Physics Letters B 788 (2019) 464–467

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

The interior volume calculation for an axially symmetric black hole

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A R T I C L E I N F O

Article history:
Received 15 November 2018
Accepted 30 November 2018
Available online 3 December 2018
Editor: M. Cvetič

A B S T R A C T

Since an axially symmetric metric is much more complicated than a spherically symmetric metric, the largest hypersurface that corresponds to the interior volume of a black hole proposed by Christodoulou and Rovelli, cannot be found easily. Analogous to a Schwarzschild black hole, the particular hypersurface at constant r in a Kerr black hole has been selected directly. Using the extrinsic curvature of the hypersurface, we investigate how the hypersurface deviates from the largest hypersurface as the black hole angular momentum increases. It is shown that the hypersurface is very close to the largest hypersurface and the spinning of spacetime has pretty limited influence. Therefore, the hypersurface can be regarded as the largest hypersurface approximately in a Kerr black hole. So the volume of the hypersurface can be regarded as the interior volume of a Kerr black hole. Subsequently, we also calculate the interior volume of a Kerr–Newman black hole.

1. Introduction

In general relativity, it is a subtle issue to define the interior volume of a black hole which has a particular physical significance, and this is related to the choice of a special spacelike hypersurface. Parikh [1] suggests that a reasonable definition of interior volume inside a black hole should be a slicing-invariant volume, which has been discussed deeply in the following works [2–8]. Recently, Christodoulou and Rovelli [9] have extended the definition of a sphere volume from the flat spacetime to the curved spacetime. So, a new definition of the interior volume in a spherically symmetric black hole has been proposed. Based on this definition, the interior volume of a Schwarzschild black hole has been calculated. The results show that the interior volume mainly comes from the spacelike hypersurface at $r = \frac{3}{2}M$ at the late advanced time v. Furthermore, the interior volume increases linearly with the advanced time v, which can be expressed as

$$V \sim 3\sqrt{3}\pi M^2 v. \tag{1}$$

Subsequently, Bengtsson and Jakobsson [10] extended the interior volume definition from a spherically symmetric black hole to an axially symmetric black hole. Based on the definition, the interior volume of a Kerr black hole has been calculated. Since a Kerr metric is much more complicated than a Schwarzschild metric, the method proposed by Christodoulou and Rovelli to calculate the interior volume of a black hole cannot be used directly. They chose an arbitrary hypersurface at constant r inside a Kerr black hole technically, and calculate the volume expression using this hypersurface. Furthermore, they demonstrate that the volume expression has a maximal value when the coordinate r takes a special value. This can be regarded as the interior volume of a Kerr black hole approximately. Based on this literature, we use the method of extrinsic curvature to demonstrate more clearly that the interior volume calculation for a Kerr black hole is reasonable. Then, taking the same approach, we calculate the interior volume of a Kerr–Newman black hole and demonstrate that this method is reasonable for a Kerr–Newman black hole too.

The organization of the paper is as follows. In section 2, we briefly review the interior volume calculation for a Schwarzschild black hole and a Kerr black hole. After that, thinking of the extrinsic curvature, we demonstrate that the method to calculate the interior volume of a Kerr black hole is reasonable. In section 3, we continue to calculate the interior volume of a Kerr–Newman black hole using the same approach. The paper ends with conclusions in section 4.

2. The interior volume of a Kerr black hole

The definition of interior volume of a spherically symmetric black hole has been proposed by Christodoulou and Rovelli [9]. This definition can be expressed as that the volume of the largest

