

Gradient flow representation of the four-dimensional $\mathcal{N} = 2$ super Yang–Mills supercurrent

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In K. Hieda, A. Kasai, H. Makino, and H. Suzuki, Prog. Theor. Exp. Phys. **2017**, 063B03 (2017), a properly normalized supercurrent in the four-dimensional (4D) $\mathcal{N} = 1$ super Yang–Mills theory (SYM) that works within on-mass-shell correlation functions of gauge-invariant operators is expressed in a regularization-independent manner by employing the gradient flow. In the present paper, this construction is extended to the supercurrent in the 4D $\mathcal{N} = 2$ SYM. The so-constructed supercurrent will be useful, for instance, for fine tuning of lattice parameters toward the supersymmetric continuum limit in future lattice simulations of the 4D $\mathcal{N} = 2$ SYM.
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1. Introduction and summary

In quantum field theory, lattice regularization enables nonperturbative computation from first principles but it breaks preferred symmetries in the target theory (such as the chiral and spacetime symmetries) quite often. For this reason, nonperturbative computation in supersymmetric field theory from first principles, especially on the basis of the lattice [1,2], is generally difficult, requiring intricate fine tuning of lattice parameters toward the supersymmetric continuum limit.

For the supersymmetric continuum limit, the lattice parameters should be tuned so that the Ward–Takahashi (WT) relations associated with the supersymmetry (SUSY) hold up to the finite lattice spacing effect. More specifically, one imposes the conservation law of the supercurrent—the Noether current associated with SUSY. See Refs. [3–6] for theoretical studies on this issue and Ref. [7] for a recent actual numerical study in the four-dimensional (4D) $\mathcal{N} = 1$ super Yang–Mills (SYM) theory. To carry out this program, however, one has to determine not only lattice parameters but also the supercurrent at the same time because the expression of the Noether current can be nontrivial when the regularization breaks the associated symmetry. This fact further complicates the situation.

Having the above problem in mind, in Ref. [8] the authors constructed a regularization-independent expression of the supercurrent in the 4D $\mathcal{N} = 1$ SYM by employing the gradient flow [9–15]. The idea is that since composite operators of fields evolved by the flow automatically become finite renormalized operators [12,13] (see also Ref. [16]), the expression of the supercurrent in terms of flowed fields is independent of regularization (in the limit in which the UV cutoff is removed); thus the expression is universal. In this way, one can have, a priori, an expression that becomes *automatically* the properly normalized conserved supercurrent in the continuum limit.

This type of construction of the Noether current by the gradient flow was first considered for the energy–momentum tensor—the Noether current associated with the translational invariance [17,18]; see also Ref. [19]. Although in conventional lattice gauge theory fine tuning for the restoration of the translational invariance is not necessary because of lattice symmetries, the construction of the associated Noether current, i.e., the energy–momentum tensor, is still intricate due to the lack of the corresponding spacetime symmetry [20,21]. The energy–momentum tensor carries important physical information and the construction in Refs. [17,18] has been applied to lattice simulations in Refs. [22–30]; see also Ref. [31].¹

In this paper, as a natural extension of the study of Ref. [8], the construction of the supercurrent in the 4D $\mathcal{N} = 2$ SYM [36–38] is considered.² For this theory, a highly nontrivial consistent low-energy description has been known [39]. Thus it is of great interest to investigate its low-energy physics by complementary nonperturbative techniques such as the lattice. See Refs. [40–45] for lattice formulations which are designed to simplify the fine tuning.

Since our analysis in this paper is rather lengthy, we summarize the basic line of reasoning and the main results in this section.

Our strategy is basically identical to that of Ref. [8]: First, we need the expression of the properly normalized conserved supercurrent with a certain regularization. As noted in Ref. [8], this is already an intricate problem because there is no known regularization that manifestly preserves the gauge symmetry and SUSY. We adopt the dimensional regularization for computational ease. Also, since we take the Wess–Zumino (WZ) gauge [46], having the application to actual lattice simulations in mind, this gauge choice and the associated gauge fixing and ghost–anti-ghost terms also break SUSY. The would-be SUSY WT relations are thus full of SUSY-breaking terms. Only after adding appropriate counterterms to the action and appropriate rearrangements of terms under the renormalization, the SUSY emerges in the WT relations. Although this realization of SUSY in renormalized theory should occur from the general argument (see Ref. [47] and references cited therein), as demonstrated in Ref. [8] for the 4D $\mathcal{N} = 1$ SYM to the one-loop order,³ and as we will see in this paper through Sects. 2–6 for the 4D $\mathcal{N} = 2$ SYM to the one-loop order, the realization of SUSY after the renormalization appears miraculous. Although our argument is parallel to that of Ref. [8], the required computational labor is much higher because of the presence of the scalar field. The required Feynman diagrams for the operator renormalization are collected in Appendix C.

In Sect. 7, from the results obtained to that point we determine the expression of the properly normalized conserved supercurrent that works within on-mass-shell correlation functions of gauge-invariant operators to the one-loop order. In Sect. 8, we introduce the gradient flow and the small flow time expansion [12], the expansion with respect to the flow time t . For the 4D $\mathcal{N} = 2$ SYM, in addition to the flow of the gauge and fermion fields frequently considered in the literature, we have to include the flow of the scalar field. For this, we use a simple flow equation following the discussion of Ref. [48]. See also Refs. [49–51] for related arguments. We then compute the wave function renormalization of the flowed fields; although we can almost borrow the

¹ A similar construction has also been considered for fermion bilinear operators [32,33], including the (axial) vector current and (pseudo-)scalar density [34], and has also been applied in lattice simulations in Refs. [23,35].

² This is a natural extension in the sense that we need the asymptotic freedom for the construction; in the 4D $\mathcal{N} = 2$ SYM, all the interactions are governed by the asymptotic-free gauge coupling.

³ This is the first explicit demonstration to our knowledge.

result of Ref. [18], the contribution of the scalar field has to be computed anew as summarized in Table 1.⁴ We then compute the small flow time expansion [12] of composite operators relevant to the representation of the supercurrent. The computation of the small flow time expansion has been presented many times in above references (see also Ref. [55]), but the presence of the scalar field greatly increases the number of required flow Feynman diagrams; the diagrams are summarized in Appendix C.

By substituting the small time expansion obtained in Sect. 8 in the supercurrent from Sect. 7, we have the representation of the supercurrent in terms of the flowed fields. The uncomputed higher-order $O(t)$ terms in the small flow time expansion are neglected by taking the limit $t \rightarrow 0$. At the same time, in the representation of the supercurrent, a renormalization group argument shows that one can use the running gauge coupling $\bar{g}(\mu)$ in which the renormalization scale μ is identified with $1/\sqrt{8t}$. In the $t \rightarrow 0$ limit, therefore, $\bar{g}(\mu) \rightarrow 0$ because of the asymptotic freedom and this justifies the perturbative computation assumed so far.

In this way, we have the supercurrent

$$\begin{aligned}
& \tilde{S}_\mu^{\text{imp}}(x) \\
&= \lim_{t \rightarrow 0} \left(\left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[-\ln \pi - \frac{9}{4} + \frac{1}{2} \ln(432) \right] \right\} \left(-\frac{1}{4\bar{g}} \right) \sigma_{\rho\sigma} \gamma_\mu \dot{\chi}^a G_{\rho\sigma}^a \right. \\
&\quad - \frac{\bar{g}}{(4\pi)^2} C_2(G) \gamma_\nu \dot{\chi}^a G_{\nu\mu}^a \\
&\quad + \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[-\frac{19}{4} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \frac{1}{2\sqrt{2}} \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) (P_+ D_\nu \dot{\chi}^a \dot{\phi}^a - P_- D_\nu \dot{\chi}^a \dot{\phi}^{\dagger a}) \\
&\quad - \frac{3}{\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) (P_+ D_\mu \dot{\chi}^a \dot{\phi}^a - P_- D_\mu \dot{\chi}^a \dot{\phi}^{\dagger a}) \\
&\quad + \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[\frac{1}{2} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \left(-\frac{1}{\sqrt{2}} \right) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) (P_+ \dot{\chi}^a D_\nu \dot{\phi}^a - P_- \dot{\chi}^a D_\nu \dot{\phi}^{\dagger a}) \\
&\quad + \frac{1}{\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) \gamma_5 D_\nu \dot{\chi}^a (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \\
&\quad + \frac{1}{2\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) \gamma_5 \dot{\chi}^a D_\nu (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \\
&\quad - \frac{1}{4} \frac{\bar{g}^3}{(4\pi)^2} C_2(G) f^{abc} \gamma_5 \gamma_\mu \dot{\chi}^a \dot{\phi}^{\dagger b} \dot{\phi}^c \Big), \tag{1.1}
\end{aligned}$$

⁴ Flow equations in supersymmetric theories that are alternative to our choice are given in Refs. [52,53]. There is interesting indication that no wave function renormalization is necessary [54] if one employs the flow equation of Ref. [52].

and its Dirac conjugate

$$\begin{aligned}
& \tilde{S}_\mu^{\text{imp}}(\chi) \\
&= \lim_{t \rightarrow 0} \left(\left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[-\ln \pi - \frac{9}{4} + \frac{1}{2} \ln(432) \right] \right\} \left(-\frac{1}{4\bar{g}} \right) \overset{\circ}{\chi}^a \gamma_\mu \sigma_{\rho\sigma} G_{\rho\sigma}^a \right. \\
&\quad + \frac{\bar{g}}{(4\pi)^2} C_2(G) \overset{\circ}{\chi}^a \gamma_\nu G_{\nu\mu}^a \\
&\quad + \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[-\frac{19}{4} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \left(-\frac{1}{2\sqrt{2}} \right) (D_\nu \overset{\circ}{\chi}^a P_+ \overset{\circ}{\phi}^a - D_\nu \overset{\circ}{\chi}^a P_- \overset{\circ}{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
&\quad + \frac{3}{\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) (D_\mu \overset{\circ}{\chi}^a P_+ \overset{\circ}{\phi}^a - D_\mu \overset{\circ}{\chi}^a P_- \overset{\circ}{\phi}^{\dagger a}) \\
&\quad + \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \left[\frac{1}{2} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \frac{1}{\sqrt{2}} (P_+ \overset{\circ}{\chi}^a D_\nu \overset{\circ}{\phi}^a - P_- \overset{\circ}{\chi}^a D_\nu \overset{\circ}{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
&\quad - \frac{1}{\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) D_\nu \overset{\circ}{\chi}^a \gamma_5 (\overset{\circ}{\phi}^a + \overset{\circ}{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
&\quad - \frac{1}{2\sqrt{2}} \frac{\bar{g}^2}{(4\pi)^2} C_2(G) \overset{\circ}{\chi}^a \gamma_5 D_\nu (\overset{\circ}{\phi}^a + \overset{\circ}{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
&\quad + \left. \frac{1}{4} \frac{\bar{g}^3}{(4\pi)^2} C_2(G) f^{abc} \overset{\circ}{\chi}^a \gamma_\mu \gamma_5 \overset{\circ}{\phi}^{\dagger b} \overset{\circ}{\phi}^c \right). \tag{1.2}
\end{aligned}$$

These are our main results in this paper.⁵ In these expressions, \bar{g} denotes the running gauge coupling in the minimal subtraction (MS) scheme $\bar{g}(\mu)$ in which the renormalization scale μ is set to $\mu = 1/\sqrt{8t}$. The beta function in

$$\mu \frac{d\bar{g}(\mu)}{d\mu} = \beta(\bar{g}(\mu)) \tag{1.3}$$

is $\beta(g) = -2g^3 C_2(G)/(4\pi)^2$ to all orders in perturbation theory [56–60]. In the right-hand sides of Eqs. (1.1) and (1.2), the fields χ , $\bar{\chi}$, B_μ ($G_{\mu\nu}$ is the field strength corresponding to B_μ), ϕ , and ϕ^\dagger are fields evolved from the corresponding fields in the original 4D $\mathcal{N} = 2$ SYM, ψ , $\bar{\psi}$, A_μ , φ , and φ^\dagger by flow equations Eqs. (8.1)–(8.7) to the flow time t . The “ring” \circ above a field implies that the normalization of the field is changed as in Eqs. (8.14)–(8.19); this prescription avoids the explicit wave function renormalization [18]. In this way, the expressions in Eqs. (1.1) and (1.2) are manifestly finite and are independent of regularization. In particular, Eqs. (1.1) and (1.2) can be used with the lattice regularization; we believe that one can use the representation for the fine tuning and/or for extracting some low-energy physics associated with the supercurrent.

⁵ Our notational convention is summarized in Appendix A.

In Appendix B, we summarize the parity and charge conjugation symmetries that are quite helpful in the actual calculation.

