

U(1) gauge vector field on a codimension-2 brane

Chun-E Fu,^{a,1} Yuan Zhong^a and Yu-Xiao Liu^b

^a*School of Science, Xi'an Jiaotong University,
28 Xianning West Road, Xi'an 710049, China*

^b*Institute of Theoretical Physics & Research Center of Gravitation, Lanzhou University,
222 South Tianshui Road, Lanzhou 730000, China*

E-mail: fuche13@mail.xjtu.edu.cn, zhongy@mail.xjtu.edu.cn,
liuyx@lzu.edu.cn

ABSTRACT: In this paper, we obtain a gauge invariant effective action for a bulk massless U(1) gauge vector field on a brane with codimension two by using a general Kaluza-Klein (KK) decomposition for the field. It suggests that there exist two types of scalar KK modes to keep the gauge invariance of the action for the massive vector KK modes. Both the vector and scalar KK modes can be massive. The masses of the vector KK modes $m^{(n)}$ contain two parts, $m_1^{(n)}$ and $m_2^{(n)}$, due to the existence of the two extra dimensions. The masses of the two types of scalar KK modes $m_\phi^{(n)}$ and $m_\varphi^{(n)}$ are related to the vector ones, i.e., $m_\phi^{(n)} = m_1^{(n)}$ and $m_\varphi^{(n)} = m_2^{(n)}$. Moreover, we derive two Schrödinger-like equations for the vector KK modes, for which the effective potentials are just the functions of the warp factor.

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¹Corresponding author.

Contents

1	Introduction	1
2	KK decomposition and the effective action	2
3	KK modes of the U(1) gauge vector field	5
3.1	The equations of KK modes	5
3.2	The orthonormality condition	9
3.3	The gauge invariant effective brane action	9
4	Further discussions	10
5	Conclusion	11

1 Introduction

If there are extra dimensions, the physical world will be surely more interesting. For example, in the Kaluza-Klein (KK) theory with one extra spatial dimension, the four-dimensional (4D) gravity and electromagnetism can be unified. In the Arkani-Hamed-Dimopoulos-Dvali (ADD) theory [1], the hierarchy problem is reconsidered to be related with large extra dimensions. Especially, in the Randall-Sundrum (RS) brane world theory [2, 3], the warped space-time outside our four-dimensional world leads to some new and dramatic phenomenologies. That's why the extra dimension and brane world theories are paid for more and more attention [4–14].

One of the most interesting things is that there will be KK modes for various bulk fields with extra dimensions. In the brane world theory, the zero modes can be regarded as particles living in four-dimensional space-time, while the massive KK modes might reveal the mysteries of extra dimensions. Thus, there has been many papers focus on these KK modes [15–33].

In this paper, we will discuss the KK modes of U(1) gauge vector field in a brane model with codimension two. It is known that the localization of the vector field is difficult in five-dimensional brane world model, and so there are many literatures in order to find localization mechanism [34–41]. In this paper, we would like to think about another question for the KK modes of a vector field, i.e., how to keep the invariance of the brane effective action. It may be argued that the invariance of the action can be guaranteed by the Higgs mechanism. But here we begin with a massless vector field in the bulk space-time, and only consider the influence of the extra dimensions on the KK modes.

For the zero modes, the action is straightforward gauge invariant not only in the bulk but also on the brane. For the massive vector KK modes, it is not easy to obtain a

gauge invariant effective action on the brane. Fortunately, in our recently work, we have investigated a new localization mechanism of an arbitrary q -form field on p -brane with codimension one [26]. We discovered that with a general KK decomposition for the bulk q -form field there is a gauge invariant effective action for the KK modes. For example, by the new localization mechanism for a 1-form bulk field (vector field), we can get a gauge invariant effective action for the massive vector KK modes, which are coupled with some massless scalar KK modes. This is similar to the Higgs mechanism which enables a massless vector to obtain mass. The difference is that in this brane world model with codimension one, the vector KK modes obtain masses due to the existence of extra dimension.

Therefore, it is reasonable to think that on a brane with codimension two the vector KK modes would get two parts of masses from the two extra dimensions. In this work, we would like to verify this idea. What interesting is that we also obtain a gauge invariance effective action on the brane. This effective action indicates that the vector KK modes really obtain their masses from the two extra dimensions. In addition, there are two types of massive scalar KK modes.

Our discussion is based on the following line-element:

$$ds^2 = e^{2A(y,z)}(\hat{g}_{\mu\nu}dx^\mu dx^\nu + dy^2 + dz^2), \tag{1.1}$$

where the warp factor $A(y, z)$ is a function of the two extra dimensional coordinates y and z , and $\hat{g}_{\mu\nu}$ is the induced metric on the brane. The bulk fields can propagate along the two directions outside the brane, and play the role of KK modes on the brane. In the following section we will start our work with a general KK decomposition for the bulk vector field. Then we will try to find two groups of equations of motion (EOM) for the KK modes of the field, and will discover some interesting results by comparing these EOMs.

