

Letter

A lattice formulation of the $\mathcal{N} = 2$ supersymmetric SYK modelMitsuhiro Kato^{1,*}, Makoto Sakamoto², and Hiroto So³¹*Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan*²*Department of Physics, Kobe University, Nada-ku, Hyogo 657-8501, Japan*³*Department of Physics, Ehime University, Bunkyo-chou 2-5, Matsuyama 790-8577, Japan*

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 We construct the $\mathcal{N} = 2$ supersymmetric SYK model on a 1D (Euclidean time) lattice. One nilpotent supersymmetry is exactly realized on the lattice in the use of the cyclic Leibniz rule.

Subject Index B10, B17, B38

The Sachdev–Ye–Kitaev (SYK) model [1,2] and its generalizations have been attracting much attention in several contexts.¹ In these analyses, the large- N limit is fully utilized to obtain an effective theory in the IR region. Finite- N analysis, however, is also interesting to see, e.g., some sort of stringy corrections for AdS/CFT-type correspondence.

For finite- N analysis beyond perturbation, one approach is the exact diagonalization of the Hamiltonian [5] based on the realization of fermion operators by gamma matrices. Another approach may be lattice formulation on Euclidean time. The latter seems convenient for the calculation of multi-point correlation functions of the operators with distinct times, especially for comparison with results from the effective field theory of bi-local collective modes. Also, if higher-dimensional extension becomes possible, a Monte Carlo simulation will be numerically less expensive than exact diagonalization of the Hamiltonian. So we concentrate on lattice formulation in the present letter. Among others, we will focus on lattice formulation of the supersymmetric generalization [6] of the SYK model. This is actually highly non-trivial, since realizing supersymmetry on a lattice is a very difficult task [7].

The present authors have been studying a way for realizing the nilpotent subalgebra of supersymmetry on a lattice by using the cyclic Leibniz rule (CLR) [8,9]. In the present letter, as an application of the CLR, we will construct an $\mathcal{N} = 2$ supersymmetric SYK model [6] on the lattice. As will be seen, one of the two supersymmetries is exactly realized on the lattice thanks to the CLR.

Let us consider the $\mathcal{N} = 2$ supersymmetric SYK model [6] whose Hamiltonian H is given by the anti-commutator of two nilpotent supercharges Q and \bar{Q} :

$$H = \{Q, \bar{Q}\}, \quad Q^2 = 0, \quad \bar{Q}^2 = 0, \quad (1)$$

¹ For recent reviews see, e.g., Refs. [3,4] and references therein.

where supercharges are defined by N complex fermions $\psi^i, \bar{\psi}^i$ ($i = 1, \dots, N$) and complex random couplings C_{ijk}, \bar{C}_{ijk} with totally anti-symmetric indices:

$$Q = \frac{i}{3!} C_{ijk} \psi^i \psi^j \psi^k, \quad \bar{Q} = \frac{i}{3!} \bar{C}_{ijk} \bar{\psi}^i \bar{\psi}^j \bar{\psi}^k. \quad (2)$$

These fermions satisfy the following anti-commutation relations:

$$\{\psi^i, \psi^j\} = 0, \quad \{\bar{\psi}^i, \bar{\psi}^j\} = 0, \quad \{\psi^i, \bar{\psi}^j\} = \delta^{ij}. \quad (3)$$

The random couplings have non-zero second moments under a quenched average:

$$\overline{C_{ijk} \bar{C}_{ijk}} = \frac{2J}{N^2} \quad (4)$$

with a characteristic constant J , which controls the strength of the interaction.