2. Four-dimensional $\mathcal{N} = 2$ SYM in the WZ gauge

2.1. The actions and SUSY transformations

The Euclidean action of the $\mathcal{N} = 2$ SYM in the WZ gauge is given by $S = \int d^D x \mathcal{L}$,⁶ where

$$\begin{aligned} \mathcal{L} = & \frac{1}{4g_0^2} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}^a \mathcal{D} \psi^a + D_\mu \varphi^{\dagger a} D_\mu \varphi^a - \frac{1}{2} g_0^2 f^{abc} f^{ade} \varphi^{\dagger b} \varphi^c \varphi^{\dagger d} \varphi^e \\ & + \sqrt{2} g_0 f^{abc} \bar{\psi}^a \left(P_+ \varphi^b - P_- \varphi^{\dagger b} \right) \psi^c. \end{aligned} \quad (2.1)$$

In this expression, ψ ($\bar{\psi}$) is the Dirac fermion field and φ (φ^\dagger) the complex scalar field; $F_{\mu\nu}^a$ is the field strength of the gauge field A_μ^a , $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$. All fields belong to the adjoint representation. In four dimensions, i.e. when $D = 4$, the action S is invariant under the following SUSY transformations:

$$\delta_\xi A_\mu^a = \frac{1}{2} g_0 (\bar{\xi} \gamma_\mu \psi^a - \bar{\psi}^a \gamma_\mu \xi), \quad (2.2)$$

$$\delta_\xi \varphi^a = \frac{1}{\sqrt{2}} (-\bar{\xi} P_- \psi^a + \bar{\psi}^a P_- \xi), \quad (2.3)$$

$$\delta_\xi \varphi^{\dagger a} = \frac{1}{\sqrt{2}} (\bar{\xi} P_+ \psi^a - \bar{\psi}^a P_+ \xi), \quad (2.4)$$

$$\delta_\xi \psi^a = -\frac{1}{4g_0} \sigma_{\mu\nu} \xi F_{\mu\nu}^a - \frac{1}{\sqrt{2}} \gamma_\mu P_+ \xi D_\mu \varphi^a + \frac{1}{\sqrt{2}} \gamma_\mu P_- \xi D_\mu \varphi^{\dagger a} - \frac{1}{2} g_0 \gamma_5 \xi f^{abc} \varphi^{\dagger b} \varphi^c, \quad (2.5)$$

$$\delta_\xi \bar{\psi}^a = \frac{1}{4g_0} \bar{\xi} \sigma_{\mu\nu} F_{\mu\nu}^a - \frac{1}{\sqrt{2}} \bar{\xi} \gamma_\mu P_- D_\mu \varphi^a + \frac{1}{\sqrt{2}} \bar{\xi} \gamma_\mu P_+ D_\mu \varphi^{\dagger a} - \frac{1}{2} g_0 \bar{\xi} \gamma_5 f^{abc} \varphi^{\dagger b} \varphi^c. \quad (2.6)$$

The easiest way to derive these formulas is dimensional reduction from the 6D $\mathcal{N} = 1$ SYM that possesses a much simpler structure [38].

To apply the perturbation theory, we also introduce the gauge-fixing and ghost–anti-ghost terms by

$$S_{\text{gf}} = \frac{\lambda_0}{2g_0^2} \int d^D x \partial_\mu A_\mu^a \partial_\nu A_\nu^a, \quad (2.7)$$

$$S_{\bar{c}c} = -\frac{1}{g_0^2} \int d^D x \bar{c}^a \partial_\mu D_\mu c^a, \quad (2.8)$$

where λ_0 is the bare gauge-fixing parameter.

2.2. Supercurrent and SUSY-breaking terms

To derive SUSY WT relations, we consider the SUSY transformations in Eqs. (2.2)–(2.6) with localized transformation parameters, $\xi \rightarrow \xi(x)$ and $\bar{\xi} \rightarrow \bar{\xi}(x)$. Under these, the D -dimensional action S changes as

$$\delta_\xi S = - \int d^D x \left[\bar{\xi} (\partial_\mu S_\mu + X_{\text{Fierz}}) + (\partial_\mu \bar{S}_\mu + \bar{X}_{\text{Fierz}}) \xi \right], \quad (2.9)$$

⁶ Here, we assume the spacetime dimension is D to apply the dimensional regularization.

where the supercurrents are given by

$$S_\mu = -\frac{1}{4g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a F_{\rho\sigma}^a + \frac{1}{\sqrt{2}}\gamma_\nu\gamma_\mu P_+\psi^a D_\nu\varphi^a - \frac{1}{\sqrt{2}}\gamma_\nu\gamma_\mu P_-\psi^a D_\nu\varphi^{\dagger a} + \frac{1}{2}g_0 f^{abc}\gamma_5\gamma_\mu\psi^a\varphi^{\dagger b}\varphi^c, \quad (2.10)$$

$$\bar{S}_\mu = -\frac{1}{4g_0}\bar{\psi}^a\gamma_\mu\sigma_{\rho\sigma}F_{\rho\sigma}^a - \frac{1}{\sqrt{2}}\bar{\psi}^a\gamma_\mu\gamma_\nu P_+D_\nu\varphi^a + \frac{1}{\sqrt{2}}\bar{\psi}^a\gamma_\mu\gamma_\nu P_-D_\nu\varphi^{\dagger a} - \frac{1}{2}g_0 f^{abc}\bar{\psi}^a\gamma_\mu\gamma_5\varphi^{\dagger b}\varphi^c, \quad (2.11)$$

while the breaking terms (due to the breaking of the Fierz identity in $D \neq 4$ dimensions) are

$$X_{\text{Fierz}} = \frac{1}{2}g_0 f^{abc}\gamma_\mu\psi^a\bar{\psi}^b\gamma_\mu\psi^c + \frac{1}{2}g_0 f^{abc}\gamma_5\psi^a\bar{\psi}^b\gamma_5\psi^c - \frac{1}{2}g_0 f^{abc}\psi^a\bar{\psi}^b\psi^c, \quad (2.12)$$

$$\bar{X}_{\text{Fierz}} = -\frac{1}{2}g_0 f^{abc}\bar{\psi}^a\gamma_\mu\psi^b\bar{\psi}^c\gamma_\mu - \frac{1}{2}g_0 f^{abc}\bar{\psi}^a\gamma_5\psi^b\bar{\psi}^c\gamma_5 + \frac{1}{2}g_0 f^{abc}\bar{\psi}^a\psi^b\bar{\psi}^c. \quad (2.13)$$

The above supercurrents would be regarded as ‘‘canonical’’ ones. From the perspective of the conformal or scale symmetry of the classical theory, Eq. (2.1), for $D = 4$, it would be natural to use ‘‘improved’’ supercurrents defined by

$$S_\mu^{\text{imp}} \equiv S_\mu + \frac{\sqrt{2}}{3}\sigma_{\mu\nu}\partial_\nu(P_+\psi^a\varphi^a - P_-\psi^a\varphi^{\dagger a}), \quad (2.14)$$

$$\bar{S}_\mu^{\text{imp}} \equiv \bar{S}_\mu - \frac{\sqrt{2}}{3}\partial_\nu(\bar{\psi}^a P_+\varphi^a - \bar{\psi}^a P_-\varphi^{\dagger a})\sigma_{\nu\mu}. \quad (2.15)$$

It can be seen that, noting that $\gamma_\mu\sigma_{\rho\sigma}\gamma_\mu = 0$ holds for $D = 4$, these are γ -traceless,

$$\gamma_\mu S_\mu^{\text{imp}} = \bar{S}_\mu^{\text{imp}}\gamma_\mu = 0, \quad (2.16)$$

under equations of motion. Note that the terms added in Eqs. (2.14) and (2.15) do not have the total divergence and thus $\partial_\mu S_\mu = \partial_\mu S_\mu^{\text{imp}}$ and $\partial_\mu \bar{S}_\mu = \partial_\mu \bar{S}_\mu^{\text{imp}}$.

Through the following analyses, however, we find that S_μ^{imp} and \bar{S}_μ^{imp} are *not* finite operators and they can be rendered finite by adding further terms that are proportional to equations of motion, as

$$\tilde{S}_\mu^{\text{imp}} \equiv S_\mu^{\text{imp}} - \frac{1}{2\sqrt{2}}\gamma_\mu(P_-D\psi^a\varphi^a - P_+D\psi^a\varphi^{\dagger a} - \sqrt{2}g_0 f^{abc}\gamma_5\psi^a\varphi^{\dagger b}\varphi^c), \quad (2.17)$$

$$\tilde{\bar{S}}_\mu^{\text{imp}} \equiv \bar{S}_\mu^{\text{imp}} + \frac{1}{2\sqrt{2}}(\bar{\psi}^a \overleftarrow{D} P_-\varphi^a - \bar{\psi}^a \overleftarrow{D} P_+\varphi^{\dagger a} - \sqrt{2}g_0 f^{abc}\bar{\psi}^a\gamma_5\varphi^{\dagger b}\varphi^c)\gamma_\mu. \quad (2.18)$$

Here, the added term in Eq. (2.17) has the structure

$$(\varphi \text{ or } \varphi^\dagger) \times (\text{the equation of motion of } \psi). \quad (2.19)$$

The effect of the insertion of such a term in a correlation function can be deduced by the infinitesimal change of variable $\bar{\psi}$ in the functional integral (i.e., the Schwinger–Dyson equation), where the variation is proportional to φ or φ^\dagger ; the associated Jacobian is unity because $\bar{\psi}$ and φ (or φ^\dagger) are independent integration variables. Then, *if* the wave function renormalization factors for $\bar{\psi}$ and φ (or φ^\dagger) are the same, then the Schwinger–Dyson equation will show that the combination in

Eq. (2.19) is a finite operator. The fact is that the wave function renormalization factors differ as Eqs. (5.6) and (5.7) show, and the terms added in Eq. (2.17) is diverging and cancels divergences in S_μ^{imp} . A similar remark applies to the added term in Eq. (2.18).

Some calculation shows that these finite supercurrents enjoy extremely simple forms:

$$\begin{aligned}\tilde{S}_\mu^{\text{imp}} = & -\frac{1}{4g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a F_{\rho\sigma}^a \\ & + \frac{1}{2\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)(P_+D_\nu\psi^a\varphi^a - P_-D_\nu\psi^a\varphi^{\dagger a}) \\ & - \frac{1}{\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)(P_+\psi^a D_\nu\varphi^a - P_-\psi^a D_\nu\varphi^{\dagger a}),\end{aligned}\quad (2.20)$$

$$\begin{aligned}\tilde{\tilde{S}}_\mu^{\text{imp}} = & -\frac{1}{4g_0}\bar{\psi}^a\gamma_\mu\sigma_{\rho\sigma}F_{\rho\sigma}^a \\ & - \frac{1}{2\sqrt{2}}(D_\nu\bar{\psi}^a P_+\varphi^a - D_\nu\bar{\psi}^a P_-\varphi^{\dagger a})\left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right) \\ & + \frac{1}{\sqrt{2}}(\bar{\psi}^a P_+D_\nu\varphi^a - \bar{\psi}^a P_-D_\nu\varphi^{\dagger a})\left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right).\end{aligned}\quad (2.21)$$

At $D = 4$, these currents are manifestly γ -traceless *without any use of the equation of motion* because $\gamma_\mu\sigma_{\rho\sigma}\gamma_\mu = 0$ and $\gamma_\mu[(1/3)\sigma_{\mu\nu} - \delta_{\mu\nu}] = 0$ for $D = 4$.

The gauge-fixing and ghost–anti-ghost terms also break SUSY. We define the ghost and anti-ghost fields as SUSY singlets. Then

$$\delta_\xi S_{\text{gf}} = -\int d^Dx (\bar{\xi}X_{\text{gf}} + \bar{X}_{\text{gf}}\xi),\quad (2.22)$$

$$\delta_\xi S_{c\bar{c}} = -\int d^Dx (\bar{\xi}X_{c\bar{c}} + \bar{X}_{c\bar{c}}\xi),\quad (2.23)$$

where

$$X_{\text{gf}} = \frac{\lambda_0}{2g_0}\gamma_\mu\psi^a\partial_\mu\partial_\nu A_\nu^a,\quad (2.24)$$

$$\bar{X}_{\text{gf}} = -\frac{\lambda_0}{2g_0}\bar{\psi}^a\gamma_\mu\partial_\mu\partial_\nu A_\nu^a,\quad (2.25)$$

and

$$X_{c\bar{c}} = \frac{1}{2g_0}f^{abc}\partial_\mu\bar{c}^a c^b\gamma_\mu\psi^c,\quad (2.26)$$

$$\bar{X}_{c\bar{c}} = -\frac{1}{2g_0}f^{abc}\partial_\mu\bar{c}^a c^b\bar{\psi}^c\gamma_\mu.\quad (2.27)$$

We note that $X_{\text{gf}} + X_{c\bar{c}}$ (and also $\bar{X}_{\text{gf}} + \bar{X}_{c\bar{c}}$) are BRS exact:

$$X_{\text{gf}} + X_{c\bar{c}} = \delta_B\frac{1}{2g_0}\partial_\mu\bar{c}^a\gamma_\mu\psi^a,\quad (2.28)$$

where the BRS transformation δ_B is defined by

$$\delta_B A_\mu^a = D_\mu c^a,\quad \delta_B c^a = -\frac{1}{2}f^{abc}c^b c^c,\quad (2.29)$$

$$\delta_B \bar{c}^a = \lambda_0 \partial_\mu A_\mu^a, \quad (2.30)$$

$$\delta_B \psi^a = -f^{abc} c^b \psi^c, \quad \delta_B \bar{\psi}^a = -f^{abc} c^b \bar{\psi}^c, \quad (2.31)$$

$$\delta_B \varphi^a = -f^{abc} c^b \varphi^c, \quad \delta_B \varphi^{\dagger a} = -f^{abc} c^b \varphi^{\dagger c}. \quad (2.32)$$

Because of Eq. (2.28), $X_{\text{gf}} + X_{c\bar{c}}$ does not contribute in correlation functions of gauge-invariant operators.

