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Reduction of order and Fadeev–Jackiw formalism in generalized electrodynamics

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Abstract

The aim of this work is to discuss some aspects of the reduction of order formalism in the context of the Fadeev–Jackiw symplectic formalism, both at the classical and the quantum level. We start by reviewing the symplectic analysis in a regular theory (a higher derivative massless scalar theory), both using the Ostrogradsky prescription and also by reducing the order of the Lagrangian with an auxiliary field, showing the equivalence of these two approaches. The interpretation of the degrees of freedom is discussed in some detail. Finally, we perform the similar analysis in a singular higher derivative gauge theory (the Podolsky electrodynamics), in the reduced order formalism: we claim that this approach have the advantage of clearly separating the symplectic structure of the model into a Maxwell and a Proca (ghost) sector, thus complementing the understanding of the degrees of freedom of the theory and simplifying calculations involving matrices.

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1. Introduction

Constrained systems are a basic tool for theoretical research in different contexts such as gauge theories and the field theory approach for gravity, for example. The pioneers in this treatment were Dirac and Bergmann (DB) [1–3] whose works established the standard method to study theories with constraints, providing generalized brackets appropriate to quantize these systems. When the dynamics of a singular Lagrangian formulated in configuration space is translated to a Hamiltonian formulation in phase space, the first constraints that appear, from the definition of the canonical momenta, are called the Dirac primary constraints. The condition that these should not change over time (consistency condition) may generate additional constraints, called secondary constraints, for which consistency conditions are again applied, and so forth. From this iterated process we obtain a complete set of constraints, which we may classify as being of first or second class, according to the vanishing or not of their canonical Poisson brackets. This Dirac–Bergmann algorithm, including its classification of constraints, has a meaning associated with the physical degrees of freedom [4]. This provides a first approach to the connection between classical and quantum dynamics, the classical dynamics described in the phase space by the observables and (Poisson\Dirac) brackets, and the quantum dynamics described in Hilbert space by the operators and commutator\anti-commutators. A second approach begins in a study by Dirac about the connection between a classical dynamics described in configuration space and its resulting quantum description, where we see the emergence of a very important object called the transition amplitude [5]. Feynman later used Dirac's idea to describe the quantum Lagrangian mechanics with the path integral formalism [6]; afterwards, an elegant variational principle [7] of the quantum action was developed by Schwinger, utilizing as a guide the Heisenberg description [8].

The need to describe the interactions of nature along the lines of a relativistic dynamics leads us to build a covariant language with gauge symmetry [9], which has more degrees of freedom than the physical ones, hence the necessity of introducing constrains. The connections between classical and quantum physical systems with constraints, in a functional formalism, was first formulated by Faddeev (for first class constraints), and later extended by Senjanovic (including second class constraints) [10]. The quantization procedure of a gauge theory is in principle possible for the physical degrees of freedom only, and thus we lose the explicit covariance of the equations: in order to maintain it at the quantum level, Faddeev, Popov and DeWitt built a method in which additional, non-physical ghost fields, are introduced [11].

The canonical quantization gained new life with the Fadeev–Jackiw (FJ) method, developed in the 1980's [12]. The (FJ) formalism pursues a classical geometric treatment based on the symplectic structure of the phase space and it is only applied to first order Lagrangians. The 2-form symplectic matrix associated with the reduced Lagrangian allows us to obtain the generalized brackets in the reduced phase space without the need to follow Dirac's method step by step [13]. The (FJ) method has some very useful properties, such as not needing to distinguish the types of constraints and the Dirac's conjecture, and therefore evoked much attention. Barcelos and Wotzasek introduced one procedure of dealing with constraints in the (FJ) method [14,15]; on the other hand, despite the quantization being essentially canonical, the path integral quantization was also constructed in [16,17]. We can find in the literature many studies of the equivalence between the (DB) and (FJ) formalisms [18–21], which can be proved in many (but not all) cases.

When Ostrogradski constructed Lagrangian theories with higher order derivatives in classical mechanics, a new field of research was opened [22,23]. Bopp, Podolsky and Schwed [24] proposed generalized electrodynamics in an endeavor to get rid of the infinites in quantum electrodynamics (QED), starting from a higher order Lagrangian, corresponding to the usual QED Lagrangian augmented by a term quadratic in the divergence of the field-strength tensor, which by dimensional reasons introduces a free parameter that can be identified as the Podolsky's mass m. This modification gives the correct (finite) expression for the self-force of charged particles, as shown by Frenkel, and interesting effects produced by the presence of external sources [25,26]. At the quantum level, higher derivative theories have in general the property of better behaved (or even absent) ultraviolet divergences in a sense closely related to the Pauli–Villars–Rayski regularization scheme [27,28], but also sometimes exhibit Hamiltonians without a lower limit [29] due to the presence of states with negative norm (ghosts), leading to the breakdown of unitarity [30]. Several procedures to avoid this problem have been already been studied [31], one approach being a careful investigation of the analytic structure of the Green functions as discussed in [33–35]. Another way to implement terms with higher order derivatives without breaking the stability of the theory has recently been proposed using the concept of Lagrangian anchors [32], an extension of the Noether theorems in the sense that one defines a class of conserved quantities associated with a given symmetry. For instance, the symmetry due to time translations will lead to two conserved Hamiltonians, one of them will have regularizing properties but will break the stability because the energy is not bounded from below, whereas the other recovers the stability but loses the regularizing properties. This leads to a new perspective on the unitarity problem of higher derivative theories, that makes use of the formalisms of reduction of order [36,37] and the concepts of complexation of the Lagrangian [38].

Given the advantages in using the (FJ) formalism to deal with the constraint structure of gauge theories, it is natural to apply this formalism to the quantization of gauge theories with higher order derivatives. In doing so, one needs to bring the Lagrangian to a first order form, and the two most known ways to do so are either by extending the number of the canonical momenta (the Ortogradsky formalism) or by reducing the order of derivatives using auxiliary fields (which we call the reduced order formalism). The first approach was considered in [39], where the BRST quantization of the higher derivative Podolski electrodynamics was described, using the (FJ) method to deal with the constraints. On the other hand, recently the same model was also considered in the reduced order formalism [36], but using the Dirac procedure for the constraints analysis. In this work, we will work with this model also within the reduced order formalism, but using the (FJ) formalism to work through the constraints, showing that this approach has a very nice property, which is the clear separation of degrees of freedom during the calculations, neatly separating the non-massive sector from the massive (ghost) one. We believe this is therefore the optimal approach to deal with higher derivative gauge theories, which should also be extended to more complicated cases such as non-Abelian theories.