The above supercharges are cubic in fermions, and thereby the Hamiltonian has quartic interactions. This can be generalized by taking any odd number of fermions in the supercharges. Hereafter we use generalized supercharges with q fermions. The corresponding action in Euclidean time is defined by

$$S = \int dt \left\{ \bar{\psi}^i \partial_t \psi^i - \bar{b}^i b^i + \frac{i^{\frac{q-1}{2}}}{(q-1)!} C_{j_1 j_2 \dots j_q} \bar{b}^{j_1} \psi^{j_2} \dots \psi^{j_q} + \frac{i^{\frac{q-1}{2}}}{(q-1)!} \bar{C}_{j_1 j_2 \dots j_q} b^{j_1} \bar{\psi}^{j_2} \dots \bar{\psi}^{j_q} \right\}. \quad (5)$$

Here we have introduced complex auxiliary variables b^i, \bar{b}^i in order to realize the supersymmetry linearly and make the action off-shell invariant. C and \bar{C} are q -index generalizations of the random couplings. The supersymmetry transformation for each variable is

$$\begin{aligned} \delta_Q \psi^i &= 0, & \delta_{\bar{Q}} \psi^i &= \bar{b}^i, \\ \delta_Q b^i &= 0, & \delta_{\bar{Q}} b^i &= \partial_t \bar{\psi}^i, \\ \delta_Q \bar{\psi}^i &= b^i, & \delta_{\bar{Q}} \bar{\psi}^i &= 0, \\ \delta_Q \bar{b}^i &= \partial_t \psi^i, & \delta_{\bar{Q}} \bar{b}^i &= 0, \end{aligned} \quad (6)$$

where we omit the transformation parameters, so that δ_Q and $\delta_{\bar{Q}}$ should be treated as Grassmann-odd quantities.

Now let us make a lattice version of Eqs. (5) and (6). First, we replace the variables $\psi^i(t), \bar{\psi}^i(t), b^i(t)$, and $\bar{b}^i(t)$ with the lattice variables $\psi_n^i, \bar{\psi}_n^i, b_n^i$, and \bar{b}_n^i , where n stands for a lattice site. Then the supersymmetry transformation should become

$$\begin{aligned} \delta_Q \psi_n^i &= 0, & \delta_{\bar{Q}} \psi_n^i &= \bar{b}_n^i, \\ \delta_Q b_n^i &= 0, & \delta_{\bar{Q}} b_n^i &= (\nabla^{(T)} \bar{\psi}^i)_n, \\ \delta_Q \bar{\psi}_n^i &= b_n^i, & \delta_{\bar{Q}} \bar{\psi}_n^i &= 0, \\ \delta_Q \bar{b}_n^i &= (\nabla^{(T)} \psi^i)_n, & \delta_{\bar{Q}} \bar{b}_n^i &= 0, \end{aligned} \quad (7)$$

where $\nabla^{(T)}$ is an appropriate difference operator. We use superscript (T) in order to distinguish it from the difference operator in the action for which we use $\nabla^{(A)}$.

Next, we construct a lattice action in the following form:

$$S = S_{\text{kin}} + S_M + S_{\bar{M}}, \quad (8)$$

$$S_{\text{kin}} = \bar{\psi}_n^i \nabla_{nm}^{(A)} \psi_m^i - \bar{b}_n^i b_n^i, \quad (9)$$

$$S_M = \frac{i^{\frac{q-1}{2}}}{(q-1)!} C_{j_1 j_2 \dots j_q} M_{n_1 n_2 \dots n_q} \bar{b}_{n_1}^{j_1} \psi_{n_2}^{j_2} \dots \psi_{n_q}^{j_q}, \quad (10)$$

$$S_{\bar{M}} = \frac{i^{\frac{q-1}{2}}}{(q-1)!} \bar{C}_{j_1 j_2 \dots j_q} \bar{M}_{n_1 n_2 \dots n_q} b_{n_1}^{j_1} \bar{\psi}_{n_2}^{j_2} \dots \bar{\psi}_{n_q}^{j_q}, \quad (11)$$

where repeated lattice site indices are summed. $M_{n_1 n_2 \dots n_q}$ and $\bar{M}_{n_1 n_2 \dots n_q}$ are complex coefficients that define the multiple product of variables. Note that the last $q-1$ site indices of M and \bar{M} are totally symmetric. This action gives Eq. (5) in the naive continuum limit as long as M and \bar{M} go to 1 and ∇ goes to ∂_t . Requiring the invariance of the action under either transformation $\delta_{\mathcal{Q}}$ or $\delta_{\bar{\mathcal{Q}}}$ in Eq. (7), we obtain the conditions that should be satisfied by $\nabla^{(A)}$, $\nabla^{(T)}$, M , and \bar{M} . For example, if we require $\delta_{\bar{\mathcal{Q}}}$ invariance, then we have