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https://doi.org/10.1016/j.physletb.2018.11.056

hypersurface inside the black hole bounded by two-sphere in the event horizon is just the interior volume of the black hole. Based on the definition, the interior volume of a Schwarzschild black hole which is formed by the collapse of a spherically symmetric object is calculated. The result shows that the largest hypersurface can be divided into three parts. The first part is a null hypersurface at constant advanced time v, which connects the event horizon to the spacelike hypersurface at $r = \frac{3}{2}M$. The second part is a long stretch at nearly constant radius $r = \frac{3}{2}M$. The third part is a hypersurface which connects the hypersurface at $r = \frac{3}{2}M$ to the center of the collapsing object $r = 0$. The first part does not contribute to the interior volume because it is a null hypersurface. The third part is entirely inside the collapsing object. Since the spacetime inside the collapsing object acquires a timelike Killing vector field, this part is a spacelike hypersurface in this region. So, the third part can contribute finite volume to the interior volume of the black hole. However, this contribution can be neglected in the late advanced time because the second part can increase linearly with the advanced time. It means that the spacelike hypersurface at $r = \frac{3}{2}M$ can be regarded as the largest hypersurface inside a Schwarzschild black hole at the late advanced time. Therefore, according to the definition of the interior volume, the volume of hypersurface at $r = \frac{3}{2}M$ is the interior volume of a Schwarzschild black hole.

Subsequently, the definition of interior volume has been extended from a spherically symmetric black hole to an axially symmetric black hole. Because the Kerr spacetime is spinning, the interior volume is different from the spherical symmetry case. The Kerr metric is much more complicated and it is difficult to calculate the interior volume of a Kerr black hole directly using the method in Ref. [9]. However, a particular hypersurface inside a Kerr black hole can be selected. This hypersurface is also formed by three parts. The first part is "close to null" just inside the two-sphere in the event horizon, and joins the second part $r = constant$ hypersurface all the way down to the third part which is in the matter filled region. The hypersurface is closed up at the center of the collapsing object $r = 0$. The volume of the first part is zero because it is null hypersurface. The third part contributes finite volume to the interior volume. This part can be neglected in the late advanced time because the second part increases with the advanced time. Therefore, the contribution to the interior volume of a Kerr black hole mainly comes from the spacelike hypersurface at $r = constant$. Based on this idea, the volume expression of an arbitrary hypersurface at $r = constant$ inside a Kerr black hole is given in Ref. [10] as

$$V_\Sigma = 2\pi v\sqrt{2Mr - r^2 - a^2}\left(\sqrt{r^2+a^2} + \frac{r^2}{2a}\ln\frac{\sqrt{r^2+a^2}+a}{\sqrt{r^2+a^2}-a}\right). \quad (2)$$

The interior volume of a Kerr black hole is regarded as the maximal value of Eq. (2) when r takes a special value r_s. According to the definition, the volume of the largest hypersurface bounded by two-sphere in the event horizon is just the interior volume of a black hole. Hence, we should demonstrate that the spacelike hypersurface at $r = r_s$ is the largest hypersurface in a Kerr black hole.

In principle, using the method proposed by Christodoulou and Rovelli, the volume of the largest hypersurface should be obtained naturally. However, this approach cannot be used directly because the Kerr metric has a cross product term. Fortunately, Refs. [11] and [12] have shown that if the trace of extrinsic curvature of a hypersurface vanishes, the variation of its volume function is automatically zero. In other words, if the trace of extrinsic curvature of a hypersurface is equal to zero, the hypersurface is the largest hypersurface. So, we can use the trace of extrinsic curvature to demonstrate whether the hypersurface at $r = r_s$ is the largest hypersurface. The trace of extrinsic curvature of an arbitrary hypersurface at constant r inside the Kerr black hole can be expressed as [13]

$$K = -\frac{\sqrt{2}\left[(m-r)a^2\cos 2\theta + a^2(m-3r) + 2r^2(3m-2r)\right]}{\left[a^2\cos 2\theta + a^2 + 2r^2\right]^2\sqrt{-\frac{a^2+r(r-2m)}{a^2\cos 2\theta + a^2 + 2r^2}}}. \quad (3)$$

