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Solving Heun's equation using conformal blocks

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Abstract

It is known that the classical limit of the second order BPZ null vector decoupling equation for the simplest two 5-point degenerate spherical conformal blocks yields: (i) the normal form of the Heun equation with the complex accessory parameter determined by the 4-point classical block on the sphere, and (ii) a pair of the Floquet type linearly independent solutions. A key point in a derivation of the above result is the classical asymptotic of the 5-point degenerate blocks in which the so-called heavy and light contributions decouple. In the present work the semi-classical heavy–light factorization of the 5-point degenerate conformal blocks is studied. In particular, a mechanism responsible for the decoupling of the heavy and light contributions is identified. Moreover, it is shown that the factorization property yields a practical method of computation of the Floquet type Heun's solutions. Finally, it should be stressed that tools analyzed in this work have a broad spectrum of applications, in particular, in the studies of spectral problems with the Heun class of potentials, sphere–torus correspondence in 2d CFT, the KdV theory, the connection problem for the Heun equation and black hole physics. These applications are main motivations for the present work.

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1. Introduction

1.1. The Heun equation

To begin with let us consider the Heun equation, i.e., the Fuchsian equation with four singularities. The most familiar form of the Heun equation reads as follows [1][1]

$$\frac{d^2\Phi}{dz^2}+\left[\frac{\gamma}{z}+\frac{\omega}{z-1}+\frac{\epsilon}{z-x}\right]\frac{d\Phi}{dz}+\frac{\alpha\beta z-\mathsf{Q}}{z(z-1)(z-x)}\Phi=0. \tag{1.1}$$

Eq. (1.1) has four regular singular points at $0, 1, x$ and ∞. In (1.1) it is assumed the condition $\epsilon=\alpha+\beta-\gamma-\omega+1$ needed to ensure regularity of the point at ∞. The complex number Q is called the accessory parameter. Another representation of the Heun equation known as the *normal form* looks as follows [2]

$$\frac{d^2\Psi}{dz^2}-\left[\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-x}+\frac{D}{z^2}+\frac{E}{(z-1)^2}+\frac{F}{(z-x)^2}\right]\Psi=0. \tag{1.2}$$

Eqs. (1.1) and (1.2) are linked by the substitution $\Psi(z)=z^{\gamma/2}(z-1)^{\omega/2}(z-x)^{\epsilon/2}\,\Phi(z)$, where

$$A=-\frac{\gamma\omega}{2}-\frac{\gamma\epsilon}{2x}+\frac{\mathsf{Q}}{x},\qquad B=\frac{\gamma\omega}{2}-\frac{\omega\epsilon}{2(x-1)}-\frac{\mathsf{Q}-\alpha\beta}{x-1},$$
$$C=\frac{\gamma\epsilon}{2x}+\frac{\omega\epsilon}{2(x-1)}-\frac{x\alpha\beta-\mathsf{Q}}{x(x-1)}$$

and

$$D=\tfrac{1}{2}\gamma\left(\tfrac{1}{2}\gamma-1\right),\qquad E=\tfrac{1}{2}\omega\left(\tfrac{1}{2}\omega-1\right),\qquad F=\tfrac{1}{2}\epsilon\left(\tfrac{1}{2}\epsilon-1\right).$$

In the present paper we study a realization of the Heun equation in a two-dimensional conformal field theory (CFT_2) and discuss some of its consequences. In CFT_2, the Heun equation occurs in the normal form (1.2) usually written as

$$\frac{d^2\Psi}{dz^2}+\left[\frac{\delta_1}{z^2}+\frac{\delta_2}{(z-x)^2}+\frac{\delta_3}{(1-z)^2}+\frac{\delta_1+\delta_2+\delta_3-\delta_4}{z(1-z)}+\frac{x(1-x)c_2(x)}{z(z-x)(1-z)}\right]\Psi=0. \tag{1.3}$$

From (1.2) and (1.3) one has

$$\begin{aligned}&-A=\delta_1+\delta_2+\delta_3-\delta_4+c_2(x)(x-1),\qquad -B=\delta_4-\delta_1-\delta_2-\delta_3-c_2(x)x,\\ &-C=c_2(x),\quad -D=\delta_1,\quad -E=\delta_3,\quad -F=\delta_2.\end{aligned} \tag{1.4}$$

Here we just only briefly announce that eq. (1.3) emerges in *Liouville field theory* (LFT) and in a model independent *chiral* CFT_2. These two contexts are related to computations of certain *conformal blocks*. For this reason, before we express our concrete goals and motivations of the present work, we would like to spell out basic information about the *quantum* and *classical* conformal blocks. In particular, we will list some interesting current research topics related to them.

[1] Eq. (1.1) is called now the Heun equation in honor of its first investigator — German mathematician Karl Heun (1859–1929).

1.2. Quantum and classical conformal blocks

Let $C_{g,n}$ be the Riemann surface with genus g and n punctures. The basic objects of any two-dimensional conformal field theory living on C_g [3,4] are the n-point correlation functions of primary physical vertex operators defined on $C_{g,n}$. Any correlation function can be factorized according to the pattern given by the pant decomposition of $C_{g,n}$ and written as a sum (or an integral for theories with a continuous spectrum) which includes the terms consisting of the holomorphic and anti-holomorphic conformal blocks times the 3-point functions of the model for each pair of pants. The Virasoro conformal block $\mathcal{F}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ on $C_{g,n}$ depends on the cross ratios of the vertex operators locations denoted symbolically by Z and on the $3g-3+n$ intermediate conformal weights Δ_p. Moreover, it depends on the n external conformal weights Δ_i and on the central charge c.

In the operator formalism [5,6] the conformal blocks on the Riemann sphere are defined as matrix elements (vacuum expectation values) of radially ordered compositions of the primary chiral vertex operators (CVO's) acting between Verma modules — the highest weight representations of the Virasoro algebra. Conformal blocks on the torus are traced cylinder matrix elements of 2d Euclidean 'space–time' translation operators and CVO's insertions, i.e., 'chiral partition functions'. Conformal blocks are fully determined by the underlying conformal symmetry. These functions possess an interesting, although not yet completely understood analytic structure. In general, they can be expressed only as a formal power series and no closed formula is known for its coefficients.

Among the issues concerning conformal blocks which are still not fully understood there is the problem of their *classical limit* [7]. This is the limit in which all parameters of the conformal blocks tend to infinity in such a way that their ratios are fixed:

$$\Delta_i,\ \Delta_p,\ c \longrightarrow \infty, \qquad \frac{\Delta_i}{c} = \frac{\Delta_p}{c} = \text{const.}$$

For the standard parametrization of the central charge $c = 1 + 6Q^2$, where $Q = b + \frac{1}{b}$ and for 'heavy' weights $(\Delta_p, \Delta_i) = \frac{1}{b^2}(\delta_p, \delta_i)$ with $\delta_p, \delta_i = \mathcal{O}(b^0)$ the classical limit corresponds to $b \to 0$. There exist many convincing arguments that in the classical limit the conformal blocks behave exponentially with respect to Z:

$$\mathcal{F}_{c,\Delta_p}[\Delta_i](\mathsf{Z}) \overset{b\to 0}{\sim} \mathrm{e}^{\frac{1}{b^2} f_{\delta_p}[\delta_i](\mathsf{Z})}.$$

The functions $f_{\delta_p}[\delta_i](\mathsf{Z})$ are known as the classical conformal blocks [7,8].

Conformal blocks may also include the 'light' conformal weights Δ_{light} which appear as contributions from the 'light' chiral vertex operators $V_{\Delta_{\text{light}}}(y)$. The light conformal weights are defined by the property $\lim_{b\to 0} b^2 \Delta_{\text{light}} = 0$. It is known (but not proven in general) that light insertions have no influence to the classical limit, i.e., do not contribute to the classical blocks:

$$\left\langle \prod_{i=1}^{n} V_{\Delta_{\text{heavy}}}(z_i) V_{\Delta_{\text{light}}}(y) \right\rangle \overset{b\to 0}{\sim} \Psi(y)\, \mathrm{e}^{\frac{1}{b^2} f_{\delta_p}[\delta_i](\mathsf{Z})}. \tag{1.5}$$

Recently, a considerable progress in the theory of conformal blocks and their applications has been achieved. This is mainly due to the discovery of the so-called AGT correspondence [9]. The AGT conjecture states that the Liouville field theory correlators on the Riemann surface $C_{g,n}$ with genus g and n punctures can be identified with the partition functions of a class $T_{g,n}$ of four-dimensional $\mathcal{N} = 2$ supersymmetric SU(2) quiver gauge theories. A significant part of the AGT conjecture is an exact correspondence between the Virasoro blocks on $C_{g,n}$ and the instanton sectors of the Nekrasov partition functions of the gauge theories $T_{g,n}$. Soon after its discovery, the AGT hypothesis has been extended, in particular, (i) to the correspondence between conformal Toda correlators and SU(N) gauge theories partition functions [10]; (ii) to the relation (cf. [11]) between *irregular conformal blocks* [12–14] and Nekrasov's instanton partition functions for 'non-conformal' $\mathcal{N}=2$, SU(2) super Yang–Mills theories.

The irregular conformal blocks discovered by Gaiotto were introduced in his work [11] as products of some new states belonging to the Hilbert space of CFT_2. These novel irregular Gaiotto states are kind of coherent vectors for some Virasoro generators. It is also known that irregular blocks can be obtained from standard (regular) conformal blocks in properly defined *decoupling limits* of the external conformal weights, cf. [12,13]. Furthermore, the Gaiotto vectors can be understood as a result of suitable defined *collision limit* of locations of vertex operators in their operator product expansion (OPE), cf. [14].

Interestingly, the classical limit exists also for irregular blocks and consistently defines the *classical irregular blocks*. This claim first time has appeared in [15] as a result of non-conformal AGT relations and yet another new duality, the so-called Bethe/gauge correspondence [16].

Let us recall that the AGT correspondence works at the level of the quantum Liouville field theory. At this point it arises the question as to what happens if we proceed to the classical limit of the Liouville theory. It turns out that the semi-classical limit of the LFT correlation functions [7] corresponds to the Nekrasov–Shatashvili (NS) limit of the Nekrasov partition functions [16]. In particular, a consequence of that correspondence is that the classical conformal blocks can be identified with the instanton sectors of the *effective twisted superpotentials* [17,18].[2] The latter quantities determine the low energy effective dynamics of the two-dimensional gauge theories restricted to the so-called Ω-background. The twisted superpotentials play also a pivotal role in another duality, the aforementioned Bethe/gauge correspondence [16] that maps supersymmetric vacua of the $\mathcal{N}=2$ theories to Bethe states of quantum integrable systems (QIS). A result of that duality is that the twisted superpotentials are identified with the *Yang–Yang functions* which describe spectra of the corresponding quantum integrable systems. Hence, combining the classical–NS limit of the AGT duality and the Bethe/gauge correspondence one thus gets the triple correspondence:

CFT_2	2d $\mathcal{N}=2$ SU(2) SYM	2-particle QIS
Classical Virasoro blocks	twisted superpotentials	spectra of Schrödinger operators

which links the classical Virasoro blocks to the SU(2) twisted superpotentials and then to spectra of some Schrödinger operators. Indeed, let us note that 2-particle QIS are nothing but quantum-mechanical systems. The above correspondence can be extended to the following:

CFT_2	2d $\mathcal{N}=2$ SU(N) SYM	N-particle QIS
Classical Toda blocks	twisted superpotentials	Yang–Yang functions

if we take the classical–NS limit of the generalized AGT conjecture [10].

[2] On the relation between the quantization of the Hitchin system, SYM theories, CFT_2 and the geometric Langlands program, see also [19].

Concluding, the interest in the classical conformal blocks and their uses has recently dramatically increased. Almost every day one may find in a literature more new, additional to those described above, fascinating contexts, in which these functions emerge. Some of these topics are: spectral problems for some Schrödinger operators [15,20–22]; the Painlevé VI equation [23]; the KdV equation [24]; the isomonodromic deformation problem [25]; S-duality in $\mathcal{N}=2$ super Yang–Mills theories [26]; entanglement entropy in CFT_2 and $\mathrm{CFT}_2/\mathrm{AdS}_3$ holography [27, 28]; quantum chaos [29]; the connection problem for the Heun equation and scattering in black hole backgrounds [30–32]; black holes, holography and information paradox [33]; semi-classical spectra and restrictions on holographic two-dimensional CFT's [28,34]; holographic interpretation of classical blocks and classical bootstrap [34–41].

1.3. Objectives, motivations, outline

As has been already mentioned, in two-dimensional conformal field theory the Heun equation emerges in two circumstances.

First, eq. (1.3) one gets in the classical limit from the BPZ null vector decoupling equation [3] for the Liouville 5-point function of primary fields with one light degenerate field $\mathsf{V}_{\Delta_{2,1}}(z,\bar{z})$, where $\Delta_{2,1}=-\frac{1}{2}-\frac{3}{4}b^2$. The accessory parameter $c_2(x)$ in eq. (1.3) is determined in this framework by the classical Liouville action on the 4-punctured sphere.