$$\delta_{\bar{\mathcal{Q}}} S_{\text{kin}} = -\bar{\psi}_n^i (\nabla_{nm}^{(A)} + \nabla_{mn}^{(T)}) \bar{b}_m^i = 0, \quad (12)$$

$$\delta_{\bar{\mathcal{Q}}} S_M = \frac{i^{\frac{q-1}{2}}}{(q-1)!} C_{j_1 j_2 \dots j_q} M_{n_1 n_2 \dots n_q} \sum_{k=2}^q (-1)^{k-2} \bar{b}_{n_1}^{j_1} \psi_{n_2}^{j_2} \dots \psi_{n_{k-1}}^{j_{k-1}} \bar{b}_{n_k}^{j_k} \psi_{n_{k+1}}^{j_{k+1}} \dots \psi_{n_q}^{j_q} = 0, \quad (13)$$

$$\delta_{\bar{\mathcal{Q}}} S_{\bar{M}} = \frac{i^{\frac{q-1}{2}}}{(q-1)!} \bar{C}_{j_1 j_2 \dots j_q} \bar{M}_{n_1 n_2 \dots n_q} \nabla_{mn_1}^{(T)} \bar{\psi}_{n_1}^{j_1} \bar{\psi}_{n_2}^{j_2} \dots \bar{\psi}_{n_q}^{j_q} = 0. \quad (14)$$

The first condition (12) is satisfied if the difference operator ∇ meets

$$\nabla_{nm}^{(A)} + \nabla_{mn}^{(T)} = 0. \quad (15)$$

The second condition (13) is satisfied if we ensure that

$$M \text{ is totally symmetric for all } q \text{ indices.} \quad (16)$$

The third condition (14) is satisfied if \bar{M} meets

$$\sum_{\text{permutation of } \{n_1 \dots n_q\}} \nabla_{mn_1}^{(T)} \bar{M}_{mn_2 \dots n_q} = 0. \quad (17)$$

The summation in Eq. (17) can be reduced to that in the only cyclic permutation due to the totally symmetric nature of the last $q-1$ indices. Thus this relation is nothing but the cyclic Leibniz rule (CLR) [8,9].

It should be stressed that the CLR (17) is not an abstract relation but has many concrete solutions. For example, if we simply take $\nabla^{(A)} = \nabla^{(T)}$, then

$$M_{lmn} = \delta_{l,m} \delta_{l,n}, \quad (18)$$

$$\bar{M}_{lmn} = \frac{1}{6} (2\delta_{l,m-1} \delta_{l,n-1} + \delta_{l,m+1} \delta_{l,n-1} + \delta_{l,m-1} \delta_{l,n+1} + 2\delta_{l,m+1} \delta_{l,n+1}), \quad (19)$$

$$\nabla_{mn} = \frac{1}{2} (\delta_{m+1,n} - \delta_{m-1,n}) \quad (20)$$

is one of the ultralocal² solutions of Eqs. (15), (16), and (17) for $q = 3$. Systematic construction of the solutions for the CLR with a symmetric difference operator can be found in Ref. [10].