This work is organized as follows. In Sec. 2, we review the main conceptual aspects of Fadeev–Jackiw formalism. In Sec. 3 we apply the (FJ) symplectic analysis to a simple higher derivative scalar model in both Ostrogradsky and reduced order formalisms, discussing their equivalence at classical and quantum level, as well as the interpretation of their degrees of freedom. In Sec. 4 we present the reduced order version of the (FJ) formalism in a singular higher derivative gauge theory (Podolsky electrodynamics). Sec. 5 contains our conclusions and perspectives.

2. Review of Fadeev–Jackiw formalism

We start with a brief review of the elementary aspects of the (FJ) formalism. Starting with a Lagrangian $L(q_i, \dot{q}_i)$, by means of a Lagrange transformation we define the canonical momenta

$p_i = \frac{\partial L}{\partial \dot{q}^i}$ and Hamiltonian $H = p_i \dot{q}^i - L$. With the aim of writing the symplectic structure, the Lagrangian has to be cast as a first order expression in the velocities $\dot{\xi}_i$, where hereafter ξ_i represents the set of all the canonical variables in the theory (at this point, ξ_i corresponds to the set of the q_i and p_i). More explicitly, the Lagrangian has to be brought up to the form

$$L(\xi, \dot{\xi}) = a_i(\xi)\dot{\xi}^i - V(\xi_i)\,, \tag{1}$$

where we identify $a_i(\xi) = p_i$ and $V = H$. The equations of motion are derived as usual from the principle of least action,

$$\delta S = \int dt \left[\frac{\partial L}{\partial \xi^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}^i}\right)\right]\delta\xi^i = 0\,, \tag{2}$$

where $S = \int dt L$. Taking into account the explicit form of L as a linear function in $\dot{\xi}^i$ given in (1), we have $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\xi}^i}) = \dot{a}_i(\xi) = \frac{\partial a_i}{\partial \xi_j}\dot{\xi}_j$ as well as $\frac{\partial L}{\partial \xi^i} = \frac{\partial a_j}{\partial \xi^i}\dot{\xi}^j - \frac{\partial V}{\partial \xi^i}$. Finally, introducing the symplectic matrix f_{ij},

$$f_{ij} \doteq \frac{\partial a_j}{\partial \xi^i} - \frac{\partial a_i}{\partial \xi^j}\,, \tag{3}$$

we can rewrite the equations of motion as

$$f_{ij}\dot{\xi}^j = \frac{\partial V}{\partial \xi^i}\,. \tag{4}$$

In the regular case, f_{ij} has an inverse f^{ij}, and this last equation can immediately be solved for the velocities $\dot{\xi}^i$ as follows,

$$\dot{\xi}^i = f^{ij}\frac{\partial V}{\partial \xi^j} = \left\{\xi^i, V\right\}_P = \left\{\xi^i, \xi^j\right\}_P \frac{\partial V}{\partial \xi^j}\,, \tag{5}$$

on the other hand, when f_{ij} is singular, there is no inverse matrix since $\det[f] = 0$, thus establishing the existence of zero modes. This can be seen clearly by considering the problem of eigenvalues and eigenvectors

$$[f]v_a = \omega_a v_a, \tag{6}$$

$$\det[f - \omega_a I] = 0, \tag{7}$$

from which it follows that $\det[f] = \prod_a \omega_a$. Hence if $[f]$ is singular, $\det[f] = 0$ and we have null eigenvalues. Let $\{v_n\}$ be the set of linearly independent null eigenvectors: when we multiply Eq. (4) by each of the v_n we obtain

$$v_n[f][\dot{\xi}] = v_n \frac{\partial V}{\partial [\xi]} = \Omega_n^{(1)} = 0\,, \tag{8}$$

which represents an initial set of constraints on the dynamics. They can be enforced by means of Lagrange multipliers $\lambda_n^{(1)}$, augmenting the initial Lagrangian by the term $\sum_n \lambda_n^{(1)}\Omega_n^{(1)}$. Alternatively, taking into account that the constraint does not evolve in time ($\dot{\Omega} = 0$) and that the Lagrangian is defined up to total time derivatives, it follows that a term such as

$$\frac{d(\lambda_n^{(1)}\Omega_n^{(1)})}{dt} = \dot{\lambda}_n^{(1)}\Omega_n^{(1)} + \lambda_n^{(1)}\dot{\Omega}_n^{(1)} \tag{9}$$

does not modify the dynamics, so we can actually write a first iterated Lagrangian as

$$L^{(1)} = a_i^{(1)}\dot{\xi}^i + \sum_n \dot{\lambda}_n^{(1)}\Omega_n^{(1)} - V^{(1)}\,, \tag{10}$$

where

$$V^{(1)} = V|_{\Omega^{(1)}=0}\,. \tag{11}$$

At this point, one can enlarge the set of canonical variables ξ_i including the $\lambda_n^{(1)}$. A new iteration can be started, taking $L^{(1)}$ as the initial Lagrangian, and the procedure continues until a non singular symplectic matrix f_{ij} is obtained – a process which, in the case of gauge theories, involves also the inclusion of gauge fixing conditions into the Lagrangian.

After a nonsingular symplectic matrix f_{ij} is obtained at the end of the (FJ) procedure, the transition amplitude is written as [17]

$$Z = \int D\xi\sqrt{\det[f]}\exp[iS]\,. \tag{12}$$

The crucial point to understand the previous equation is based in the Darboux theorem, which states that by an appropriate change of canonical coordinates ($\xi_i \to \xi_i'$), we can write the symplectic part of the Lagrangian, in the canonical form, as

$$L(\dot{Q}_i, P_i) = P_i\dot{Q}^i - H(Q_i, P_i)\,, \tag{13}$$

where

$$P_i\dot{Q}^i = \frac{1}{2}\omega^{ij}\dot{\xi}_i'\xi_j'\,,$$

Q^i and P_i being canonical variables obeying the standard Poisson algebra, and $[\omega]$ the anti-symmetric block matrix,

$$[\omega] = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{14}$$

In fact, Eq. (13) can be written in the following form,

$$L(\xi') = a'^i\dot{\xi}_i' - H(\xi')\,, \tag{15}$$

where

$$a'^i = \frac{1}{2}\xi_j'\omega^{ji}\,. \tag{16}$$

Here we can identify the arbitrary vector potential (one-form) as

$$a' = a'^i d\xi_i', \tag{17}$$

whose associated field strength (two-form) is given by

$$da' = \frac{\partial a'^i}{\partial \xi_j'}d\xi_i'd\xi_j' \tag{18}$$

$$= \frac{1}{2}\left(\frac{\partial \xi_i'}{\partial \xi_a}\omega^{ij}\frac{\partial \xi_j'}{\partial \xi_b}\right)d\xi_a d\xi_b\,. \tag{19}$$

