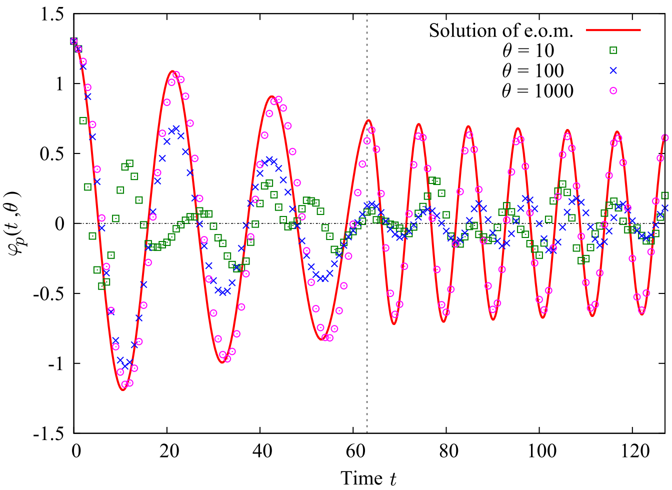

Fig. 1. Evolution of the averaged field variable $\varphi_p(t,\theta)$ from $t_I = 0$ with increasing θ. A pulse electric field is imposed around $t = t_0$. The boundary condition at $t = t_I$ is specified as an outgoing form: $\varphi_p(t,\theta) \propto e^{i\mathcal{E}_{in}t_I}$.

$$\equiv \langle \hat{\phi}^\dagger_{p'}(t',\theta)\hat{\phi}_p(t,\theta)\rangle$$
$$= \lim_{\theta\to\infty}\langle \phi^-_p(t,\theta)\bar{\phi}^+_{p'}(t',\theta) + \psi^-_p(t,\theta)\bar{\psi}^+_{p'}(t',\theta)\rangle_\eta, \tag{28}$$

$$D_{-+}(t,\boldsymbol{p};t',\boldsymbol{p}')$$
$$\equiv \langle \hat{\phi}_p(t,\theta)\hat{\phi}^\dagger_{p'}(t',\theta)\rangle$$
$$= \lim_{\theta\to\infty}\langle \phi^+_p(t,\theta)\bar{\phi}^-_{p'}(t',\theta) + \psi^+_p(t,\theta)\bar{\psi}^-_{p'}(t',\theta)\rangle_\eta \tag{29}$$

where T and $\tilde{\text{T}}$, respectively, denote the time and reversed-time ordered products. By using the explicit solutions (7) and (8), we can show that these propagators are equivalent to those defined in Ref. [34]. Thus, we can regard $\psi^\pm_p(t,\theta)$ as the positive and negative energy fields along the backward time path and our formulation encompasses the precise structure of the perturbation theory in the Schwinger–Keldysh formalism.

For the rest of this paper, we will address the relation to the classical statistical simulation [24]. Let us consider a numerical simulation with fixed values of $\phi(t_I,\theta)$ and $\dot{\phi}(t_I,\theta)$ [or $\phi(t_I+\Delta t,\theta)$] to solve Eq. (2). We then perform the η-average except at $t = t_I$ and $t_I + \Delta t$. Taking the θ-average can significantly stabilize the θ-oscillation and reduce the computational cost. More specifically, the θ-averaged field as defined by

$$\varphi_p(t,\theta) \equiv \theta^{-1}\int_0^\theta d\theta' \phi_p(t,\theta'), \tag{30}$$

approaches the solution of the equation of motion (14). We can clearly confirm it in Fig. 1 in the presence of an electric field pulsed around $t = t_0$, which is chosen specifically as

$$\boldsymbol{A}(t) = \left(0, 0, \frac{E_0}{\mathfrak{w}}\left[\tanh \mathfrak{w}(t-t_0) + 1\right]\right). \tag{31}$$