3. SUSY WT relations

3.1. SUSY WT relations in bare quantities

In what follows, we consider SUSY WT relations following from the identities

$$\left\langle \delta_\xi \left[\begin{array}{c} A_v^b(y) \\ \varphi^b(y) \\ \varphi^{\dagger b}(y) \end{array} \right] \bar{\psi}^c(z) \right\rangle = 0 \quad (3.1)$$

and

$$\left\langle \delta_{\bar{\xi}} \left[\bar{\psi}^b(y) c^c(z) \bar{c}^d(w) \right] \right\rangle = 0. \quad (3.2)$$

In these identities, the parameters of the SUSY transformation, ξ and $\bar{\xi}$, are promoted to local functions, $\xi(x)$ and $\bar{\xi}(x)$. The variation of the action $S + S_{\text{gf}} + S_{c\bar{c}}$ produces the combination $\partial_\mu S_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x)$ as the coefficient of $\bar{\xi}(x)$; recall Eqs. (2.9), (2.22), and (2.23). The difference between $\tilde{S}_\mu^{\text{imp}}(x)$ and $S_\mu^{\text{imp}}(x)$ in Eq. (2.17) produces another contact term that can be determined by considering the variations, $\delta \bar{\psi}^a(x) = -1/(2\sqrt{2})\varepsilon(x)\gamma_\mu [P_- \varphi^a(x) - P_+ \varphi^{\dagger a}(x)]$ and $\delta(\text{other fields}) = 0$, because the difference is proportional to the equation of motion. In this way, we have

$$\begin{aligned} & \left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] A_v^b(y) \bar{\psi}^c(z) \right\rangle \\ &= -\delta(x-y) \frac{1}{2} g_0 \left\langle \gamma_\nu \psi^b(y) \bar{\psi}^c(z) \right\rangle \\ & \quad - \delta(x-z) \frac{1}{4g_0} \left\langle A_v^b(y) \sigma_{\rho\sigma} F_{\rho\sigma}^c(z) \right\rangle + \delta(x-z) \frac{1}{2} g_0 \left\langle A_v^b(y) \gamma_5 f^{cde} \varphi^{\dagger d}(z) \varphi^e(z) \right\rangle \\ & \quad + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle A_v^b(y) \gamma_\rho \left[P_- D_\rho \varphi^c(z) - P_+ D_\rho \varphi^{\dagger c}(z) \right] \right\rangle \\ & \quad - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle A_v^b(y) \gamma_\mu \left[P_- \varphi^c(z) - P_+ \varphi^{\dagger c}(z) \right] \right\rangle, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] \varphi^b(y) \bar{\psi}^c(z) \right\rangle \\ &= \delta(x-y) \frac{1}{\sqrt{2}} \left\langle P_- \psi^b(y) \bar{\psi}^c(z) \right\rangle \\ & \quad - \delta(x-z) \frac{1}{4g_0} \left\langle \varphi^b(y) \sigma_{\rho\sigma} F_{\rho\sigma}^c(z) \right\rangle + \delta(x-z) \frac{1}{2} g_0 \left\langle \varphi^b(y) \gamma_5 f^{cde} \varphi^{\dagger d}(z) \varphi^e(z) \right\rangle \end{aligned}$$

$$\begin{aligned}
& + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle \varphi^b(y) \gamma_\rho \left[P_- D_\rho \varphi^c(z) - P_+ D_\rho \varphi^{\dagger c}(z) \right] \right\rangle \\
& - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle \varphi^b(y) \gamma_\mu \left[P_- \varphi^c(z) - P_+ \varphi^{\dagger c}(z) \right] \right\rangle,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] \varphi^{\dagger b}(y) \bar{\psi}^c(z) \right\rangle \\
& = -\delta(x-y) \frac{1}{\sqrt{2}} \left\langle P_+ \psi^b(y) \bar{\psi}^c(z) \right\rangle \\
& \quad - \delta(x-z) \frac{1}{4g_0} \left\langle \varphi^{\dagger b}(y) \sigma_{\rho\sigma} F_{\rho\sigma}^c(z) \right\rangle + \delta(x-z) \frac{1}{2} g_0 \left\langle \varphi^{\dagger b}(y) \gamma_5 f^{cde} \varphi^{\dagger d}(z) \varphi^e(z) \right\rangle \\
& \quad + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle \varphi^{\dagger b}(y) \gamma_\rho \left[P_- D_\rho \varphi^c(z) - P_+ D_\rho \varphi^{\dagger c}(z) \right] \right\rangle \\
& \quad - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle \varphi^{\dagger b}(y) \gamma_\mu \left[P_- \varphi^c(z) - P_+ \varphi^{\dagger c}(z) \right] \right\rangle.
\end{aligned} \tag{3.5}$$

From Eq. (3.2), on the other hand, we have

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] \bar{\psi}^b(y) c^c(z) \bar{c}^d(w) \right\rangle \\
& = -\delta(x-y) \frac{1}{4g_0} \left\langle \sigma_{\rho\sigma} F_{\rho\sigma}^b(y) c^c(z) \bar{c}^d(w) \right\rangle + \delta(x-y) \frac{1}{2} g_0 \left\langle \gamma_5 f^{bef} \varphi^{\dagger e}(y) \varphi^f(y) c^c(z) \bar{c}^d(w) \right\rangle \\
& \quad + \delta(x-y) \frac{1}{\sqrt{2}} \left\langle \gamma_\rho \left[P_- D_\rho \varphi^c(z) - P_+ D_\rho \varphi^{\dagger c}(z) \right] c^c(z) \bar{c}^d(w) \right\rangle \\
& \quad - \partial_\mu^x \delta(x-y) \frac{1}{2\sqrt{2}} \left\langle \gamma_\mu \left[P_- \varphi^b(y) - P_+ \varphi^{\dagger b}(y) \right] c^c(z) \bar{c}^d(w) \right\rangle.
\end{aligned} \tag{3.6}$$

These are *identities* holding exactly under the dimensional regularization. In what follows, we will rewrite these identities in terms of renormalized quantities to the one-loop order and find SUSY WT relations among renormalized quantities. Then, using these SUSY WT relations, we will determine the form of a properly normalized supercurrent.

4. The effect of X_{Fierz}

Before going into the problem of renormalization, we analyze the effect of X_{Fierz} in SUSY WT relations. X_{Fierz} arises from the breaking of the Fierz identity at $D \neq 4$ and thus it vanishes in classical theory at $D = 4$. It survives, however, in quantum theory through UV divergences. From the calculation of one-loop diagrams in Fig. 1 in the Feynman gauge $\lambda_0 = 1$, we have

$$\begin{aligned}
& \left\langle X_{\text{Fierz}}(x) \begin{Bmatrix} A_\alpha^b(y) \\ \varphi^b(y) \\ \varphi^{\dagger b}(y) \end{Bmatrix} \bar{\psi}^c(z) \right\rangle \\
& = \frac{g_0^2}{(4\pi)^2} C_2(G) \delta^{bc} \frac{\Gamma(D/2)^2}{\Gamma(D)} \Gamma(2-D/2) (-1)(D-4) \int_p e^{ip(x-y)} \int_q e^{iq(x-z)} \\
& \quad \times \left(\frac{p^2}{4\pi} \right)^{D/2-2} \begin{Bmatrix} g_0(-p^2 \gamma_\alpha + \not{p} p_\alpha) \\ \frac{1}{\sqrt{2}} \frac{1}{D-2} [(D-1) + 3\gamma_5] p^2 \\ -\frac{1}{\sqrt{2}} \frac{1}{D-2} [(D-1) - 3\gamma_5] p^2 \end{Bmatrix} \frac{1}{p^2} \frac{1}{i\cancel{q}}.
\end{aligned} \tag{4.1}$$

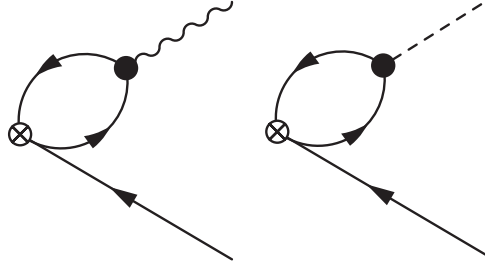


Fig. 1. One-loop Feynman diagrams containing X_{Fierz} .

We see that the effect of X_{Fierz} is proportional to $D - 4$ as expected, but the UV divergences of the diagrams cancel this factor. We can identify its effect for $D \rightarrow 4$ as an insertion of a finite local operator:

$$X_{\text{Fierz}} \xrightarrow{D \rightarrow 4} \frac{g_0^2}{(4\pi)^2} C_2(G) \left(\frac{1}{3g_0} \gamma_\nu \psi^a \partial_\mu F_{\mu\nu}^a - \frac{1}{\sqrt{2}} P_+ \psi^a \partial_\mu \partial_\mu \varphi^{\dagger a} + \frac{1}{\sqrt{2}} P_- \psi^a \partial_\mu \partial_\mu \varphi^a \right). \quad (4.2)$$

(Consideration of the correlation function $\langle X_{\text{Fierz}}(x) \bar{\psi}^b(y) c^c(z) \bar{c}^d(w) \rangle$ does not provide any new information.) We can see that this finite effect of X_{Fierz} in WT relations can be removed by adding a local counterterm to the original action as $S \rightarrow S + \int d^D x \mathcal{L}'$, where

$$\mathcal{L}' \equiv \frac{g_0^2}{(4\pi)^2} C_2(G) \left(-\frac{1}{6g_0^2} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} \partial_\mu \varphi^a \partial_\mu \varphi^a + \frac{1}{2} \partial_\mu \varphi^{\dagger a} \partial_\mu \varphi^{\dagger a} \right). \quad (4.3)$$

Interestingly, this counterterm breaks the global axial $U(1)$ symmetry, $\psi \rightarrow e^{i\gamma_5 \alpha} \psi$, $\bar{\psi} \rightarrow \bar{\psi} e^{i\gamma_5 \alpha}$, $\varphi \rightarrow e^{-2i\alpha} \varphi$, and $\varphi^\dagger \rightarrow e^{2i\alpha} \varphi^\dagger$, that the original action in Eq. (2.1) possesses. The appearance of such a term in quantum theory is, however, not unexpected because the dimensional regularization does not preserve this axial $U(1)$ symmetry.

In what follows, we assume the presence of the counterterm S' . This implies that we can forget about X_{Fierz} in WT relations; this is to be understood throughout the following discussion.

5. Renormalization

In this section, we will work out the renormalization in the one-loop order. We set

$$D \equiv 4 - 2\epsilon, \quad (5.1)$$

and assume the MS scheme. We use the abbreviation

$$\Delta \equiv \frac{g^2}{(4\pi)^2} C_2(G) \frac{1}{\epsilon}, \quad (5.2)$$

where g is the renormalized gauge coupling. Since we work in the one-loop approximation, we always neglect terms of the order $O(\Delta^2)$. In particular, under the product with Δ , we can always neglect the difference between bare and renormalized quantities.

The present 4D $\mathcal{N} = 2$ SYM is, if the complex scalar field is neglected, identical to the 4D $\mathcal{N} = 1$ SYM except that the fermion field is Dirac instead of Majorana. So, for diagrams that do not contain scalar lines, we can borrow the results of Ref. [8], possibly doubling the fermionic degrees of freedom. This somewhat reduces our labor, but there still remain many diagrams that contain scalar lines; see Appendix C. We will work in the Feynman gauge $\lambda_0 = 1$.

5.1. Parameters, elementary fields, and some composite operators

For parameters and elementary fields, to the one-loop order, the renormalization is accomplished by

$$g_0 = \mu^\epsilon (1 - \Delta)g, \quad (5.3)$$

$$\lambda_0 = \lambda, \quad (5.4)$$

$$A_\mu^a = (1 - \Delta)A_{\mu R}^a, \quad (5.5)$$

$$\begin{Bmatrix} \psi^a \\ \bar{\psi}^a \end{Bmatrix} = (1 - \Delta) \begin{Bmatrix} \psi_R^a \\ \bar{\psi}_R^a \end{Bmatrix}, \quad (5.6)$$

$$\begin{Bmatrix} \varphi^a \\ \varphi^{a\dagger} \end{Bmatrix} = \begin{Bmatrix} \varphi_R^a \\ \varphi_R^{a\dagger} \end{Bmatrix}, \quad (5.7)$$

$$c^a = \left(1 - \frac{1}{2}\Delta\right) c_R^a, \quad (5.8)$$

$$\bar{c}^a = (1 - \Delta)\bar{c}_R^a, \quad (5.9)$$

where the quantities on the right-hand side are renormalized ones.

For some gauge-covariant composite operators that appear on the right-hand sides of SUSY WT relations, after some calculation, we find

$$F_{\mu\nu}^a = \left(1 - \frac{5}{2}\Delta\right) (\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) + \left(1 - \frac{11}{4}\Delta\right) \{f^{abc} A_\mu^b A_\nu^c\}_R, \quad (5.10)$$

$$f^{abc} \varphi^{\dagger b} \varphi^c = (1 - \Delta) \{f^{abc} \varphi^{\dagger b} \varphi^c\}_R, \quad (5.11)$$

$$D_\mu \varphi^a = \left(1 - \frac{3}{2}\Delta\right) \partial_\mu \varphi_R^a + \left(1 - \frac{15}{8}\Delta\right) \{f^{abc} A_\mu^b \varphi^c\}_R, \quad (5.12)$$

$$D_\mu \varphi^{\dagger a} = \left(1 - \frac{3}{2}\Delta\right) \partial_\mu \varphi_R^{\dagger a} + \left(1 - \frac{15}{8}\Delta\right) \{f^{abc} A_\mu^b \varphi^{\dagger c}\}_R, \quad (5.13)$$

where $\{\mathcal{O}\}_R$ denotes the renormalized composite operator corresponding to a bare composite operator \mathcal{O} .

