

a -theorem at large N_f

Oleg Antipin,¹ Nicola Andrea Dondi,² Francesco Sannino,² and Anders Eller Thomsen²
¹*Rudjer Boskovic Institute, Division of Theoretical Physics, Bijenička 54, HR-10000 Zagreb, Croatia*
²*CP³-Origins & Danish IAS, University of Southern Denmark, Campusvej 55, Odense M—DK-5230, Denmark*

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We determine the Jack and Osborn a -function and related metric for gauge-fermion theories to leading order in the large number of fermions and to all orders in the gauge coupling, demonstrating that the strong a -theorem is violated for the minimal choice of the a -function.

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Quantum field theory (QFT) is the language chosen by nature to describe its fundamental laws with the renormalization group (RG) flow connecting physics at different energy scales. Remarkably this flow is thought to be irreversible as encompassed by Cardy's proposal of the 4-dimensional a -theorem [1], originally inspired by the 2-dimensional proof of the Zamolodchikov c -theorem [2]. The main idea is that one can, in principle, define a monotonically decreasing function from the ultraviolet (UV) to the infrared (IR) along the flow.

Our goal is to push forward the state of the art by computing the a -function to all orders in the couplings, for nonsupersymmetric field theories, exploiting the large number of flavors limit [3].

In order to define the relevant quantities we start by considering a generic theory defined via the bare Lagrangian $\mathcal{L}_0(g_0^i, \phi_0)$ for a set of fundamental fields denoted by ϕ_0 . We restrict ourselves to the case in which the couplings, g_0^i , are associated to marginal operators. The theory is extended to curved background, changing the metric $\eta_{\mu\nu} \rightarrow \gamma_{\mu\nu}(x)$ so that it is classically invariant under $\text{diff} \times \text{Weyl}$, and to space-time-dependent couplings $g_0^i \rightarrow g_0^i(x)$.

Within the extended theory the (renormalized) couplings act as sources for associated composite operators, $\mathcal{O}_i = \frac{\delta S}{\delta g^i}$. If we appropriately renormalize the vacuum energy functional, $W = W[\gamma_{\mu\nu}, g^i]$, such that it is finite, we can extract renormalized composite operator correlators through functional derivatives of W . To achieve this, usual renormalization is insufficient: the spacetime dependence of the couplings induces new divergences that have to be canceled by new counter terms (CTs) proportional to coupling

derivatives. These of course vanish in the limit of constant sources.

We restrict ourselves to only listing the field-independent CTs. The new Lagrangian in $d = 4 - \epsilon$ reads [4]

$$\tilde{\mathcal{L}}_0(g_0^i, \phi_0) = \mathcal{L}_0(g_0^i, \phi_0) + \mu^{-\epsilon} \lambda(g) \cdot \mathcal{R}. \quad (1)$$

Of all 4-dimensional CTs built from metric and couplings that can appear in $\lambda \cdot \mathcal{R}$ we are mainly interested in the following ones:

$$\begin{aligned} \lambda \cdot \mathcal{R} \supset & \lambda_a E_4 + \frac{1}{2} \mathcal{G}_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} + \frac{1}{2} \mathcal{A}_{ij} \nabla^2 g^i \nabla^2 g^j \\ & + \frac{1}{2} \mathcal{B}_{ijk} \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k, \end{aligned} \quad (2)$$

where E_4 is the Euler density and $G_{\mu\nu}$ is the Einstein tensor. In the minimal subtraction (MS) scheme, the CT coefficients only contain poles in ϵ ; there are no finite parts. These terms are needed to renormalize specific contact divergences in composite operator correlators, so they can be written directly in terms of the renormalized couplings, g^i .

In this framework, RG transformations are deeply interconnected with Weyl rescalings:

$$\gamma_{\mu\nu} \rightarrow e^{-2\sigma} \gamma_{\mu\nu}, \quad \phi \rightarrow e^{\sigma \Delta_\phi} \phi. \quad (3)$$

We know that for a general QFT in curved space the Weyl symmetry, when present at the classical level, is anomalous. We take $\{g^i\}$ to couple to the set of all marginal operators defined at $g^i = 0$, so there exists a source transformation law such that the field-dependent part of the action is invariant under Weyl transformation [5,6]. The associated operator $\Delta_\sigma \equiv \int d^d x \sigma(x) [2\gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} - \hat{\beta}^i \frac{\delta}{\delta g^i}]$ acting on W transforms the field-independent CTs