On the other hand, the degenerate 5-point blocks in the Liouville correlation function fulfill the same null vector decoupling equation as the physical correlator itself. Applying to this equation the asymptotic (1.5) with $n=5$ and $\Delta_{\text{light}}=\Delta_{2,1}$ one gets the normal form Heun equation with the two linearly independent solutions (Ψ_+,Ψ_-) known as the *path-multiplicative* or *Floquet type solutions*, cf. [23,42]. Here, the Heun accessory parameter $c_2(x)$ is a complex-valued function determined by the classical 4-point block on the sphere. In fact, one obtains the following claim:

Proposition 1.1. *The accessory parameter $c_2(x)$ in the Heun equation (1.3) is given by the derivative of the classical 4-point block on the Riemann sphere w.r.t. the modular parameter x,*

$$c_2(x)=\frac{\partial}{\partial x}\,f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x),\qquad \delta:=\tfrac{1}{4}\left(1-\lambda^2\right) \tag{1.6}$$

iff the two linearly independent solutions $(\Psi_+(z),\Psi_-(z))$ of the equation (1.3) are of the Floquet type and have diagonal monodromy along a curve $\gamma(0,x)$ encircling both 0 and x, and the corresponding monodromy matrix $\mathrm{M}_{\gamma(0,x)}$ obeys $\mathrm{Tr}\,\mathrm{M}_{\gamma(0,x)}=-2\cos(\pi\lambda)$.

Recall that the path-multiplicative (Floquet) solutions of the Heun equation are related to two singularities, for example, $z=0$ and $z=1$. Let γ be a smoothly parametrized closed contour encircling these points: $\gamma: z=\varphi(s)$ for $s\in[0,1]$, $\varphi(0)=\varphi(1)=z_0$. A Floquet solution $y(z)$ is defined as a solution satisfying $y(\varphi(1))=\mathrm{e}^{2\pi i\sigma}y(\varphi(0))$. So, it is a solution which is multiplied by a constant factor $\mathrm{e}^{2\pi i\sigma}$ if we pass around a simple closed contour in the z-plane that encircles two of the three singularities $0,1,x$. The value σ is called a *Floquet exponent* or *path exponent* or *characteristic exponent*. It depends on the parameters of the Heun equation [42]. Within the CFT realization the characteristic exponent σ is determined by the intermediate classical conformal weight δ.

To get the claim above one can use perturbative methods (see [43,44], where the implication $\Leftarrow$ was shown) and a standard conformal field theory machinery. It should be stressed that the CFT tools, such as the OPE, allows to find monodromy properties of the solutions $(\Psi_+(z), \Psi_-(z))$ without their explicit computation. However, it would be interesting to know if these solutions can be calculated within CFT_2. For this reason in the present paper we study a mechanism of the semi-classical 'heavy–light' factorization (1.5) in the case of the degenerate 5-point blocks. Our aim here is to answer the question whether the formula (1.5) for $n = 5$ and $\Delta_{\rm light} = \Delta_{2,1}$ really allows to explicitly compute the Floquet type Heun's solutions.

Indeed, little is known about a concrete form of the Floquet type solutions (cf. the end of subsection 3.2.1 in [42]) and an elaboration of any method useful for practical numerical calculation is of great importance. In particular, in [42] one can read that the path-multiplicative Heun's functions 'theoretically can be constructed as series'

$$\sum_{-\infty}^{+\infty} c_n f_{\sigma+n}(z) \tag{1.7}$$

with conjectured form of the coefficients $f_{\sigma+n}(z)$ given by the hypergeometric function $f_{\sigma+n}(z) = z^{\sigma}\,{}_2\mathsf{F}_1(\cdot;\cdot;\cdot;z)$, where dots denote dependence on the combinations of parameters of the Heun equation and the characteristic exponent σ. However, further one can find in [42] that 'in the general case, the form of the dependence of the path exponents σ on the parameters of the Heun equation is lacking, and hence expansions (1.7) cannot be used for practical numerical calculations.'

The answer to the above question allows to make contact with the perturbative techniques used to solve the monodromy problem for the Heun equation determining the classical 4-point block, cf. [43,44]. Both, the CFT and complementary perturbative methods can be capitalized in concrete applications. In particular, it seems very promising to use the CFT methods in the study of scalar perturbations of certain black hole (BH) backgrounds described by the Heun equation. This particular application is our main motivation for the present work. We spell out it in more details below.

It is known that the Klein–Gordon equation in the Kerr–AdS_5 background [45] can be reduced to the two (angular and radial) Heun equations by a separation of variables [46].[3] Having in mind two contexts in which the Heun equation appears in 2-dimensional CFT one may consider an idea of the formal correspondence between the dynamics of a scalar field in Kerr–AdS_5 and the Liouville theory/chiral CFT_2. Such concept has recently occurred in [48].[4] First, it would be nice to explore this relation at the formal level, i.e. to complete a dictionary and to describe formal implications, as it has been partially done in [48]. Interestingly, this identification has many common features with what is known as the Kerr/CFT correspondence [49]. It would be very interesting to examine whether it is an accidental similarity or indeed something deeper.

Then, one may ask about its possible application in the computation of the so-called BH quasi-normal modes (QNMs), cf. [50]. These quantities are certain *complex frequency modes* that resonate when the BH is weakly perturbed, e.g. by the weak scalar field. The quasi-normal modes are important for studying a stability of the black holes. They are also useful in the study of a plasma through the finite temperature AdS/CFT correspondence. For the Kerr–AdS_5 BH a

[3] In fact, a bit more general situation can be considered, than those in [46], namely, for the metric of the 5-dimensional *charged* AdS BH, cf. e.g. [47]. Such BH solution contains the electric charge Q in addition and for $Q = 0$ reduces to the metric discussed in [45,46].

[4] See also [30–32].

preliminary analysis shows that the parameter in the radial and angular Heun equations, which can be interpreted as the frequency of the BH oscillations, is determined by the complex accessory parameter, and so, by the *calculable* complex-valued classical 4-point block $f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$. Using such an identification one can express the frequency as a function of the other parameters in the radial/angular Heun equation and the characteristic exponent related to δ. However, to interpret the frequency parameter as the BH frequency QNMs the scalar field, and hence, its radial and angular parts have to fulfill physically appropriate boundary conditions at the horizon (purely ingoing waves) and at the spatial infinity (purely outgoing waves).[5] Therefore, one of the key points that needs to be clarified here is to answer the question whether the solutions available within CFT_2 obey these boundary conditions.

The above ideas should work also for other BH backgrounds. For instance, the radial equation for the scalar perturbation of the Kerr–NUT–(A)dS BH in four dimensions is reducible to the Heun equation. Next, *confluent* Heun equations determine the scalar perturbations of the Kerr metric in four dimensions.[6] It is an interesting question of how to get the latter, i.e., the confluent Heun equation together with its certain solutions entirely within the formalism of CFT_2. Perhaps one needs to analyze the collision and classical limits of the BPZ equation for the 5-point blocks. In such a way the quantum and classical irregular conformal blocks should come to the game, cf. [31].[7]

Closely related to the scattering problems in the black hole backgrounds is the connection problem for the Heun and confluent Heun equations, cf. e.g. [42,51]. It is a question of how to express any of the pairs of linearly independent solutions of eq. (1.1) around some singular point 0 or 1 or x or ∞ as a linear combination of any one of the other three such pairs. Recently some progress in the solution of the connection problem has been achieved mainly due to the application of the so-called 'isomonodromic approach' and results relating regular and irregular conformal blocks to Painlevé VI and Painlevé V τ-functions, cf. [30–32].

The structure of the paper is as follows. In subsection 2.1 a well-known fact is reminded, namely, it is shown how the CFT machinery determines a monodromy of the Heun's solutions referred to in the Proposition 1.1. In subsection 2.2 the Heun equation is studied within purely mathematical representation theoretic formalism of the conformal blocks. The formula for the linearly independent solutions of the normal form Heun equation in terms of the classical limit of the conformal blocks is derived. In subsection 3.1 the heavy–light factorization property of the degenerate 5-point blocks is studied. This analysis leads to a computation of the limit defining the Floquet type Heun's functions and in fact yields a new method of the calculation of the latter. A concrete example of the path-multiplicative Heun function computed in this way is presented there and in the appendix B. In subsection 3.2 the $x \to 0$ limit of the solution extracted from the conformal blocks is computed. This calculation confirms that the 'CFT solution' of the Heun equation has for $x \to 0$ the expected form in terms of a hypergeometric function. In subsection 3.3 a comparison with the perturbative techniques is discussed. In section 4 conclusions of the present work and open problems for further research are collected.

5 Exactly, the presence of the horizon implies that the boundary value problem, which must be solved to determine BH QNMs, is non-hermitian and associated eigenvalues (= QNMs) are complex.

6 These equations are known in the literature as the Teukolsky equations (cf. [42] and refs. therein).

7 Moreover, it is reasonable to expect that there exists a monodromy problem for the confluent Heun equation whose solution matches the CFT results.

2. The Heun equation in CFT$_2$

2.1. Classical limit of BPZ equation for the degenerate five-point function

Let us consider the projected 5-point function on the sphere in the diagonal theory ($\Delta_i = \bar{\Delta}_i$):

$$G_{\Delta}(z,x) := \Big\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathsf{V}_{-\frac{b}{2}}(z,\bar{z})\mathsf{P}_{\Delta,\Delta}\mathsf{V}_2(x,\bar{x})\mathsf{V}_1(0,0)\Big\rangle,$$

where $\mathsf{V}_{\alpha=-\frac{b}{2}}$ is the degenerate field with the conformal weight

$$\Delta_{-\frac{b}{2}} = \Delta_{\alpha=-\frac{b}{2}} = \alpha(Q-\alpha) = -\frac{1}{2}-\frac{3}{4}b^2, \qquad Q = b+\frac{1}{b}$$

and V_i's are the four heavy primary operators ($\Delta_i = b^{-2}\,\delta_i$, $\delta_i = \mathcal{O}(1)$). The function $G_{\Delta}(z,x)$ satisfies the following null vector decoupling (NVD) equation [3]:

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial z^2} - b^2\left(\frac{1}{z}-\frac{1}{1-z}\right)\frac{\partial}{\partial z}\right]G_{\Delta}(z,x)\\
&\quad = -b^2\left[\frac{\Delta_1}{z^2}+\frac{\Delta_2}{(z-x)^2}+\frac{\Delta_3}{(1-z)^2}+\frac{\Delta_1+\Delta_2+\Delta_3+\Delta_{-\frac{b}{2}}-\Delta_4}{z(1-z)}\right.\\
&\qquad \left. + \frac{x(1-x)}{z(z-x)(1-z)}\frac{\partial}{\partial x}\right]G_{\Delta}(z,x).
\end{aligned} \tag{2.8}$$

Our aim now is to consider the classical limit of the equation above. The key point here is an observation that in the limit $b \to 0$ only the operator with weight $\Delta_{-\frac{b}{2}}$ remains light ($\Delta_{-\frac{b}{2}} = \mathcal{O}(1)$) and its presence in the correlation function has no influence on the classical dynamics. Then, for $b \to 0$

$$G_{\Delta}(z,x) \sim \Psi(z)\,\mathrm{e}^{-\frac{1}{b^2}\left(S^{\rm cl}(\delta_4,\delta_3,\delta)+S^{\rm cl}(\delta,\delta_2,\delta_1)-f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)-\bar{f}_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{x})\right)}. \tag{2.9}$$

Indeed, assuming that the light field does not contribute to the classical limit we are left with the projected 4-point function of the heavy operators:

$$\begin{aligned}
&\Big\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathsf{P}_{\Delta,\Delta}\mathsf{V}_2(x,\bar{x})\mathsf{V}_1(0,0)\Big\rangle\\
&\quad = C(\Delta_4,\Delta_3,\Delta)\,C(\Delta,\Delta_2,\Delta_1)\,\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)\bar{\mathcal{F}}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](\bar{x})\\
&\quad \overset{b\to 0}{\sim} \mathrm{e}^{-\frac{1}{b^2}\left(S^{\rm cl}(\delta_4,\delta_3,\delta)+S^{\rm cl}(\delta,\delta_2,\delta_1)-f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)-\bar{f}_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{x})\right)}.
\end{aligned}$$

The quantities $S^{\rm cl}(\delta_3,\delta_2,\delta_1)$, known as the classical 3-point actions, are the classical limits of the structure constants $C\big(\Delta_i,\Delta_j,\Delta_k\big)$. The functions

$$\begin{aligned}
\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) &= \Delta_4 - \underset{1}{\overset{\Delta_3}{|}} \overset{\Delta}{\overline{\qquad\quad}} \underset{x}{\overset{\Delta_2}{|}} - \Delta_1\\
&= x^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{n>0}x^n\,\mathcal{F}^{(n)}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\right)
\end{aligned} \tag{2.10}$$

and

$$f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) = \lim_{b\to 0} b^2 \log \mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \tag{2.11}$$

are quantum, and classical 4-point blocks on the sphere respectively. The substitution of eq. (2.9) into the NVD eq. (2.8) leads, within the limit $b \to 0$, to the normal form of the Heun equation (1.3) with the accessory parameter $c_2(x)$ determined by the classical 4-point block, according to (1.6), cf. [23].

Alternatively, the parameter $c_2(x)$ can be found once the solution of certain Bethe-like saddle point equation defining the twisted superpotential of the SU(2) $\mathrm{N}_f = 4$ $\mathcal{N} = 2$ SYM theory is known, cf. [52].

Let us stress that the holomorphic accessory parameter $c_2(x)$ is related to that which emerges in the uniformization theory of the 4-punctured sphere. The real-valued accessory parameter for $C_{0,4}$ can be found by taking the classical limit of the NVD equations obeyed by the physical Liouville 5-point function on the sphere with $\mathsf{V}_{\alpha=-\frac{b}{2}}$, cf. [7,8,52,53]. The latter is nothing but the projected correlation function discussed above integrated over the Liouville field theory spectrum $\Delta = \Delta_\alpha$, $\alpha = \frac{1}{2}Q + i\mathbb{R}^+$ with the structure constant $C(\Delta_3, \Delta_2, \Delta_1)$ identified as the Liouville 3-point function [7,54].