2 KK decomposition and the effective action

In this paper, we consider the model of brane world with codimension two. The line-element is given by (1.1). The action for a bulk massless U(1) gauge vector field is

$$\begin{aligned} S &= -\frac{1}{4} \int d^6x \sqrt{-g} Y^{M_1 M_2} Y_{M_1 M_2} \\ &= -\frac{1}{4} \int d^6x \sqrt{-g} \left(Y^{\mu_1 \mu_2} Y_{\mu_1 \mu_2} + 2Y^{\mu_1 z} Y_{\mu_1 z} + 2Y^{\mu_1 y} Y_{\mu_1 y} + 2Y^{yz} Y_{yz} \right), \end{aligned} \tag{2.1}$$

where $Y_{M_1 M_2} = \frac{1}{2}(\partial_{M_1} X_{M_2} - \partial_{M_2} X_{M_1})$ is the field strength of the vector field X_M . The equations of motion for the bulk field are given by

$$\partial_{\mu_1}(\sqrt{-g}Y^{\mu_1\mu_2}) + \partial_z(\sqrt{-g}Y^{z\mu_2}) + \partial_y(\sqrt{-g}Y^{y\mu_2}) = 0, \tag{2.2}$$

$$\partial_{\mu_1}(\sqrt{-g}Y^{\mu_1 z}) + \partial_y(\sqrt{-g}Y^{yz}) = 0, \tag{2.3}$$

$$\partial_{\mu_1}(\sqrt{-g}Y^{\mu_1 y}) + \partial_z(\sqrt{-g}Y^{zy}) = 0. \tag{2.4}$$

This action (2.1) is surely gauge invariant. With a KK decomposition of the vector field and a dimension reduction, we will obtain the effective actions for the vector KK modes.

It is easy to show that the effective action of the vector zero mode is also gauge invariant, while the ones for the massive KK modes are not without some mechanisms [42–45].

We have found one way to get the gauge invariant actions for the massive vector KK modes on a brane with codimension one, where our aim was in fact to solve the Hodge duality on the brane for the q -form field [26]. We know that, with a given KK decomposition of a bulk field, the effective actions on the brane for the KK modes of the field can be finally derived. Different KK decompositions will lead to different effective actions for the KK modes. Usually, one would like to choose a gauge to simplify the calculation before doing the KK decomposition for the vector field [46]. We think that the choice of the gauge for the vector field in fact influences the KK decomposition. Therefore, in order to obtain a gauge invariant effective brane action, one should start with a general KK decomposition without using a gauge fixing.

Thus, we begin with a general KK decomposition for the bulk massless vector field:

$$X_{\mu_1}(x_\mu, y, z) = \sum_n \hat{X}_{\mu_1}^{(n)}(x^\mu) W_1^{(n)}(y, z) e^{a_1 A(y,z)}, \quad (2.5)$$

$$X_z(x_\mu, y, z) = \sum_n \phi^{(n)}(x^\mu) W_2^{(n)}(y, z) e^{a_2 A(y,z)}, \quad (2.6)$$

$$X_y(x_\mu, y, z) = \sum_n \varphi^{(n)}(x^\mu) W_3^{(n)}(y, z) e^{a_3 A(y,z)}, \quad (2.7)$$

where $W_1^{(n)}(y, z)$, $W_2^{(n)}(y, z)$, and $W_3^{(n)}(y, z)$ are only the functions of extra dimensions y, z , and a_1, a_2 , and a_3 can be taken as any arbitrary constants. Note that we always use (n) and (n') to label KK modes in this paper.

With the above KK decompositions, we can derive the effective actions for the KK modes $\hat{X}_{\mu_1}^{(n)}(x^\mu)$, $\phi^{(n)}(x^\mu)$, and $\varphi^{(n)}(x^\mu)$:

$$\begin{aligned} S &= -\frac{1}{4} \int d^6 x \sqrt{-g} Y^{M_1 M_2} Y_{M_1 M_2} \\ &= -\frac{1}{4} \int d^6 x \sqrt{-g} \left(Y^{\mu_1 \mu_2} Y_{\mu_1 \mu_2} + 2Y^{\mu_1 z} Y_{\mu_1 z} + 2Y^{\mu_1 y} Y_{\mu_1 y} + 2Y^{yz} Y_{yz} \right), \\ &= -\frac{1}{4} \sum_n \sum_{n'} \int d^4 x \sqrt{-\hat{g}} \left[I_1^{(nn')} \hat{Y}_{\mu_1 \mu_2}^{(n)} \hat{Y}^{\mu_1 \mu_2 (n')} + (I_2^{(nn')} + I_4^{(nn')}) \hat{X}_{\mu_1}^{(n)} \hat{X}^{\mu_1 (n')} \right. \\ &\quad + I_3^{(nn')} \partial_{\mu_1} \phi^{(n)} \partial^{\mu_1} \phi^{(n')} - I_6^{(nn')} \left(\partial_{\mu_1} \phi^{(n)} \hat{X}^{\mu_1 (n')} + \hat{X}_{\mu_1}^{(n)} \partial^{\mu_1} \phi^{(n')} \right) \\ &\quad + I_5^{(nn')} \partial_{\mu_1} \varphi^{(n)} \partial^{\mu_1} \varphi^{(n')} - I_8^{(nn')} \left(\partial_{\mu_1} \varphi^{(n)} \hat{X}^{\mu_1 (n')} + \hat{X}_{\mu_1}^{(n)} \partial^{\mu_1} \varphi^{(n')} \right) \\ &\quad \left. + I_7^{(nn')} \phi^{(n)} \phi^{(n')} + I_9^{(nn')} \varphi^{(n)} \varphi^{(n')} - I_{10}^{(nn')} \left(\phi^{(n)} \varphi^{(n')} + \varphi^{(n)} \phi^{(n')} \right) \right], \quad (2.8) \end{aligned}$$