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$$\begin{aligned}\Delta_\sigma W &= \Delta_\sigma \int d^d x \sqrt{\gamma} \mu^{-\epsilon} \lambda \cdot \mathcal{R} \\ &= \int d^d x \sqrt{\gamma} \mu^{-\epsilon} (\sigma \beta_\lambda \cdot \mathcal{R} + \partial_\mu \sigma \mathcal{L}^\mu).\end{aligned}\quad (4)$$

Here $\mu \frac{d}{d\mu} g^i = \hat{\beta}^i = -\rho^i g^i \epsilon + \beta^i(g)$.¹ The σ -dependent part of Eq. (4) has a tensor expansion analogous to (2):

$$\begin{aligned}\beta_\lambda \cdot \mathcal{R} \supset a E_4 + \frac{1}{2} \chi_{ij}^g \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} + \frac{1}{2} \chi_{ij}^a \nabla^2 g^i \nabla^2 g^j \\ + \frac{1}{2} \chi_{ijk}^b \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k.\end{aligned}\quad (5)$$

Since $[\mu \frac{\partial}{\partial \mu} + \int d^d x 2\gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}}] W = 0$ by dimensional analysis, for a global ($\sigma = \text{const}$) transformation, the integrand of (4) becomes

$$\left[\epsilon - \int d^d y \hat{\beta}^i(y) \frac{\delta}{\delta g^i(y)} \right] \lambda \cdot \mathcal{R} = \beta_\lambda \cdot \mathcal{R}.\quad (6)$$

From this we derive, e.g.,

$$\chi_{ij}^g = (\epsilon - \hat{\beta}^\ell \partial_\ell) \mathcal{G}_{ij} - \mathcal{G}_{\ell j} \partial_i \hat{\beta}^\ell - \mathcal{G}_{i\ell} \partial_j \hat{\beta}^\ell,\quad (7)$$

$$\chi_{ij}^a = (\epsilon - \hat{\beta}^\ell \partial_\ell) \mathcal{A}_{ij} - \mathcal{A}_{\ell j} \partial_i \hat{\beta}^\ell - \mathcal{A}_{i\ell} \partial_j \hat{\beta}^\ell,\quad (8)$$

which we will need later.

Since $\Delta_\sigma W$ has to be finite by construction, χ^g and χ^a must be finite too. This implies that (7) and (8) can be interpreted as RG equations for the CTs \mathcal{G}_{ij} and \mathcal{A}_{ij} . The same reasoning is independently valid for the term proportional to $\partial_\mu \sigma$ in (4). The constraints originating from the requirement of finiteness² were found in [4], and lead to nontrivial relations between CTs. First off

$$8\partial_i \tilde{a} = (\chi_{ij}^g + (\partial_i w_j - \partial_j w_i)) \beta^j, \quad 8\tilde{a} \equiv 8a + w_i \beta^i,\quad (9)$$

where w_i is a 1-form parametrizing a renormalization scheme redundancy, and \tilde{a} coincides with a at fixed points (FPs). It follows that \tilde{a} satisfies the gradient flow equation

$$8\mu \frac{d\tilde{a}}{d\mu} = 8\beta^i \partial_i \tilde{a} = \chi_{ij}^g \beta^i \beta^j,\quad (10)$$

which suggests viewing χ_{ij}^g as a metric in the space of couplings.

Under the assumption that the metric χ_{ij}^g is positive definite \tilde{a} decreases monotonically along the RG flow. The positivity of the metric χ_{ij}^g has been established at

¹ ρ^i is defined such that $g_0^i \mu^{-\rho^i \epsilon}$ is dimensionless, and its index does not count in the summation convention.

²These relations can be also recovered by imposing the Weyl variation to be Abelian: $[\Delta_{\sigma_1}, \Delta_{\sigma_2}] W = 0$.

sufficiently small couplings to the highest known order in perturbation theory, and it has been *conjectured* to hold nonperturbatively. Up to an arbitrariness in the definition of χ^g (to be discussed later), this constitutes the *strong version* of the *a*-theorem *conjecture*, see e.g., [7]. In contrast, the *weak version* of the *a*-theorem has been *proven* [8] and states that the quantity $\Delta \tilde{a} = (\tilde{a}_{\text{UV}} - \tilde{a}_{\text{IR}}) = \Delta a = (a_{\text{UV}} - a_{\text{IR}}) > 0$ for any RG flow between physical FPs, where $\tilde{a}_{\text{UV(IR)}}$ is evaluated at the corresponding FPs.