Monodromy properties of the solutions

Let us consider monodromy properties of the independent solutions $\Psi_\pm$ of eq. (1.3) with the accessory parameter given by (1.6). It turns out that the substitution (2.9) automatically fixes the solutions $\Psi_\pm$ to be of the Floquet type with the characteristic exponent σ determined by the intermediate classical conformal weight $\delta = \frac{1}{4}(1 - \lambda^2)$ [23]. First, note that the functions on both sides of eq. (2.9) should have the same monodromy properties along the contour encircling the points 0 and x. Secondly, from a completeness of the intermediate states follows that the monodromy properties w.r.t. z of the 5-point function $G_\Delta(z, x)$ along a curve encircling both 0 and x are the same as the monodromy of the 4-point correlator:

$$\left\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathsf{V}_{\Delta_{-\frac{b}{2}}}(z,\bar{z})\mathsf{V}_{\Delta_\alpha}(0,0)\right\rangle \tag{2.12}$$

for a curve encircling 0. The z dependence (i.e., an overall prefactor) of the correlator (2.12) can be read off from (holomorphic part of) the OPE:

$$\begin{aligned}\mathsf{V}_{-\frac{b}{2}}(z)\mathsf{V}_\alpha(0) &= z^{\Delta_+ - \Delta_{-\frac{b}{2}} - \Delta_\alpha} C\big(\Delta_+, \Delta_{-b/2}, \Delta_\alpha\big)\Big[\mathsf{V}_{\alpha_+}(0) + \text{descendants}\Big]\\ &\quad + z^{\Delta_- - \Delta_{-\frac{b}{2}} - \Delta_\alpha} C\big(\Delta_-, \Delta_{-b/2}, \Delta_\alpha\big)\Big[\mathsf{V}_{\alpha_-}(0) + \text{descendants}\Big],\end{aligned}$$

where $\Delta_\pm \equiv \Delta_{\alpha_\pm}$, $\alpha_\pm = \alpha \pm \frac{b}{2}$, and explicitly,

$$\begin{aligned}\Delta_+ &= \Delta_\alpha - \alpha b + \frac{b^2}{4} + \frac{1}{2},\\ \Delta_- &= \Delta_\alpha + \alpha b - \frac{3b^2}{4} - \frac{1}{2}.\end{aligned}$$

Assuming that the intermediate weight Δ_α is heavy, i.e.,

$$\alpha = \frac{1}{2b}(1-\lambda) \quad \Leftrightarrow \quad \lim_{b\to 0} b^2\Delta_\alpha = \frac{1}{4}\left(1-\lambda^2\right) = \delta$$

then in the limit $b \to 0$ one gets $\Delta_{\pm} - \Delta_{-\frac{b}{2}} - \Delta_{\alpha} \to \frac{1}{2}(1 \pm \lambda)$. Therefore, in the space of solutions of eq. (1.3) there exist basis solutions $\Psi_{\pm}(z) \propto z^{\frac{1}{2}(1\pm\lambda)}$ which analytically continued in z along the path encircling the points 0 and x satisfy the condition:

$$\Psi_{\pm}(\mathrm{e}^{2\pi i}z) = -\mathrm{e}^{\pm i\pi\lambda}\Psi_{\pm}(z). \tag{2.13}$$

This corresponds to the monodromy matrix with trace equal $-2\cos(\pi\lambda)$.

As a final remark in this subsection let us notice that knowing monodromies (2.13) of the solutions of eq. (1.3) one can determine the classical 4-point block solving the Riemann–Hilbert problem formulated as follows: adjust $c_2(x)$ in such a way that the eq. (1.3) admits solutions with the monodromy around 0 and x given by (2.13). The latter was Zamolodchikov's idea which allowed him to find (i) a large classical intermediate weight behavior of the classical 4-point block, (ii) a large quantum intermediate weight behavior of the quantum 4-point block and its nome $q(x)$ expansion [55].

2.2. *Classical limit of BPZ equation for the degenerate five-point blocks*

Let $\mathcal{V}_{c,\Delta}^{(n)}$ denote the vector space generated by all vectors of the form:

$$|\,\Delta_I^n\,\rangle = L_{-I}|\,\Delta\,\rangle \equiv L_{-k_1}\ldots L_{-k_{\ell(I)}}|\,\Delta\,\rangle, \qquad n = k_1 + \ldots + k_{\ell(I)} =: |I|, \tag{2.14}$$

where $I = (k_1 \geq \ldots \geq k_{\ell(I)} \geq 1)$ is a partition of n,[8] L_n's are the Virasoro generators obeying

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{2.15}$$

and $|\,\Delta\,\rangle$ is the highest weight state with the following property:

$$L_0|\,\Delta\,\rangle = \Delta|\,\Delta\,\rangle, \qquad L_n|\,\Delta\,\rangle = 0, \quad \forall\, n > 0. \tag{2.16}$$

The representation of the Virasoro algebra on the space:

$$\mathcal{V}_{c,\Delta} = \bigoplus_{n=0}^{\infty} \mathcal{V}_{c,\Delta}^{(n)}, \qquad \mathcal{V}_{c,\Delta}^{(0)} = \mathbb{R}|\,\Delta\,\rangle$$

defined by the relations (2.15), (2.16) is called the Verma module with the central charge c and the highest weight Δ. It is clear that $\dim \mathcal{V}_{c,\Delta}^{(n)} = p(n)$, where $p(n)$ is the number of partitions of n (with the convention $p(0) = 1$). On $\mathcal{V}_{c,\Delta}^{(n)}$ exists symmetric bilinear form $\langle\,\cdot\,|\,\cdot\,\rangle$ uniquely defined by the relations $\langle\,\Delta\,|\,\Delta\,\rangle = 1$ and $(L_n)^{\dagger} = L_{-n}$.

Let $\mathcal{V}_{\Delta}$ be the Verma module with the highest weight state $|\,\nu_{\Delta}\,\rangle$. We define the chiral vertex operator (CVO) as the linear map

$$V_{\infty\;z\;0}^{\Delta_3\Delta_2\Delta_1} : \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_1} \longrightarrow \mathcal{V}_{\Delta_3}$$

such that for all $|\,\xi_2\,\rangle \in \mathcal{V}_{\Delta_2}$ the operator

$$V(\xi_2|z) \equiv V_{\infty\;z\;0}^{\Delta_3\Delta_2\Delta_1}(|\,\xi_2\,\rangle \otimes \cdot\,) : \mathcal{V}_{\Delta_1} \longrightarrow \mathcal{V}_{\Delta_3}$$

satisfies the following conditions:

[8] We will use the notation $I \vdash n$.

$$[L_n, V(\nu_2|z)] = z^n\left(z\frac{\partial}{\partial z} + (n+1)\Delta_2\right)V(\nu_2|z)\,, \qquad n\in\mathbb{Z} \tag{2.17}$$

$$V(L_{-1}\xi_2|z) = \frac{\partial}{\partial z}V(\xi_2|z)\,, \tag{2.18}$$

$$V(L_n\xi_2|z) = \sum_{k=0}^{n+1}\binom{n+1}{k}(-z)^k\left[L_{n-k}, V(\xi_2|z)\right], \qquad n>-1, \tag{2.19}$$

$$\begin{aligned} V(L_{-n}\xi_2|z) &= \sum_{k=0}^{\infty}\binom{n-2+k}{n-2} z^k\, L_{-n-k}\, V(\xi_2|z) \\ &\quad + (-1)^n\sum_{k=0}^{\infty}\binom{n-2+k}{n-2} z^{-n+1-k}\; V(\xi_2|z)\; L_{k-1}, \quad n>1 \end{aligned} \tag{2.20}$$

and

$$\left\langle\,\nu_{\Delta_3}\,|V\left(\nu_{\Delta_2}|z\right)|\,\nu_{\Delta_1}\right\rangle = z^{\Delta_3-\Delta_2-\Delta_1}. \tag{2.21}$$

The commutation relation (2.17) defines the primary vertex operator corresponding to the highest weight state $|\,\nu_2\,\rangle\in\mathcal{V}_{\Delta_2}$. Eqs. (2.18)–(2.20) characterize the descendant CVO's.

Let $|\,0\,\rangle$ denote the vacuum state, i.e., the highest weight state in the vacuum module with the highest weight $\Delta=0$. Conformal blocks on the Riemann sphere are defined as the matrix elements

$$\langle\,0\,|V_{\Delta_n}(z_n)\ldots V_{\Delta_1}(z_1)|\,0\,\rangle \tag{2.22}$$

of compositions of the primary chiral vertex operators[9]:

$$V_{\Delta_j}(z)\;\equiv\; V\left(\nu_j|z\right)\;\equiv\; V^{\alpha_j}_{\alpha_k,\alpha_i}(z):\mathcal{V}_{\Delta_i}\longrightarrow\mathcal{V}_{\Delta_k}, \qquad \Delta_l=\Delta_{\alpha_l}=\alpha_l(Q-\alpha_l),$$

cf. [5,6]. By inserting projection operators[10] $\mathbb{P}_{\tilde{\Delta}_p}$ on the intermediate conformal weights $\tilde{\Delta}_p$, $p=1,\ldots,n-3$ into the internal channels of the spherical blocks (2.22) one gets the latter in terms of the formal power series. Coefficients of these series consist of three-point blocks (forms ρ defined below) and inverses $(G^{(n)}_{c,\Delta})^{IJ}$ of the Gram matrix

$$\left(G^{(n)}_{c,\Delta}\right)_{IJ} \;=\; \langle\,\Delta^n_I\,|\,\Delta^n_J\,\rangle. \tag{2.23}$$

For a given triple Δ_1, Δ_2, Δ_3 of conformal weights we define the trilinear map

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\;0}:\mathcal{V}_{\Delta_3}\otimes\mathcal{V}_{\Delta_2}\otimes\mathcal{V}_{\Delta_1}\longrightarrow\mathbb{C}$$

induced by the matrix element of a single chiral vertex operator

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\;0}(\xi_3,\xi_2,\xi_1)\;=\;\langle\,\xi_3\,|V(\xi_2|z)\,|\,\xi_1\,\rangle\,, \qquad \forall\,|\,\xi_i\,\rangle\in\mathcal{V}_{\Delta_i}, \quad i=1,2,3.$$

The form $\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\;0}$ (three-point block) is uniquely determined by the conditions (2.17)–(2.20). In particular,

[9] We adopt here three equivalent notations for primary CVO's.

[10] $\mathbb{P}_{\tilde{\Delta}_p}$ are identity operators in $\mathcal{V}_{c,\tilde{\Delta}_p}$ built out of the basis vectors (2.14) and their duals (cf. calculations below).

1. for L_0-eingenstates[11] $L_0|\,\xi_i\,\rangle = \Delta_i(\xi_i)|\,\xi_i\,\rangle$, $i = 1, 2, 3$ one gets

$$\rho_{\infty\;z\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\xi_3, \xi_2, \xi_1) = z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\rho_{\infty\;1\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\xi_3, \xi_2, \xi_1)\,; \tag{2.24}$$

2. for basis vectors $\nu_{i,I} \equiv |\,\nu_{\Delta_i,I}\,\rangle \in \mathcal{V}_{\Delta_i}$, $i = 1, 2, 3$ one finds

$$\begin{aligned}
&\rho_{\infty\;1\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_{3,I}, \nu_2, \nu_1) = \gamma_{\Delta_3}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_I ,\\
&\rho_{\infty\;1\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_3, \nu_2, \nu_{1,I}) = \gamma_{\Delta_1}\left[\begin{smallmatrix}\Delta_2\\ \Delta_3\end{smallmatrix}\right]_I ,\\
&\rho_{\infty\;1\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_3, \nu_{2,I}, \nu_1) = (-1)^{|I|}\gamma_{\Delta_2}\left[\begin{smallmatrix}\Delta_1\\ \Delta_3\end{smallmatrix}\right]_I ,
\end{aligned} \tag{2.25}$$

where for a given partition $I = (k_1, \ldots, k_{\ell(I)})$, $k_i \geq k_j \geq 1$, $i < j$,

$$\gamma_{\Delta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_I \equiv \prod_{i=1}^{\ell(I)}\left(\Delta + k_i\Delta_2 - \Delta_1 + \sum_{i<j}^{\ell(I)} k_j\right). \tag{2.26}$$

Let us stress once again that using the three-point blocks

$$\begin{matrix} & & \Delta_2 & & \\ & & | & & \\ \Delta_3 & — & z & — & \Delta_1\end{matrix} \;\equiv\; \rho_{\infty\;z\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_3, \nu_2, \nu_1)$$

and the inverse $(G_{c,\Delta}^{(n)})^{IJ}$ of the Gram matrix (2.23) one can construct the matrix elements of various compositions of the primary chiral vertex operators.[12] For instance, the four-point block in the s channel reads as follows

$$\begin{aligned}
&\begin{matrix} & & \Delta_3 & & & & \Delta_2 & & \\ & & | & & & & | & & \\ \Delta_4 & — & z_3 & — & \Delta_s & — & z_2 & — & \Delta_1\end{matrix}\\
&= \sum_{I,J}\rho_{\infty\;z_3\;0}^{\Delta_4\,\Delta_3\,\Delta_s}(\nu_4, \nu_3, \nu_{s,I})\Big(G_{c,\Delta_s}\Big)^{IJ}\rho_{\infty\;z_2\;0}^{\Delta_s\,\Delta_2\,\Delta_1}(\nu_{s,J}\,,\;\nu_2\,,\;\nu_1).
\end{aligned} \tag{2.27}$$

Precisely, eqs. (2.25) and (2.27) yield a function introduced in (2.10),

$$\begin{aligned}
&\begin{matrix} & & \Delta_3 & & & & \Delta_2 & & \\ & & | & & & & | & & \\ \Delta_4 & — & 1 & — & \Delta_s & — & x & — & \Delta_1\end{matrix}\\
&= x^{\Delta_s-\Delta_2-\Delta_1}\left(1 + \sum_{n=1}^{\infty}x^n\sum_{|I|=|J|=n}\gamma_{\Delta_s}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_I\Big(G_{c,\Delta_s}^{(n)}\Big)^{IJ}\gamma_{\Delta_s}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_J\right).
\end{aligned}$$

[11] Note that for the basis vectors $\{|\,\nu_{i,I}\,\rangle\}$ one has $\Delta_i(\nu_{i,I}) = \Delta_i + |I|$.