5.2. X_{gf} and $X_{c\bar{c}}$

From the renormalization factors in Eqs. (5.3)–(5.9) and Eq. (2.36) of Ref. [8] (divided by 2), and the calculation of the divergent part of diagrams A01–A06 in Appendix C, we have the operator renormalization

$$\begin{aligned} X_{gf} + X_{c\bar{c}} &= (1 + \Delta)X_{gfR} + (1 - \Delta)X_{c\bar{c}R} \\ &+ \Delta \partial_\mu \left[-\frac{1}{4g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a (\partial_\rho A_{\sigma R}^a - \partial_\sigma A_{\rho R}^a) \right] \\ &+ \Delta \left(-\frac{1}{g} \right) \gamma_\nu \psi_R^a \partial_\mu \partial_\mu A_{\nu R}^a \\ &+ \Delta \frac{3}{8g} \sigma_{\mu\nu} \not{\partial} \psi_R^a (\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) \\ &+ \Delta \frac{1}{8g} \partial_\mu \left[(A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \not{\partial} \psi_R^a \right] \end{aligned}$$

$$\begin{aligned}
& + \Delta \left(-\frac{1}{\sqrt{2}} \right) \partial_\mu \left[(P_+ \partial_\nu \varphi_R^a - P_- \partial_\nu \varphi_R^{\dagger a}) \gamma_\nu \gamma_\mu \psi_R^a \right] \\
& + \Delta \left(-\frac{1}{2\sqrt{2}} \right) (P_+ \varphi_R^a - P_- \varphi_R^{\dagger a}) \partial_\mu \partial_\mu \psi_R^a \\
& + \Delta \frac{1}{4} g f^{abc} \gamma_\mu \psi_R^a (\varphi_R^{\dagger b} \overleftrightarrow{\partial}_\mu \varphi_R^c) \\
& + \Delta \left(-\frac{3}{4} \right) g f^{abc} \gamma_5 \gamma_\mu \psi_R^a \partial_\mu (\varphi_R^{\dagger b} \varphi_R^c) \\
& + \Delta (-1) g f^{abc} \gamma_5 \not{\partial} \psi_R^a \varphi_R^{\dagger b} \varphi_R^c \\
& + \Delta \mathcal{H}_1,
\end{aligned} \tag{5.14}$$

where $X_{\text{gf}R}$ and $X_{c\bar{c}R}$ are renormalized finite operators whose tree-level forms coincide with X_{gf} and $X_{c\bar{c}}$, respectively. In the last line, \mathcal{H}_1 is the abbreviation of possible ‘‘higher order terms,’’ which include the following (schematically written) types of operators:

$$O(\psi_R A_R^2) + O(\psi_R A_R \varphi_R) + O(\psi_R \varphi_R^3) + O(\psi_R^3). \tag{5.15}$$

Our present calculation of diagrams in Appendix C cannot determine the coefficients of operators of these forms; they include, for instance, $\Delta f^{abc} \partial_\mu \psi_R A_{\mu R}^b \varphi_R^{\dagger c}$, $\Delta f^{abc} f^{ade} \psi_R^b \varphi_R^c \varphi_R^{\dagger d} \varphi_R^e$, and $\Delta f^{abc} \psi_R \bar{\psi}_R^b \psi_R^c$, etc.

One can confirm that Eq. (5.14) can be further rewritten in the following form:

$$\begin{aligned}
& X_{\text{gf}} + X_{c\bar{c}} \\
& = (1 + \Delta) (X_{\text{gf}R} + X_{c\bar{c}R}) \\
& \quad + \Delta \partial_\mu \tilde{S}_\mu^{\text{imp}} + \Delta \frac{1}{8g} \partial_\mu \left[(A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \frac{\delta S^t}{\delta \bar{\psi}^a} \right] \\
& \quad + \Delta g \gamma_\mu \psi_R^a \frac{\delta S^t}{\delta A_\mu^a} \\
& \quad + \Delta \left[\frac{3}{8g} \sigma_{\mu\nu} (\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) - g \gamma_5 \{ f^{abc} \varphi^{\dagger b} \varphi^c \}_R \right. \\
& \quad \quad \left. - \frac{3}{2\sqrt{2}} \gamma_\mu (\partial_\mu \varphi_R^a P_- - \partial_\mu \varphi_R^{\dagger a} P_+) \right] \frac{\delta S^t}{\delta \bar{\psi}^a} \\
& \quad + \Delta (-\sqrt{2}) P_- \psi_R^a \frac{\delta S^t}{\delta \varphi^a} + \Delta \sqrt{2} P_+ \psi_R^a \frac{\delta S^t}{\delta \varphi^{\dagger a}} \\
& \quad + \Delta \mathcal{H}_2,
\end{aligned} \tag{5.16}$$

where

$$S^t \equiv S + S_{\text{gf}} + S_{c\bar{c}} \tag{5.17}$$

is the total action.⁷ In deriving this form, we have noted the relation $\Delta g \gamma_\mu \psi_R^a \delta S_{c\bar{c}} / (\delta A_\mu^a) = -2\Delta X_{c\bar{c}R}$. The higher-order terms, \mathcal{H}_2 in this expression, which have the same form as Eq. (5.15), differ from \mathcal{H}_1 in Eq. (5.14).

5.3. Supercurrent

Combining the result of Ref. [8], the renormalization factors in Eqs. (5.3)–(5.7), and the calculation of diagrams A01, A02, B02, B03, B04, C01, C02, and C03 in Appendix C (diagrams A03–A06, B16–B19, C06, and C07 also potentially contribute, but it turns out that these diagrams do not give divergences), we find

$$\begin{aligned}
\tilde{S}_\mu^{\text{imp}} &= \tilde{S}_{\mu R}^{\text{imp}} + \Delta \frac{1}{4g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a (\partial_\rho A_{\sigma R}^a - \partial_\sigma A_{\rho R}^a) \\
&\quad + \Delta \frac{3}{4g} \left(\frac{1}{3} \sigma_{\mu\sigma} - \delta_{\mu\sigma} \right) \gamma_\rho \psi_R^a (\partial_\rho A_{\sigma R}^a - \partial_\nu A_{\sigma R}^a) \\
&\quad + \Delta \left(-\frac{1}{8g} \right) (A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \frac{\delta S^t}{\delta \bar{\psi}^a} \\
&\quad + \Delta \mathcal{H}_{3\mu} \\
&= \tilde{S}_{\mu R}^{\text{imp}} + \Delta \left(-\frac{1}{8g} \right) (A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \frac{\delta S^t}{\delta \bar{\psi}^a} + \Delta \mathcal{H}_{3\mu}, \tag{5.18}
\end{aligned}$$

where $\tilde{S}_{\mu R}^{\text{imp}}$ is a renormalized composite operator to the one-loop order and \mathcal{H}_3 are again higher-order terms of the form of Eq. (5.15). The last equality follows from the identity at $D = 4$:

$$\left(\frac{1}{3} \sigma_{\mu\sigma} - \delta_{\mu\sigma} \right) \gamma_\rho \mathcal{A}_{\rho\sigma} = -\frac{1}{3} \sigma_{\rho\sigma} \gamma_\mu \mathcal{A}_{\rho\sigma}, \tag{5.19}$$

where $\mathcal{A}_{\rho\sigma}$ is a quantity that is anti-symmetric in $\rho \leftrightarrow \sigma$.

From Eqs. (5.16) and (5.18), we have

$$\begin{aligned}
&\partial_\mu \tilde{S}_\mu^{\text{imp}} + X_{\text{gf}} + X_{c\bar{c}} \\
&= (1 + \Delta) \left(\partial_\mu \tilde{S}_{\mu R}^{\text{imp}} + X_{\text{gf}R} + X_{c\bar{c}R} \right) \\
&\quad + \Delta g \gamma_\mu \psi_R^a \frac{\delta S^t}{\delta A_\mu^a} \\
&\quad + \Delta \left[\frac{3}{8g} \sigma_{\mu\nu} (\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) - g \gamma_5 \{ f^{abc} \varphi^{\dagger b} \varphi^c \}_R \right. \\
&\quad \quad \left. - \frac{3}{2\sqrt{2}} \gamma_\mu (\partial_\mu \varphi_R^a P_- - \partial_\mu \varphi_R^{\dagger a} P_+) \right] \frac{\delta S^t}{\delta \bar{\psi}^a} \\
&\quad + \Delta (-\sqrt{2}) P_- \psi_R^a \frac{\delta S^t}{\delta \varphi^a} + \Delta \sqrt{2} P_+ \psi_R^a \frac{\delta S^t}{\delta \varphi^{\dagger a}} \\
&\quad + \Delta \partial_\mu \mathcal{H}_{3\mu} + \Delta \mathcal{H}_2. \tag{5.20}
\end{aligned}$$

⁷ In principle, we should include the counterterm of Eq. (4.3) to the total action, but its effect in Eq. (5.16) is $O(\Delta^2)$ and negligible.

We thus observe that, to the one-loop order, the combination $\partial_\mu \tilde{S}_\mu^{\text{imp}} + X_{\text{gf}} + X_{c\bar{c}}$ appearing in the SUSY WT relations of Eqs. (3.3)–(3.6) (except X_{Fierz} , which was already treated in Sect. 4) is a linear combination of $\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}$, $X_{\text{gf}R}$, and $X_{c\bar{c}R}$, up to terms that are proportional to equations of motion and higher-order terms. This structure is common to the case of the 4D $\mathcal{N} = 1$ SYM and we thus expect that the following argument will be similar to that of Ref. [8].

6. SUSY WT relations in renormalized quantities

We now rewrite the SUSY WT relations of Eqs. (3.3)–(3.6) in terms of renormalized quantities by using Eq. (5.20) and the renormalization factors in Eqs. (5.3)–(5.13). Let us illustrate the calculation in some detail by taking Eq. (3.3) as the example: After substituting Eq. (5.20) in the left-hand side of Eq. (3.3), we note the Schwinger–Dyson equations:

$$\left\langle \mathcal{F}_\mu^a(x) \frac{\delta S^t}{\delta A_\mu^a(x)} A_\nu^b(y) \bar{\psi}^c(z) \right\rangle = \delta(x-y) \left\langle \mathcal{F}_\nu^b(y) \bar{\psi}^c(z) \right\rangle, \quad (6.1)$$

$$\left\langle \mathcal{F}^a(x) \frac{\delta S^t}{\delta \bar{\psi}^a(x)} \begin{Bmatrix} A_\alpha^b(y) \\ \varphi^b(y) \\ \varphi^{\dagger b}(y) \end{Bmatrix} \bar{\psi}^c(z) \right\rangle = \delta(x-z) \left\langle \begin{Bmatrix} A_\alpha^b(y) \\ \varphi^b(y) \\ \varphi^{\dagger b}(y) \end{Bmatrix} \mathcal{F}^c(z) \right\rangle, \quad (6.2)$$

$$\left\langle \mathcal{F}^a(x) \frac{\delta S^t}{\delta \varphi^a(x)} \varphi^b(y) \bar{\psi}^c(z) \right\rangle = \delta(x-y) \left\langle \mathcal{F}^b(y) \bar{\psi}^c(z) \right\rangle, \quad (6.3)$$

$$\left\langle \mathcal{F}^a(x) \frac{\delta S^t}{\delta \varphi^{\dagger a}(x)} \varphi^{\dagger b}(y) \bar{\psi}^c(z) \right\rangle = \delta(x-y) \left\langle \mathcal{F}^b(y) \bar{\psi}^c(z) \right\rangle. \quad (6.4)$$

Then, the left-hand side of Eq. (3.3) becomes

$$\begin{aligned} & (1 - \Delta) \left\langle \left[\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \Delta \mathcal{H}_4(x) \right] A_{\nu R}^b(y) \bar{\psi}_R^c(z) \right\rangle \\ & - 2\Delta \delta(x-y) \left\langle \left(-\frac{1}{2} \right) g \left\langle \gamma_\nu \psi_R^b(y) \bar{\psi}_R^c(z) \right\rangle \right\rangle \\ & - \frac{3}{2} \Delta \delta(x-z) \left\langle \left(-\frac{1}{4g} \right) \left\langle A_{\nu R}^b(y) \sigma_{\rho\sigma} \left[\partial_\rho A_{\sigma R}^c(z) - \partial_\sigma A_{\rho R}^c(z) \right] \right\rangle \right\rangle \\ & - 2\Delta \delta(x-z) \frac{1}{2} g \left\langle A_{\nu R}^b(y) \gamma_5 \{ f^{cde} \varphi^{\dagger d} \varphi^e \}_R(z) \right\rangle \\ & - \frac{3}{2} \Delta \delta(x-z) \frac{1}{\sqrt{2}} \left\langle A_{\nu R}^b(y) \gamma_\rho \left[P_- \partial_\rho \varphi_R^c(z) - P_+ \partial_\rho \varphi_R^{\dagger c}(z) \right] \right\rangle, \end{aligned} \quad (6.5)$$

where

$$\mathcal{H}_4 = \partial_\mu \mathcal{H}_{3\mu} + \mathcal{H}_2 + O(\Delta). \quad (6.6)$$