where we have assumed that $W_1^{(n)}(y, z)$, $W_2^{(n)}(y, z)$, and $W_3^{(n)}(y, z)$ satisfy the following

orthonormality conditions:

$$I_1^{(nn')} \equiv \int dy dz W_1^{(n)} W_1^{(n')} = \delta_{nn'}, \quad (2.9a)$$

$$I_3^{(nn')} \equiv \frac{1}{2} \int dy dz W_2^{(n)} W_2^{(n')} = 2\delta_{nn'}, \quad (2.9b)$$

$$I_5^{(nn')} \equiv \frac{1}{2} \int dy dz W_3^{(n)} W_3^{(n')} = 2\delta_{nn'}, \quad (2.9c)$$

and the other constants are given by

$$I_2^{(nn')} \equiv \frac{1}{2} \int dy dz \partial_y(W_1^{(n)} e^{-A}) \partial_y(W_1^{(n')} e^{-A}) e^{2A} < \infty, \quad (2.10a)$$

$$I_4^{(nn')} \equiv \frac{1}{2} \int dy dz \partial_z(W_1^{(n)} e^{-A}) \partial_z(W_1^{(n')} e^{-A}) e^{2A} < \infty, \quad (2.10b)$$

$$I_6^{(nn')} \equiv \frac{1}{2} \int dy dz W_2^{(n)} \partial_z(W_1^{(n')} e^{-A}) e^A < \infty, \quad (2.10c)$$

$$I_8^{(nn')} \equiv \frac{1}{2} \int dy dz W_3^{(n)} \partial_y(W_1^{(n')} e^{-A}) e^A < \infty, \quad (2.10d)$$

$$I_7^{(nn')} \equiv \frac{1}{2} \int dy dz \partial_y(W_2^{(n)} e^{-A}) \partial_y(W_2^{(n')} e^{-A}) e^{2A} < \infty, \quad (2.10e)$$

$$I_9^{(nn')} \equiv \frac{1}{2} \int dy dz \partial_z(W_3^{(n)} e^{-A}) \partial_z(W_3^{(n')} e^{-A}) e^{2A} < \infty, \quad (2.10f)$$

$$I_{10}^{(nn')} \equiv \frac{1}{2} \int dy dz \partial_y(W_2^{(n)} e^{-A}) \partial_z(W_3^{(n')} e^{-A}) e^{2A} < \infty. \quad (2.10g)$$

We have taken $a_1 = a_2 = a_3 = -1$, and defined $\hat{Y}_{\mu_1\mu_2}^{(n)} \equiv \frac{1}{2}(\partial_{\mu_1} \hat{X}_{\mu_2}^{(n)} - \partial_{\mu_2} \hat{X}_{\mu_1}^{(n)})$.

From the effective action (2.8), we can analyze the following useful information:

- There are three types of KK modes, the vector $\hat{X}_{\mu}^{(n)}$, and two types of scalar ones $\phi^{(n)}$, $\varphi^{(n)}$. The constant symmetric matrixes $2(I_2^{(nn)} + I_4^{(nn)})$, $\frac{1}{2}I_7^{(nn)}$ and $\frac{1}{2}I_9^{(nn)}$ are the mass matrixes of the vector $\hat{X}_{\mu_1}^{(n)}(x^\mu)$, and the scalar modes $\phi^{(n)}(x^\mu)$ and $\varphi^{(n)}(x^\mu)$, respectively;
- The asymmetric matrixes $I_6^{(nn')}$ and $I_8^{(nn')}$ describe the coupling between the vector and scalar KK modes, and $I_{10}^{(nn')}$ between two scalar modes;
- In the effective action (2.8), except for the term $\hat{Y}_{\mu_1\mu_2}^{(n)} \hat{Y}^{\mu_1\mu_2(n')}$, if the other terms can be rewritten as the form

$$(\partial_\mu \phi^{(n)} - C \hat{X}_\mu^{(n)})^2, \quad (\partial_\mu \varphi^{(n)} - C \hat{X}_\mu^{(n)})^2, \quad (\phi^{(n)} - \varphi^{(n)})^2, \quad (2.11)$$

the action is gauge invariant under the transformation

$$\hat{X}_\mu^{(n)} \rightarrow \hat{X}_\mu^{(n)} + \partial_\mu \rho^{(n)}, \quad \phi^{(n)} \rightarrow \phi^{(n)} + C\rho^{(n)}, \quad \varphi^{(n)} \rightarrow \varphi^{(n)} + C\rho^{(n)}, \quad (2.12)$$

where C is a constant and $\rho^{(n)}$ a scalar field.