[12] It should be stressed that the compositions are well defined when the weights in the internal channels are non-degenerate.

Curious class of CVO's form the degenerate operators [3] $V_{\Delta_{rs}}(z)$ having conformal weights:

$$\Delta_{r,s} = \frac{Q^2}{4} - \frac{1}{4}\left(rb + \frac{s}{b}\right)^2, \qquad Q = b + \frac{1}{b}, \qquad r, s \in \mathbb{N}$$

being zeros of the Kac determinant [56–60]:

$$\det\left(G^{(n)}_{c,\Delta}\right)_{IJ} = \det\langle\, \Delta^n_I \,|\, \Delta^n_J \,\rangle = \text{const.}(n) \times \prod_{1\le rs\le n} (\Delta - \Delta_{r,s})^{p(n-rs)}.$$

Let us consider the degenerate chiral vertex operators of the form

$$V^{(\pm)}_{\Delta_{2,1}}(z) \equiv V^{-b/2}_{\beta_\pm,\beta}(z), \;\; \Delta_{2,1} = \Delta_{-b/2} = -\frac{1}{2} - \frac{3}{4}b^2. \tag{2.28}$$

If $\beta_\pm = \beta \pm \frac{b}{2}$ then

1. the CVO's $V^{(\pm)}_{\Delta_{2,1}}(z)$ satisfy the differential equation

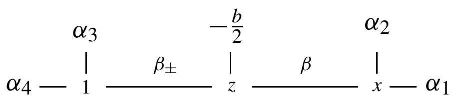
$$\frac{1}{b^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} V^{(\pm)}_{\Delta_{2,1}}(z) + :\mathrm{T}V^{(\pm)}_{\Delta_{2,1}}(z): = 0\,, \tag{2.29}$$

where

$$:\mathrm{T}V_\Delta(z): \equiv \sum_{n\leqslant -2} z^{-n-2} L_n V_\Delta(z) + \sum_{n\geqslant -1} V_\Delta(z) L_n z^{-n-2}\,; \tag{2.30}$$

2. conformal blocks with $V^{(\pm)}_{\Delta_{2,1}}(z)$ obey certain partial differential equations (PDE's), in particular, the 5-point degenerate conformal blocks[13]:

$$\mathscr{F}_\pm(z,x) \equiv \langle \alpha_4 | V^{\alpha_3}_{\alpha_4,\beta_\pm}(1) V^{-b/2}_{\beta_\pm,\beta}(z) V^{\alpha_2}_{\beta,\alpha_1}(x) | \alpha_1 \rangle$$

$=$ 

obey the following PDE[14]:

$$\left[\frac{\partial^2}{\partial z^2} - b^2\left(\frac{1}{z} - \frac{1}{1-z}\right)\frac{\partial}{\partial z} + b^2\left(\frac{\Delta_1}{z^2} + \frac{\Delta_2}{(z-x)^2} + \frac{\Delta_3}{(1-z)^2} + \frac{\Delta_1+\Delta_2+\Delta_3+\Delta_{-\frac{b}{2}}-\Delta_4}{z(1-z)}\right) + b^2\frac{x(1-x)}{z(z-x)(1-z)}\frac{\partial}{\partial x}\right]\mathscr{F}_\pm(z,x) = 0. \tag{2.31}$$

In what follows we will compute the classical limit of eqs. (2.31). Let us assume that all 'momenta' α_i, β are heavy, i.e.[15]:

$$\alpha_i = \frac{\eta_i}{b}\,, \qquad \beta = \frac{\eta}{b} \tag{2.32}$$

---

13 Here $|\,\alpha_i\,\rangle \equiv |\,\Delta_{\alpha_i}\,\rangle$ and the operator-state correspondence is assumed.

14 Eqs. (2.31) are nothing but eq. (2.8) rewritten for conformal blocks.

15 Here β is the same as α in the previous subsection and $2\eta = 1 - \lambda$.

then, in the limit $b \to 0$ one gets

$$\mathscr{F}_{\pm}(z,x) \overset{b\to 0}{\sim} \Psi_{\pm}(\infty,1,z,x,0)\, \mathrm{e}^{\frac{1}{b^2} f_{\delta}\left[\substack{\delta_3\,\delta_2\\ \delta_4\,\delta_1}\right](x)}, \tag{2.33}$$

where $f_{\delta}\left[\substack{\delta_3\,\delta_2\\ \delta_4\,\delta_1}\right](x)$ is the classical spherical 4-point block (2.11) with $\delta = \lim_{b\to 0} b^2\Delta_\beta = \eta(1-\eta)$, $\delta_i = \lim_{b\to 0} b^2\Delta_i = \eta_i(1-\eta_i)$ and

$$\Psi_{\pm}(\infty,1,z,x,0) = \lim_{b\to 0} \frac{\langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta_\pm}(1)V^{-b/2}_{\beta_\pm,\beta}(z)V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle}{\langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta}(1)V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle}. \tag{2.34}$$

To see that the semi-classical asymptotic of $\mathscr{F}_{\pm}(z,x)$ has the factorized form (2.33) let us define for $Z = (z_4, z_3, z)$ the ratio (cf. appendix C.2 in [61])

$$\Psi_{\pm,n}(Z) \equiv \frac{\langle 0|V^{\alpha_4}_{0,\alpha_4}(z_4)V^{\alpha_3}_{\alpha_4,\beta_\pm}(z_3)V^{-b/2}_{\beta_\pm,\beta}(z)|\beta^n_I\rangle}{\langle 0|V^{\alpha_4}_{0,\alpha_4}(z_4)V^{\alpha_3}_{\alpha_4,\beta}(z_3)|\beta^n_I\rangle}, \tag{2.35}$$

where $|\,\beta^n_I\,\rangle \equiv |\,\Delta^n_{\beta,I}\,\rangle$ are the basis vectors of the form (2.14). Let us note that the numerator in (2.35) is a four-point block of two generic primary CVO's, a degenerate primary CVO and a general descendant CVO, respectively. On the other hand, the denominator in (2.35) is a three-point block built out of two generic primary CVO's and the generic descendant.

One can see that the functions $\Psi_{\pm,n}(Z)$ are light in the limit $b \to 0$, i.e., $\Psi_{\pm,n}(Z) \sim \mathcal{O}(b^0)$. Let us employ the notation $\mathsf{X} := V^{\alpha_4}_{0,\alpha_4}(z_4)V^{\alpha_3}_{\alpha_4,\beta_\pm}(z_3)$ and consider first the case $n = 0$,

$$\Psi_{\pm,0}(Z) \;=\; \frac{\langle\, 0\,|\,\mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z)|\,\Delta_\beta\,\rangle}{\langle\, 0\,|\,\mathsf{X}\,|\,\Delta_\beta\,\rangle}\,. \tag{2.36}$$

Note that the denominator of $\Psi_{\pm,0}(Z)$ for $Z = (\infty, 1, z)$ equals one, cf. (2.21). Then, the functions $\Psi_{\pm,0}(\infty,1,z)$ are nothing but the degenerate four-point blocks, which are well known to be given by the hypergeometric function ${}_2\mathsf{F}_1$, e.g.:

$$\begin{aligned}\Psi_{+,0}(\infty,1,z) = {}& z^{\Delta_{\beta_+}-\Delta_{2,1}-\Delta_\beta}(1-z)^{\Delta(\alpha_3-\frac{b}{2})-\Delta_3-\Delta_{2,1}}\\ &\times {}_2\mathsf{F}_1\left(b(\alpha_3-\alpha_4+\bar\beta-\tfrac{b}{2}), b(\alpha_3+\alpha_4-\beta-\tfrac{b}{2}); b(2\bar\beta-b); z\right),\end{aligned}$$

where $\bar\alpha_i = Q - \alpha_i$ and $\Delta(\alpha - \frac{b}{2}) - \Delta_\alpha - \Delta_{2,1} = b\alpha$. In the classical limit α_i's and β scale like (2.32) therefore the functions $\Psi_{\pm,0}(\infty,1,z)$ are really light in this limit.[16] Next, let us choose a basis state of the form $L_{-k}|\,\Delta_\beta\,\rangle$. Then,

$$\Psi_{\pm,k}(Z) \;=\; \frac{\langle\, 0\,|\,\mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z)L_{-k}|\,\Delta_\beta\,\rangle}{\langle\, 0\,|\,\mathsf{X}\,L_{-k}|\,\Delta_\beta\,\rangle}\,. \tag{2.37}$$

[16] Let's remind that a relation between the four-point block in arbitrary locations of the vertex operators and the block in $(z_4, z_3, z_2, z_1) = (\infty, 1, z, 0)$ is of the form

$$\begin{aligned}&\langle\, 0\,|V_{\Delta_4}(z_4)V_{\Delta_3}(z_3)V_{\Delta(-\frac{b}{2})}(z)V_{\Delta_\beta}(z_1)|\,0\,\rangle\\ &= (z_4-z_1)^{\Delta_3+\Delta_{-\frac{b}{2}}-\Delta_4-\Delta_\beta}(z_4-z)^{-2\Delta_{-\frac{b}{2}}}(z_4-z_3)^{\Delta_{-\frac{b}{2}}+\Delta_\beta-\Delta_4-\Delta_3}\\ &\quad\times(z_3-z_1)^{\Delta_4-\Delta_\beta-\Delta_{-\frac{b}{2}}-\Delta_3}\langle\,\Delta_4\,|V_{\Delta_3}(1)V_{\Delta(-\frac{b}{2})}(z)|\,\Delta_\beta\,\rangle\end{aligned}$$

as it follows from the $\mathrm{SL}(2,\mathbb{C})$ invariance of the vacuum state.

Using (2.36) and (2.17) one can compute

$$
\begin{aligned}
\langle 0 | \mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z) L_{-k} | \Delta_\beta \rangle &= \sum_{i=4,3,z} \left[\frac{(k-1)\Delta_i}{z_i^k} - \frac{1}{z_i^{k-1}} \partial_i \right] \langle 0 | \mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z) | \Delta_\beta \rangle \\
&= \sum_{i=4,3,z} \left[\frac{(k-1)\Delta_i}{z_i^k} - \frac{1}{z_i^{k-1}} \partial_i \right] \Psi_{\pm,0}(Z) \langle 0 | \mathsf{X} | \Delta_\beta \rangle \\
&= \Psi_{\pm,0}(Z) \langle 0 | \mathsf{X} L_{-k} | \Delta_\beta \rangle \\
&\quad + z^{-k}(k-1)\Delta_{-\frac{b}{2}} \Psi_{\pm,0}(Z) \langle 0 | \mathsf{X} | \Delta_\beta \rangle \\
&\quad - \sum_{i=4,3,z} \left(z_i^{1-k} \partial_i \Psi_{\pm,0}(Z) \right) \langle 0 | \mathsf{X} | \Delta_\beta \rangle ,
\end{aligned}
$$

where $\partial_i := \partial_{z_i}$, $z_z := z$, $\Delta_z := \Delta_{-\frac{b}{2}}$. Dividing both sides of this equation by $\langle 0 | \mathsf{X} L_{-k} | \Delta_\beta \rangle$ and taking into account the definition (2.37) one gets[17]

$$
\begin{aligned}
\Psi_{\pm,k}(Z) = \Psi_{\pm,0}(Z) + z^{-k}(k-1)\Delta_{-\frac{b}{2}} \Psi_{\pm,0}(Z) \frac{\langle 0 | \mathsf{X} | \Delta_\beta \rangle}{\langle 0 | \mathsf{X} L_{-k} | \Delta_\beta \rangle} \\
- \sum_{i=4,3,z} \left(z_i^{1-k} \partial_i \Psi_{\pm,0}(Z) \right) \frac{\langle 0 | \mathsf{X} | \Delta_\beta \rangle}{\langle 0 | \mathsf{X} L_{-k} | \Delta_\beta \rangle} .
\end{aligned}
$$

The above calculation can be simply generalized to any basis state (2.14).