The fact that the action $S = \int a_i d\xi^i - \int H dt$ is invariant under canonical transformations leads us to define the symplectic matrix as

$$f_{ij} \doteq \frac{\partial \xi'_i}{\partial \xi_a} \omega_{ij} \frac{\partial \xi'_j}{\partial \xi_b}, \tag{20}$$

while, by the Schwinger variational principle of quantum action, $\delta Z = \langle \delta \hat{S} \rangle = \delta S Z$, we have

$$Z = \int DQDP \exp\left[i \int dt \big(P_i \dot{Q}^i - H(Q_i, P_i) \big) \right], \tag{21}$$

or, in other words, $Z = \int D\xi' \exp[iS']$, where $S' = \int a'_i d\xi'^i - \int H' dt$. Therefore, by a canonical transformation ($\xi'_i \to \xi_i$), we write

$$Z = \int D\xi \det\left(\frac{\partial \xi'^i}{\partial \xi_j} \right) \exp[iS], \tag{22}$$

wherein we see that

$$\det\left(\frac{\partial \xi'^i}{\partial \xi_j} \right) = \sqrt{\det\left(\frac{\partial \xi'_i}{\partial \xi_a} \omega_{ij} \frac{\partial \xi'_j}{\partial \xi_b} \right)} = \sqrt{\det[f]}. \tag{23}$$

As stated in [17], it is important that the final result actually does not depend on the explicit form of the transformation ($\xi'_i \to \xi_i$), but only on the symplectic structure of the theory, which is solved by the (FJ) procedure.

3. Toy model as a prof of concept

In this section, we consider a rather simple higher derivative theory, based on a massless real scalar field. The aim is to gain insight in the physical interpretations of such theories, and to present in a simpler setting the procedure to be considered in connection to the Podolsky electrodynamics in the next section. We will work out both the Ostrogradsky and the reduced order approach, and we will explicitly verify that both routes lead to the same quantum theory. In the literature, the first order form of higher derivative theories was explored in [40]. The connections between the Ostrogradsky formalism (starting with a fourth order Lagrangian), the reduction of order formalism with an auxiliary field (starting with a first order Lagrangian, directly suitable to the application of the (FJ) method), and the final first order description (Hamiltonian), should be such that in any description we have the same propagating degrees of freedom, which in the present case are two: one being the original massless and the other one, massive, whose physical interpretation is of a ghost (unphysical) mode. We will also briefly comment on some recent ideas on how to interpret the presence of this ghost mode.

3.1. Ostrogradsky formalism

We begin with the Lagrangian density $\mathcal{L}_{Ostro}$,

$$\mathcal{L}_{Ostro} = \frac{1}{2} \partial_\mu \phi \left(1 + \frac{\Box}{m^2} \right) \partial^\mu \phi, \tag{24}$$

so the corresponding, fourth order equation of motion is given by

$$\Box (\Box + m^2) \phi = 0. \tag{25}$$

According to the Noether theorem, the conserved quantity corresponding to the time translation invariance of the action is the Hamiltonian density

$$\mathcal{H}_{Ostro} = \pi \partial_0 \phi + P \partial_0 Q - \mathcal{L}_{Ostro} \,, \tag{26}$$

where the additional canonical coordinate $Q = \partial_0 \phi$ was introduced to account for the higher order time derivatives. The canonical momenta are given by

$$\pi = \left(1 + \frac{\Box}{m^2}\right)\partial_0 \phi , \;\; P = -\frac{\Box}{m^2}\phi \,, \tag{27}$$

and therefore

$$\mathcal{H}_{Ostro}(\phi, Q; \pi, P) = \pi Q - \frac{1}{2}m^2 P^2 - P\partial_k \partial^k - \frac{1}{2}D^2 - \partial_k \phi \partial^k \phi \,. \tag{28}$$

The first order Lagrangian can be written as

$$\mathcal{L}_{Ostro} = \pi \partial_0 \phi + P \partial_0 Q - \mathcal{H}_{Ostro} \,, \tag{29}$$

where the canonical one form of the symplectic variables $\xi = (\phi, \pi, Q, P)$ corresponds to

$$a_\phi = \pi , \quad a_\pi = 0, \quad a_Q = P, \quad a_P = 0 \,. \tag{30}$$

Therefore, we obtain the symplectic matrix $f_{ij} = \frac{\partial a_j}{\partial \xi^i} - \frac{\partial a_i}{\partial \xi^j}$,

$$[f] = \left[\begin{array}{c|cccc} & \phi & \pi & Q & P \\ \hline \phi & 0 & -1 & 0 & 0 \\ \pi & 1 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 & -1 \\ P & 0 & 0 & 1 & 0 \end{array}\right] \delta^3(\vec{x} - \vec{y}) \,, \tag{31}$$

and, as $\det[f] = 1$, the inverse matrix exists, and can be readily obtained as $[f]^{-1} = -[f]$. As a consequence, the fundamental non null Poisson brackets read

$$\left\{\phi(x), \pi(y)\right\}_P = \delta^3(\vec{x} - \vec{y}), \quad \left\{Q(x), P(y)\right\}_P = \delta^3(\vec{x} - \vec{y}). \tag{32}$$

Now, going to the quantum language, the transition amplitude is given in view of Eq. (12), as

$$\begin{aligned} Z_{Ostro} &= \int D\phi D\pi DQ DP \exp\left\{i \int d^4x [\pi \partial_0 \phi + P\partial_0 Q - \mathcal{H}_{Ostro}]\right\} \\ &= \int D\phi D\pi DQ DP \exp\left\{i \int d^4x \left[\pi \partial_0 \phi + P \partial_0 Q - \pi Q + \frac{1}{2}m^2 P^2 + P\partial_k \partial^k \phi + \right.\right. \\ &\quad \left.\left. + \frac{1}{2}Q^2 \partial_k \phi \partial^k \phi\right]\right\}. \end{aligned} \tag{33}$$

After integration in $DQD\pi$, and completing the squares we obtain as our final result the gaussian functional,

$$Z_{Ostro} = \int D\phi DP \exp\left\{i \int d^4x \left[\frac{1}{2}m^2\left(P + \frac{\partial_\mu \partial^\mu \phi}{m^2}\right)^2 - \left(\frac{\Box}{m}\phi\right)^2 - \frac{1}{2}\phi\Box\phi\right]\right\} \tag{34}$$

$$= N \int D\phi \exp\left\{-i \int d^4x \, \phi \Box \left(1 + \frac{\Box}{m^2}\right)\phi\right\} \tag{35}$$

$$= N \det\left[-\frac{1}{16}\Box\left(\Box + m^2\right)\right]. \tag{36}$$

3.2. Reduced order with an auxiliary field

Instead of dealing with the higher derivatives via the Ostrograsdky method, one may also introduce an auxiliary field Z, starting with the Lagrangian

$$\mathcal{L}_{red} = \frac{1}{2}\phi\Box Z - \frac{1}{8}m^2\phi\phi + \frac{1}{4}m^2\phi Z - \frac{1}{8}m^2 ZZ\,, \tag{37}$$

whose corresponding equations of motion are given by

$$\left(1+2\frac{\Box}{m^2}\right)\phi = Z,\ \left(1+2\frac{\Box}{m^2}\right)Z = \phi\,. \tag{38}$$