We can guess that if the effective action (2.8) is really invariant, there must be some relationship between the mass parameters and the coupling constants. Through deriving the equations the KK modes satisfied, we will find these relationships.

3 KK modes of the U(1) gauge vector field

To derive the equations of the KK modes, we will compare two groups of EOMs for the KK modes. One group of EOMs is obtained from the effective action (2.8), and another is from substituting the KK decomposition into eqs. (2.2)–(2.4).

To simplify the calculation, we now introduce four mass parameters for the vector and scalar KK modes:

$$I_2^{(nn)} = \frac{1}{2} m_1^{(n)2}, \quad (3.1)$$

$$I_4^{(nn)} = \frac{1}{2} m_2^{(n)2}, \quad (3.2)$$

$$I_7^{(nn)} = 2 m_\phi^{(n)2}, \quad (3.3)$$

$$I_9^{(nn)} = 2 m_\varphi^{(n)2}, \quad (3.4)$$

where $m_v^{(n)} \equiv \sqrt{m_1^{(n)2} + m_2^{(n)2}}$ are the masses of the vector KK modes, and $m_\phi^{(n)}$ and $m_\varphi^{(n)}$ are the masses of the two scalar modes $\phi^{(n)}$ and $\varphi^{(n)}$, respectively.

3.1 The equations of KK modes

Firstly, from the effective action (2.8), we obtain three equations:

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(I_1^{(nn')} \sqrt{-\hat{g}} \hat{Y}^{\mu_1 \mu_2 (n')} \right) - (I_2^{(nn')} - I_4^{(nn')}) \hat{X}^{\mu_2 (n')} + I_6^{(nn')} \partial^{\mu_2} \phi^{(n')} + I_8^{(nn')} \partial^{\mu_2} \varphi^{(n')} = 0, \quad (3.5)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(I_3^{(nn')} \sqrt{-\hat{g}} \partial^{\mu_1} \phi^{(n')} - I_6^{(nn')} \sqrt{-\hat{g}} \hat{X}^{\mu_1 (n')} \right) - I_7^{(nn')} \phi^{(n')} + I_{10}^{(nn')} \varphi^{(n')} = 0, \quad (3.6)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(I_5^{(nn')} \sqrt{-\hat{g}} \partial^{\mu_1} \varphi^{(n')} - I_8^{(nn')} \sqrt{-\hat{g}} \hat{X}^{\mu_1 (n')} \right) - I_9^{(nn')} \varphi^{(n')} + I_{10}^{(nn')} \phi^{(n')} = 0. \quad (3.7)$$

Secondly, by inserting the KK decomposition (2.5)–(2.7) into eqs. (2.2)–(2.4), we have

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(\sqrt{-\hat{g}} \hat{Y}^{\mu_1 \mu_2 (n)} \right) + \lambda_1 \hat{X}^{\mu_2 (n)} + \lambda_2 \hat{X}^{\mu_2 (n)} - \lambda_3 \partial^{\mu_2} \varphi^{(n)} - \lambda_4 \partial^{\mu_2} \phi^{(n)} = 0, \quad (3.8)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(\sqrt{-\hat{g}} \partial^{\mu_1} \phi^{(n)} - \lambda_5 \sqrt{-\hat{g}} \hat{X}^{\mu_1 (n)} \right) + \lambda_6 \phi^{(n)} - \lambda_7 \varphi^{(n)} = 0, \quad (3.9)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left(\sqrt{-\hat{g}} \partial^{\mu_1} \varphi^{(n)} - \lambda_8 \sqrt{-\hat{g}} \hat{X}^{\mu_1 (n)} \right) - \lambda_9 \phi^{(n)} + \lambda_{10} \varphi^{(n)} = 0, \quad (3.10)$$

where we define ten coefficients:

$$\begin{aligned} \lambda_1 &\equiv \frac{\partial_y \left(\partial_y (W_1^{(n)} e^{-A}) e^{2A} \right) e^{-A}}{2W_1^{(n)}}, & \lambda_3 &\equiv \frac{\partial_y (W_3^{(n)} e^A) e^{-A}}{2W_1^{(n)}}, \\ \lambda_2 &\equiv \frac{\partial_z \left(\partial_z (W_1^{(n)} e^{-A}) e^{2A} \right) e^{-A}}{2W_1^{(n)}}, & \lambda_4 &\equiv \frac{\partial_z (W_2^{(n)} e^A) e^{-A}}{2W_1^{(n)}}, \\ \lambda_6 &\equiv \frac{\partial_y \left[\partial_y (W_2^{(n)} e^{-A}) e^{2A} \right] e^{-A}}{W_2^{(n)}}, & \lambda_5 &\equiv \frac{\partial_z (W_1^{(n)} e^{-A}) e^A}{W_2^{(n)}}, \\ \lambda_7 &\equiv \frac{\partial_y \left[\partial_z (W_3^{(n)} e^{-A}) e^{2A} \right] e^{-A}}{W_2^{(n)}}, & \lambda_8 &\equiv \frac{\partial_y (W_1^{(n)} e^{-A}) e^A}{W_3^{(n)}}, \\ \lambda_9 &\equiv \frac{\partial_z \left[\partial_y (W_2^{(n)} e^{-A}) e^{2A} \right] e^{-A}}{W_3^{(n)}}, & \lambda_{10} &\equiv \frac{\partial_z \left[\partial_z (W_3^{(n)} e^{-A}) e^{2A} \right] e^{-A}}{W_3^{(n)}}. \end{aligned}$$

Then we compare eqs. (3.5)–(3.7) with (3.8)–(3.10).

- We first focus on the terms containing $I_2^{(nn')}$, $I_4^{(nn')}$, $I_7^{(nn')}$ and $I_9^{(nn')}$. Through the comparison, it is found

$$I_2^{(nn')} = -\delta_{nn'} \lambda_1, \quad (3.11)$$

$$I_4^{(nn')} = -\delta_{nn'} \lambda_2, \quad (3.12)$$

$$I_7^{(nn')} = -2\delta_{nn'} \lambda_6, \quad (3.13)$$

$$I_9^{(nn')} = -2\delta_{nn'} \lambda_{10}, \quad (3.14)$$

which leads to four Schrödinger-like equations:

$$[-\partial_y^2 + P_1] W_1^{(n)}(y, z) = m_1^{(n)2} W_1^{(n)}(y, z), \quad (3.15)$$

$$[-\partial_z^2 + P_2] W_1^{(n)}(y, z) = m_2^{(n)2} W_1^{(n)}(y, z), \quad (3.16)$$

$$[-\partial_y^2 + P_1] W_2^{(n)}(y, z) = m_\phi^{(n)2} W_2^{(n)}(y, z), \quad (3.17)$$

$$[-\partial_z^2 + P_2] W_3^{(n)}(y, z) = m_\phi^{(n)2} W_3^{(n)}(y, z), \quad (3.18)$$

with $P_1(y)$ and $P_2(z)$ the effective potentials

$$P_1(y) = \partial_y^2 A(y, z) + \partial_y A(y, z) \partial_y A(y, z), \quad (3.19)$$

$$P_2(z) = \partial_z^2 A(y, z) + \partial_z A(y, z) \partial_z A(y, z). \quad (3.20)$$

Given the solution of the background, the masses and wave functions of the vector KK modes can be solved through eqs. (3.15) and (3.16). But for the two types of scalar KK modes, the equations (3.17) and (3.18) only tell us the behaviors of their wave functions along one of the extra dimensions. Continuing to compare the two groups of EOMs, we will find more equations.

- Considering the terms about $I_6^{(nn')}$, $I_8^{(nn')}$ and $I_{10}^{(nn')}$, it is also easy to get some equations from the comparison. But for convenience we introduce three constants $C_1^{(n)}, C_2^{(n)}, C_3^{(n)}$ to simplify the calculation, and let

$$I_6^{(nn')} = C_1^{(n)} \delta_{n'n}, \quad (3.21)$$

$$I_8^{(nn')} = C_2^{(n)} \delta_{n'n}, \quad (3.22)$$

$$I_{10}^{(nn')} = C_3^{(n)} \delta_{n'n}. \quad (3.23)$$

So we can obtain

$$-2 C_1^{(n)} W_1^{(n)} = \partial_z (W_2^{(n)} e^A) e^{-A}, \quad (3.24)$$

$$\frac{1}{2} C_1^{(n)} W_2^{(n)} = \partial_z (W_1^{(n)} e^{-A}) e^A, \quad (3.25)$$

$$-2 C_2^{(n)} W_1^{(n)} = \partial_y (W_3^{(n)} e^A) e^{-A}, \quad (3.26)$$

$$\frac{1}{2} C_2^{(n)} W_3^{(n)} = \partial_y (W_1^{(n)} e^{-A}) e^A, \quad (3.27)$$

$$-\frac{1}{2} C_3^{(n)} W_2^{(n)} = \partial_y \left[\partial_z (W_3^{(n)} e^{-A}) e^{2A} \right] e^{-A}, \quad (3.28)$$

$$-\frac{1}{2} C_3^{(n)} W_3^{(n)} = \partial_z \left[\partial_y (W_2^{(n)} e^{-A}) e^{2A} \right] e^{-A}. \quad (3.29)$$

These equations show the relationships between the wave functions $W_1^{(n)}(y, z)$, $W_2^{(n)}(y, z)$ and $W_3^{(n)}(y, z)$. We can further simplify them.