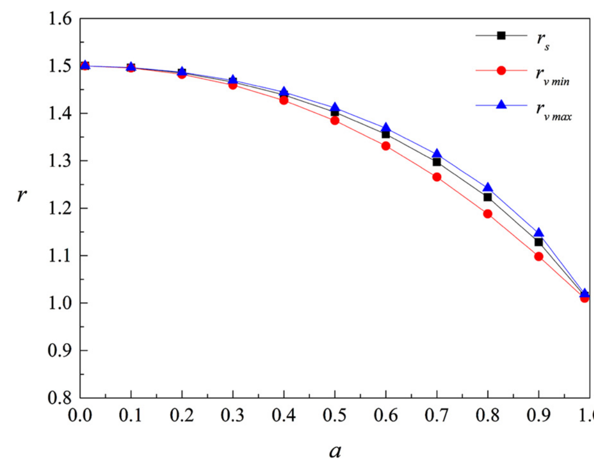

Fig. 1. The value of r_s and the range of r_v changes with the angular momentum a as $M = 1$.

Based on it, we calculate the value of r_v which corresponds to $K = 0$ and compare it with the value of r_s. Furthermore, the values of r_s and r_v change with the angular momentum. We have also investigated how the spinning of spacetime affects the hypersurface at $r = r_s$ in a Kerr black hole. The results are shown in Fig. 1.

It is shown that the value of r_s and the range of r_v vary with the angular momentum a. Since Eq. (3) contains the coordinate θ, the value of r_v should vary with θ. However, for a Kerr black hole, the coordinate θ ranges from 0 to π and it is centered exactly on $\theta = \frac{\pi}{2}$. So, at each specific value of a, the value of r_v ranges from $r_{v\,min}$ to $r_{v\,max}$ as θ varies from 0 to $\frac{\pi}{2}$, and the value of r_s falls within the range of r_v. It means that the hypersurface at $r = r_s$ is not the largest hypersurface in a Kerr black hole, because the largest hypersurface extrinsic curvature should be precisely equal to zero at any value of coordinate θ. Actually, the volume expression of the hypersurface at $r = r_s$ cannot accurately represent the interior volume of a Kerr black hole mentioned in Ref. [9]. However, since the value of r_s falls within the range of r_v, the hypersurface at $r = r_s$ can be approximately regarded as the largest hypersurface.

On the other hand, when $a = 0$, r_s, $r_{v\,min}$ and $r_{v\,max}$ have the same value at $r = 1.5$. This situation corresponds to the Schwarzschild case. It is shown that, in a Schwarzschild black hole, the largest hypersurface is exactly at $r = \frac{3}{2}M$ and the extrinsic curvature of the hypersurface is precisely equal to zero. As the value of a increases, the values of r_s, $r_{v\,min}$ and $r_{v\,max}$ decrease and the difference between $r_{v\,min}$ and $r_{v\,max}$ increases. As the angular momentum increases, the hypersurface at $r = r_s$ gradually moves away from the event horizon and deviates from the largest hypersurface in the mean time. At about $a = 0.8$, the hypersurface at $r = r_s$ deviates most from the largest hypersurface. After that, the difference between $r_{v\,min}$ and $r_{v\,max}$ decreases, and r_s, $r_{v\,min}$ and $r_{v\,max}$ will merge into one point when $a = 1$. Here the Kerr black hole approaches to the extremal black hole, it leads the inner hori-