Another consistency relation, derived with the same procedure as (9), turns out to be particularly useful:

$$\chi_{ij}^g = -2\chi_{ij}^a + \bar{\chi}_{ijk}^a \beta^k - \beta^\ell \partial_\ell V_{ij} - \partial_i \beta^\ell V_{\ell j} - \partial_j \beta^\ell V_{i\ell},\quad (11)$$

where

$$\begin{aligned}V_{ij} &\equiv \rho^k g^k \bar{\mathcal{A}}_{ijk}^{(1)}, \quad \bar{\mathcal{A}}_{ijk} \equiv \partial_k \mathcal{A}_{ij} - \frac{1}{2} \mathcal{B}_{ikj} - \frac{1}{2} \mathcal{B}_{jki}, \\ \bar{\chi}_{ijk}^a &\equiv (\epsilon - \hat{\beta}^\ell \partial_\ell) \bar{\mathcal{A}}_{ijk} - \bar{\mathcal{A}}_{\ell jk} \partial_i \hat{\beta}^\ell - \bar{\mathcal{A}}_{i\ell k} \partial_j \hat{\beta}^\ell - \bar{\mathcal{A}}_{ij\ell} \partial_k \hat{\beta}^\ell,\end{aligned}\quad (12)$$

and $\mathcal{A}_{ijk}^{(1)}$ is the residue of the $1/\epsilon$ pole. Thus, χ^g can be computed from CTs needed to renormalize contact divergences of marginal operators.

In this article, we consider Yang-Mills theories with N_f vectorlike fermions in the $1/N_f$ expansion. We compute the metric χ_{ij}^g and the *a*-function \tilde{a} to leading order (LO) in $1/N_f$ but to all orders in the gauge coupling.

I. PRELIMINARIES TO DETERMINE THE METRICS

Using the finiteness of n -point functions in the renormalized theory we show how to determine the CTs \mathcal{A}_{ij} and \mathcal{B}_{ijk} in regular flat space and constant g^i . We start with the 2-point function

$$\frac{\delta}{\delta g^i(x)} \frac{\delta}{\delta g^j(y)} W = i \langle [\mathcal{O}_i(x)] [\mathcal{O}_j(y)] \rangle + \left\langle \frac{\delta [\mathcal{O}_j(y)]}{\delta g^i(x)} \right\rangle,\quad (13)$$

which must be finite. Working with the flat metric $\gamma_{\mu\nu} = \eta_{\mu\nu}$ and following [4], we define

$$[\mathcal{O}_i]^c \equiv [\mathcal{O}_i] |_{\partial_\mu g=0} = \partial_i h^a \mathcal{O}_a^0.\quad (14)$$

The last equality embodies that the standard constant-coupling operators $[\mathcal{O}_i]^c$ can always be expanded in terms of some functions $h^a(g^i)$ and coupling independent operators \mathcal{O}_a^0 . Only the CTs in $\tilde{\mathcal{L}}_0$ have dependence on $\partial_\mu g^i$, so when the limit of spacetime-independent couplings is taken, we have

$$\frac{\delta[\mathcal{O}_j(y)]}{\delta g^i(x)} = K_{ij}^k [\mathcal{O}_k]^c \delta^d(x-y) + \mu^{-\epsilon} \mathcal{A}_{ij} \partial^4 \delta^d(x-y), \quad (15)$$

where $K_{ij}^k \equiv \partial_i \partial_j h^a \frac{\partial g^k}{\partial h^a}$. Also in this limit, the Fourier-transformed 2-point function is defined by

$$\Gamma_{ij}(p) = i \int d^d x e^{-ip \cdot x} \langle [\mathcal{O}_i(x)]^c [\mathcal{O}_j(0)]^c \rangle. \quad (16)$$

The renormalized and finite 2-point function from (13) takes the form

$$\Gamma_{ij}^R(p) = \Gamma_{ij}(p) + \mu^{-\epsilon} \mathcal{A}_{ij} (p^2)^2 + K_{ij}^k \langle [\mathcal{O}_k]^c \rangle. \quad (17)$$

This relation can be used to extract the \mathcal{A}_{ij} CT from the momentum-dependent part of $\Gamma_{ij}(p)$ in flat space and constant couplings.