- With the Schrödinger-like equation (3.16) and eq. (3.25), we have $C_1^{(n)} = m_2^{(n)2}$. Then with (3.24) we can find another Schrödinger-like equation for $W_2(y, z)^{(n)}$. It is similar to $C_2^{(n)}$ and $W_3^{(n)}(y, z)$. We list the result:

$$C_1^{(n)2} = m_2^{(n)2}, \quad (3.30)$$

$$C_2^{(n)2} = m_1^{(n)2}, \quad (3.31)$$

and

$$[-\partial_z^2 + P_3] W_2^{(n)}(y, z) = m_2^{(n)2} W_2^{(n)}(y, z), \quad (3.32)$$

$$[-\partial_y^2 + P_4] W_3^{(n)}(y, z) = m_1^{(n)2} W_3^{(n)}(y, z), \quad (3.33)$$

where P_3 and P_4 are the effective potentials:

$$P_3 = \partial_z A \partial_z A - \partial_z^2 A, \quad (3.34)$$

$$P_4 = \partial_y A \partial_y A - \partial_y^2 A. \quad (3.35)$$

Now there are two more Schrödinger-like equations for $W_2^{(n)}(y, z)$ and $W_3^{(n)}(y, z)$. Moreover, it is known that $m_2^{(n)}$ and $m_1^{(n)}$ are related to the masses of the vector KK modes, and the wave functions $W_2^{(n)}(y, z)$, $W_3^{(n)}(y, z)$ describe the scalar KK modes. But they appear in (3.32) and (3.33) at the same time, which implies that there may be some relationship between the vector KK modes and the scalar ones.

– Equations (3.28) and (3.29) show a relationship between $W_2^{(n)}(y, z)$ and $W_3^{(n)}(y, z)$. As eqs. (3.13) and (3.14) read

$$\begin{aligned} m_\phi^{(n)2} W_2^{(n)}(y, z) &= -\partial_y \left[\partial_y (W_2^{(n)}(y, z) e^{-A}) e^{2A} \right] e^{-A}, \\ m_\varphi^{(n)2} W_3^{(n)}(y, z) &= -\partial_z \left[\partial_z (W_3^{(n)}(y, z) e^{-A}) e^{2A} \right] e^{-A}, \end{aligned}$$

we can substitute $W_2^{(n)}(y, z), W_3^{(n)}(y, z)$ into eqs. (3.28) and (3.29), and get

$$\frac{C_3^{(n)}}{2m_\phi^{(n)2}} \partial_y (W_2^{(n)}(y, z) e^{-A}) = \partial_z (W_3^{(n)}(y, z) e^{-A}), \quad (3.36)$$

$$\frac{C_3^{(n)}}{2m_\varphi^{(n)2}} \partial_z (W_3^{(n)}(y, z) e^{-A}) = \partial_y (W_2^{(n)}(y, z) e^{-A}). \quad (3.37)$$

It is easily found that

$$C_3^{(n)2} = 4m_\phi^{(n)2} m_\varphi^{(n)2}. \quad (3.38)$$

Equations (3.36) and (3.37) show the relationship between $W_2^{(n)}(y, z)$ and $W_3^{(n)}(y, z)$ more explicitly.

We remember that $W_2^{(n)}(y, z)$ and $W_3^{(n)}(y, z)$ are both related to $W_1^{(n)}(y, z)$ according to (3.24)–(3.27). Thus eqs. (3.28) and (3.29) also read as

$$\frac{C_3^{(n)} e^{2A}}{2C_1^{(n)} C_2^{(n)}} \partial_z \left[\partial_y (W_3^{(n)}(y, z) e^A) e^{-2A} \right] = \partial_y \left[\partial_z (W_3^{(n)}(y, z) e^{-A}) e^{2A} \right], \quad (3.39)$$

$$\frac{C_3^{(n)} e^{2A}}{2C_1^{(n)} C_2^{(n)}} \partial_y \left[\partial_z (W_2^{(n)}(y, z) e^A) e^{-2A} \right] = \partial_z \left[\partial_y (W_2^{(n)}(y, z) e^{-A}) e^{2A} \right], \quad (3.40)$$

and eqs. (3.36) and (3.37) read as

$$\frac{C_3^{(n)}}{2m_\phi^{(n)2}} \partial_y \left(\frac{2}{C_1} \partial_z (W_1^{(n)} e^{-A}) \right) = \partial_z \left(\frac{2}{C_2} \partial_y (W_1^{(n)} e^{-A}) \right), \quad (3.41)$$

$$\left(\frac{2}{C_2} \partial_y (W_1^{(n)} e^{-A}) \right) = \partial_y \left(\frac{2}{C_1} \partial_z (W_1^{(n)} e^{-A}) \right). \quad (3.42)$$

Now, the interesting results can be obtained. From eqs. (3.39) and (3.40), we get

$$C_3^{(n)} = 2 C_1^{(n)} C_2^{(n)}. \quad (3.43)$$

Then from (3.41) and (3.42), we find

$$m_\phi^{(n)2} = m_1^{(n)2}, \quad (3.44)$$

$$m_\varphi^{(n)2} = m_2^{(n)2}. \quad (3.45)$$

These results imply that the masses of the scalar KK modes are related to those of the vector KK modes. If the vector KK modes are massive, the scalar ones must be also massive. This is different from that in a brane with one extra dimension, where the accompanying scalar KK modes are always massless [26].