On the other hand, the right-hand side of Eq. (3.3) becomes, after using Eqs. (5.3)–(5.13),

$$\begin{aligned} & (1 - 3\Delta) \delta(x-y) \left\langle \left(-\frac{1}{2} \right) g \left\langle \gamma_\nu \psi_R^b(y) \bar{\psi}_R^c(z) \right\rangle \right\rangle \\ & + \left(1 - \frac{5}{2} \Delta \right) \delta(x-z) \left\langle \left(-\frac{1}{4g} \right) \left\langle A_{\nu R}^b(y) \sigma_{\rho\sigma} \left[\partial_\rho A_{\sigma R}^c(z) - \partial_\sigma A_{\rho R}^c(z) + \mathcal{H}'(z) \right] \right\rangle \right\rangle \end{aligned}$$

$$\begin{aligned}
& + (1 - 3\Delta)\delta(x - z)\frac{1}{2}g\left\langle A_{\nu R}^b(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
& + \left(1 - \frac{5}{2}\Delta\right)\delta(x - z)\frac{1}{\sqrt{2}}\left\langle A_{\nu R}^b(y)\gamma_\rho\left[P_-\partial_\rho\varphi_R^c(z) - P_+\partial_\rho\varphi_R^{\dagger c}(z) + \mathcal{H}'(z)\right]\right\rangle \\
& + (1 - \Delta)(-1)\partial_\mu^x\delta(x - z)\frac{1}{2\sqrt{2}}\left\langle A_{\nu R}^b(y)\gamma_\mu\left[P_-\varphi_R^c(z) - P_+\varphi_R^{\dagger c}(z)\right]\right\rangle, \tag{6.7}
\end{aligned}$$

where \mathcal{H}' is an abbreviation of ‘‘higher-order terms’’ of the following (schematically written) form:

$$O(A_R^2) + O(A_R\varphi_R) + O(\varphi_R^3) + O(\psi_R^2). \tag{6.8}$$

We will be sloppy about the indices of \mathcal{H}' in order to avoid the expressions becoming unnecessarily complicated. Then, transposing the last four lines in Eq. (6.5), the left-hand side of the identity, to the right-hand side, i.e., Eq. (6.7), we find that both sides have precisely the same overall factor $1 - \Delta$. In this way, we finally have

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \Delta\mathcal{H}_4(x) \right] A_{\nu R}^b(y)\bar{\psi}_R^c(z) \right\rangle \\
& = \delta(x - y)\left(-\frac{1}{2}\right)g\left\langle \gamma_\nu\psi_R^b(y)\bar{\psi}_R^c(z) \right\rangle \\
& \quad + \delta(x - z)\left(-\frac{1}{4g}\right)\left\langle A_{\nu R}^b(y)\sigma_{\rho\sigma}\left[\partial_\rho A_{\sigma R}^c(z) - \partial_\sigma A_{\rho R}^c(z) + \mathcal{H}'(z)\right]\right\rangle \\
& \quad + \delta(x - z)\frac{1}{2}g\left\langle A_{\nu R}^b(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
& \quad + \delta(x - z)\frac{1}{\sqrt{2}}\left\langle A_{\nu R}^b(y)\gamma_\rho\left[P_-\partial_\rho\varphi_R^c(z) - P_+\partial_\rho\varphi_R^{\dagger c}(z) + \mathcal{H}'(z)\right]\right\rangle \\
& \quad - \partial_\mu^x\delta(x - z)\frac{1}{2\sqrt{2}}\left\langle A_{\nu R}^b(y)\gamma_\mu\left[P_-\varphi_R^c(z) - P_+\varphi_R^{\dagger c}(z)\right]\right\rangle. \tag{6.9}
\end{aligned}$$

Starting from Eq. (3.4), a similar calculation shows that

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \Delta\mathcal{H}_4(x) \right] \varphi_R^b(y)\bar{\psi}_R^c(z) \right\rangle \\
& = \delta(x - y)\frac{1}{\sqrt{2}}\left\langle P_-\psi_R^b(y)\bar{\psi}_R^c(z) \right\rangle \\
& \quad - \delta(x - z)\frac{1}{4g}\left\langle \varphi_R^b(y)\sigma_{\rho\sigma}\left[\partial_\rho A_{\sigma R}^c(z) - \partial_\sigma A_{\rho R}^c(z) + \mathcal{H}'(z)\right]\right\rangle \\
& \quad + \delta(x - z)\frac{1}{2}g\left\langle \varphi_R^b(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
& \quad + \delta(x - z)\frac{1}{\sqrt{2}}\left\langle \varphi_R^b(y)\gamma_\rho\left[P_-\partial_\rho\varphi_R^c(z) - P_+\partial_\rho\varphi_R^{\dagger c}(z) + \mathcal{H}'(z)\right]\right\rangle \\
& \quad - \partial_\mu^x\delta(x - z)\frac{1}{2\sqrt{2}}\left\langle \varphi_R^b(y)\gamma_\mu\left[P_-\varphi_R^c(z) - P_+\varphi_R^{\dagger c}(z)\right]\right\rangle, \tag{6.10}
\end{aligned}$$

and, for Eq. (3.5), similarly:

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \Delta\mathcal{H}_4(x) \right] \varphi_R^{\dagger b}(y) \bar{\psi}_R^c(z) \right\rangle \\
&= -\delta(x-y) \frac{1}{\sqrt{2}} \left\langle P_+ \psi_R^b(y) \bar{\psi}_R^c(z) \right\rangle \\
&\quad - \delta(x-z) \frac{1}{4g} \left\langle \varphi_R^{\dagger b}(y) \sigma_{\rho\sigma} \left[\partial_\rho A_{\sigma R}^c(z) - \partial_\sigma A_{\rho R}^c(z) + \mathcal{H}'(z) \right] \right\rangle \\
&\quad + \delta(x-z) \frac{1}{2} g \left\langle \varphi_R^{\dagger b}(y) \gamma_5 \{ f^{cde} \varphi^{\dagger d} \varphi^e \}_R(z) \right\rangle \\
&\quad + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle \varphi_R^{\dagger b}(y) \gamma_\rho \left[P_- \partial_\rho \varphi_R^c(z) - P_+ \partial_\rho \varphi_R^{\dagger c}(z) + \mathcal{H}'(z) \right] \right\rangle \\
&\quad - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle \varphi_R^{\dagger b}(y) \gamma_\mu \left[P_- \varphi_R^c(z) - P_+ \varphi_R^{\dagger c}(z) \right] \right\rangle. \tag{6.11}
\end{aligned}$$

Also, Eq. (3.6) yields

$$\begin{aligned}
& \left\langle \left[\partial_\mu \tilde{S}_{\mu R}^{\text{imp}}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \Delta\mathcal{H}_4(x) \right] \bar{\psi}_R^b(y) c_R^c(z) \bar{c}_R^d(w) \right\rangle \\
&= -\delta(x-y) \frac{1}{4g} \left\langle \sigma_{\rho\sigma} \left[\partial_\rho A_{\sigma R}^b(y) - \partial_\sigma A_{\rho R}^b(y) + \mathcal{H}'(y) \right] c_R^c(z) \bar{c}_R^d(w) \right\rangle \\
&\quad + \delta(x-y) \frac{1}{2} g \left\langle \gamma_5 \{ f^{bef} \varphi^{\dagger e} \varphi^f \}_R(y) c_R^c(z) \bar{c}_R^d(w) \right\rangle \\
&\quad + \delta(x-y) \frac{1}{\sqrt{2}} \left\langle \gamma_\rho \left[P_- \partial_\rho \varphi_R^c(y) - P_+ \partial_\rho \varphi_R^{\dagger c}(y) + \mathcal{H}'(y) \right] c_R^c(z) \bar{c}_R^d(w) \right\rangle \\
&\quad - \partial_\mu^x \delta(x-y) \frac{1}{2\sqrt{2}} \left\langle \gamma_\mu \left[P_- \varphi_R^b(y) - P_+ \varphi_R^{\dagger b}(y) \right] c_R^c(z) \bar{c}_R^d(w) \right\rangle. \tag{6.12}
\end{aligned}$$

Thus, in all the above WT relations, we have observed that the combination

$$\partial_\mu \tilde{S}_{\mu R}^{\text{imp}} + X_{\text{gf}R} + X_{c\bar{c}R} + \Delta\mathcal{H}_4 \tag{6.13}$$

generates *properly normalized super transformations on renormalized elementary fields*. The existence of such a finite operator would be expected on general grounds (i.e., SUSY should be free from the anomaly). Nevertheless, the validity of renormalized SUSY WT relations in the WZ gauge that we have observed appears miraculous, because it resulted from nontrivial renormalization/mixing of various composite operators.

7. Properly normalized supercurrent

We have observed that the combination in Eq. (6.13) generates the correct super transformations on renormalized elementary fields. It is by no means obvious if the combination in Eq. (6.13) also generates correct renormalized SUSY transformations on renormalized *composite operators*. To answer this, we will need further complicated analyses of SUSY WT relations containing composite operators. Therefore, as in Ref. [8], we will be satisfied by finding the form of a properly normalized supercurrent that works within the on-mass-shell correlation functions containing gauge-invariant operators. By “on-mass-shell,” we mean that all (renormalized) composite operators including the combination in Eq. (6.13) are separated from each other in position space. In such on-mass-shell correlation functions, we can still regard the combination in Eq. (6.13) as properly normalized

because no UV divergence associated with composite operators colliding at an equal point arises. In what follows, we show that an insertion of the combination in Eq. (6.13) in such correlation functions reduces to

$$\partial_\mu \tilde{S}_\mu^{\text{imp}}, \quad (7.1)$$

where $\tilde{S}_\mu^{\text{imp}}$ is the supercurrent in Eq. (2.20) (its conjugate is given by $\tilde{\tilde{S}}_\mu^{\text{imp}}$ in Eq. (2.21)). This also implies the conservation law of the current $\tilde{S}_\mu^{\text{imp}}$ in on-mass-shell correlation functions, because for on-mass-shell correlation functions there will be no contact terms, such as the right-hand side of Eq. (6.9).

Now, in on-mass-shell correlation functions, equations of motion identically hold. Under tree-level equations of motion, Eq. (5.16) then reduces to

$$X_{\text{gf}} + X_{c\bar{c}} = X_{\text{gf}R} + X_{c\bar{c}R} + \Delta \left(\partial_\mu \tilde{S}_\mu^{\text{imp}} + X_{\text{gf}} + X_{c\bar{c}} \right) + \Delta \mathcal{H}_2. \quad (7.2)$$

Moreover, since $\partial_\mu \tilde{S}_\mu^{\text{imp}} + X_{\text{gf}} + X_{c\bar{c}} = 0$ under tree-level equations of motion, we can further set

$$X_{\text{gf}} + X_{c\bar{c}} = X_{\text{gf}R} + X_{c\bar{c}R} + \Delta \mathcal{H}_2 \quad (7.3)$$

in on-mass-shell correlation functions. This, however, identically vanishes in correlation functions with gauge-invariant operators, because $X_{\text{gf}} + X_{c\bar{c}}$ is BRS exact, as noted in Eq. (2.28). Thus, in on-mass-shell correlation functions with gauge-invariant operators, the combination in Eq. (6.13) can be replaced by

$$\partial_\mu \left(\tilde{S}_{\mu R}^{\text{imp}} + \Delta \mathcal{H}_{3\mu} \right), \quad (7.4)$$

where we have used Eq. (6.6). Then, going back to Eq. (5.18), under tree-level equations of motion, we see that

$$\tilde{S}_\mu^{\text{imp}} = \tilde{S}_{\mu R}^{\text{imp}} + \Delta \mathcal{H}_{3\mu}. \quad (7.5)$$

This is the current appearing in Eq. (7.4), and shows the above Eq. (7.1).

The bottom line of the above very lengthy one-loop analysis is that, in on-mass-shell correlation functions that contain gauge-invariant operators only, the combination in Eq. (6.13), which generates correct renormalized SUSY transformations on renormalized elementary fields, is replaced by $\partial_\mu \tilde{S}_\mu^{\text{imp}}$. This shows that to the one-loop order, the bare supercurrent and its conjugate,

$$\tilde{S}_\mu^{\text{imp}}, \quad \tilde{\tilde{S}}_\mu^{\text{imp}}, \quad (7.6)$$

can be regarded as the properly normalized supercurrents in on-mass-shell correlation functions containing only gauge-invariant operators.

We now express these currents by fields defined by the gradient flow.

8. Gradient flow and the small flow time expansion

8.1. Flow equations and the wave function renormalization of flowed fields

Our flow equations for the gauge field and the fermion field are standard ones [9–13]: Let $t \geq 0$ be the flow time, for the gauge field,⁸

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \quad (8.1)$$

where

$$D_\mu \equiv \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t, x) \equiv \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)], \quad (8.2)$$

and, for the fermion fields,

$$\partial_t \chi(t, x) = D_\mu D_\mu \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \quad (8.3)$$

$$\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) \overleftarrow{D}_\mu \overleftarrow{D}_\mu, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x), \quad (8.4)$$

where

$$D_\mu \equiv \partial_\mu + [B_\mu, \cdot], \quad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - [\cdot, B_\mu]. \quad (8.5)$$

The fields $B_\mu(t, x)$, $\chi(t, x)$, and $\bar{\chi}(t, x)$ are referred to as flowed fields throughout this paper. Although we often call the above one-parameter evolution the gradient flow for historical reasons, the right-hand side of these equations is not the equation of motion of the corresponding field and thus in this sense the evolution is not gradient flow defined by the functional derivative of the action S . This point does not matter, however, for the application in the present paper.

For the scalar field, we adopt

$$\partial_t \phi(t, x) = D_\mu D_\mu \phi(t, x), \quad \phi(t = 0, x) = \varphi(x), \quad (8.6)$$

$$\partial_t \phi^\dagger(t, x) = \phi^\dagger(t, x) \overleftarrow{D}_\mu \overleftarrow{D}_\mu, \quad \phi^\dagger(t = 0, x) = \varphi^\dagger(x). \quad (8.7)$$

This is also not the gradient flow in the narrow sense. In this case, for the renormalizability of the flowed scalar field, it is important not to include other terms of the equation of motion (such as the term arising from the Yukawa coupling) on the right-hand side of the flow equations. We refer the reader to Ref. [48] for the renormalizability of the flow in the scalar field theory. See also Ref. [49,50] for related studies.