So, one sees that the functions $\Psi_{\pm,n}(Z)$ are light in the limit $b \to 0$. Second property of $\Psi_{\pm,n}(Z)$ (already seen in the calculation above) is that the 'lightness property' of the ratios of the type (2.35) does not depend on the level n. Indeed, using (2.35) one may find

$$
\begin{aligned}
\langle 0 | \mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z) L_{-m} | \beta^n_I \rangle &= \sum_{i=4,3,z} \left[\frac{(m-1)\Delta_i}{z_i^m} - \frac{1}{z_i^{m-1}} \partial_i \right] \langle 0 | \mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z) | \beta^n_I \rangle \\
&= \sum_{i=4,3,z} \left[\frac{(m-1)\Delta_i}{z_i^m} - \frac{1}{z_i^{m-1}} \partial_i \right] \Psi_{\pm,n}(Z) \langle 0 | \mathsf{X} | \beta^n_I \rangle \\
&= \Psi_{\pm,n}(Z) \langle 0 | \mathsf{X} L_{-m} | \beta^n_I \rangle \\
&\quad + z^{-m}(m-1)\Delta_{-\frac{b}{2}} \Psi_{\pm,n}(Z) \langle 0 | \mathsf{X} | \beta^n_I \rangle \\
&\quad - \sum_{i=4,3,z} \left(z_i^{1-m} \partial_i \Psi_{\pm,n}(Z) \right) \langle 0 | \mathsf{X} | \beta^n_I \rangle .
\end{aligned}
$$

As before dividing both sides of this equation by $\langle 0 | \mathsf{X} L_{-m} | \beta^n_I \rangle$ we see that the 'shifted' ratio

$$
\begin{aligned}
\Psi_{\pm,n+m}(Z) := \frac{\langle 0 | \mathsf{X}\, V^{-b/2}_{\beta_\pm,\beta}(z) L_{-m} | \beta^n_I \rangle}{\langle 0 | \mathsf{X} L_{-m} | \beta^n_I \rangle} = \Psi_{\pm,n}(Z) \\
+ z^{-m}(m-1)\Delta_{-\frac{b}{2}} \Psi_{\pm,n}(Z) \frac{\langle 0 | \mathsf{X} | \beta^n_I \rangle}{\langle 0 | \mathsf{X} L_{-m} | \beta^n_I \rangle} \\
- \sum_{i=4,3,z} \left(z_i^{1-m} \partial_i \Psi_{\pm,n}(Z) \right) \frac{\langle 0 | \mathsf{X} | \beta^n_I \rangle}{\langle 0 | \mathsf{X} L_{-m} | \beta^n_I \rangle}
\end{aligned}
\tag{2.38}
$$

[17] Note that for $z_4 = \infty$ we have $\langle 0 | \mathsf{X} | \Delta_\beta \rangle = z_3^{\Delta_4 - \Delta_3 - \Delta_\beta}$ and $\langle 0 | \mathsf{X} L_{-k} | \Delta_\beta \rangle = z_3^{\Delta_4 - \Delta_3 - \Delta_\beta - k}$ (cf. (2.21)).

is also light.

Using the fact that the 'lightness property' of the ratios (2.35) does not depend on the level n and taking into account that the generic four-point block with heavy weights exponentiates in the classical limit,

$$\mathcal{F}_{c,\Delta_\beta}\left[{}^{\Delta_3\;\Delta_2}_{\Delta_4\;\Delta_1}\right](x) \equiv \langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta}(1)V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle \overset{b\to 0}{\sim} \exp\left\{\frac{1}{b^2}f_\delta\left[{}^{\delta_3\;\delta_2}_{\delta_4\;\delta_1}\right](x)\right\},$$

one can compute

$$\begin{aligned}
\mathscr{F}_{\pm}(z,x) &= \langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta_\pm}(1)V^{-b/2}_{\beta_\pm,\beta}(z)\mathbb{P}_\beta V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle \\
&= \sum_{n\geq 0}\sum_{I,J\vdash n}\left(G^{(n)}_\beta\right)^{IJ}\langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta_\pm}(1)V^{-b/2}_{\beta_\pm,\beta}(z)|\beta^n_I\rangle\langle\beta^n_J|V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle \\
&= \sum_{n\geq 0}\sum_{I,J\vdash n}\left(G^{(n)}_\beta\right)^{IJ}\Psi_{\pm,n}(Z)\,\langle\alpha_4|V^{\alpha_3}_{\alpha_4,\beta}(1)|\beta^n_I\rangle\langle\beta^n_J|V^{\alpha_2}_{\beta,\alpha_1}(x)|\alpha_1\rangle \\
&\overset{b\to 0}{\sim} \Psi_{\pm}(\infty,1,z,x,0)\exp\left\{\frac{1}{b^2}f_\delta\left[{}^{\delta_3\;\delta_2}_{\delta_4\;\delta_1}\right](x)\right\}.
\end{aligned}$$

In eqs. above $(G^{(n)}_\beta)^{IJ}$ is the inverse of the Gram matrix $(G^{(n)}_\beta)_{IJ} = \langle\,\Delta(\beta)^n_I\,|\,\Delta(\beta)^n_J\,\rangle$.[18] The above calculation defines the functions $\Psi_\pm(Z)$, so one can write

$$\Psi_\pm(\infty,1,z,x,0) = \lim_{b\to 0}\frac{\mathscr{F}_\pm(z,x)}{\mathcal{F}_{c,\Delta_\beta}\left[{}^{\Delta_3\;\Delta_2}_{\Delta_4\;\Delta_1}\right](x)}\,. \tag{2.39}$$

Turning to the problem of the classical limit of eqs. (2.31) it is easy to see that the limit $b\to 0$ taken from eqs. (2.31) after the substitution (2.33) yields the normal form of the Heun ordinary differential equation (1.3) with a pair of Floquet type linearly independent solutions given by eq. (2.39).[19]

In the next section we will look once again into depths of eq. (2.33) in order to explicitly compute the limit (2.39).

3. Heavy–light factorization and Floquet type Heun's solutions

3.1. Path-multiplicative solutions from conformal blocks

In this subsection we analyze the way the degenerate five-point blocks factorize in the classical limit. Without loss of generality, let us focus on $\mathscr{F}_+$. The analysis of the case $\mathscr{F}_-$ is analogous. Its expansion in terms of intermediate states reads

$$\begin{aligned}
\mathscr{F}_+(z,x) &= \langle\Delta_4|V_{\Delta_3}(1)\mathbb{P}_{\Delta_{\beta_+}}V_{\Delta_{2,1}}(z)\mathbb{P}_{\Delta_\beta}V_{\Delta_2}(x)|\Delta_1\rangle \\
&= x^{\Delta_\beta-\Delta_2-\Delta_1}z^{\Delta_{\beta_+}-\Delta_{2,1}-\Delta_\beta}\sum_{m,n\geq 0}x^n z^{m-n}\underbrace{\langle m;\,\Delta_{\beta_+}|V_{\Delta_{2,1}}(1)|n;\,\Delta_\beta\rangle}_{=:\,A_{m,n}},
\end{aligned} \tag{3.40}$$

[18] Here and below we use the equivalent notation $\Delta_\beta \equiv \Delta(\beta)$.

[19] For consistency of our calculations one can check that $\lim_{b\to 0} b^2\partial_z\Psi_\pm = 0$ and $\lim_{b\to 0} b^2\partial_x\Psi_\pm = 0$.

where we used the following notation

$$V_{\Delta_a}(x)|\Delta_b\rangle = x^{\Delta-\Delta_a-\Delta_b}\sum_{n\geq 0} x^n |n;\Delta\rangle\,,$$
$$|n;\Delta\rangle := \sum_{I\vdash n}\beta_\Delta\left[\begin{smallmatrix}\Delta_a\\ \Delta_b\end{smallmatrix}\right]^I L_{-I}|\Delta\rangle,\quad \beta_\Delta\left[\begin{smallmatrix}\Delta_a\\ \Delta_b\end{smallmatrix}\right]^I := \sum_{J\vdash n}\left(G_\Delta^{(n)}\right)^{IJ}\gamma_\Delta\left[\begin{smallmatrix}\Delta_a\\ \Delta_b\end{smallmatrix}\right]_J\,. \tag{3.41}$$

Formulas for bra vectors can be inferred from those above. The formula (3.40) can be given yet another form that is more useful for further study, namely

$$\mathscr{F}_+(z,x) = z^{\Delta_{\beta+}-\Delta_{2,1}-\Delta_\beta}\mathcal{F}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)\sum_{m\in\mathbb{Z}} z^m \chi_m(x), \tag{3.42a}$$

where

$$\chi_{m>0}(x) := \frac{\sum_{k\geq 0} x^k A_{m+k,k}}{\sum_{k\geq 0} x^k \mathcal{F}^{(k)}}\,,\qquad \chi_{m<0}(x) := \frac{\sum_{k\geq 0} x^{k-m} A_{k-m,k}}{\sum_{k\geq 0} x^k \mathcal{F}^{(k)}}\,,$$
$$\chi_{m=0}(x) := \frac{\sum_{k\geq 0} x^k A_{k,k}}{\sum_{k\geq 0} x^k \mathcal{F}^{(k)}}\,. \tag{3.42b}$$

The prefactor of eq. (3.42a) is the four-point conformal block defined as

$$\mathcal{F}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) = x^{\Delta_\beta-\Delta_2-\Delta_1}\sum_{n\geq 0} x^n \mathcal{F}^{(n)}\,, \tag{3.43a}$$

and

$$\mathcal{F}^{(n)} := \mathcal{F}^{(n)}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] := \sum_{I\vdash n}\gamma_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_I \beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^I\,. \tag{3.43b}$$

Let us consider for definiteness $\chi_{m>0}$. The matrix element $A_{m+n,n}$ has the following structure

$$A_{m+n,n} = \sum_{M\vdash n}\langle n+m;\Delta_{\beta+}|V_{\Delta_{2,1}}(z)L_{-M}|\Delta_\beta\rangle\beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^M\Big|_{z\to 1}\,. \tag{3.44}$$

Developing the matrix element under the sum yields

$$\begin{aligned}&\langle m+n;\Delta_{\beta+}|V_{\Delta_{2,1}}(z)L_{-M}|\Delta_\beta\rangle\\ &=\sum_{s=0}^{\ell(M)}\sum_{1\leq i_1<\ldots<i_s\leq\ell(M)}(-1)^s\langle m+n;\Delta_{\beta+}|L_{-M_s^\complement}\mathrm{ad}_{-M_s}V_{\Delta_{2,1}}(z)|\Delta_\beta\rangle\,,\end{aligned} \tag{3.45}$$

where $M_s^\complement$ is a complement of a partition M_s such that $M_s^\complement\cup M_s = M$ for any s. Explicitly, for $1\leq i_1<\ldots<i_s\leq\ell(M)$ they read

$$M_s^\complement = (k_1,\ldots,k_{i_1-1},k_{i_1+1},\ldots,k_{i_s-1},k_{i_s+1},\ldots k_{\ell(M)})\,,\qquad M_s = (k_{i_1},\ldots,k_{i_s})\,.$$

In particular $M_0^\complement = M_{\ell(M)} = M$, and $M_0 = M_{\ell(M)}^\complement = \{0\}$. We also denoted

$$\mathrm{ad}_{-M_s}V_{\Delta_{2,1}}(z) := \Big[L_{-k_{i_s}},\ldots\Big[L_{-k_{i_1}},V_{\Delta_{2,1}}(z)\Big]\Big].$$

In order to get further insight in to the formula (3.45) we notice that operating the Virasoro algebra element L_{-m} on the state $\langle\Delta; n|$ yields

$$\langle\Delta; n|L_{-m} = (\Delta + m\Delta_b - \Delta_a + n - m)\langle\Delta; n-m|\,, \quad m \leq n.$$

The generalization of this formula to $L_{-M} := L_{-k_1}\cdots L_{-k_{\ell(M)}}$, $|M| = m \leq n$ reads

$$\langle\Delta; n|L_{-M} = \gamma_{\Delta+n-m}\left[{}^{\Delta_b}_{\Delta_a}\right]_M \langle\Delta; n-m|\,. \tag{3.46}$$

Using the above result to the matrix element in eq. (3.45) and noticing that $|M_s^{\complement}| + |M_s| = n$ we obtain the general form of the term that contributes to the expansion (3.45)

$$\begin{aligned}&(-1)^s\langle m+n;\, \Delta_{\beta_+}|L_{-M_s^{\complement}}\mathrm{ad}_{-M_s}V_{\Delta_{2,1}}(z)|\Delta_\beta\rangle\big|_{z\to 1}\\ &= \mathcal{F}^{(m+|M_s|)}_{\Delta_{\beta_+}}\left[{}^{\Delta_3\;\Delta_{2,1}}_{\Delta_4\;\;\Delta_\beta}\right]\gamma_{\Delta_{\beta_+}+m+|M_s|}\left[{}^{\Delta_3}_{\Delta_4}\right]_{M_s^{\complement}}\gamma_{\Delta_\beta}\left[{}^{\Delta_{2,1}}_{\Delta_{\beta_+}+m+|M_s|}\right]_{M_s}.\end{aligned} \tag{3.47}$$