3.2 The orthonormality condition

We have found the equations for the KK modes, but only the KK modes satisfying the orthonormality conditions (2.9) could be localized on the brane. The three orthonormality conditions (2.9a)–(2.9c) are not independent because of the relationships (3.25) and (3.27). With the orthonormality condition (2.9a), the other two orthonormality conditions can be naturally derived. For example,

$$\begin{aligned}
 I_3^{(nn)} &\equiv \frac{1}{2} \int dydz W_2^{(n)} W_2^{(n)} = \frac{1}{2} \int dydz \frac{4}{C_1^{(n)2}} \left(\partial_z (W_1^{(n)} e^{-A}) \right)^2 e^{2A}, \\
 &= \frac{1}{2} \int dydz \frac{4}{C_1^{(n)2}} W_1^{(n)}(y, z) [-\partial_z^2 + P_2] W_1^{(n)}(y, z), \\
 &= \frac{1}{2} \int dydz \frac{4}{C_1^{(n)2}} m_2^{(n)2} W_1^{(n)2}(y, z) = 2 \int dydz W_1^{(n)2}(y, z), \tag{3.46}
 \end{aligned}$$

which is nothing but $I_5^{(nn)}$. This means that if there are some localized vector KK modes, there will be some localized scalar ones.

With the relationship between $W_1^{(n)}, W_2^{(n)}$ and $W_3^{(n)}$ and the orthonormality conditions (2.9) one can check that if the orthonormality conditions (2.9) are satisfied, the constants $I_2^{(nn)}, I_4^{(nn)}, I_6^{(nn)}, I_7^{(nn)}-I_{10}^{(nn)}$ are finite, which are consistent with our assumption (2.10).

3.3 The gauge invariant effective brane action

With all above discussions, we now check the gauge invariance of the effective brane action. We have obtained that $I_6^{(nn)} = C_1^{(n)}, I_8^{(nn)} = C_2^{(n)}, I_{10}^{(nn)} = 2 C_1^{(n)} C_2^{(n)}$, thus the effective action can be rewritten as:

$$\begin{aligned}
 S_{\text{eff}} &= -\frac{1}{4} \sum_n \int d^4x \sqrt{-\hat{g}} \left[\hat{Y}_{\mu_1\mu_2}^{(n)} \hat{Y}_{(n)}^{\mu_1\mu_2} + \left(\frac{1}{2} m_1^2 + \frac{1}{2} m_2^2 \right) \hat{X}_{\mu_1}^{(n)} \hat{X}_{(n)}^{\mu_1} \right. \\
 &\quad + 2 \partial_{\mu_1} \phi^{(n)} \partial^{\mu_1} \phi^{(n)} - C_1^{(n)} (\partial_{\mu_1} \phi^{(n)} \hat{X}_{(n)}^{\mu_1} + \hat{X}_{\mu_1}^{(n)} \partial^{\mu_1} \phi^{(n)}) \\
 &\quad + 2 \partial_{\mu_1} \varphi^{(n)} \partial^{\mu_1} \varphi^{(n)} - C_2^{(n)} (\partial_{\mu_1} \varphi^{(n)} \hat{X}_{(n)}^{\mu_1} + \hat{X}_{\mu_1}^{(n)} \partial^{\mu_1} \varphi^{(n)}) \\
 &\quad \left. + 2 C_2^{(n)2} \phi^{(n)} \phi^{(n)} + 2 C_1^{(n)2} \varphi^{(n)} \varphi^{(n)} - 4 C_1^{(n)} C_2^{(n)} \phi^{(n)} \varphi^{(n)} \right], \\
 &= -\frac{1}{4} \sum_n \int d^4x \sqrt{-\hat{g}} \hat{Y}_{\mu_1\mu_2}^{(n)} \hat{Y}_{(n)}^{\mu_1\mu_2} \\
 &\quad - \frac{1}{2} \sum_n \int d^4x \sqrt{-\hat{g}} \left[\left(\partial_\mu \phi^{(n)} - \frac{1}{2} C_1^{(n)} \hat{X}_\mu^{(n)} \right)^2 + \left(\partial_\mu \varphi^{(n)} - \frac{1}{2} C_2^{(n)} \hat{X}_\mu^{(n)} \right)^2 \right] \\
 &\quad - \frac{1}{2} \sum_n \int d^4x \sqrt{-\hat{g}} (C_2^{(n)} \phi^{(n)} - C_1^{(n)} \varphi^{(n)})^2.
 \end{aligned}$$

We can see that the effective action is invariant under the gauge transformation

$$\hat{X}_\mu \rightarrow \hat{X}_\mu + \partial_\mu \rho, \tag{3.47}$$

$$\phi \rightarrow \phi + \frac{1}{2} C_1 \rho, \quad \varphi \rightarrow \varphi + \frac{1}{2} C_2 \rho, \tag{3.48}$$

where ρ is a scalar field.