These set of coupled equations are equivalent to Eq. (25), as can be seen by direct substitution. The canonical Hamiltonian is given by $\mathcal{H}_{red} = \pi\,\partial_0\phi + \theta\,\partial_0 Z - \mathcal{L}_{red}$ with the respective canonical momenta defined as

$$\pi \doteq \frac{\partial\mathcal{L}_{red}}{\partial(\partial_0\phi)} = -\frac{1}{2}\partial_0 Z\,, \tag{39}$$

$$\theta \doteq \frac{\partial\mathcal{L}_{red}}{\partial(\partial_0 Z)} = -\frac{1}{2}\partial_0\phi\,, \tag{40}$$

or, more explicitly,

$$\mathcal{H}_{red} = -2\pi\theta + \frac{1}{2}\partial_i\phi\partial^i Z + \frac{1}{8}m^2\phi\phi - \frac{1}{4}m^2\phi Z + \frac{1}{8}m^2 ZZ\,. \tag{41}$$

Therefore the canonical one form of the symplectic variables $\xi = (\phi, \pi, Z, \theta)$ is given by

$$a_\phi = \pi, \quad a_\pi = 0, \quad a_Z = \theta, \quad a_\theta = 0\,, \tag{42}$$

and the corresponding symplectic matrix is

$$[f] = \left[\begin{array}{c|cccc} & \phi & \pi & Z & \theta \\ \hline \phi & 0 & -1 & 0 & 0 \\ \pi & 1 & 0 & 0 & 0 \\ Z & 0 & 0 & 0 & -1 \\ \theta & 0 & 0 & 1 & 0 \end{array}\right]\delta^3(\vec{x}-\vec{y})\,, \tag{43}$$

which again is a non-singular, unitary determinant matrix, with inverse $[f]^{-1} = -[f]$. The corresponding fundamental non null Poisson brackets are

$$\{\phi(x), \pi(y)\}_P = \delta^3(\vec{x}-\vec{y}), \quad \{Z(x), \theta(y)\}_P = \delta^3(\vec{x}-\vec{y}). \tag{44}$$

Quantization is achieved by calculating the transition amplitude which in this case reads

$$Z_{red} = \int D\phi D\pi\, DZ D\theta \exp\left\{i\int d^4x[\pi\,\partial_0\phi + \theta\,\partial_0 Z - \mathcal{H}_{red}]\right\} \tag{45}$$

$$= \int D\phi D\pi\, DZ D\theta \exp\left\{i\int d^4x\left[\pi\,\partial_0\phi + \theta\,\partial_0 Z + 2\pi\theta - \frac{1}{2}\partial_i\phi\partial^i Z - \frac{1}{8}m^2\phi\phi + \frac{1}{4}m^2\phi Z - \frac{1}{8}m^2 ZZ\right]\right\}. \tag{46}$$

Integrating in $D\pi\, D\theta$, one obtains

$$Z_{red} = \int D\phi DZ \exp\left\{i \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu Z - \frac{1}{8}m^2\phi\phi + \frac{1}{4}m^2\phi Z - \frac{1}{8}m^2 ZZ\right]\right\},$$

and therefore

$$Z_{red} = \int D\phi DZ \exp\left\{i \int d^4x \left[\,\phi \quad Z\,\right]\begin{bmatrix} -\frac{m^2}{8} & (\frac{\Box}{4}+\frac{m^2}{8}) \\ (\frac{\Box}{4}+\frac{m^2}{8}) & -\frac{m^2}{8}\end{bmatrix}\begin{bmatrix}\phi \\ Z\end{bmatrix}\right\}, \tag{47}$$

$$= N \int D\phi \exp\left\{-i \int d^4x \phi\Box\left(1+\frac{\Box}{m^2}\right)\phi\right\}, \tag{48}$$

which reduces to the determinant of the square matrix appearing in Eq. (47). The determinant of course involves both the discrete matrix indices as well as the continuous spacetime indices (coordinates): calculating explicitly the first part gives

$$\det\begin{bmatrix} -\frac{m^2}{8} & (\frac{\Box}{4}+\frac{m^2}{8}) \\ (\frac{\Box}{4}+\frac{m^2}{8}) & -\frac{m^2}{8}\end{bmatrix} = \det\left[-\frac{1}{16}\Box(\Box+m^2)\right], \tag{49}$$

which agrees with Eq. (36). We therefore verify that the equivalence between the classical equations of motion in the Ostrogradsky and Reduction of order prescriptions, seen in Eqs. (4) and (38), hold also at the quantum level, when we compare the transition amplitude obtained in both prescriptions.

3.3. Characterization of the degrees of freedom

It is a common feature of higher derivatives theories to present additional, non physical degrees of freedom. This can be clearly seen in the present model. We choose to use the reduced order formalism as discussed in the previous subsection. The coupled equations of motion for the ϕ and Z field, given in Eq. (38), can be written in matrix notation as

$$M\begin{pmatrix}\phi \\ Z\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \tag{50}$$

where

$$M = \begin{pmatrix} (1+\frac{2}{m^2}\Box) & -1 \\ -1 & (1+\frac{2}{m^2}\Box)\end{pmatrix}. \tag{51}$$

The dynamics can be rewritten in order to make manifest the fact that it involves two independent degrees of freedom. At the matrix level, this amounts to the problem of diagonalization of the matrix M. The eigenvalues of M are determined by the equation $\det[M - \lambda I] = 0$, whose solutions are

$$\lambda_\pm = \left(1+\frac{2}{m^2}\Box\right) \mp 1\,. \tag{52}$$

So the matrix M is unitarily equivalent to a matrix describing two degrees of freedom, one being massless, and the other massive. Indeed, by means of a linear transformation,

$$\phi = \alpha A + \beta B\,, \tag{53}$$

$$Z = \alpha A - \beta B\,, \tag{54}$$

the Lagrangian in Eq. (37) can be brought to the following form,

$$\mathcal{L}'_{red} = \alpha^2\left[\frac{1}{2}A\Box A\right] - \beta^2\left[\frac{1}{2}B\Box B + \frac{{m_p}^2}{2}B^2\right]. \quad (55)$$

This last equation explicitly separates the two degrees of freedom present in the model. For real α and β, clearly the B mode appears with a "wrong sign" in the Lagrangian, and will in fact violate the stability of the Hamiltonian. Therefore, B should be interpreted as a non physical (ghost) degree of freedom.