Continuing on to the 3-point function, it is given by

$$\begin{aligned} \frac{\delta}{\delta g^i(x)} \frac{\delta}{\delta g^j(y)} \frac{\delta}{\delta g^k(z)} W = & -\langle [\mathcal{O}_i(x)] [\mathcal{O}_j(y)] [\mathcal{O}_k(z)] \rangle \\ & + i \sum_{\text{cyc}} \left\langle \frac{\delta[\mathcal{O}_j(y)]}{\delta g^i(x)} [\mathcal{O}_k(z)] \right\rangle \\ & + i \left\langle \frac{\delta^2[\mathcal{O}_k(z)]}{\delta g^i(x) \delta g^j(y)} \right\rangle, \end{aligned} \quad (18)$$

the sum being over cyclic permutations of i, j, k and x, y, z . The Fourier-transformed 3-point function is defined by

$$\begin{aligned} \Gamma_{ijk}(p, q) = & - \int d^d x d^d y e^{-ip \cdot x} \\ & \times e^{-iq \cdot y} \langle [\mathcal{O}_i(x)]^c [\mathcal{O}_j(y)]^c [\mathcal{O}_k(0)]^c \rangle, \end{aligned} \quad (19)$$

to allow for computations in momentum space. The CTs of the renormalized 3-point function (18) can be determined from the CTs (2) and the relation (15). Taking for simplicity $\langle [\mathcal{O}_i]^c \rangle = 0$, the finite 3-point function with constant couplings and flat metric is

$$\begin{aligned} \Gamma_{ijk}^R(p, q) = & \Gamma_{ijk}(p, q) + K_{ij}^\ell \Gamma_{\ell k}(p+q) + K_{jk}^\ell \Gamma_{\ell i}(p) \\ & + K_{ki}^\ell \Gamma_{\ell j}(q) + \mu^{-\epsilon} (\bar{\mathcal{A}}_{ijk} p^2 q^2 + \bar{\mathcal{A}}_{jki} q^2 (p+q)^2 \\ & + \bar{\mathcal{A}}_{kij} (p+q)^2 p^2) + \frac{1}{2} \mu^{-\epsilon} (\mathcal{B}_{ijk} (p+q)^4 \\ & + \mathcal{B}_{jki} p^4 + \mathcal{B}_{kij} q^4). \end{aligned} \quad (20)$$

From here $\bar{\mathcal{A}}_{ijk}$ can be extracted by, say, considering the term proportional to $p^2 q^2$, which does not receive contributions from any other CTs.

This sets up our computation of the \mathcal{A}_{ij} and $\bar{\mathcal{A}}_{ijk}$ CTs which, through (8) and (11), will allow us to obtain the metric χ_{ij}^g .

II. LARGE N_f METRIC AND *a*-FUNCTION

Consider a theory with large number of fermions, N_f , charged under a simple (non-)Abelian gauge group;

$$\mathcal{L}_0 = -\frac{1}{4g_0^2} F_{0,\mu\nu}^a F_0^{a,\mu\nu} + \sum_{n=1}^{N_f} i \bar{\Psi}_{n,0} \gamma_\mu (\partial^\mu - i A_0^\mu) \Psi_{n,0}. \quad (21)$$

Since the theory has a single (gauge) coupling we can suppress all coupling indices i, j, k, \dots on the CTs. We now move to determine the leading $1/N_f$ contribution to the 2- and 3-point correlation functions to all orders in perturbation theory, keeping the coupling $K \equiv g^2 N_f T_R / (4\pi^2)$ fixed in order to prepare for the large N_f limit.