A remarkable feature of the “gradient” flow is that any composite operator of flowed fields for $t > 0$ becomes UV finite (i.e., automatically renormalized) under the conventional parameter renormalization such as Eqs. (5.3) and (5.4),⁹ and wave function renormalizations of *elementary* flowed fields [12,13]. See also Ref. [16]. Moreover, the flowed gauge field does not need the wave function renormalization [12]. One-loop calculations in the “Feynman gauge” $\lambda_0 = \alpha_0 = 1$ yield

$$\left\langle B_\mu^a(t, x) B_\nu^b(s, y) \right\rangle = g_0^2 \delta^{ab} \delta_{\mu\nu} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2} (1 + 2\Delta) + \text{finite part}, \quad (8.8)$$

⁸ The term that is proportional to the “gauge-fixing parameter” α_0 is introduced to simplify the perturbative treatment of the gauge degrees of freedom. Although this term breaks the gauge covariance, it can be shown that any gauge-invariant quantity is independent of α_0 [11,12]. This gauge-breaking term is thus physically irrelevant.

⁹ It is shown that the parameter α_0 in Eq. (8.1) does not receive the renormalization.

$$\left\langle \chi^a(t, x) \bar{\chi}^b(s, y) \right\rangle = \delta^{ab} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{i\cancel{p}} (1 - 4\Delta) + \text{finite part}, \quad (8.9)$$

$$\left\langle \phi^a(t, x) \phi^{\dagger b}(s, y) \right\rangle = \delta^{ab} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2} (1 - 2\Delta) + \text{finite part}, \quad (8.10)$$

where Δ is defined by Eq. (5.2). For perturbation calculation of the correlation function of flowed fields, we refer the reader to Refs. [11–13, 17, 18]. In Eq. (8.8), we see that the one-loop divergence is actually removed by the one-loop gauge-coupling renormalization of Eq. (5.3), and the flowed gauge field does not need the wave function renormalization. For other fields, setting

$$\chi_R(t, x) = Z_\chi^{1/2} \chi(t, x), \quad \bar{\chi}_R(t, x) = Z_\chi^{1/2} \bar{\chi}(t, x), \quad (8.11)$$

$$\phi_R(t, x) = Z_\phi^{1/2} \phi(t, x), \quad \phi_R^\dagger(t, x) = Z_\phi^{1/2} \phi^\dagger(t, x), \quad (8.12)$$

we see, to the one-loop order,

$$Z_\chi = 1 + 4\Delta, \quad Z_\phi = 1 + 2\Delta. \quad (8.13)$$

8.2. Ringed fields

Although the wave function renormalization of flowed fields renders all composite operators finite, the wave function renormalization factors themselves depend on the regularization, and this is not satisfactory from the perspective of a universal representation of composite operators. To avoid this point, we introduce the following ringed fields, following Ref. [18]:

$$\dot{\chi}(t, x) \equiv \sqrt{\frac{-2 \dim(G)}{(4\pi)^2 t^2 \langle \bar{\chi}^a(t, x) \overleftrightarrow{D} \chi^a(t, x) \rangle}} \chi(t, x), \quad (8.14)$$

$$\dot{\bar{\chi}}(t, x) \equiv \sqrt{\frac{-2 \dim(G)}{(4\pi)^2 t^2 \langle \bar{\chi}^a(t, x) \overleftrightarrow{D} \chi^a(t, x) \rangle}} \bar{\chi}(t, x). \quad (8.15)$$

The wave function renormalization factor is canceled out in the ringed fields, and any composite operator of the ringed fields becomes finite without an explicit wave function renormalization. The expectation value $\langle \bar{\chi}(t, x) \overleftrightarrow{D} \chi(t, x) \rangle$ in the denominator does not vanish. In fact, to the one-loop order (this includes the contribution of scalar fields in addition to the result of Ref. [18]),

$$\begin{aligned} & \left\langle \bar{\chi}^a(t, x) \overleftrightarrow{D} \chi^a(t, x) \right\rangle \\ &= \frac{-2 \dim(G)}{(4\pi)^2 t^2} \left\{ (8\pi t)^\epsilon + \frac{g_0^2}{(4\pi)^2} C_2(G) \left[-\frac{4}{\epsilon} - 8 \ln(8\pi t) - \frac{3}{2} + \ln(432) \right] \right\}. \end{aligned} \quad (8.16)$$

Similarly, for the scalar fields, we introduce [51]

$$\dot{\phi}(t, x) \equiv \sqrt{\frac{\dim(G)}{2(4\pi)^2 t \langle \phi^{\dagger a}(t, x) \phi^a(t, x) \rangle}} \phi(t, x), \quad (8.17)$$

$$\dot{\phi}^\dagger(t, x) \equiv \sqrt{\frac{\dim(G)}{2(4\pi)^2 t \langle \phi^{\dagger a}(t, x) \phi^a(t, x) \rangle}} \phi^\dagger(t, x). \quad (8.18)$$

Table 1. Contribution of each diagram in Appendix C to Eq. (8.19) in units of $\frac{\dim(G)}{2(4\pi)^2 t} \frac{g_0^2}{(4\pi)^2} C_2(G)$.

Diagram	
E02	$\frac{2}{\epsilon} + 4 \ln(8\pi t) + 6$
E03	$\frac{2}{\epsilon} + 4 \ln(8\pi t) + 6$
E04	$-2 - 4 \ln 2 + 6 \ln 3$
E05	$12 \ln 2 - 6 \ln 3$
E06	$-\frac{4}{\epsilon} - 8 \ln(8\pi t) - 6$
E07	$-\frac{2}{\epsilon} - 4 \ln(8\pi t) - 7$

For the following calculations, we need to compute the expectation value $\langle \phi^{\dagger a}(t, x) \phi^a(t, x) \rangle$. By using the integration formula, Eq. (B2) of Ref. [18], we have the numbers tabulated in Table 1.

In total,

$$\begin{aligned} & \left\langle \phi^{\dagger a}(t, x) \phi^a(t, x) \right\rangle \\ &= \frac{\dim(G)}{2(4\pi)^2 t} \left\{ \frac{1}{1-\epsilon} (8\pi t)^\epsilon + \frac{g_0^2}{(4\pi)^2} C_2(G) \left[-\frac{2}{\epsilon} - 4 \ln(8\pi t) - 3 + 8 \ln 2 \right] \right\}. \end{aligned} \quad (8.19)$$

8.3. Small flow time expansion

In this subsection we present the computation of the small flow time expansion [12] of composite operators that are relevant to the construction of the supercurrents in Eqs. (2.20) and (2.21). In what follows, we set

$$\xi(t) \equiv \frac{g_0^2}{(4\pi)^2} C_2(G) (8\pi t)^{2-D/2}. \quad (8.20)$$

The calculations of the small flow time expansion are presented in Refs. [8,12,17,18,32,33,55], and we refer the reader to these references for the actual computation. In particular, the background field method developed in Ref. [55] is very powerful and was applied to the computation of the supercurrent in the 4D $\mathcal{N} = 1$ SYM in Ref. [8]. In this paper, however, we do not use this method because the presence of the scalar field reduces the simplicity of the method; we thus use the standard diagrammatic expansion as in Refs. [12,18]. The diagrams relevant to the computation of the supercurrent are collected in Appendix C;¹⁰ our convention for the flow Feynman diagram is also summarized at the beginning of Appendix C.

Now, for diagrams *without* external scalar lines, we can literally use the results of Eqs. (3.7), (3.30), (3.31), and (3.32) from Ref. [8], because there is no one-loop diagram that contains a scalar loop

¹⁰ If only the “topology” of the diagram is concerned, there also exist other diagrams which are not included in Appendix C. We carefully confirmed that those omitted diagrams give only higher-order contributions in the small flow time expansion. Examples of such diagrams are B13, B14, and B15, which do not contribute to the following expansion.

and has gauge external lines only. For diagrams with external scalar lines, diagrams A01–A06 in Appendix C contribute. After some calculation, we have

$$\begin{aligned}
& \frac{1}{g_0} \chi^a(t, x) G_{\mu\nu}^a(t, x) \\
&= \left[1 + \frac{-2}{D-4} \xi(t) \right] \frac{1}{g_0} \psi^a(x) F_{\mu\nu}^a(x) \\
&+ \xi(t) \left\{ \frac{2}{(D-4)(D-2)} \frac{1}{g_0} [\gamma_\mu \gamma_\rho \psi^a(x) F_{\rho\nu}^a(x) - \gamma_\nu \gamma_\rho \psi^a(x) F_{\rho\mu}^a(x)] \right. \\
&\quad \left. + \frac{4}{(D-4)(D-2)D} \frac{1}{g_0} \sigma_{\rho\sigma} \sigma_{\mu\nu} \psi^a(x) F_{\rho\sigma}^a(x) \right\} \\
&+ \xi(t) \sqrt{2} \left\{ \frac{4}{(D-4)(D-2)D} \gamma_\rho \gamma_\mu \gamma_\nu [P_+ \psi^a(x) D_\rho \varphi^a(x) - P_- \psi^a(x) D_\rho \varphi^{\dagger a}(x)] \right. \\
&\quad + \frac{-2}{(D-2)D} \gamma_\nu [P_+ \psi^a(x) D_\mu \varphi^a(x) - P_- \psi^a(x) D_\mu \varphi^{\dagger a}(x)] \\
&\quad + \frac{-2}{(D-4)(D-2)} \gamma_\nu [P_+ D_\mu \psi^a(x) \varphi^a(x) - P_- D_\mu \psi^a(x) \varphi^{\dagger a}(x)] \\
&\quad + \frac{2(D+4)}{(D-2)D(D+2)} \gamma_\nu \gamma_5 D_\mu \psi^a(x) [\varphi^a(x) + \varphi^{\dagger a}(x)] \\
&\quad \left. + \frac{2}{(D-2)(D+2)} \gamma_\nu \gamma_5 \psi^a(x) D_\mu [\varphi^a(x) + \varphi^{\dagger a}(x)] \right\} - (\mu \leftrightarrow \nu) \\
&+ \xi(t) \frac{8}{(D-4)(D-2)D} g_0 f^{abc} \sigma_{\mu\nu} \gamma_5 \psi^a(x) \varphi^{\dagger b}(x) \varphi^c(x) + O(t). \tag{8.21}
\end{aligned}$$

For $\chi^a(t, x) D_\mu \phi^a(t, x)$, diagrams B01–B20 and C01–C07 in Appendix C contribute and we have

$$\begin{aligned}
& \chi^a(t, x) D_\mu \phi^a(t, x) \\
&= \left[1 + \frac{2(D-1)}{(D-4)(D-2)} \xi(t) \right] \psi^a(x) D_\mu \varphi^a(x) \\
&+ \xi(t) \left\{ \frac{2}{(D-4)(D-2)} \sigma_{\mu\nu} \psi^a(x) D_\nu \varphi^a(x) \right. \\
&\quad + \frac{2(D-1)}{(D-4)D} D_\mu \psi^a(x) \varphi^a(x) \\
&\quad \left. + \frac{-2}{(D-4)D} \sigma_{\mu\nu} D_\nu \psi^a(x) \varphi^a(x) \right\} \\
&+ \xi(t) \left\{ \frac{4}{(D-4)D} P_- \psi^a(x) D_\mu \varphi^a(x) \right. \\
&\quad + \frac{8}{(D-4)(D-2)D} \sigma_{\mu\nu} P_- \psi^a(x) D_\nu \varphi^a(x) \\
&\quad + \frac{4}{(D-4)(D-2)} P_- D_\mu \psi^a(x) \varphi^a(x) \\
&\quad \left. + \frac{-4}{(D-2)(D+2)} P_- \psi^a(x) D_\mu [\varphi^a(x) + \varphi^{\dagger a}(x)] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{-4(D+4)}{(D-2)D(D+2)} P_- D_\mu \psi^a(x) \left[\varphi^a(x) + \varphi^{\dagger a}(x) \right] \Big\} \\
& + \xi(t) \sqrt{2} \left\{ \frac{-2}{(D-4)(D-2)D} \frac{1}{g_0} \gamma_\mu \sigma_{\rho\sigma} P_- \psi^a(x) F_{\rho\sigma}^a(x) \right. \\
& \quad + \frac{8}{(D-4)(D-2)D} \frac{1}{g_0} \gamma_\nu P_- \psi^a(x) F_{\mu\nu}^a(x) \\
& \quad + \frac{-2(D+4)}{(D-4)(D-2)D} g_0 f^{abc} \gamma_\mu P_- \psi^a(x) \varphi^{\dagger b}(x) \varphi^c(x) \\
& \quad \left. + \frac{-2}{(D-2)D} g_0 f^{abc} \gamma_\mu \gamma_5 \psi^a(x) \varphi^{\dagger b}(x) \varphi^c(x) \right\} + O(t). \quad (8.22)
\end{aligned}$$

From diagrams B01, B04, B06, B08, B10, and B12, we have

$$\begin{aligned}
\chi^a(t, x) \phi^a(t, x) &= \left[1 + \frac{4(D-1)}{(D-4)(D-2)} \xi(t) \right] \psi^a(x) \varphi^a(x) \\
& + \xi(t) \left\{ \frac{8}{(D-4)(D-2)} P_- \psi^a(x) \varphi^a(x) \right. \\
& \quad \left. + \frac{-8}{(D-2)D} P_- \psi^a(x) \left[\varphi^a(x) + \varphi^{\dagger a}(x) \right] \right\} + O(t). \quad (8.23)
\end{aligned}$$

Using the relation $\partial_\mu(\chi^a \phi^a) = (D_\mu \chi^a) \phi^a + \chi^a D_\mu \phi^a$, we can also deduce the small flow time expansion of $(D_\mu \chi^a) \phi^a$ from Eqs. (8.22) and (8.23).