Observe that the first factor in the above equation is $m + |M_s|$-th coefficient of the four-point conformal block with a degenerate weight $\Delta_{2,1}$. As it has been already mentioned, the latter is well known to be related to hypergeometric function ${}_2\mathsf{F}_1$, namely

$$\begin{aligned}\mathcal{F}_{\Delta_{\beta_+}}\left[{}^{\Delta_3\;\Delta_{2,1}}_{\Delta_4\;\;\Delta_\beta}\right](z) &= z^{\Delta_{\beta_+}-\Delta_{2,1}-\Delta_\beta}\sum_{n\geq 0} z^n\mathcal{F}^{(n)}_{\Delta_{\beta_+}}\left[{}^{\Delta_3\;\Delta_{2,1}}_{\Delta_4\;\;\Delta_\beta}\right]\\ &= z^{\Delta_{\beta_+}-\Delta_{2,1}-\Delta_\beta}(1-z)^{\Delta(\alpha_3-\frac{b}{2})-\Delta_3-\Delta_{2,1}}\\ &\quad\times {}_2\mathsf{F}_1\left(b(\alpha_3-\alpha_4+\bar\beta-\tfrac{b}{2}), b(\alpha_3+\alpha_4-\beta-\tfrac{b}{2}); b(2\bar\beta-b); z\right),\end{aligned}$$

where $\bar\alpha_i = Q - \alpha_i$. Thus, the coefficient of the conformal block can be read off from the above relation, which yields ($\Delta(\alpha - \frac{b}{2}) - \Delta_\alpha - \Delta_{2,1} = b\alpha$)

$$\begin{aligned}\mathcal{F}^{(m+|M_s|)}_{\Delta_{\beta_+}}\left[{}^{\Delta_3\;\Delta_{2,1}}_{\Delta_4\;\;\Delta_\beta}\right] &= \sum_{k=0}^{m+|M_s|}(-1)^{m+|M_s|-k}\\ &\times\frac{(b\alpha_3)_{m+|M_s|-k}\left(b(\alpha_3-\alpha_4+\bar\beta-\frac{b}{2})\right)_k\left(b(\alpha_3+\alpha_4-\beta-\frac{b}{2})\right)_k}{(m+|M_s|-k)!k!\left(b(2\bar\beta-b)\right)_k}.\end{aligned} \tag{3.48}$$

Combining eqs. (3.47), (3.45) and (3.44) we arrive at the explicit form of coefficient $A_{m+n,n}$, and hence, the nominator of $\chi_m(x)$ that, when written in a suggestive form, reads

$$\begin{aligned}\sum_{n\geq 0}x^n A_{m+n,n} &= \sum_{n\geq 0}\sum_{M\vdash n}\sum_{s=0}^{\ell(M)}\sum_{1\leq i_1<\ldots<i_s\leq\ell(M)} x^{|M_s^{\complement}|}\gamma_{\Delta_{\beta_+}+m+|M_s|}\left[{}^{\Delta_3}_{\Delta_4}\right]_{M_s^{\complement}}\beta_{\Delta_\beta}\left[{}^{\Delta_2}_{\Delta_1}\right]^M\\ &\times x^{|M_s|}\mathcal{F}^{(m+|M_s|)}_{\Delta_{\beta_+}}\left[{}^{\Delta_3\;\Delta_{2,1}}_{\Delta_4\;\;\Delta_\beta}\right]\gamma_{\Delta_\beta}\left[{}^{\Delta_{2,1}}_{\Delta_{\beta_+}+m+|M_s|}\right]_{M_s}.\end{aligned}$$

Instead of summing over levels n we can rearrange the sum so that it runs over $|M_s^{\complement}|$ and $|M_s|$ separately. This rearrangement has an effect in the appearance of the symmetry factor multiplying $\beta_{\Delta_\beta}\left[{}^{\Delta_2}_{\Delta_1}\right]^M$ that comes from different ways one subpartition is immersed into the other. For instance, keeping summation index $|M_s^{\complement}|$ and partition $M_s^{\complement}$ fixed while summing over $|M_s|$ we find that there are many terms with the same shape of total partition $M = M_s^{\complement}\cup M_s$ that differ only by the distribution of parts of the partition $M_s^{\complement}$ between parts of M_s which occurs provided both have equal parts. As a result of this reshuffling of terms in the series we get ($|M_s^{\complement}| \equiv |I| = r$, $|M_s| \equiv |J| = u$)

$$\sum_{n\geq 0} x^n A_{m+n,n} = \sum_{u\geq 0} x^u \sum_{J\vdash u} \left(\sum_{r\geq 0} x^r \sum_{I\vdash r} \tbinom{I\cup J}{I}\, \gamma_{\Delta_{\beta_+}+m+u} \left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_I \beta_{\Delta_\beta} \left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{I\cup J} \right) \times \mathcal{F}^{(m+u)}_{\Delta_{\beta_+}} \left[\begin{smallmatrix}\Delta_3 & \Delta_{2,1}\\ \Delta_4 & \Delta_\beta\end{smallmatrix}\right] \gamma_{\Delta_\beta} \left[\begin{smallmatrix}\Delta_{2,1}\\ \Delta_{\beta_+}+m+u\end{smallmatrix}\right]_J ,$$

where $\binom{I\cup J}{I}$ is the number of ways the partition I is immersed into the partition $I \cup J$. As it may be inferred it equals

$$\tbinom{I\cup J}{I} := \prod_{i\geq 1} \binom{m_i(I\cup J)}{m_i(I)}, \tag{3.49}$$

where $m_i(I \cup J)$ is a multiplicity of a part i in a partition $I \cup J$, whereas $m_i(I)$ denotes a multiplicity of i in a partition I. The term in the parentheses can be rewritten in yet another form

$$\begin{aligned}\sum_{n\geq 0} x^n A_{m+n,n} &= \sum_{u\geq 0} x^u \mathcal{F}^{(m+u)}_{\Delta_{\beta_+}} \left[\begin{smallmatrix}\Delta_3 & \Delta_{2,1}\\ \Delta_4 & \Delta_\beta\end{smallmatrix}\right] \\ &\times \sum_{J\vdash u} \left(\sum_{r\geq 0} x^r \sum_{I\vdash r} \gamma_{\Delta_{\beta_+}+m+u} \left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_I \beta_{\Delta_\beta} \left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^I \tbinom{I\cup J}{I} \frac{\beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{I\cup J}}{\beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{I} \beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{J}} \right) \\ &\times \gamma_{\Delta_\beta} \left[\begin{smallmatrix}\Delta_{2,1}\\ \Delta_{\beta_+}+m+u\end{smallmatrix}\right]_J \beta_{\Delta_\beta} \left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^J .\end{aligned} \tag{3.50}$$

Observe, that the expression in the parentheses resembles the contribution to the conformal block (see eq. (3.43b)) with the intermediate weight Δ_β shifted as $\Delta_{\beta_+} = \Delta_\beta + \frac{1}{2} + \frac{b^2}{4} - b\beta$, multiplied by ratio of components of β forms. In the classical limit in which weights scale as

$$\begin{aligned}&\Delta_i \sim \delta_i b^{-2}, \text{ for } i = 1, \ldots, 4, \quad \Delta_\beta := \beta\left(b + b^{-1} - \beta\right) \sim \delta b^{-2}, \\ &\beta \sim \eta b^{-1}, \quad c \sim 6b^{-2} ,\end{aligned}$$

the term exponentiates to the classical conformal block provided the mentioned ratio decouples. This occurs if and only if the quotient of components of β forms has the following asymptotic behavior

$$\boxed{\frac{\beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{I\cup J}}{\beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{I} \beta_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^{J}} \overset{b\to 0}{\sim} \frac{1}{\binom{I\cup J}{I}}\left(1 + \mathcal{O}(b^2)\right).} \tag{3.51}$$

As we have checked in a few cases (cf. appendix A), this indeed takes place. It means that the expression in parentheses in eq. (3.50) in the classical limit looses the dependence on J and therefore decouples from the series over u in the classical limit. The decoupled series exponentiates to the classical block, so that we are finally left with the function

$$\sum_{n\geq 0} x^n A_{m+n,n} \overset{b\to 0}{\sim} \mathrm{e}^{\frac{1}{b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)} h_m(x) ,$$

where

$$h_m(x) := \sum_{s\geq 0} h_{m,s}(\delta_1,\dots,\delta_4,\delta)x^s = \lim_{b\to 0}\chi_m(x)\,,$$

and

$$h_{m,n}(\delta_1,\dots,\delta_4,\delta) = \mathcal{F}^{(m+n)}_{\Delta_{\beta_+}}\left[\begin{smallmatrix}\Delta_3 & \Delta_{2,1}\\ \Delta_4 & \Delta_\beta\end{smallmatrix}\right]\mathcal{F}^{(n)}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_{2,1} & \Delta_2\\ \Delta_{\beta_+}+m+n & \Delta_1\end{smallmatrix}\right]\Big|_{b=0}+\dots$$

becomes the coefficient of the Heun's function in vicinity of $x=0$. The Heun's function obtained as a limit $b\to 0$ of (3.42a) reads[20]

$$\Psi_+(z,x) := \lim_{b\to 0}\frac{\mathscr{F}_+(z,x)}{\mathcal{F}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)} = z^{1-\eta}\sum_{m\in\mathbb{Z}} z^m h_m(x)\,, \tag{3.52}$$

where the coefficients for $m\geq 0$ are given in the appendix B. As for the part of the Heun's function with $m<0$ we get

$$\sum_{k\geq 0} A_{k-m,k}x^{k-m} = x^{|m|}\sum_{k\geq 0} A_{k+|m|,k}x^k,$$

so that

$$h_{m<0}(x) = x^{-m}\sum_{s\geq 0} h_{-m,s}x^s\,, \tag{3.53}$$

and the coefficients are the same as for $m\geq 0$ and are presented in eqs. (2.65a–2.65c).

3.2. *Hypergeometric limit*

Solutions in the limit $x\to 0$

Let us consider the ratio of the matrix element (3.42a) and the four-point conformal block which we denote as

$$\mathscr{H}_+(z,x) := \frac{\mathscr{F}_+(z,x)}{\mathcal{F}_{\Delta_\beta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)} = z^{b\bar{\beta}}\sum_{m\in\mathbb{Z}}\chi_m(x)z^m\,, \tag{3.54}$$

in vicinity of the point $x=0$. As it follows from eq. (3.42b) at this point the only contribution to $\mathscr{H}$ comes from χ_m with $m\geq 0$. Expansion of this coefficient about $x=0$ yields

$$\chi_{m\geq 0}(x) = \frac{\sum\limits_{k\geq 0} x^k A_{m+k,k}}{1+\sum\limits_{k>0} x^k\mathcal{F}^{(n)}} = A_{m,0} + \left(A_{m+1,1} - \mathcal{F}^{(1)}A_{m,0}\right)x + \mathcal{O}(x^2)\,.$$

Hence, at this point it amounts to

$$\chi_{m>0}(0) = A_{m,0} = \langle m;\Delta_{\beta_+}|V_{\Delta_{2,1}}(1)|\Delta_\beta\rangle = \mathcal{F}^{(m)}_{\Delta_{\beta_+}}\left[\begin{smallmatrix}\Delta_3 & \Delta_{2,1}\\ \Delta_4 & \Delta_\beta\end{smallmatrix}\right], \tag{3.55}$$

so that, taking into account the relationship of the four-point conformal block with one degenerate weight with the hypergeometric function, $\mathscr{H}_+$ from eq. (3.54) amounts to

[20] Note that $1-\eta = \frac{1}{2}(1+\lambda) \Rightarrow \Psi_+$ has correct monodromy, cf. eq. (2.13).

$$\begin{aligned}\mathscr{H}_{+}(z,0) &= \mathcal{F}_{\Delta_{\beta+}}\left[\begin{smallmatrix}\Delta_3 & \Delta_{2,1}\\ \Delta_4 & \Delta_\beta\end{smallmatrix}\right](z)\\ &= z^{b\bar{\beta}}(1-z)^{b\alpha_3}{}_2\mathsf{F}_1\big(b(\alpha_3-\alpha_4+\bar{\beta}-\tfrac{b}{2})\,,b(\alpha_3+\alpha_4-\beta+\tfrac{b}{2})\,;b(2\bar{\beta}-b)\,;z\big)\,.\end{aligned} \tag{3.56}$$

Since at the classical limit parameters α_i and β scale as $\alpha_i \sim \eta_i/b$, $\beta \sim \eta/b$, we can immediately take the limit $b \to 0$ and obtain explicit form of the Heun's function at $x = 0$ that reads

$$\lim_{b\to 0}\mathscr{H}_{+}(z,0) = z^{1-\eta}(1-z)^{\eta_3}{}_2\mathsf{F}_1\big(\eta_3-\eta_4-\eta+1\,,\eta_3+\eta_4-\eta\,;2(1-\eta)\,;z\big)\,. \tag{3.57}$$

The hypergeometric equation vs. the Heun equation

Since the accessory parameter (1.6) takes the explicit form

$$c_2(x) = (\delta-\delta_1-\delta_2)\frac{1}{x} + \sum_{n>0} n x^{n-1} f^{(n)}\,, \tag{3.58}$$

the content of the square bracket in eq. (1.3) amounts in the limit $x \to 0$ to

$$\begin{aligned}&\frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(1-z)^2} + \frac{\delta_1+\delta_2+\delta_3-\delta_4}{z(1-z)} + \frac{x(1-x)c_2(x)}{z(z-x)(1-z)}\\ &\xrightarrow{x\to 0} \frac{z\delta_3+(z-1)\,(z\delta_4-\delta)}{(z-1)^2z^2}\,,\end{aligned} \tag{3.59}$$

so that we finally arrive at the following equation

$$\mathcal{A}\Psi := \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{z\delta_3+(z-1)z\delta_4-(z-1)\delta}{(z-1)^2z^2}\right]\Psi = 0\,. \tag{3.60}$$

This equation can be transformed into the form of the hypergeometric equation. Namely, the transformation $U\mathcal{A}U^{-1}U\Psi = \tilde{\mathcal{A}}\tilde{\Psi}$, where $U = z^{-a}(1-z)^{-b}$ yields

$$\begin{aligned}\tilde{\mathcal{A}} &= \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2(a-(a+b)z)}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}\\ &\quad+\frac{(1-z)z\left[(a+b)\big(1-(a+b)\big)-\delta_4\right]-(1-z)\big(a(1-a)-\delta\big)-z\big(b(1-b)-\delta_3\big)}{(z-1)^2z^2}\,.\end{aligned}$$