The presence of ghosts is a longstanding issue in the quantization of higher derivative models. Recently, it has been pointed out that, at least in the free case, these ghosts could be reinterpreted as physical particles after a proper complexification: this was discussed for the Pais–Uhlenbeck oscillator in [38]. In the present case, one may note that the choice

$$\begin{aligned} \phi &= A + iB\,, \\ Z &= A - iB\,, \end{aligned} \quad (56)$$

recovers the stability of the Hamiltonian.

If we try to interpret the imaginary part of the field ϕ as a massive physical degree of freedom, so that both A and B are real degrees of freedom associated with the real and imaginary parts of the field ϕ, it may seem that by complexifying the original Lagrangian we are increasing the degrees of freedom to four (complex ϕ and Z fields). Actually, the balance in the degrees of freedom can be preserved with the introduction the condition $Z = \bar{\phi}$ by means of a Lagrange multiplier λ into the reduced order complex scalar Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\phi\Box Z + \frac{1}{8}m^2\phi\phi - \frac{1}{4}m^2\phi Z + \frac{1}{8}m^2 ZZ + \lambda(Z - \bar{\phi}), \quad (57)$$

ϕ and Z being now complex fields. The equations of motion are given by

$$\left(1 + \frac{2}{m^2}\Box\right)\phi = Z - \frac{4}{m^2}\lambda\,, \quad (58)$$

$$\left(1 + \frac{2}{m^2}\Box\right)Z = \phi\,, \quad (59)$$

$$Z = \bar{\phi}\,, \quad (60)$$

which can be combined and brought into the form

$$\Box\left(1 + \frac{1}{m^2}\Box\right)\phi = 0\,, \quad (61)$$

$$\Box(\phi + Z) = 0\,, \quad (62)$$

$$\left(\Box + m^2\right)(\phi - Z) = 0\,, \quad (63)$$

where we conclude that $\lambda = 0$, $\phi = A + iB$ and $Z = A - iB$. Substituting this in (57), we end up with

$$\mathcal{L} = \frac{1}{2}A\Box A + \frac{1}{2}B\Box B + \frac{1}{2}m^2 B^2. \quad (64)$$

In summary: as ϕ and Z are complex fields we start with four degrees of freedom described by the complex Lagrangian (57), while the higher derivative real scalar theory has only two degrees of freedom. We match the number of degrees of freedom in both formulation by enforcing the condition $Z = \bar{\phi}$ via a Lagrange multiplier.

A more general prescription to quantize higher derivative theories, circumventing the problem of the stability of the Hamiltonian, have been discussed in [32], using the concept of Lagrangian anchors. Essentially, it involves an extension of the Noether theorems, defining a class of conserved quantities associated with a given symmetry. For time translations, this procedure can lead to different conserved quantities which could be in principle be identified with a Hamiltonian, some of them would have regularizing properties but will break the stability because the energy is not bounded from below, whereas the other recovers the stability but loses the regularizing properties, seen in the self-energy of a particles and ultraviolet divergences. It would be an interesting endeavor to investigate this approach for more involved models, something that we will not try in this work.

4. HD Podolsky theory in the (FJ) formalism

Although the (FJ) formalism does not implement major changes in the quantization process of a regular theory, in a singular theory there might be considerable simplifications when adopting the symplectic formalism instead of the usual (DB) algorithm. We apply the (FJ) method to discuss the quantization of the Podolsky electrodynamics [41] but, differently from what was done in [39], we start by writing the theory in the reduction of order formalism, by means of the introduction of an additional auxiliary field B^μ, following [36]. We will show that this technique allows us to write the symplectic matrix in a block structure, thus clearly separating the Maxwell and Proca sectors. This makes the treatment of the different degrees of freedom of the model particularly simple and clear.

Concretely, we start with,

$$\mathcal{L}_{red} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{a^2}{2}B_\mu B^\mu + a^2\partial_\mu B_\nu F^{\mu\nu}\,, \tag{65}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{66}$$

Up to surface terms, we can also write

$$\mathcal{L}_{red} = \frac{1}{2}A^\mu(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu - \frac{a^2}{2}B_\mu B^\mu - a^2 B^\mu(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\mu\,, \tag{67}$$

which leads directly to the coupled equations of motion

$$(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu = a^2(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B^\nu\,, \tag{68}$$

and

$$(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu = -B_\mu\,. \tag{69}$$

A direct consequence of the last equation is that $\partial_\mu B^\mu = 0$. Additionally, one may decouple the previous two equations, obtaining

$$\big(1 + a^2\Box\big)(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu = 0;\;\; \big(1 + a^2\Box\big)B_\mu = 0\,. \tag{70}$$

Classically the reduced order Lagrangian density $\mathcal{L}_{red}$ is equivalent to the following Ostrogradsky Lagrangian density, up to surface terms,

$$\mathcal{L}_{Ostro} = -\frac{1}{4}F^{\mu\nu}\big(1 + a^2\Box\big)F_{\mu\nu} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho}\,. \tag{71}$$

Also, the classical coupled equations of motion (68) and (69) can be written as

$$\begin{bmatrix} T_{\mu\nu} & a^2 T^{\nu}_{\mu} \\ T^{\mu}_{\nu} & -\eta^{\mu\nu} \end{bmatrix} \begin{bmatrix} A^{\nu} \\ B_{\nu} \end{bmatrix} = 0, \tag{72}$$

wherein we have the definition $T_{\mu\nu} \doteq \eta_{\mu\nu}\Box - \partial_\nu\partial_\nu$. Implicitly in the analysis we have a problem of eigenvalues and eigenvectors and the diagonalization of a matrix since

$$\det\begin{bmatrix} T_{\mu\nu} & a^2 T^{\nu}_{\mu} \\ T^{\mu}_{\nu} & -\eta^{\mu\nu} \end{bmatrix} = \det\begin{bmatrix} T_{\mu\nu} & 0 \\ 0 & -(T^{\mu\nu} + a^2\eta^{\mu\nu}) \end{bmatrix} = -3\Box\left(1 + a^2\Box\right), \tag{73}$$

making explicit the Maxwell (the $T_{\mu\nu}$ factor) and Proca (the $-(T^{\mu\nu} + a^2\eta^{\mu\nu})$ factor) physical degrees of freedom (2+3, respectively) of the theory, as well as the problem of instability due to the negative sign of the massive mode.