4 Further discussions

There are some questions needed to be further discussed.

- About the ansatz (1.1)

In a model of brane with codimension two, extra dimensions can be assumed to be both compact [20], or one of them is compact and another is non-compact [6, 45, 47–49], or both non-compact [21], where the ansatz was supposed as

$$ds^2 = B(z)^2 [A(y)^2 (\eta_{\mu\nu} dx^\mu dx^\nu + dy^2) + dz^2]. \tag{4.1}$$

The interesting thing is that in either the compact case [20] or the non-compact case [21], the fermion generations can be obtained. In this paper, motivated by (4.1) we proposed a similar but more general non-compact metric (1.1). Although in this work, we only study the localization of a bulk U(1) gauge vector field, we plan to investigate the fermion generations like ref. [21] in our future work.

On the other hand, the reason we use this new ansatz (1.1) instead of others existing in the literature in codimension 2 is that if we use a metric like (4.1), and perform the procedure as before, instead of getting the relationship like (3.43), we will obtain some constraint equations like $\partial_z W_3 = 0, W_3 = 0$ and $\partial_z W_2 = 0, W_2 = 0$, which will make $W_1 = 0$. Therefore, in order to avoid the above unreasonable constraint on the wave functions of the KK modes we suppose a more general ansatz (1.1). However, we need further work to investigate how to build the brane with ansatz (1.1). For example, we may consider a complex scalar field and suppose some typical form of (1.1) as $A(y, z) = A_1(y)A_2(z)$ or $A(y, z) = A_1(y) + A_2(z)$.

- About the applications of our result

We briefly discuss an application of our result. Begin with the interaction between bulk fermions and gauge boson [50, 51] in our brane model

$$S_I = \int d^4x dy dz \sqrt{-g} (-e_6) \bar{\Psi}(x, y, z) \Gamma^M X_M(x, y, z) \Psi(x, y, z), \tag{4.2}$$

where e_6 is a 6D coupling constant, we can make a dimensional reduction for this action. Since there are three types of KK modes (i.e., one type of vector modes and two types of scalar ones) for the bulk U(1) gauge field and one type of fermion KK modes for the bulk fermion, we will have three kinds of interactions between the KK modes in the four-dimensional effective action derived from the fundamental one (4.2).

The coupling between the fermion zero mode and the vector zero mode will recover the usual four-dimensional Coulomb's law, and the couplings between the fermion zero mode and the massive vector KK modes will lead to correction to Coulomb's law [50, 51]. The Yukawa couplings between the fermion zero mode and the scalar modes will supply the masses of the four-dimensional fermions. The appearance of scalar modes differentiates the present work from the previous, where no scalar modes appear because gauge fixing is usually adopted before KK decomposition.

5 Conclusion

In this work, we discussed a U(1) gauge vector field on a brane with codimension two. It was found that there are three types of KK modes, one vector and two types of scalars. In many previous papers, the scalar KK modes always were ignored because of some gauge fixing for the bulk vector field. In this paper, we did not choose any gauge for the bulk field, and did a general KK decomposition. We found a gauge invariant effective action on the brane, where the scalar KK modes play an important role.

We first used the general KK decomposition and some orthonormality conditions to obtain the effective action on the brane, which just contains the vector KK modes and two types of scalar ones. Further, by comparing two groups of EOMs, we found the equations of the KK modes, from which the mass spectra and the wave functions can be calculated for a given background solution.

In this paper we only focused on some general discussion of the KK modes, which does not depend on the special solution of the background. Here, we give a simple summary:

- The KK modes including the vector and scalar ones satisfy a series of Schrödinger-like equations. But these equations are not independent, as there are some relationship between the wave functions of the KK modes;
- The masses of the vector KK modes are obtained from the two extra dimensions. The masses of the two types of scalars $m_\phi^{(n)}$ and $m_\varphi^{(n)}$ are related to the vector ones through $m_\phi^{(n)} = m_1^{(n)}$ and $m_\varphi^{(n)} = m_2^{(n)}$;
- The effective brane action is gauge invariant. The vector KK modes couple with the two types of scalar ones, and the two types of scalar KK modes also couple to each other. Because of these couplings the gauge invariance is guaranteed.

In fact in the brane world with codimension one, we have found that the effective brane action of the 1-form field is gauge invariant [26]. But the scalar KK modes, which couple with the vector ones, are all massless. Here the brane have one more extra dimension, the scalar KK modes also obtain masses. It is expected that in models with more extra dimensions there will be more types of massive scalar KK modes.

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