The operator $\tilde{\mathcal{A}}$ can be brought to the canonical hypergeometric form

$$\mathcal{H} := z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (\gamma-(\alpha+\beta+1)z)\frac{\mathrm{d}}{\mathrm{d}z} - \alpha\beta\,,$$

by multiplying by $z(1-z)$, redefining $\delta = \eta(1-\eta)$, $\delta_3 = \eta_3(1-\eta_3)$ and setting

$$a(1-a)-\eta(1-\eta) = 0\,,\quad b(1-b)-\eta_3(1-\eta_3) = 0\,. \tag{3.61}$$

The solution of the above equations reads

$$a = \eta \vee a = 1-\eta \quad \text{and} \quad b = \eta_3 \vee b = 1-\eta_3 \tag{3.62}$$

such that we obtain

$$\bar{\mathcal{A}} = z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\,(a-(a+b)z)\,\frac{\mathrm{d}}{\mathrm{d}z} + ((a+b)(1-(a+b))-\delta_4)\,.$$

Depending on the choice of the solution (3.62) of eqs. (3.61) we get the relationship between the classical block parameters η, η_3, η_4 and those from the hypergeometric operator $\mathcal{H}$, i.e., α, β, γ. For $a = 1 - \eta$ and $b = \eta_3$ we have

$$\alpha = \eta_3 - \eta_4 - \eta + 1, \quad \beta = \eta_3 + \eta_4 - \eta, \quad \gamma = 2(1 - \eta)\,.$$

Hence, the solution for $a = 1 - \eta$ and $b = \eta_3$ canonical at $z = 0$ is the hypergeometric function ${}_2\mathsf{F}_1$ with the following parameters

$$\tilde{\Psi}_+ = {}_2\mathsf{F}_1\left(\eta_3 - \eta_4 - \eta + 1, \eta_3 + \eta_4 - \eta; 2(1 - \eta); z\right)\,.$$

Thus, the original function being a solution to eq. (3.60) takes the form

$$\begin{aligned}\Psi_+ &= U^{-1}\tilde{\Psi}_+\\ &= z^{1-\eta}(1-z)^{\eta_3}{}_2\mathsf{F}_1\left(\eta_3 - \eta_4 - \eta + 1, \eta_3 + \eta_4 - \eta; 2(1 - \eta); z\right)\\ \text{for}\quad & a = 1 - \eta,\ b = \eta_3\,.\end{aligned} \tag{3.63}$$

The above solution has been labeled with Ψ_+ to match the one in eq. (3.57) that stems from the five-point degenerate block in the classical limit with a given choice of fusion rule. The second solution of the equation (3.60) reads

$$\begin{aligned}\Psi_- &= U^{-1}\tilde{\Psi}_-\\ &= z^{\eta}(1-z)^{\eta_3}{}_2\mathsf{F}_1\left(\eta + \eta_3 - \eta_4, \eta + \eta_3 + \eta_4 - 1; 2\eta; z\right) \quad \text{for} \quad a = \eta,\ b = \eta_3\,.\end{aligned}$$

3.3. *Comparison with other methods*

The path-multiplicative solutions of the Heun equation can be computed by making use of the perturbative methods. Such technics have been developed lately by Menotti in [43] to solve the monodromy problem determining the complex-valued accessory parameter in the Heun equation. An elegant and useful generalization of the perturbative approach has been recently developed in [62] and used to calculate the monodromy representation of the (generalized) Heun's *opers*.

In [43] (see also [44]) the author study the monodromy problem for the eq.

$$y''(z) + \mathcal{Q}(z)y(z) = 0, \tag{3.64}$$

where

$$\mathcal{Q}(z) = \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(1-z)^2} + \frac{\delta_1 + \delta_2 + \delta_3 - \delta_4}{z(1-z)} + \frac{\mathcal{C}(x)}{z(z-x)(1-z)}$$

and $\delta_i = \frac{1}{4}(1 - \lambda_i^2)$. The 'potential' $\mathcal{Q}(z)$ in eq. (3.64) is almost the same as in eq. (1.3). The only difference is in the definition of the accessory parameter, namely, $\mathcal{C}(x) = x(1-x)c_2(x)$. Further, it is assumed that $\mathcal{C}(0) = \delta - \delta_1 - \delta_2$. This assumption is precisely met by $x(1-x)c_2(x)$, according to the formula (3.58). The monodromy problem considered in [43] is the same as stated in the last paragraph of subsection 2.1, i.e., the accessory parameter $\mathcal{C}(x)$ must be adjusted so that the monodromy of the fundamental solutions along a contour encircling both 0 and x has the trace $-2\cos(\pi\lambda)$.

To reconstruct the accessory parameter $\mathcal{C}(x)$ as a power series in x, a specific contour including both the origin 0 and x is chosen in [43] for the monodromy calculation. Then, the appropriate fundamental solutions $(y^{(1)}(z), y^{(2)}(z))$ are computed in the form $y^{(i)}(z) = y_0^{(i)}(z) + y_1^{(i)}(z)x + \ldots$ by perturbing the Heun equation in small x. More precisely, the method relies on an expanding of $\mathcal{Q}(z)$ in small x, $\mathcal{Q} = \mathcal{Q}_0 + x\mathcal{Q}_1 + x^2\mathcal{Q}_2 + \mathcal{O}(x^3)$, where in particular one gets

$$\mathcal{Q}_0 = \frac{\delta}{z^2} + \frac{\delta_3}{(z-1)^2} + \frac{\delta_4 - \delta_3 - \delta}{z(z-1)} ,$$
$$\mathcal{Q}_1 = \frac{2\delta_2 - \mathcal{C}'(0)}{z^2(z-1)} + \frac{2\delta_2 - \mathcal{C}(0)}{z^3(z-1)} ,$$
$$\mathcal{Q}_2 = -\frac{\mathcal{C}''(0)}{2z^2(z-1)} + \frac{3\delta_2 - \mathcal{C}'(0)}{z^3(z-1)} - \frac{3\delta_2 + \mathcal{C}(0)}{z^4(z-1)} .$$

The calculation starts with the solutions $(y_0^{(1)}(z), y_0^{(2)}(z))$ to the zeroth-order equation $y''(z) + \mathcal{Q}_0(z)y(z) = 0$ and then subsequent corrections in powers of x are computed. The accessory parameter $\mathcal{C}(x)$ thus obtained exactly matches the formula (3.58).

In order to compare the method quoted above with the calculations made in previous subsections let us point out that the zeroth-order equation is nothing but the eq. (3.60) whose solutions are given in terms of the hypergeometric functions. In refs. [43,44] these zeroth-order solutions are built out of the hypergeometric functions canonical at $z = 1$. An argument that appears in [43] for such a choice is a reasonable observation that ‘working near $z = 0$ is difficult due to the singular nature of the kernel’ $\mathcal{Q}(z)$. On the other hand, we have obtained that in the limit $x \to 0$ the solutions extracted from the conformal blocks yield the zeroth-order solutions built out of the hypergeometric functions canonical at $z = 0$. In the present work we leave this discrepancy without an explanation. It requires a better understanding of a mechanism of an analytic continuation of the solutions parallelly within perturbative and CFT approaches.

4. Concluding remarks

In the present paper we have studied the mechanism of the heavy–light factorization which occurs in the classical limit of conformal blocks with the heavy and light contributions. We have examined the factorization property in the case of the simplest 5-point degenerate spherical conformal blocks $\mathscr{F}_{\pm}$. Our goal was to answer the question whether this semi-classical asymptotical behavior of $\mathscr{F}_{\pm}$ determines the linearly independent Floquet type solutions of the normal form Heun equation. Recall that the Heun equation in its normal form can be obtained in the classical limit from the null vector decoupling equations obeyed by $\mathscr{F}_{\pm}$. Analyzing the semi-classical factorization in the case under consideration we have identified the mechanism responsible for the decoupling of the heavy and light contributions. In particular, a crucial observation made in the present work is the limit (3.51) which we have checked in many cases. This leads to an interesting novel way of computing the path-multiplicative Heun's functions (see the result written down in appendix B). Indeed, if the observation (3.51) is true then an analysis performed in subsection 3.1 yields a practical method of computation of the Floquet type Heun's solutions which is suitable for numerical calculations. In subsection 3.2 we have computed the $x \to 0$ limit of the Heun's solution within CFT framework and have confirmed that this has an expected form in terms of the hypergeometric function. A demonstration of a complete compatibility of our approach with the perturbative methods requires further study of the mechanism of the analytic continuation of the solutions. We aim to analyze this problem soon.

Finally, let us stress that methods developed in this work have immediate applications in black hole physics problems listed in the introduction. We plan to examine these contexts very soon. Moreover, we plan to continue an exploration of the correspondence between the Heun equation, the classical limit of the null vector decoupling equation for the 5-point function/blocks and the BC_1 Inozemtsev integrable model. The latter is nothing but the Schrödinger spectral problem for a class of elliptic so-called Treibich–Verdier potentials built out of the Weierstrass elliptic $\wp$-function. This spectral problem is yet one more incarnation of the Heun equation and on the other hand it can be obtained in the classical limit from the null vector decoupling equation written in elliptic variables living on the torus (cf. [63] and refs. therein). It turns out that this identification has some unspoken so far intriguing consequences for: (i) the theory of elliptic solitons, i.e., the solutions of the Korteweg–de Vries equation given by the above-mentioned Treibich–Verdier potentials; (ii) the study of modular properties of $\mathcal{N}=2$ gauge theories; (iii) the correspondence between the sphere and torus correlation functions in the Liouville theory as well as between the sphere and torus conformal blocks.

Appendix A. Ratio of beta forms in the classical limit

In what follows we present the results of computations of asymptotic of the ratio of coefficients of β in the limit $b\to 0$ given in eq. (3.51) which we denote here as

$$R_{I\cup J} := \frac{\beta_{\Delta_\beta}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}^{I\cup J}}{\beta_{\Delta_\beta}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}^{I}\beta_{\Delta_\beta}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}^{J}}.$$

- $|I\cup J|=2,\ |I|=1,\ |J|=1$

$$R_{(1)\cup(1)} = \frac{1}{2} + \frac{-2\left(3\delta^2+(5\delta-3)\delta_2\right)\delta_1+(5\delta-3)\delta_1^2+(\delta-\delta_2)\left(\delta(\delta+3)+(3-5\delta)\delta_2\right)}{4\delta(4\delta+3)\left(\delta-\delta_1+\delta_2\right)^2}b^2 + \mathcal{O}(b^4).$$

The value of $\binom{(1)\cup(1)}{(1)}=2$.

- $|I\cup J|=3,\ |I|=2,\ |J|=1$
  - ▶ $I=(2),\ J=(1)$

$$R_{(2)\cup(1)} = 1 + \frac{2\left(\delta_1-\delta_2\right)\left(-2\left(5\delta^2+(7\delta-6)\delta_2\right)\delta_1+(7\delta-6)\delta_1^2+(\delta-\delta_2)\left(3\delta(\delta+2)+(6-7\delta)\delta_2\right)\right)}{3\delta(\delta+2)\left(\delta-\delta_1+\delta_2\right)\left(-3\delta_1^2+2\left(\delta+3\delta_2\right)\delta_1+(\delta-\delta_2)\left(\delta+3\delta_2\right)\right)}b^2 + \mathcal{O}(b^4).$$

The value of $\binom{(2)\cup(1)}{(2)}=1$.

  - ▶ $I=(1,1),\ J=(1)$

$$R_{(1,1)\cup(1)} = \frac{1}{3} + \frac{\left(-2\left(3\delta^2+(5\delta-3)\delta_2\right)\delta_1+(5\delta-3)\delta_1^2+(\delta-\delta_2)\left(\delta(\delta+3)+(3-5\delta)\delta_2\right)\right)}{3\delta(4\delta+3)\left(\delta-\delta_1+\delta_2\right)^2}b^2 + \mathcal{O}(b^4).$$

The value of $\binom{(1,1)\cup(1)}{(1,1)} = 3$.

- $|I \cup J| = 4$, $|I| = 3$, $|J| = 1$
  - ► $I = (3)\,,\; J = (1)$

$$R_{(3)\cup(1)} = 1 + \mathcal{O}(b^2)\,, \qquad \binom{(3)\cup(1)}{(3)} = 1\,.$$

  - ► $I = (2,1)\,,\; J = (1)$

$$R_{(2,1)\cup(1)} = \tfrac{1}{2} + \mathcal{O}(b^2)\,, \qquad \binom{(2,1)\cup(1)}{(2,1)} = 2\,.$$

  - ► $I = (1,1,1)\,,\; J = (1)$

$$R_{(1,1,1)\cup(1)} = \tfrac{1}{4} + \mathcal{O}(b^2)\,, \qquad \binom{(1,1,1)\cup(1)}{(1,1,1)} = 4\,.$$

In the last item for $|I \cup J| = 4$, because of their sizable forms, we have omitted the terms of order $\mathcal{O}(b^2)$.