Due to the fact that $\mathcal{L}_{red}$ is of second order, we can define the usual canonical momenta

$$\pi^i = \frac{\partial\mathcal{L}_{red}}{\partial\dot{A}_i} = F^{0i} + a^2\left(\partial^0 B^i - \partial^i B^0\right), \quad \theta^i = \frac{\partial\mathcal{L}_{red}}{\partial\dot{B}_i} = a^2 F^{0i}, \tag{74}$$

leading to

$$\mathcal{L}_{red} = \frac{1}{a^2}\pi^i\theta_i - \frac{1}{2a^4}\theta^i\theta_i - \frac{1}{4}F^{ij}F_{ij} + a^2\partial_i B_j F^{ij} - \frac{a^2}{2}B_\mu B^\mu .$$

By a Legendre transform, we obtain the canonical Hamiltonian

$$\mathcal{H}_{red} = \pi_i\dot{A}^i + \theta_i\dot{B}^i - \mathcal{L}_{red} , \tag{75}$$

which, up to surface terms, leads to

$$\begin{aligned} \mathcal{H}_{red}(A_i, \pi_i, B_i, \theta_i, A_0, B_0) = {} & \frac{1}{a^2}\pi^i\theta_i + \frac{1}{2a^4}\theta^i\theta_i + \frac{1}{4}F^{ij}F_{ij} - a^2\partial_i B_j F^{ij} \\ & + \frac{a^2}{2}B_\mu B^\mu - A^0\partial^i\pi_i - B^0\partial^i\theta_i . \end{aligned} \tag{76}$$

We can now construct the symplectic structure in the (FJ) formalism, starting by writing

$$\mathcal{L}_{red} = \frac{1}{a^2}\pi_i\dot{A}^i + \theta_i\dot{B}^i - \mathcal{V}^{(0)} , \tag{77}$$

where

$$\mathcal{V}^{(0)} = \pi^i\theta_i + \frac{1}{2a^4}\theta^i\theta_i + \frac{1}{4}F^{ij}F_{ij} - a^2\partial_i B_j F^{ij} + \frac{a^2}{2}B_\mu B^\mu - A^0\partial^i\pi_i - B^0\partial^i\theta_i . \tag{78}$$

The symplectic variables are up to this point $\xi = (A_i, \pi_i, B_i, \theta_i, A_0, B_0)$ and the canonical one-form is given by

$$a_{A_i} = \pi_i, \quad a_{\pi_i} = 0, \quad a_{B_i} = \theta_i, \quad a_{\theta_i} = 0, \quad a_{A_0} = 0, \quad a_{B_0} = 0 , \tag{79}$$

therefore, the symplectic matrix can be written as

$$[f] = \left[\begin{array}{c|cccccc} & A_j & \pi_j & B_j & \theta_j & A_0 & B_0 \\ \hline A_i & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 \\ \pi_i & \delta_{ij} & 0 & 0 & 0 & 0 & 0 \\ B_i & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 \\ \theta_i & 0 & 0 & \delta_{ij} & 0 & 0 & 0 \\ A_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\delta^3(\vec{x} - \vec{y}). \tag{80}$$

Clearly, $\det[f]=0$ signaling a singular system, as expected. The following eigenvectors have null eigenvalues,

$$u=\big(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},u^{13},0\big)\,, \tag{81}$$

$$v=\big(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},0,v^{14}\big)\,, \tag{82}$$

and the respective constraint equations are

$$\Omega_1=\int dxdy\,u\frac{\delta\mathcal{V}^{(0)}(y)}{\delta A_0(x)}=\int dx\,u^{13}\partial^i\pi_i=0\,, \tag{83a}$$

$$\Omega_2=\int dxdy\,v\frac{\delta\mathcal{V}^{(0)}(y)}{\delta B_0(x)}=\int dx\,u^{13}\big(a^2B_0-\partial^i\theta_i\big)=0\,. \tag{83b}$$

We enforce the previous constraint equations into $\mathcal{L}_{red}$ using Lagrange multipliers,

$$\mathcal{L}_{red}=\pi_i\dot{A}^i+\theta_i\dot{B}^i+\dot{\lambda}^a\Omega_a-\mathcal{V}^{(2)},\quad a=1,2\,, \tag{84}$$

where

$$\mathcal{V}^{(2)}=\mathcal{V}^{(0)}|_{\Omega_a=0}=\frac{1}{a^2}\pi^i\theta_i+\frac{1}{2a^4}\theta^i\theta_i+\frac{1}{4}F^{ij}F_{ij}-a^2\partial_iB_jF^{ij}-\frac{a^2}{2}B_0B^0+\frac{a^2}{2}B_iB^i\,. \tag{85}$$

So from this augmented symplectic structure, we have the following one form vectors

$$\begin{aligned}&a^{(2)}_{A_i}=\pi_i,\quad a^{(2)}_{\pi_i}=0,\quad a^{(2)}_{B_0}=0,\quad a^{(2)}_{B_i}=\theta_i,\quad a^{(2)}_{\theta_i}=0,\\ &a^{(2)}_{\lambda_1}=\partial^i\pi_i,\quad a^{(2)}_{\lambda_2}=\big(a^2B_0-\partial^i\theta_i\big)\,.\end{aligned} \tag{86}$$

At this point, when calculating the symplectic matrix, one realizes the main advantage in the present formalism, since $[f]$ turns out to be a block diagonal matrix

$$[f]=\begin{bmatrix}[M] & \mathbf{0}\\ \mathbf{0} & [P]\end{bmatrix}, \tag{87}$$

where $[M]$ corresponds to the massless Maxwell sector of the theory,

$$[M]=\left[\begin{array}{c|ccc} & A_j & \pi_j & \lambda_1\\ \hline A_i & 0 & -\delta_{ij} & \partial_i\\ \pi_i & \delta_{ij} & 0 & 0\\ \lambda_1 & \partial_j & 0 & 0\end{array}\right]\delta^3(\vec{x}-\vec{y})\,, \tag{88}$$

and $[P]$ to the massive Proca sector

$$[P]=\left[\begin{array}{c|cccc} & B_j & \theta_j & B_0 & \lambda_2\\ \hline B_i & 0 & -\delta_{ij} & 0 & \partial_i\\ \theta_i & \delta_{ij} & 0 & 0 & 0\\ B_0 & 0 & 0 & 0 & -a^2\\ \lambda_2 & \partial_j & 0 & a^2 & 0\end{array}\right]\delta^3(\vec{x}-\vec{y})\,. \tag{89}$$

Needless to say, the structure of $[f]$ implies that

$$\det[f]=\det[M]\det[P]\,. \tag{90}$$

The neat separation between the Maxwell and Proca sectors is a distinctive feature of the (FJ) formalism applied to the reduced order Podolsky electrodynamics, which does not happen within the Ortogradsky formalism [39].