Although this might be the simplest solution, we would have a species doubler with it. So we propose an alternative solution:

$$M_{lmn} = \delta_{l,m}\delta_{l,n}, \quad (21)$$

$$\begin{aligned} \bar{M}_{lmn} = \frac{1}{24} & \left[2(1+r)^2\delta_{l+1,m}\delta_{l+1,n} + 2(1-r)^2\delta_{l-1,m}\delta_{l-1,n} \right. \\ & + (1-r^2)(\delta_{l-1,m}\delta_{l+1,n} + \delta_{l+1,m}\delta_{l-1,n}) \\ & + (3-r)(1+r)(\delta_{l+1,m}\delta_{l,n} + \delta_{l,m}\delta_{l+1,n}) \\ & + (3+r)(1-r)(\delta_{l-1,m}\delta_{l,n} + \delta_{l,m}\delta_{l-1,n}) \\ & \left. + 2(3+r^2)\delta_{l,m}\delta_{l,n} \right], \quad (22) \end{aligned}$$

$$\nabla_{mn}^{(T)} = \frac{1}{2} \left[\delta_{m+1,n} - \delta_{m-1,n} + r(\delta_{m+1,n} + \delta_{m-1,n} - 2\delta_{m,n}) \right], \quad (23)$$

$$\nabla_{mn}^{(A)} = \frac{1}{2} \left[\delta_{m+1,n} - \delta_{m-1,n} - r(\delta_{m+1,n} + \delta_{m-1,n} - 2\delta_{m,n}) \right]. \quad (24)$$

Here r is a real parameter and the kinetic action with this $\nabla^{(A)}$ contains the so-called Wilson term (r corresponds to the Wilson term coefficient), which lifts doublers up with a cutoff-scale mass.

For the $r = 1$ case, which corresponds to the forward difference operator, we can write down solutions with generic q :

$$\begin{aligned} M_{l n_1 \dots n_{q-1}} & = \frac{1}{q!} \sum_P (\delta_{l+1,n_1} \dots \delta_{l+1,n_{q-1}} + \delta_{l+1,n_1} \dots \delta_{l+1,n_{q-2}} \delta_{l,n_{q-1}} + \delta_{l+1,n_1} \dots \delta_{l+1,n_{q-3}} \delta_{l,n_{q-2}} \delta_{l,n_{q-1}} \\ & + \dots + \delta_{l+1,n_1} \delta_{l,n_2} \dots \delta_{l,n_{q-1}} + \delta_{l,n_1} \dots \delta_{l,n_{q-1}}) \quad (25) \end{aligned}$$

$$= \frac{1}{q!} \sum_P \sum_{k=0}^{q-1} \left[\prod_{a=1}^k \delta_{l+1,n_a} \right] \left[\prod_{b=k+1}^{q-1} \delta_{l,n_b} \right] \quad (26)$$

where P stands for the summation over all permutations of $(n_1, n_2, \dots, n_{q-1})$, and $\prod_{a=1}^0 \delta_{l+1,n_a} = 1 =$

$\prod_{b=q}^{q-1} \delta_{l,n_b}$ is understood.

Thus we have a concrete way to construct the $\delta_{\bar{Q}}$ -invariant lattice action for the $\mathcal{N} = 2$ supersymmetric SYK model. If you need δ_Q -invariant action, just exchange the roles of M and \bar{M} .

A few remarks are in order, as follows.

² Here ultralocal means the operator defined in a region with a finite extent on a lattice. Local operators on the lattice include not only ultralocal ones but also ones with infinite extent that decay exponentially.

We cannot require both δ_Q and $\delta_{\bar{Q}}$ invariance because these simultaneously impose the CLR relation and totally symmetric indices on M and \bar{M} , but there is no local solution of CLR with symmetric M or \bar{M} [8].

For $\delta_{\bar{Q}}$ invariance, M is totally symmetric, but \bar{M} is not because of CLR and locality. Therefore, \bar{M} is not a complex conjugate of M , so that the resulting action is not Hermitian. This “could-be sign problem” is of the order of $O(a)$ where a is the lattice constant, and disappears at least in the naive continuum limit.

It seems that the CLR approach is a unique way to realize supersymmetry for models of this type. The other approaches, like the method using the Nicolai map or the method of finding a nilpotent transformation without a difference operator [11–13], may not work for the model. In particular, the last term of the action (5) is $\delta_{\bar{Q}}$ -invariant but not $\delta_{\bar{Q}}$ -exact, and therefore the topological field theoretic approach cannot be applied. This supports the uniqueness of the CLR approach.

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