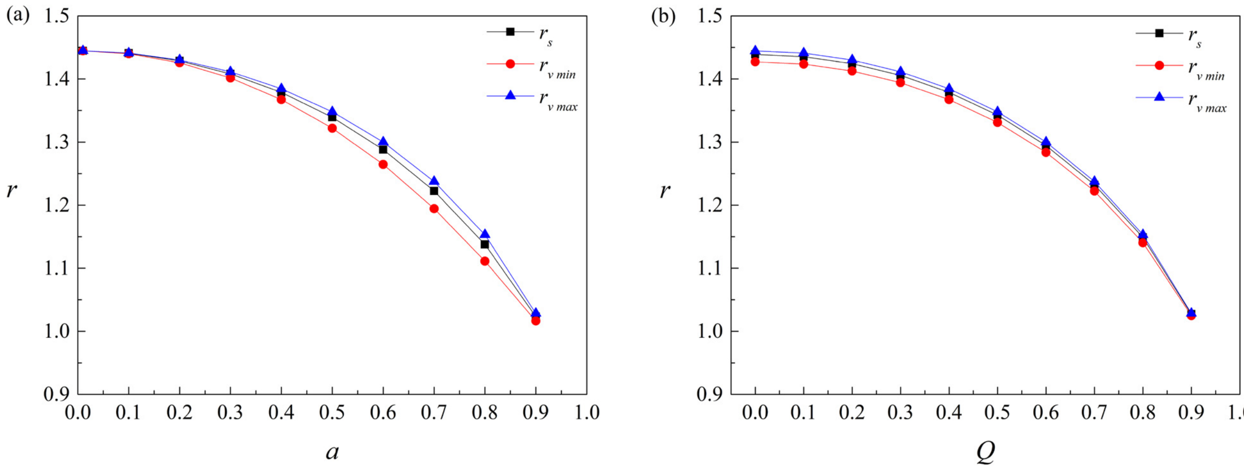

Fig. 2. (a) The value of r_s and the range of r_v vary with the angular momentum a as $M = 1$ and $Q = 0.4$. (b) The value of r_s and the range of r_v vary with the charge Q as $M = 1$ and $a = 0.4$.

zon and event horizon come together. In addition, the value of r_s is always falling within the range between $r_{v\,min}$ and $r_{v\,max}$. Although the hypersurface at $r = r_s$ is not the largest hypersurface inside a Kerr black hole, this hypersurface is very close to the largest hypersurface for any value of a. The spinning of spacetime has weak influence on the hypersurface at $r = r_s$. Hence, the hypersurface at $r = r_s$ can be regarded as the largest hypersurface inside a Kerr black hole approximately. The volume of the hypersurface at $r = r_s$ can be regarded as the interior volume of a Kerr black hole.

3. The interior volume of a Kerr–Newman black hole

For a Kerr–Newman black hole, similar to the Kerr case, the analytic expression of interior volume is still difficult to obtain. Therefore, analogous to the calculation method of the interior volume in a Kerr black hole, we also directly select a particular hypersurface inside a Kerr–Newman black hole. This hypersurface can also be divided into three parts. The central part of the hypersurface is a long stretch at nearly constant radius $r = r_s$, to which the event horizon attaches through the null hypersurface at one end, while the center of collapsing object $r = 0$ attaches to it at the other end. Based on the above statement, the interior volume of a Kerr–Newman black hole can be regarded as the volume of the hypersurface at $r = r_s$ in the late advanced time. In the following, we calculate the volume expression of an arbitrary hypersurface at $r = constant$ and demonstrate that the hypersurface at $r = r_s$ can be regarded as the largest hypersurface in a Kerr–Newman black hole approximately.

The metric of a Kerr–Newman black hole in the Eddington–Finkelstein coordinates is [14]

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\rho^2}dv^2 + 2dv\,dr + \rho^2 d\theta^2 + \frac{A\sin^2\theta}{\rho^2}d\phi^2 \\ &- 2a\sin^2\theta\,dr\,d\phi - \frac{2(2Mr - Q^2)}{\rho^2}a\sin^2\theta\,dv\,d\phi, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\rho^2 \equiv r^2 + a^2\cos^2\theta, \\ &\Delta \equiv r^2 - 2Mr + a^2 + Q^2, \\ &A \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned} \tag{5}$$