First, let us work with the Maxwell sector. As expected, $\det[M]=0$ so $[M]$ is singular, and the null eigenvector is of the form $v=(0,v_j^{\pi},v^{\lambda_1})$, $j=1,2,3$, corresponding to the constraint equation

$$\int dxdy\, v_i^{\pi}\frac{\delta\mathcal{V}^{(2)}(y)}{\delta A_i(x)}=\int dx\,\partial_i\partial_j v^{\lambda_1}\left[-\partial_i F^{ij}+\frac{a^2}{2}\partial_i\left(\partial^i B^j-\partial^j B^j\right)\right]=0. \tag{91}$$

This zero mode does not generate any additional constraints and, consequently, the symplectic matrix remains singular, which is a characteristic of gauge theories: a gauge fixing condition should be introduced in order to obtain a non singular symplectic matrix. Inspired by the form of the fourth-order equations of motion for A_μ, Eq. (70), as well as the analysis presented in [39], we use generalized Coulomb gauge fixing conditions in the form

$$A_0=0,\quad \Omega_3=\left(1+a^2\Box\right)\vec{\nabla}\vec{A}=0. \tag{92}$$

For more details on the gauge fixing of the Podolsky theory we refer the reader to [41]. When this gauge condition is included in $\mathcal{L}_{red}$ using a Lagrange multiplier $\lambda_3\Omega_3$, we obtain the following $[M]$ matrix for the Maxwell sector

$$[M]=\left[\begin{array}{c|cccc} & A_j & \pi_j & \lambda_3 & \lambda_2\\ \hline A_i & 0 & -\delta_{ij} & 0 & \partial_i\\ \pi_i & \delta_{ij} & 0 & (1+a^2\vec{\nabla}^2)\partial_i & 0\\ \lambda_3 & 0 & (1+a^2\vec{\nabla}^2)\partial_j & 0 & 0\\ \lambda_2 & \partial_j & 0 & 0 & 0\end{array}\right]\delta^3(\vec{x}-\vec{y}), \tag{93}$$

which is a regular matrix, with $\det[M]=[(1+a^2\vec{\nabla}^2)\vec{\nabla}^2]^2$, and its inverse can be calculated almost immediately

$$[M]^{-1}=\frac{1}{(1+a^2\vec{\nabla}^2)\vec{\nabla}^2}\times$$
$$\times\left[\begin{array}{c|cccc} & A_j & \pi_j & \lambda_3 & \lambda_2\\ \hline A_i & 0 & -(1+a^2\vec{\nabla}^2)\vec{\nabla}^2\delta_{ij}+\partial_i\partial_j & 0 & \partial_i\\ \pi_i & (1+a^2\vec{\nabla}^2)\vec{\nabla}^2\delta_{ij}-\partial_i\partial_j & 0 & \partial_i & 0\\ \lambda_3 & 0 & \partial_j & 0 & 1\\ \lambda_2 & \partial_j & 0 & 1 & 0\end{array}\right]\delta^3(\vec{x}-\vec{y}). \tag{94}$$

From this, one easily identifies the Dirac brackets between the dynamics variables in the generalized Lorenz gauge

$$\{A_i,\pi_j\}_D=\left[-\delta_{ij}+\frac{\partial_i\partial_j}{(1+a^2\vec{\nabla}^2)\vec{\nabla}^2}\right]\delta^3(\vec{x}-\vec{y}). \tag{95}$$

Now, we consider the Proca sector. One way to calculate the determinant of $[P]$ is to notice that, for any block matrix of the form

$$P_{n\times n}=\begin{pmatrix}A_{m\times m} & B_{m\times n-m}\\ C_{n-m\times m} & D_{m\times m}\end{pmatrix}, \tag{96}$$

if D has an inverse, the following identity holds

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}, \tag{97}$$

and therefore

$$\det P = \det\big(A - BD^{-1}C\big)\det D\,. \tag{98}$$

Applied to Eq. (89), this leads to $\det[P] = a^4$. The Proca sector is therefore regular, and we obtain the following inverse symplectic matrix

$$[P]^{-1} = \left[\begin{array}{c|cccc} & B_j & \theta_j & B_0 & \lambda_2 \\ \hline B_i & 0 & -\delta_{ij} & -\frac{1}{a^2}\partial_i & 0 \\ \theta_i & \delta_{ij} & 0 & 0 & 0 \\ B_0 & \frac{1}{a^2}\partial_j & 0 & 0 & \frac{1}{a^2} \\ \lambda_2 & 0 & 0 & -\frac{1}{a^2} & 0 \end{array}\right]\delta^3(\vec{x} - \vec{y})\,, \tag{99}$$

corresponding to the following Dirac brackets between the dynamics variables,

$$\{B_i, \theta_j\}_D = -\delta_{ij}\delta^3(\vec{x} - \vec{y})\,. \tag{100}$$

From now on we are ready to construct the quantum description of this theory. According to Eq. (12), the transition amplitude is given by

$$Z_{red} = \int DA_i D\pi_i DB_0 DB_i D\theta_i D\lambda_a \times \tag{101}$$

$$\times \sqrt{\det[M]\det[P]}\exp\left[i\int d^4x\big(\pi_i \dot{A}^i + \theta_i \dot{B}^i + \dot{\lambda}^a\Omega_a - \mathcal{V}^{(2)}\big)\right], \quad a = 1, 2, 3\,. \tag{102}$$

Identifying $\lambda_1 = A_0$, we can write

$$\begin{aligned} Z_{red} =& Na^2 \int DA_0 DA_i DB_0 DB_i D\pi_i D\theta_i \det\big[\big(1 + a^2\vec{\nabla}^2\big)\vec{\nabla}^2\big]\delta\big(\big(1 + a^2\Box\big)\vec{\nabla}\vec{A}\big)\times \\ & \times \delta\big(a^2 B_0 - \partial^i\theta_i\big)\exp\left[i\int d^4x\left(\pi_i \dot{A}^i + \theta_i \dot{B}^i + A_0\big(\partial_i\pi^i\big) - \frac{1}{a^2}\pi^i\theta_i - \frac{1}{2a^4}\theta^i\theta_i \right.\right. \\ & \left.\left. - \frac{1}{4}F^{ij}F_{ij} + a^2\partial_i B_j F^{ij} + \frac{a^2}{2}B_0 B^0 - \frac{a^2}{2}B_i B^i\right)\right]. \end{aligned} \tag{103}$$

Integration in $D\pi_i$ leads to the appearance of a delta function $\delta(F^{0i} - \frac{1}{a^2}\theta^i)$, and further integrations in $D\theta_i$ and B_0 leads to

$$\begin{aligned} Z_{red} = & Na^2 \int DA_0 DA_i DB_i \det\big[\big(1 + a^2\vec{\nabla}^2\big)\vec{\nabla}^2\big]\delta\big(\big(1 + a^2\Box\big)\vec{\nabla}\vec{A}\big)\times \\ & \times \exp\left[i\int d^4x\left(a^2 F_{0i}\partial^0 B^i - \frac{1}{2}F^{0i}F_{0i} - \frac{1}{4}F^{ij}F_{ij} + \right.\right. \\ & \left.\left. + a^2\partial_i B_j F^{ij} + \frac{a^2}{2}\partial_i F^{0i}\partial^j F_{0j} - \frac{a^2}{2}B_i B^i\right)\right]. \end{aligned} \tag{104}$$