The leading 1PI correction to the gauge field 2-point function is given by the amputated fermion loop

$$\begin{aligned} i\Pi_{\mu\nu}(p) = & ip^2 \Delta_{\mu\nu}(p) \mu^{-\epsilon} \Pi_0(p^2), \\ \Pi_0(p^2) = & -\frac{N_f T_R}{2\pi^2} \frac{\Gamma^2(2 - \frac{\epsilon}{2}) \Gamma(\frac{\epsilon}{2})}{\Gamma(4 - \epsilon)} \left(-\frac{4\pi\mu^2}{p^2} \right)^{\epsilon/2}, \end{aligned} \quad (22)$$

with the transverse projector $\Delta_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu / p^2$. One may then extract the LO contribution to the gauge field renormalization or equivalently to the coupling renormalization, setting

$$K_0 = Z_A^{-1} K, \quad Z_A = 1 - \frac{2K}{3\epsilon} + \mathcal{O}(1/N_f). \quad (23)$$

The beta function associated to the coupling is given by to LO in $1/N_f$

$$\beta(K) = \frac{2}{3} K^2 + \mathcal{O}(1/N_f). \quad (24)$$

Following this renormalization convention, the operator associated with K is found to be

$$[\mathcal{O}_K]^c = \frac{N_f S_2(R)}{16\pi^2 K^2} \mu^{-\epsilon} F_0^2 + \mathcal{O}(1/N_f). \quad (25)$$

Only the F^2 -term contributes at LO to the 2- and 3-point functions.

At LO in the $1/N_f$ expansion, 2- and 3-point correlation functions are computed by dressing the gluon propagators with fermion bubble chains as shown in Fig. 1. For the 2-point function, we have

$$\begin{aligned} \Gamma_{KK}(p) = & \frac{id(G)}{2K^2 Z_A^2} \int \frac{d^d k}{(2\pi)^d} \frac{-i\Delta^{\mu\nu}(k)}{k^2 [1 - \Pi_0(k)]} V_{\nu\rho}(k, k+p) \\ & \times \frac{-i\Delta^{\rho\sigma}(k+p)}{(k+p)^2 [1 - \Pi_0(k+p)]} V_{\sigma\mu}(k+p, k), \end{aligned} \quad (26)$$

where $V_{\mu\nu}(p, q) = p \cdot q \eta_{\mu\nu} - q_\mu p_\nu$ is the momentum dependent Feynman rule stemming from a F^2 vertex and $d(G) \equiv N_c^2 - 1$ is the number of gauge bosons in the loop.

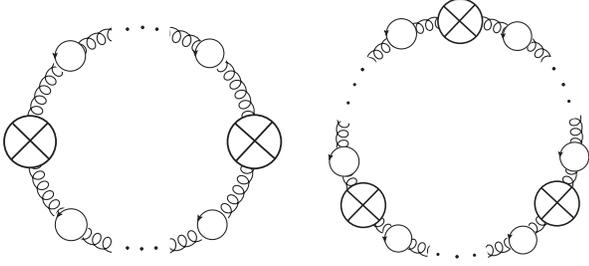


FIG. 1. LO 2- and 3-point functions. The crosses represent insertions of the $[\mathcal{O}_g]^c$ composite operator.

Here we summed over every number of bubble insertions, which for the purpose of finding the divergent part is equivalent to using dressed gauge propagators. Extracting the $(p^2)^2$ dependent part of $\Gamma_{KK}(p)$ and setting $p \rightarrow 0$, the integral may be evaluated using elementary methods. The \mathcal{A} CT is then determined from the finiteness of the renormalized 2-point function (17) to be

$$\mathcal{A} = -\frac{3d(G)}{64\pi^2 K^2} \text{div} \frac{H^a(\epsilon)}{Z_A^2 K_0}, \quad \text{where}$$

$$H^a(x) = \frac{(1 - \frac{x}{3})(240 - 240x + 90x^2 - 15x^3 + x^4)\Gamma(4-x)}{60(4-x)(6-x)\Gamma(1 + \frac{x}{2})\Gamma^3(2 - \frac{x}{2})}. \quad (27)$$

Here “div” is taken to mean the divergent part of the expression as $\epsilon \rightarrow 0$.

Similarly, we evaluate the divergent part of the $p^2 q^2$ term in the 3-point function $\Gamma_{KKK}(p, q)$ (20) which allows us to determine the $\bar{\mathcal{A}}$ CT in (12). We find

$$\bar{\mathcal{A}} = \frac{d(G)}{64\pi^2 K^3} \text{div} \frac{\bar{H}^a(\epsilon)}{Z_A^3 K_0}, \quad \text{where}$$

$$\bar{H}^a(x) = \frac{(80 - 60x + 13x^2 - x^3)x\Gamma(4-x)}{120(4-x)\Gamma(1 + \frac{x}{2})\Gamma^3(2 - \frac{x}{2})}. \quad (28)$$