On the other hand, by applying the parity transformations of Eqs. (B.1)–(B.7) to this, we infer that

$$\begin{aligned}
\chi^a(t, x) \phi^{\dagger a}(t, x) &= \left[1 + \frac{4(D-1)}{(D-4)(D-2)} \xi(t) \right] \psi^a(x) \varphi^{\dagger a}(x) \\
& + \xi(t) \left\{ \frac{8}{(D-4)(D-2)} P_+ \psi^a(x) \varphi^{\dagger a}(x) \right. \\
& \quad \left. + \frac{-8}{(D-2)D} P_+ \psi^a(x) \left[\varphi^a(x) + \varphi^{\dagger a}(x) \right] \right\} + O(t). \quad (8.24)
\end{aligned}$$

Finally, for $g_0 f^{abc} \chi^a(t, x) \phi^{\dagger b}(t, x) \phi^c(t, x)$, diagrams D01–D11 give rise to

$$\begin{aligned}
& g_0 f^{abc} \chi^a(t, x) \phi^{\dagger b}(t, x) \phi^c(t, x) \\
&= \left[1 + \frac{2(3D^2 - 6D - 8)}{(D-4)(D-2)D} \xi(t) \right] g_0 f^{abc} \psi^a(x) \varphi^{\dagger b}(x) \varphi^c(x) \\
& \quad + \xi(t) \sqrt{2} \frac{2}{(D-4)(D-2)} \gamma_\mu \left[P_+ D_\mu \psi^a(x) \varphi^a(x) + P_- D_\mu \psi^a(x) \varphi^{\dagger a}(x) \right] + O(t). \quad (8.25)
\end{aligned}$$

It is easy to invert the above relations and obtain expressions for composite operators of the unflowed fields in terms of composite operators of flowed fields to the one-loop order. For example, Eq. (8.23) yields

$$\begin{aligned}
\psi^a(x)\varphi^a(x) &= \left[1 + \frac{-4(D-1)}{(D-4)(D-2)}\xi(t) \right] \chi^a(t,x)\phi^a(t,x) \\
&\quad + \xi(t) \left\{ \frac{-8}{(D-4)(D-2)} P_- \chi^a(t,x)\phi^a(t,x) \right. \\
&\quad \left. + \frac{8}{(D-2)D} P_- \chi^a(t,x) \left[\phi^a(t,x) + \phi^{\dagger a}(t,x) \right] \right\} + O(t). \quad (8.26)
\end{aligned}$$

Similar inversions can be made for other relations.

8.4. Final steps

We substitute the relations in the small flow time expansion presented in the last subsection into the expression of the supercurrent, Eq. (2.20) [in this form, we do not need Eq. (8.25)]. Then we rewrite the expression in terms of the renormalized gauge coupling in Eq. (5.3) and the ringed fields in Sect. 8.2. Taking the limit $D \rightarrow 4$, we finally find

$$\begin{aligned}
&\tilde{S}_\mu^{\text{imp}} \\
&= \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\ln(8\pi\mu^2 t) - \frac{9}{4} + \frac{1}{2} \ln(432) \right] \right\} \left(-\frac{1}{4g} \right) \sigma_{\rho\sigma} \gamma_\mu \dot{\chi}^a G_{\rho\sigma}^a \\
&\quad - \frac{g}{(4\pi)^2} C_2(G) \gamma_\nu \dot{\chi}^a G_{\nu\mu}^a \\
&\quad + \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\frac{19}{4} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \frac{1}{2\sqrt{2}} \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) (P_+ D_\nu \dot{\chi}^a \dot{\phi}^a - P_- D_\nu \dot{\chi}^a \dot{\phi}^{\dagger a}) \\
&\quad - \frac{3}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) (P_+ D_\mu \dot{\chi}^a \dot{\phi}^a - P_- D_\mu \dot{\chi}^a \dot{\phi}^{\dagger a}) \\
&\quad + \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[\frac{1}{2} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
&\quad \quad \times \left(-\frac{1}{\sqrt{2}} \right) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) (P_+ \dot{\chi}^a D_\nu \dot{\phi}^a - P_- \dot{\chi}^a D_\nu \dot{\phi}^{\dagger a}) \\
&\quad + \frac{1}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) \gamma_5 D_\nu \dot{\chi}^a (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \\
&\quad + \frac{1}{2\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) \left(\frac{1}{3} \sigma_{\mu\nu} - \delta_{\mu\nu} \right) \gamma_5 \dot{\chi}^a D_\nu (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \\
&\quad - \frac{1}{4} \frac{g^3}{(4\pi)^2} C_2(G) f^{abc} \gamma_5 \gamma_\mu \dot{\chi}^a \dot{\phi}^{\dagger b} \dot{\phi}^c + O(t). \quad (8.27)
\end{aligned}$$

The conjugate supercurrent $\tilde{\tilde{S}}_\mu^{\text{imp}}$ is related to $\tilde{S}_\mu^{\text{imp}}$ by the charge conjugation in Eqs. (B.8)–(B.16) as $\tilde{\tilde{S}}_\mu^{\text{imp}} \rightarrow C(\tilde{S}_\mu^{\text{imp}})^T$. Using this, we have

$$\begin{aligned}
&\tilde{\tilde{S}}_\mu^{\text{imp}} \\
&= \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\ln(8\pi\mu^2 t) - \frac{9}{4} + \frac{1}{2} \ln(432) \right] \right\} \left(-\frac{1}{4g} \right) \dot{\chi}^a \gamma_\mu \sigma_{\rho\sigma} G_{\rho\sigma}^a
\end{aligned}$$

$$\begin{aligned}
& + \frac{g}{(4\pi)^2} C_2(G) \dot{\chi}^a \gamma_\nu G_{\nu\mu}^a \\
& + \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\frac{19}{4} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
& \quad \times \left(-\frac{1}{2\sqrt{2}} \right) (D_\nu \dot{\chi}^a P_+ \dot{\phi}^a - D_\nu \dot{\chi}^a P_- \dot{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
& + \frac{3}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) (D_\mu \dot{\chi}^a P_+ \dot{\phi}^a - D_\mu \dot{\chi}^a P_- \dot{\phi}^{\dagger a}) \\
& + \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[\frac{1}{2} + 4 \ln 2 + \frac{1}{2} \ln(432) \right] \right\} \\
& \quad \times \frac{1}{\sqrt{2}} (P_+ \dot{\chi}^a D_\nu \dot{\phi}^a - P_- \dot{\chi}^a D_\nu \dot{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
& - \frac{1}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) D_\nu \dot{\chi}^a \gamma_5 (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
& - \frac{1}{2\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) \dot{\chi}^a \gamma_5 D_\nu (\dot{\phi}^a + \dot{\phi}^{\dagger a}) \left(\frac{1}{3} \sigma_{\nu\mu} - \delta_{\nu\mu} \right) \\
& + \frac{1}{4} \frac{g^3}{(4\pi)^2} C_2(G) f^{abc} \dot{\chi}^a \gamma_\mu \gamma_5 \dot{\phi}^{\dagger b} \dot{\phi}^c + O(t). \tag{8.28}
\end{aligned}$$

Some remarks are in order: (1) The composite operators in Eqs. (8.27) and (8.28) are written completely in terms of the renormalized gauge coupling, the flowed gauge field, and the ringed flowed fermion and scalar fields. Thus, these operators are *manifestly finite* renormalized operators being independent of the regularization. (2) From these expressions, we have the γ -trace anomaly:¹¹

$$\begin{aligned}
\gamma_\mu \tilde{S}_\mu^{\text{imp}} &= \frac{g}{(4\pi)^2} C_2(G) \sigma_{\mu\nu} \dot{\chi}^a G_{\mu\nu}^a - \frac{3}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) (P_- \mathcal{D} \dot{\chi}^a \dot{\phi}^a - P_+ \mathcal{D} \dot{\chi}^a \dot{\phi}^{\dagger a}) \\
& + \frac{g^3}{(4\pi)^2} C_2(G) f^{abc} \gamma_5 \dot{\chi}^a \dot{\phi}^{\dagger b} \dot{\phi}^c + O(t) \\
& = \frac{g}{(4\pi)^2} C_2(G) \sigma_{\mu\nu} \dot{\chi}^a G_{\mu\nu}^a - 2 \frac{g^3}{(4\pi)^2} C_2(G) f^{abc} \gamma_5 \dot{\chi}^a \dot{\phi}^{\dagger b} \dot{\phi}^c + O(t), \tag{8.29}
\end{aligned}$$

where in the last line we have used equations of motion whose use is justified in on-mass-shell correlation functions. Note that in the tree level, the flowed fields are identical to unflowed ones up to $O(t)$ terms. (3) To the one-loop order, Eqs. (8.27) and (8.28) give the properly normalized supercurrent, as argued in Eq. (7.6). The difference between this approximation and the “would-be true supercurrent” will be $O(g_0^2)$. Since we will invoke the renormalization group improvement to be shortly discussed, this difference can be neglected in the final expressions in the $t \rightarrow 0$ limit. (4) Since the composite operators on both sides of Eq. (8.27), $\tilde{S}_\mu^{\text{imp}}(x)$ [Eq. (2.20)], $\mathcal{O}_1(t, x) \equiv \sigma_{\rho\sigma} \gamma_\mu \dot{\chi}^a(t, x) G_{\rho\sigma}^a(t, x)$, etc. are bare operators, the derivative of the coefficients $c_1(t)$ etc., where $\tilde{S}_\mu^{\text{imp}}(x) \equiv c_1(t) \mathcal{O}_1(t, x) + \dots$, with respect to the renormalization scale μ while bare quantities are

¹¹ Note that we can set $\gamma_\mu \sigma_{\rho\sigma} \gamma_\mu = \gamma_\mu [(1/3) \sigma_{\mu\nu} - \delta_{\mu\nu}] = 0$ in these finite (and thus $D = 4$) expressions.

kept fixed vanishes:

$$\left(\mu \frac{\partial}{\partial \mu}\right)_0 c_1(t) = 0. \quad (8.30)$$

By the standard argument, this implies that we can set the renormalization scale μ in $c_1(t)$ arbitrarily, if the renormalized gauge coupling g in $c_1(t)$ is replaced by the running gauge coupling $\bar{g}(\mu)$ defined by

$$\mu \frac{d\bar{g}(\mu)}{d\mu} = \beta(\bar{g}(\mu)), \quad (8.31)$$

where the beta function is $\beta(g) = -2g^3 C_2(G)/(4\pi)^2$ to all orders in perturbation theory [56–60]. In fact,

$$\begin{aligned} \frac{d}{d\mu} c_1(t)|_{g=\bar{g}(\mu)} &= \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{d\bar{g}(\mu)}{d\mu} \frac{\partial}{\partial \bar{g}(\mu)} \right] c_1(t)|_{g=\bar{g}(\mu)} \\ &= \left[\mu \frac{\partial}{\partial \mu} + \beta(\bar{g}(\mu)) \frac{\partial}{\partial \bar{g}(\mu)} \right] c_1(t)|_{g=\bar{g}(\mu)} \\ &= \left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] c_1(t) \Big|_{g=\bar{g}(\mu)} \\ &= \left(\mu \frac{\partial}{\partial \mu} \right)_0 c_1(t) \Big|_{g=\bar{g}(\mu)} \\ &= 0. \end{aligned} \quad (8.32)$$

Here, we have used the definition of the beta function

$$\beta(g) = \left(\mu \frac{\partial}{\partial \mu} \right)_0 g \quad (8.33)$$

and Eq. (8.30). We thus take $\mu = 1/\sqrt{8t}$ by using the flow time t . Then, taking the $t \rightarrow 0$ limit, we have our formulas, Eqs. (1.1) and (1.2). This completes our argument.

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Appendix A. Notational convention

Without noting otherwise, repeated indices are understood to be summed over. The spacetime dimension is denoted by $D \equiv 4 - 2\epsilon$. We use the following abbreviation for the momentum integral:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}. \quad (A.1)$$

Our Dirac matrices γ_μ , satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, are all Hermitian and for the trace over the spinor index we set $\text{tr}(1) = 4$ for any spacetime dimension D . The chiral matrix and chirality projection operators are defined by

$$\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3, \quad P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5), \quad (\text{A.2})$$

for any D ; we have

$$\text{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \begin{cases} 4\epsilon_{\mu\nu\rho\sigma}, & \mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise,} \end{cases} \quad (\text{A.3})$$

where the totally anti-symmetric tensor is normalized as $\epsilon_{0123} = 1$. We also use

$$\sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \quad (\text{A.4})$$

In the 4D $\mathcal{N} = 2$ SYM, all fields belong to the adjoint representation and the covariant derivatives for a generic field X are defined from the structure constants of the gauge group f^{abc} by

$$D_\mu X^a \equiv \partial_\mu X^a + f^{abc} A_\mu^b X^c, \quad (\text{A.5})$$

$$X^a \overleftarrow{D}_\mu \equiv X^a \overleftarrow{\partial}_\mu - X^c f^{cba} A_\mu^b. \quad (\text{A.6})$$

We also use the abbreviations $D \equiv \gamma_\mu D_\mu$ and $\overleftarrow{D} \equiv \gamma_\mu \overleftarrow{D}_\mu$. The quadratic Casimir $C_2(G)$ is defined by $f^{bXa} f^{aYb} = -C_2(G) \delta^{XY}$. A useful identity is

$$f^{cXa} f^{aYb} f^{bZc} = -\frac{1}{2} C_2(G) f^{XYZ}. \quad (\text{A.7})$$

Appendix B. The parity and charge conjugation invariance

The actions, the flow equations and the initial conditions, all elements of the present system, are invariant under the following parity transformation and charge conjugation.