Appendix B. Coefficients of the Heun's function

In this appendix we present the explicit form of numerical derivation of the coefficients of the Heun's function given as

$$\Psi_+(z,x) = z^{1-\eta}\sum_{m\in\mathbb{Z}} h_m(x) z^m, \quad h_m(x) = \sum_{s\geq 0} h_{m,s}(\delta_1,\ldots,\delta_4,\eta)x^s\,,$$

where particular values read

$$h_0(x) = 1 - \tfrac{(\delta_1-\delta_2+(\eta-1)\eta)(\eta((\eta-1)\eta-\delta_3+\delta_4-1)+2\delta_3-2\delta_4+1)}{4(\eta-1)^2\eta}x + \ldots, \tag{2.65a}$$

$$\begin{aligned} h_1(x) = {} & \tfrac{\delta_3-\delta_4+(1-\eta)\eta}{2(\eta-1)} \\ & + \tfrac{(\delta_1-\delta_2+(\eta-1)\eta)}{8(\eta-1)^2\eta(2\eta-3)}\Big((-2\delta_3+2\delta_4+1)\,\eta^3 + (8\delta_3-6\delta_4+4)\,\eta^2 \\ & + \Big(\delta_3^2 - 2\,(\delta_4+5)\,\delta_3 + (\delta_4-1)\,(\delta_4+3)\Big)\,\eta - 3\,(\delta_3-\delta_4)^{\,2} \\ & + 3\,(\delta_3+\delta_4) + \eta^4(\eta-3)\Big)x + \ldots \end{aligned} \tag{2.65b}$$

$$\begin{aligned} h_2(x) = {} & \tfrac{(5-2\delta_3+2\delta_4)\eta^2+(6\delta_3-4\delta_4-2)\eta+\delta_3^2+(\delta_4+2)(\delta_4-2\delta_3)+\eta^3(\eta-4)}{4(2\eta^2-5\eta+3)} \\ & - \tfrac{(\delta_1-\delta_2+(\eta-1)\eta)}{48(\eta-2)(\eta-1)^2\eta(2\eta-3)}\Big((-3\delta_3+3\delta_4+16)\,\eta^5 + (24\delta_3-18\delta_4-7)\,\eta^4 \\ & + \Big(3\delta_3^2 - 3\,(2\delta_4+25)\,\delta_3 + 3\delta_4\,(\delta_4+11) - 23\Big)\,\eta^3 \\ & + \Big(-21\delta_3^2 + 6\,(6\delta_4+19)\,\delta_3 - 3\delta_4\,(5\delta_4+4) + 32\Big)\,\eta^2 \\ & + \Big(-\delta_3^3 + (3\delta_4+49)\,\delta_3^2 - 3\,(\delta_4\,(\delta_4+22)+28)\,\delta_3 \\ & + \;(\delta_4-1)\,\delta_4\,(\delta_4+18) - 12)\,\eta \\ & + 12\,(2\delta_3+\delta_4) + 2\,(\delta_3-\delta_4)\,\Big(2\delta_3^2 - (4\delta_4+17)\,\delta_3 + 2\delta_4^2 + \delta_4\Big) + \eta^7 - 7\eta^6\Big)x \\ & + \ldots \end{aligned} \tag{2.65c}$$

References

[1] A. Roveaux, P. Arscott, Heun's Differential Equations, Oxford University Press, 1995.

[2] J. Olver, A.B.O. Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark et al., NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16.

[3] A. Belavin, A.M. Polyakov, A. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241 (1984) 333–380.

[4] T. Eguchi, H. Ooguri, Conformal and current algebras on general Riemann surface, Nucl. Phys. B 282 (1987) 308–328.

[5] G.W. Moore, N. Seiberg, Polynomial equations for rational conformal field theories, Phys. Lett. B 212 (1988) 451.

[6] G. Felder, J. Fröhlich, G. Keller, On the structure of unitary conformal field theory II: representation theoretic approach, Commun. Math. Phys. 130 (May 1990) 1–49.

[7] A.B. Zamolodchikov, A.B. Zamolodchikov, Structure constants and conformal bootstrap in Liouville field theory, Nucl. Phys. B 477 (1996) 577–605, arXiv:hep-th/9506136.

[8] L. Hadasz, Z. Jaskolski, M. Piatek, Classical geometry from the quantum Liouville theory, Nucl. Phys. B 724 (2005) 529–554, arXiv:hep-th/0504204.

[9] L.F. Alday, D. Gaiotto, Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, Lett. Math. Phys. 91 (2010) 167–197, arXiv:0906.3219.

[10] N. Wyllard, A_{N-1} conformal Toda field theory correlation functions from conformal $N = 2$ SU(N) quiver gauge theories, JHEP 0911 (2009) 002, arXiv:0907.2189.

[11] D. Gaiotto, Asymptotically free $N = 2$ theories and irregular conformal blocks, arXiv:0908.0307.

[12] A. Marshakov, A. Mironov, A. Morozov, On non-conformal limit of the AGT relations, Phys. Lett. B 682 (2009) 125–129, arXiv:0909.2052.

[13] V. Alba, A. Morozov, Non-conformal limit of AGT relation from the 1-point torus conformal block, JETP Lett. 90 (2009) 708–712, arXiv:0911.0363.

[14] D. Gaiotto, J. Teschner, Irregular singularities in Liouville theory and Argyres–Douglas type gauge theories, I, JHEP 1212 (2012) 050, arXiv:1203.1052.

[15] M. Piatek, A.R. Pietrykowski, Classical irregular block, $\mathcal{N} = 2$ pure gauge theory and Mathieu equation, J. High Energy Phys. 12 (2014) 032, arXiv:1407.0305.

[16] N.A. Nekrasov, S.L. Shatashvili, Quantization of Integrable Systems and Four Dimensional Gauge Theories, 2009.

[17] N. Nekrasov, A. Rosly, S. Shatashvili, Darboux coordinates, Yang–Yang functional, and gauge theory, Nucl. Phys. Proc. Suppl. 216 (2011) 69–93, arXiv:1103.3919.

[18] J. Teschner, Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I, Adv. Theor. Math. Phys. 15 (2011) 471–564, arXiv:1005.2846.

[19] J. Teschner, Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence, arXiv:1707.07873.

[20] M. Piatek, A.R. Pietrykowski, Classical limit of irregular blocks and Mathieu functions, JHEP 01 (2016) 115, arXiv:1509.08164.

[21] M. Piatek, A.R. Pietrykowski, Classical irregular blocks, Hill's equation and PT-symmetric periodic complex potentials, JHEP 07 (2016) 131, arXiv:1604.03574.

[22] G. Basar, G.V. Dunne, Resurgence and the Nekrasov–Shatashvili limit: connecting weak and strong coupling in the Mathieu and Lamé systems, JHEP 02 (2015) 160, arXiv:1501.05671.

[23] A. Litvinov, S. Lukyanov, N. Nekrasov, A. Zamolodchikov, Classical conformal blocks and Painleve VI, J. High Energy Phys. 07 (2014) 144, arXiv:1309.4700.

[24] W. He, Quasimodular instanton partition function and the elliptic solution of Korteweg–de Vries equations, Ann. Phys. 353 (2015) 150–162, arXiv:1401.4135.

[25] J. Teschner, Classical conformal blocks and isomonodromic deformations, arXiv:1707.07968.

[26] A.-K. Kashani-Poor, J. Troost, Quantum geometry from the toroidal block, J. High Energy Phys. 8 (2014) 117, arXiv:1404.7378.

[27] T. Hartman, Entanglement entropy at large central charge, arXiv:1303.6955.

[28] C.T. Asplund, A. Bernamonti, F. Galli, T. Hartman, Holographic Entanglement Entropy from 2d CFT: heavy states and local quenches, J. High Energy Phys. 2 (2015) 171, arXiv:1410.1392.

[29] D.A. Roberts, D. Stanford, Two-dimensional conformal field theory and the butterfly effect, Phys. Rev. Lett. 115 (2015) 131603, arXiv:1412.5123.

[30] B. Carneiro da Cunha, F. Novaes, Kerr–de Sitter greybody factors via isomonodromy, Phys. Rev. D 93 (2016) 024045, arXiv:1508.04046.

[31] B. Carneiro da Cunha, F. Novaes, Kerr scattering coefficients via isomonodromy, J. High Energy Phys. 11 (2015) 144, arXiv:1506.06588.

[32] F. Novaes, B. Carneiro da Cunha, Isomonodromy, Painlevé transcendents and scattering off of black holes, J. High Energy Phys. 7 (2014) 132, arXiv:1404.5188.

[33] T. Anous, T. Hartman, A. Rovai, J. Sonner, Black hole collapse in the 1/c expansion, J. High Energy Phys. 07 (2016) 123, arXiv:1603.04856.

[34] C.-M. Chang, Y.-H. Lin, Bootstrapping 2D CFTs in the semiclassical limit, J. High Energy Phys. 08 (2016) 056, arXiv:1510.02464.

[35] K.B. Alkalaev, Many-point classical conformal blocks and geodesic networks on the hyperbolic plane, J. High Energy Phys. 12 (2016) 070, arXiv:1610.06717.

[36] K.B. Alkalaev, V.A. Belavin, Holographic interpretation of 1-point toroidal block in the semiclassical limit, J. High Energy Phys. 06 (2016) 183, arXiv:1603.08440.

[37] K.B. Alkalaev, V.A. Belavin, From global to heavy–light: 5-point conformal blocks, J. High Energy Phys. 03 (2016) 184, arXiv:1512.07627.

[38] K.B. Alkalaev, V.A. Belavin, Monodromic vs geodesic computation of Virasoro classical conformal blocks, Nucl. Phys. B 904 (2016) 367–385, arXiv:1510.06685.

[39] K.B. Alkalaev, V.A. Belavin, Classical conformal blocks via AdS/CFT correspondence, J. High Energy Phys. 08 (2015) 049, arXiv:1504.05943.

[40] C.-M. Chang, Y.-H. Lin, Bootstrap, universality and horizons, J. High Energy Phys. 10 (2016) 068, arXiv:1604.01774.

[41] P. Banerjee, S. Datta, R. Sinha, Higher-point conformal blocks and entanglement entropy in heavy states, J. High Energy Phys. 05 (2016) 127, arXiv:1601.06794 [hep-th].

[42] S.Y. Slavyanov, W. Lay, Special Functions, A Unified Theory Based on Singularities, Oxford Mathematical Monographs, 2000.

[43] P. Menotti, On the monodromy problem for the four-punctured sphere, J. Phys. A 47 (41) (2014) 415201, arXiv:1401.2409 [hep-th].

[44] P. Menotti, Classical conformal blocks, Mod. Phys. Lett. A 31 (27) (2016) 1650159, arXiv:1601.04457 [hep-th].

[45] S.W. Hawking, C.J. Hunter, M. Taylor, Rotation and the AdS/CFT correspondence, Phys. Rev. D 59 (1999) 064005, arXiv:hep-th/9811056.

[46] H.K. Kunduri, J. Lucietti, Integrability and the Kerr–(A)dS black hole in five dimensions, Phys. Rev. D 71 (2005) 104021, arXiv:hep-th/0502124.

[47] A.N. Aliev, O. Delice, Superradiant instability of five-dimensional rotating charged AdS black holes, Phys. Rev. D 79 (2009) 024013, arXiv:0808.0280.

[48] J.B. Amado, B. Carneiro da Cunha, E. Pallante, On the Kerr–AdS/CFT correspondence, J. High Energy Phys. 8 (2017) 094, arXiv:1702.01016.

[49] M. Guica, T. Hartman, W. Song, A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D 80 (2009) 124008, arXiv:0809.4266.

[50] E. Berti, V. Cardoso, A.O. Starinets, Quasinormal modes of black holes and black branes, Class. Quantum Gravity 26 (2009) 163001, arXiv:0905.2975 [gr-qc].

[51] A. Castro, J.M. Lapan, A. Maloney, M.J. Rodriguez, Black hole scattering from monodromy, Class. Quantum Gravity 30 (2013) 165005, arXiv:1304.3781.

[52] F. Ferrari, M. Piatek, Liouville theory, $N = 2$ gauge theories and accessory parameters, J. High Energy Phys. 1205 (2012) 025, arXiv:1202.2149.

[53] L. Hadasz, Z. Jaskolski, Liouville theory and uniformization of four-punctured sphere, J. Math. Phys. 47 (2006) 082304, arXiv:hep-th/0604187.

[54] H. Dorn, H.J. Otto, Two and three point functions in Liouville theory, Nucl. Phys. B 429 (1994) 375–388, arXiv:hep-th/9403141.

[55] A. Zamolodchikov, Conformal symmetry in two-dimensional space: recursion representation of conformal block, Theor. Math. Phys. 73 (1987) 1088.

[56] V.G. Kac, Contravariant form for infinite dimensional lie algebras and superalgebras, in: W. Beiglbock, A. Bohm, E. Takasugi (Eds.), Group Theoretical Methods in Physics, Springer-Verlag, 1978, pp. 441–445.

[57] B.L. Feigin, D.B. Fuks, Invariant skew symmetric differential operators on the line and Verma modules over the Virasoro algebra, Funct. Anal. Appl. 16 (1982) 114–126.

[58] B. Feigin, D. Fuchs, Representations of the Virasoro algebra, in: A.M. Vershik, D.P. Zhelobenko (Eds.), Representation of Lie Groups and Related Topics, vol. 7, Gordon and Breach, London, 1990, pp. 465–554.

[59] C.B. Thorn, Computing the Kac determinant using dual model techniques and more about the no-ghost theorem, Nucl. Phys. B 248 (1984) 551.

[60] V.G. Kac, A.K. Raina, Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras, Vol. 2, World Scientific, 1987.
[61] A.L. Fitzpatrick, J. Kaplan, M.T. Walters, Universality of long-distance AdS physics from the CFT bootstrap, J. High Energy Phys. 08 (2014) 145, arXiv:1403.6829.
[62] L. Hollands, O. Kidwai, Higher length-twist coordinates, generalized Heun's opers, and twisted superpotentials, arXiv:1710.04438 [hep-th].
[63] A.-K. Kashani-Poor, J. Troost, Transformations of spherical blocks, J. High Energy Phys. 1310 (2013) 009, arXiv:1305.7408 [hep-th].