The $1/\epsilon$ pole of \mathcal{A} and $\bar{\mathcal{A}}$ can be extracted using $Z_A K_0 = K$. Since both H^a and \bar{H}^a are regular they are expanded as power series and then resummed in the coupling K at the simple pole in a manner similar to [9]. Inserting these results back into Eqs. (8), (11), and (12) yields

$$\chi^a = -\frac{d(G)}{32\pi^2 K^2} \partial_K \left[KH^a \left(\frac{2}{3} K \right) \right]$$

$$= -\frac{d(G)}{32\pi^2 K^2} \left(1 - \frac{5}{3} K + \frac{49}{108} K^2 + \dots \right),$$

$$\chi^g = \frac{d(G)}{16\pi^2 K^2} \partial_K \left[KH^a \left(\frac{2}{3} K \right) - \frac{1}{9} K^2 \bar{H}^a \left(\frac{2}{3} K \right) \right]$$

$$= \frac{d(G)}{16\pi^2 K^2} \left(1 - \frac{5}{3} K + \frac{25}{108} K^2 + \dots \right). \quad (29)$$

Our result for χ^a agrees with [10] to all orders and to $\mathcal{O}(K^2)$ with [11]. Both metrics also agree with [4] to $\mathcal{O}(K)$. Notice

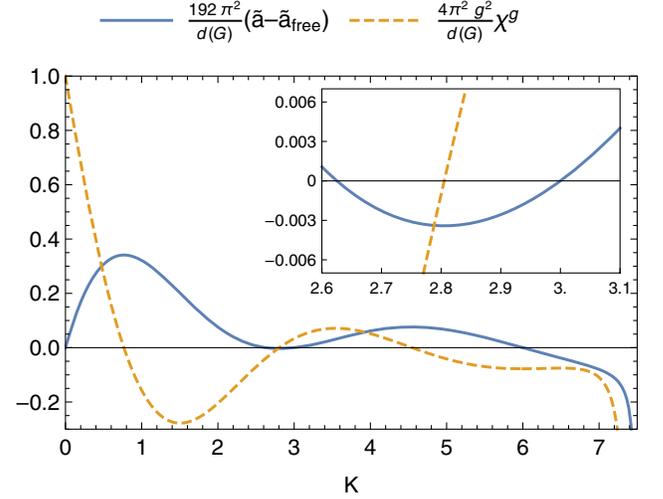


FIG. 2. The LO in $1/N_f$ metric and a -function.

that the LO result only distinguishes the Abelian and non-Abelian theory through an overall normalization because gauge self-interactions are subleading in $1/N_f$. The theories also share the LO in $1/N_f$ beta function, Eq. (24), with corresponding Landau pole. Using (9), we can now derive the LO a -function (the contribution from the one-form w_i vanishes)

$$\tilde{a} - \tilde{a}_{\text{free}} = \int dK \frac{K^2}{12} \chi^g$$

$$= \frac{d(G)}{192\pi^2} \left[KH^a \left(\frac{2}{3} K \right) - \frac{1}{9} K^2 \bar{H}^a \left(\frac{2}{3} K \right) \right], \quad (30)$$

where K is the large N_f coupling defined previously and \tilde{a}_{free} is the free field theory result:

$$\tilde{a}_{\text{free}} = \frac{1}{90(8\pi)^2} (11N_f + 62d(G)) \approx \frac{11N_f}{90(8\pi)^2}. \quad (31)$$

Finally, in Fig. 2 we plot the metric χ^g and the a -function $\tilde{a} - \tilde{a}_{\text{free}}$. We conclude that the metric is not positive definite for all values of K and thus, the a -function is not monotonic, violating the strong version of the a -theorem. For completeness, the function χ^a is plotted in Fig. 3.