The parity transformation is defined, denoting the spatial directions $\mu = 1, 2, \text{ and } 3$ by i , by

$$\psi(x) \rightarrow \gamma_0 \psi(\tilde{x}), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(\tilde{x}) \gamma_0, \quad (\text{B.1})$$

$$A_0(x) \rightarrow A_0(\tilde{x}), \quad A_i(x) \rightarrow -A_i(\tilde{x}), \quad (\text{B.2})$$

$$\phi(x) \rightarrow -\phi^\dagger(\tilde{x}), \quad \phi^\dagger(x) \rightarrow -\phi(\tilde{x}), \quad (\text{B.3})$$

$$c(x) \rightarrow c(\tilde{x}), \quad \bar{c}(x) \rightarrow \bar{c}(\tilde{x}), \quad (\text{B.4})$$

where $\tilde{x} \equiv (x_0, -x_i)$ and

$$\chi(t, x) \rightarrow \gamma_0 \chi(t, \tilde{x}), \quad \bar{\chi}(t, x) \rightarrow \bar{\chi}(t, \tilde{x}) \gamma_0, \quad (\text{B.5})$$

$$B_0(t, x) \rightarrow A_0(t, \tilde{x}), \quad B_i(t, x) \rightarrow -A_i(t, \tilde{x}), \quad (\text{B.6})$$

$$\phi(t, x) \rightarrow -\phi^\dagger(t, \tilde{x}), \quad \phi^\dagger(t, x) \rightarrow -\phi(t, \tilde{x}). \quad (\text{B.7})$$

The charge conjugation, on the other hand, is defined by

$$\psi(x) \rightarrow C \bar{\psi}^T(x), \quad \bar{\psi}(x) \rightarrow -\psi^T(x) C^{-1}, \quad (\text{B.8})$$

$$A_\mu(x) \rightarrow A_\mu(x), \quad (\text{B.9})$$

$$\varphi(x) \rightarrow -\varphi(x), \quad \varphi^\dagger(x) \rightarrow -\varphi^\dagger(x), \quad (\text{B.10})$$

$$c(x) \rightarrow c(x), \quad \bar{c}(x) \rightarrow \bar{c}(x), \quad (\text{B.11})$$

where C is the charge conjugation matrix satisfying

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \quad C^T = -C, \quad (\text{B.12})$$

and thus

$$C^{-1}\sigma_{\mu\nu} C = -\sigma_{\mu\nu}^T, \quad C^{-1}\gamma_5 C = \gamma_5^T, \quad (\text{B.13})$$

for any D ; see Appendix A of Ref. [8]. Correspondingly,

$$\chi(t, x) \rightarrow C\bar{\chi}^T(t, x), \quad \bar{\chi}(t, x) \rightarrow -\chi^T(t, x)C^{-1}, \quad (\text{B.14})$$

$$B_\mu(t, x) \rightarrow B_\mu(t, x), \quad (\text{B.15})$$

$$\phi(t, x) \rightarrow -\phi(t, x), \quad \phi^\dagger(t, x) \rightarrow -\phi^\dagger(t, x). \quad (\text{B.16})$$

Appendix C. (Flow) Feynman diagrams

Here we collect Feynman diagrams and flow Feynman diagrams [11,12] that are relevant to the computations in the main text. We basically follow the convention in Ref. [18]: The wavy line and the straight arrowed line represent the gauge field propagator and the Dirac fermion field propagator, respectively. In addition to these, in this paper the broken line represents the scalar field propagator. Doubled lines denote the corresponding heat kernels [12,13,18]. That is, the double wavy line, the double arrowed line, and the double broken line represent the gauge field heat kernel, the fermion field heat kernel, and the scalar field heat kernel, respectively. The black bullet denotes the interaction vertex in the original action, while the white circle denotes the interaction term in the flow equations [12,13,18]. The x-mark generally represents the composite operator under consideration.

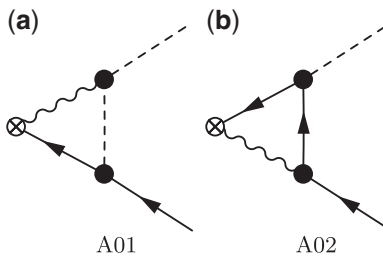


Fig. C.1. One-loop (flow) Feynman diagrams for the operator renormalization and the small flow time expansion.

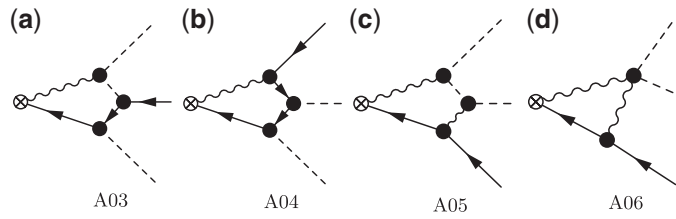


Fig. C.2. One-loop (flow) Feynman diagrams (continued).

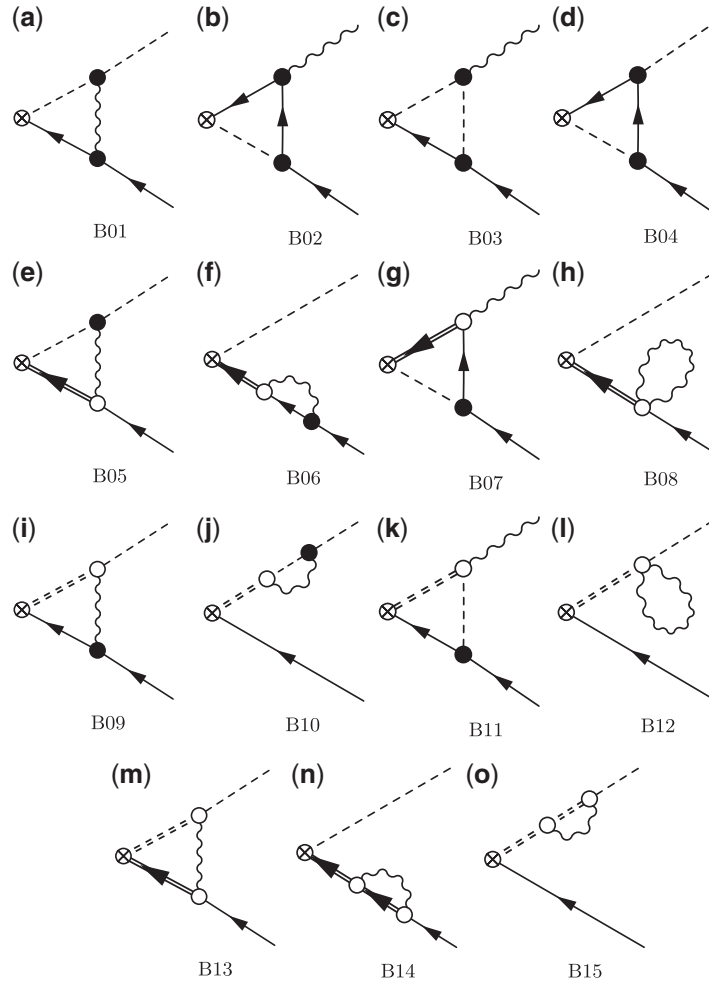


Fig. C.3. One-loop (flow) Feynman diagrams (continued).

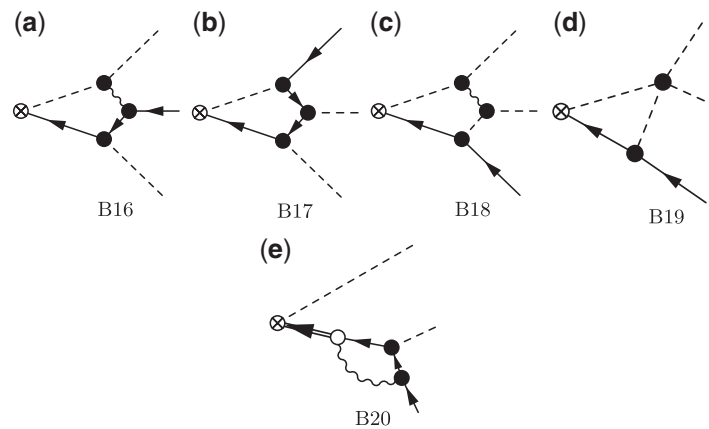


Fig. C.4. One-loop (flow) Feynman diagrams (continued).

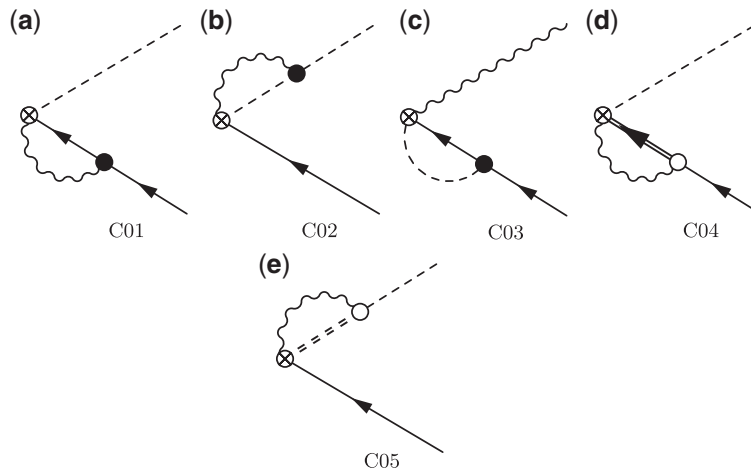


Fig. C.5. One-loop (flow) Feynman diagrams (continued).

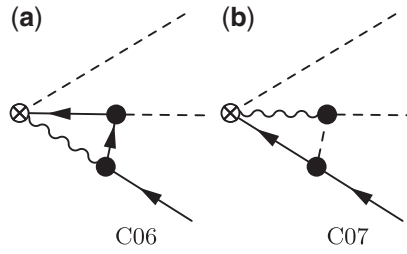


Fig. C.6. One-loop (flow) Feynman diagrams (continued).

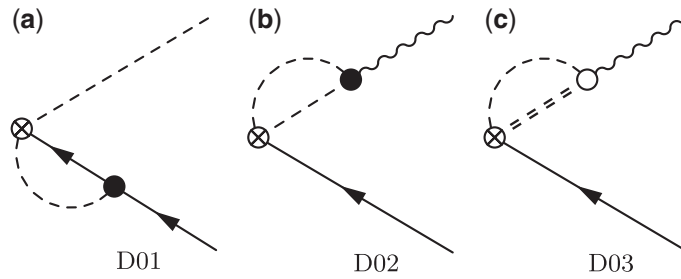


Fig. C.7. One-loop (flow) Feynman diagrams (continued).

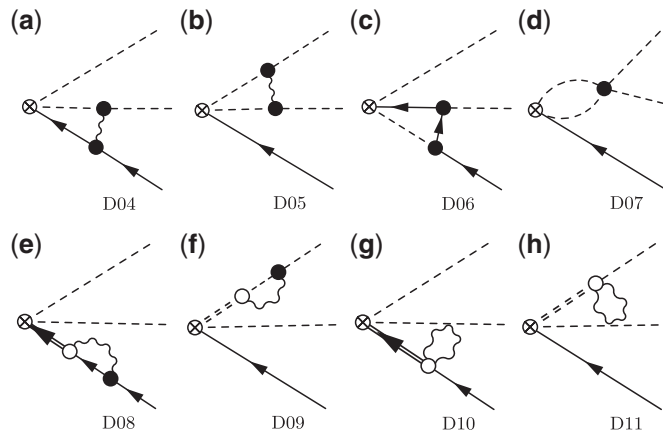


Fig. C.8. One-loop (flow) Feynman diagrams (continued).

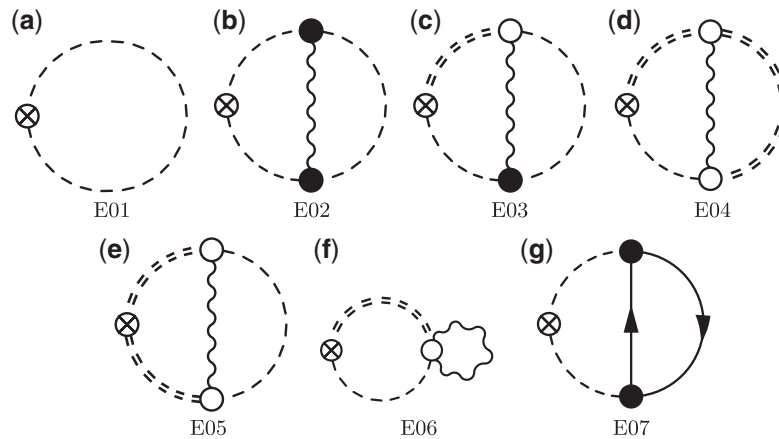


Fig. C.9. Flow Feynman diagrams relevant for the calculation in Sect. 8.2.

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