Based on the Kerr–Newman metric, the volume expression of an arbitrary hypersurface at $r = constant$ inside the black hole can be obtained as

$$V_\Sigma = 2\pi v f(M, a, Q), \tag{6}$$

where

$$\begin{aligned} f(M,a,Q) = &\sqrt{2Mr - r^2 - a^2 - Q^2} \\ &\times \left(\sqrt{r^2+a^2} + \frac{r^2}{2a}\ln\frac{\sqrt{r^2+a^2}+a}{\sqrt{r^2+a^2}-a}\right). \end{aligned} \tag{7}$$

According to Ref. [10], the maximal value of Eq. (6) when r takes the special value r_s corresponds to the interior volume of a Kerr–Newman black hole. Now, we should demonstrate that the hypersurface at $r = r_s$ is the largest hypersurface in a Kerr–Newman black hole.

We also use the method of extrinsic curvature to demonstrate it. The trace of extrinsic curvature of an arbitrary hypersurface at $r = constant$ in a Kerr–Newman black hole can be expressed as

$$K = -\frac{\sqrt{2}\left[a^2(M-r)\cos(2\theta) + a^2(M-3r) - 2r\left(Q^2 + 2r^2 - 3Mr\right)\right]}{\left[a^2\cos(2\theta) + a^2 + 2r^2\right]^2\sqrt{-\frac{a^2+r(r-2M)+Q^2}{a^2\cos(2\theta)+a^2+2r^2}}}. \tag{8}$$

Based on it, we can calculate the values of r_s and r_v at each specific value of angular momentum a and charge Q. Furthermore, the value of r_s and the range of r_v change with a and Q. We investigate how the angular momentum and charge affect r_s and r_v. The result is shown in Fig. 2.

Fig. 2 (a) shows that the value of r_s and the range of r_v vary with the angular momentum a as the charge Q is fixed, and Fig. 2 (b) shows that r_s and the range of r_v vary with the charge Q as a is fixed. From Fig. 2 (a), we can see that r_s and r_v vary with the angular momentum, it is similar to the Kerr case. The hypersurface at $r = r_s$ is very close to the largest hypersurface inside a Kerr–Newman black hole, and the spinning of spacetime has weak influence on the hypersurface. From Fig. 2 (b), we can see that the value of r_s and the difference between $r_{v\,min}$ and $r_{v\,max}$ decrease with the charge Q, and r_s also falls within the range of r_v. The charge of a Kerr–Newman black hole has weak influence on r_s and r_v. That is to say, the variation of charge has weak influence on how the hypersurface at $r = r_s$ deviates from the largest hypersurface.

So, although the hypersurface at $r = r_s$ is not the largest hypersurface in a Kerr–Newman black hole, it is very close to the largest hypersurface at any value of a and Q. The hypersurface at $r = r_s$ can be regarded as the largest hypersurface in a Kerr–Newman black hole approximately. The volume of the hypersurface at $r = r_s$ is approximately the interior volume of a Kerr–Newman black hole.

4. Conclusions

The definition of interior volume inside a black hole has been extended from a spherically symmetric black hole to an axially symmetric black hole. It is regarded as the volume of spacelike hypersurface at $r = constant$ when r takes special value r_s. Calculating the extrinsic curvature of the hypersurface, we demonstrate that the hypersurface at $r = r_s$ is very close to the largest hypersurface in the definition of interior volume, and the spinning of spacetime has weak influence on it. So, the volume of $r = r_s$ hypersurface can be regarded as the interior volume of a Kerr black hole approximately. After that, analogous to the Kerr black hole, we calculate the interior volume of a Kerr–Newman black hole. Calculating the extrinsic curvature of the hypersurface, we demonstrate that $r = r_s$ is also close to the largest hypersurface in a Kerr–Newman black hole, and the variation of charge and angular momentum have weak influence on the hypersurface. The hypersurface at $r = r_s$ can also be regarded as the largest hypersurface in a Kerr–Newman black hole approximately. The volume of $r = r_s$ hypersurface can be regarded as the interior volume of a Kerr–Newman black hole.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant No. 11235003).

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