One natural interpretation to restore the strong version of the a -theorem is that the flow towards the IR should start at K no higher than ≈ 0.8 in the UV. In fact, to this order in $1/N_f$, the underlying theory is UV incomplete and should be considered as an effective field theory that could be trusted up to a maximum value of the energy corresponding to $K \approx 0.8$. In other words monotonicity of the a -function gives us a sense of how far in the UV the theory can be pushed as an effective field theory. As the underlying theory is UV incomplete, we cannot impose the weak version of the a -theorem. We notice however that if we were to impose it, we can extend the validity of the theory

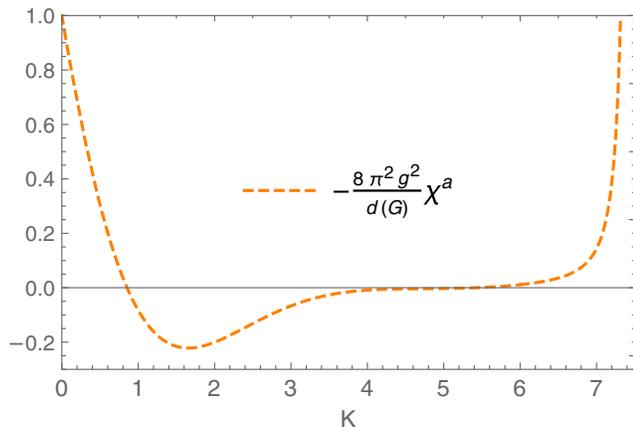


FIG. 3. The LO in $1/N_f$ function χ^a .

to $K \approx 2.6$ where the a -function becomes slightly negative. In the interval $3 < K < 6$ we find that a is positive albeit not monotonic. In between $K > 6$ and $K = 15/2$ we find that a is negative with the LO metric and the a -function having a pole at $K = 15/2$.

Different versions of the a -function can be obtained by redefining a with an arbitrary function $f(K)$ parametrizing the RG scheme change [12]: $\tilde{a}' = \tilde{a} + f(K)\beta^2$ and simultaneously modifying the metric to $\chi^{g'} = \chi^g + 8[\beta\partial_K f(K) + 2f(K)\partial_K\beta]$ so that (10) is invariant. To LO in $1/N_f$, we have

$$\begin{aligned} \tilde{a}' &= \tilde{a} + \frac{4K^4 f(K)}{9} \\ \chi^{g'} &= \chi^g + 16K \left[\frac{K\partial f(K)}{3\partial K} + \frac{4}{3}f(K) \right]. \end{aligned} \quad (32)$$

Other proposals have been used in e.g., [8,13,14]. It is not known if other versions of the a -function are monotonic outside perturbation theory. Nevertheless, we have shown that the Jack and Osborn version [4] is not monotonic.

III. SUBLEADING CORRECTIONS AND OUTLOOK

At LO in $1/N_f$, no UV fixed point can emerge and the theory is therefore at best viewed as an effective field theory. However, the situation becomes intriguing upon considering the $1/N_f$ corrections. In particular, it has been argued in favor of the existence of an interacting UVFP for gauge-fermion theories due to the interplay between the

leading and the subleading terms in $1/N_f$. Although the result is not as well established as the discovery of asymptotic safety in four dimensions in the Veneziano-Witten limit [15], it has nevertheless led to a number of phenomenological [16–18] and theoretical investigations [19,20] culminating in the conformal window 2.0 [21]. According to the studies above, the UVFP for the fundamental representation occurs at

$$\begin{aligned} K_{\text{QCD}}^* &= 4N_f T_R \alpha^* = 3 - \exp\left[-p \frac{N_f}{N_c} + k\right], \\ K_{\text{QED}}^* &= 4N_f \alpha^* = \frac{15}{2} - 0.0117e^{-15\pi^2 N_f/7}, \end{aligned} \quad (33)$$

where $p = 16T_R$ and $k = 15.86 + 2.63/N_c^2$. The UVFP is expected to appear above some critical number of flavors N_f^{crit} above which the large N_f expansion is reliable. We notice that, for both Abelian and non-Abelian theories, the UVFP occurs for $K > 0.8$ and thus the strong version of the a -theorem is necessarily violated. Intriguingly, the non-Abelian fixed point occurs very close to the apparent loss of validity of the weak a -theorem. Of course, very near to the UVFP one must include the missing $1/N_f$ corrections to the a -function. For the Abelian case the alleged UVFP occurs at a pole of the a -function.

If an UVFP exists for the non-Abelian case, only the weak version of the a -theorem survives, the reason being that the $1/N_f$ corrections cannot change the nonmonotonic character of the a -function away from the UVFP.

We elucidated the dynamics of large N_f gauge-fermion theories by determining important properties such as the metric and a -function, for the first time, to all orders in perturbation theory. Our results can be tested via first principle lattice simulations, and can be further extended to multiple-couplings theories at large N_f [22–24].

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