Physical quantities are all made dimensionless by the time step Δt and the site number along the t-axis is chosen as $N_t = 256$. The θ-axis is discretized with $\Delta\theta = 5\times 10^{-3}$ (which means that we update the θ-evolution 2×10^5 times to get the results at $\theta = 1000$). We choose $p_3 = 0$ and $\mathcal{E}_{in}(\boldsymbol{p}) = \sqrt{(p_1)^2+(p_2)^2+m^2} = 12\times(2\pi/N_t)$, so that there are 12 periods included along the t-direction from $t = 0$ to $(N_t - 1)\Delta t$ if not affected by the electric field. We postulate a short life time for the electric field: $\mathfrak{w} = 5\mathcal{E}_{in}(\boldsymbol{p})$ for a fixed momentum $\boldsymbol{p}$ and the it stands at $t_0 = 63\Delta t$ (i.e., a quarter of the whole time range).

To manifest the effect of the electric field, we specifically adopt: $|e|E_0/\mathfrak{w} = (\sqrt{3}/2)\mathcal{E}_{in}(\boldsymbol{p})$, and then $\mathcal{E}_{out}(\boldsymbol{p}) = 2\mathcal{E}_{in}(\boldsymbol{p})$. With this choice we see that the results in Fig. 1 is quite reasonable; there are 3 and 6 periods of the oscillation from $t = 0$ to t_0 and from $t = t_0$ to $2t_0$, respectively, observed in Fig. 1. We note that $\epsilon = 5\times 10^{-3}$ is used for numerical stability. On the technical level it is the most tough part to avoid unphysical "run-away" flows in θ, which is overcome here by implementing the Crank–Nicolson method [35].

We imposed an outgoing initial condition as $\varphi_p(t_I,\theta) = (1/\sqrt{2\mathcal{E}_{in}(\boldsymbol{p})})e^{-i\mathcal{E}_{in}(\boldsymbol{p})t_I}$ at $t = t_I$ and $t = t_I + \Delta t$ in our Stochastic Quantization simulation, which is the right choice to evaluate the production rate in the ordinary procedure [27,28]. Also, we numerically solved the equation of motion (14) in the presence of $\boldsymbol{A}(t)$ with the same initial condition as shown by a solid curve in Fig. 1. It is clear that the Stochastic Quantization output converges to the solution of the equation of motion as it should. It should be mentioned that the decomposition to positive- and negative-energy parts with $\eta^\pm_p(t,\theta)$ is now effectively taken into account in our procedure to impose the outgoing initial condition. Since the convergence to the solution of the equation of motion guarantees that we can reproduce correct $dN/d^3\boldsymbol{p}$, we would not explicitly evaluate it.

Let us comment on the relation to the classical statistical simulation [36–38] here. If we compute $\langle\phi_p(t)\rangle$, as seen in Fig. 1, the Stochastic Quantization leads to the solution of the equation of motion. More generally, if we are allowed to make an approximation for an operator $\mathcal{O}[\phi]$ that $\langle\mathcal{O}[\phi]\rangle_t \approx \mathcal{O}[\langle\phi\rangle_t]$ for a given initial condition, this is nothing but the calculation procedure in the classical statistical simulation. The initial state should accommodate quantum fluctuations described by the initial Wigner function, and so we should perform the ensemble average with fluctuating initial conditions in general. (For the present purpose to investigate the vacuum physics the iϵ prescription is sufficient.) We would emphasize that such a derivation of the classical statistical simulation sheds light on the structure of the approximation, e.g., the renormalization problem as addressed in Ref. [34].

5. Summary

In summary, in this work, we gave a derivation of the standard formula for the Schwinger mechanism with Stochastic Quantization. The most non-trivial part was how to prescribe the retarded propagator, in such a way that the θ-integration is properly regulated. We decomposed the stochastic variables into positive- and negative-energy parts, and this corresponds to imposing a proper initial condition in the numerical simulation. We showed that our machinery has a natural connection to the closed-time path formalism and we presented our numerical results that converge to the correct answer.

Our formulation on the basis of Stochastic Quantization has potential applications to variety of real-time physics problems. Apart from the particle production issue, one of the most interesting extensions would be the computation of the spectral functions and the transport coefficients. We are now making a progress in this direction.

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