Some algebraic manipulations are now in order. Up to a surface term, we have

$$a^2 F_{0i}\partial^0 B^i + a^2\partial_i B_j F^{ij} = a^2\partial^\nu F_{i\nu}B^i\,, \tag{105}$$

and completing the squares,

$$a^2\partial^\nu F_{i\nu}B^i - \frac{a^2}{2}B_iB^i = -\frac{a^2}{2}\big(B^i + \partial_\nu F^{i\nu}\big)^2 + \frac{a^2}{2}\partial_\nu F^{i\nu}\partial^\rho F_{i\rho}\,. \tag{106}$$

Thus, by translation invariance of the functional integral, the integration in DB_i amounts to a A^μ independent Gaussian integral, which can be incorporated in the normalization factor. As a consequence, the transition amplitude can be cast as

$$Z_{red} = N' \int DA_0 DA_i \det\big[\big(1 + a^2\vec{\nabla}^2\big)\vec{\nabla}^2\big]\delta\big(\big(1 + a^2\Box\big)\vec{\nabla}\vec{A}\big)\times \\ \times \exp\left[i\int d^4x\left(-\frac{1}{2}F^{0i}F_{0i} - \frac{1}{4}F^{ij}F_{ij} + \frac{a^2}{2}\partial_i F^{0i}\partial^j F_{0j} + \frac{a^2}{2}\partial_\nu F^{i\nu}\partial^\rho F_{i\rho}\right)\right], \tag{107}$$

where

$$N' = Na^2 \int DB_i \exp\left[-i\int d^4x\,\frac{a^2}{2}\big(B^i + \partial_\nu F^{i\nu}\big)^2\right]. \tag{108}$$

Here, we kept the seemingly dependence of N' on A^μ for clarity purposes. So we rewrite explicitly the following transition amplitude in the generalized Coulomb gauge

$$Z_{red} = N' \int DA_\mu \det\big[\big(1 + a^2\vec{\nabla}^2\big)\vec{\nabla}^2\big]\delta\big(\big(1 + a^2\Box\big)\vec{\nabla}\vec{A}\big) \\ \times \exp\left[i\int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho}\right)\right] \\ = Z_{Ostro}\,. \tag{109}$$

We therefore verify that the equivalence between the classical equations of motion in the reduction of order and Ostrogradsky prescriptions, seen in Eqs. (65) and (71), hold also at the quantum level, when we compare the transition amplitude obtained in both prescriptions.

We end this section by making some comments to further clarify the counting of the degrees of freedom in the Podolsky electrodynamics. The Ostogradsky phase space has 16 variables $(\Phi^\nu, \Gamma_\nu, \Pi^\mu, A_\mu)$ and 6 constraints, so the physical phase space has 10 variables and 5 degrees of freedom [42]. Half of the constraints are first-class and the other half are the gauge fixing conditions that transform the first-class constraints into second-class constraints (for example, imposing the generalized Coulomb gauge), such that we can determine all the Lagrange multipliers. On other hand, in the reduced order approach, the phase space has 16 variables $(A_\mu, B_\nu; \pi^\mu, \theta^\nu)$ and also 6 constraints, but these have different structure: two of these constraints are second-class from the start, two are first-class and the last two are the corresponding gauge fixing conditions. Now, in the (FJ) methodology for the reduced order theory, we obtained a symplectic matrix separated as Maxwell plus Proca in the first iteration form. In doing so, the formalism already takes into account two constraints[1] and we have to deal just with Ω_1, Ω_2 defined in Eq. (83). The Proca sector already presents itself as a regular sector, contributing three degrees of freedom, while Maxwell sector is singular, and after the introduction of two gauge fixing conditions $A_0 = \Omega_3 = 0$ (see Eq. (92)), will describe the additional two degrees of freedom of the theory.

[1] This apparent disappearance of constraints also happens when we compare the (DB) metodology and (FJ) method in first order theories with second-class constraints as in the Schrödinger or Dirac Lagrangians [18,20].

5. Final remarks

Our main objective was to discuss the use of the (FJ) formalism for higher derivatives theories, in particular showing how, when the order of the equations of motion are reduced by the introduction of auxiliary fields, the dynamics can be put in a more transparent form, with an explicit separation of the relevant degrees of freedom.

These ideas were first presented in a toy model involving a massless scalar field as the physical degree of freedom. We presented both the classical and quantum basic developments of the model, both in the Ostogradsky and the reduction of order approach, showing their equivalence, but also pointed out that, in the latter case, one can neatly disentangle the two degrees of freedom present in the model: one physical massless scalar and a ghost massive one. We also briefly discussed some recent approaches toward a consistent understanding of these ghost fields, which present themselves as a longstanding problem for higher derivative theories.

Afterwards, we discussed the Podolsky electrodynamics. This is a well known higher derivative gauge theory: the (FJ) quantization procedure have already been used for this model in the Ortogradsky formalism [39], while the reduced order formalism was also considered in [36] together with the (DB) quantization procedure. We pointed out that the combination of the reduced order with the (FJ) formalism presents itself as a simpler way to study the constraint structure and the quantization of this theory, since the relevant degrees of freedom (Maxwell+Proca) are clearly separated. Our results are consistent with the ones obtained in the other formalisms.

It is worth noticing that Podolsky electrodynamics breaks the dual symmetry [43]

$$\begin{aligned} \vec{E} &\to \vec{B} \\ \vec{B} &\to -\vec{E} \end{aligned} \tag{110}$$

that led Dirac to consider the existence of magnetic monopoles. Hence, a study of Podolsky equations in the vacuum may shed some light on the question of the existence of monopoles as two Dirac strings (solenoids) have an interaction associated with the Podolsky mass [26]. Besides, the fact that the Podolsky characteristic length is associated with the size of the electron [44] could lead us to explore, by electron-positron scattering, the existence of Maxwell → Podolsky transition from the point of view of a mechanism which breaks the dual symmetry and generate mass. These speculations derive from our ignorance associated with the mechanisms behind the self-interaction of the particles and their sizes and deserve rigorous scientific analysis.

Finally, we think that the natural next step of this investigation would be the extension of this discussion for important interacting cases (minimal coupling and sources) or the non Abelian and gravity theories [45]. These matters will be further elaborated and requires deeper investigations.

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