Spectrum of Majorana Quantum Mechanics with $O(4)^3$ Symmetry

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(Received 13 September 2018; published 10 January 2019)

We study the quantum mechanics of three-index Majorana fermions ψ^{abc} governed by a quartic Hamiltonian with $O(N)^3$ symmetry. Similarly to the Sachdev-Ye-Kitaev model, this tensor model has a solvable large-N limit dominated by the melonic diagrams. For $N = 4$ the total number of states is 2^{32}, but they naturally break up into distinct sectors according to the charges under the $U(1) \times U(1)$ Cartan subgroup of one of the O(4) groups. The biggest sector has vanishing charges and contains over 165 million states. Using a Lanczos algorithm, we determine the spectrum of the low-lying states in this and other sectors. We find that the absolute ground state is nondegenerate. If the $SO(4)^3$ symmetry is gauged, it is known from earlier work that the model has 36 states and a residual discrete symmetry. We study the discrete symmetry group in detail; it gives rise to degeneracies of some of the gauge singlet energies. We find all the gauge singlet energies numerically and use the results to propose exact analytic expressions for them.

DOI: 10.1103/PhysRevLett.122.011601

Introduction.—In recent literature there has been considerable interest in the quantum mechanical models where the degrees of freedom are fermionic tensors of rank 3 or higher [1,2]. Similarly to the Sachdev-Ye-Kitaev (SYK) model [3–5], these models have solvable large-N limit dominated by the so-called melonic diagrams [6–8]. In this limit they become solvable with the use of Schwinger-Dyson equations as were derived earlier for the SYK-like models [2,4,5,9–12]. While this spectrum of eigenstates is discrete and bounded for finite N, the low-lying states become dense for a large N leading to the (nearly) conformal behavior where it makes sense to calculate the operator scaling dimensions. In the SYK model, the number of states is $2^{N_{\rm SYK}/2}$, and numerical calculations of spectra have been carried out for rather large values of $N_{\rm SYK}$ [13–15]. They reveal a smooth distribution of energy eigenvalues, which typically has no degeneracies and is almost symmetric under $E \to -E$.

The corresponding studies of spectra in the tensor models of [1] and [2] have been carried out in [16–23], but in these cases the numerical limitations have been more severe—the number of states grows as $2^{N^3/2}$ in the $O(N)^3$ symmetric model of [2] and as 2^{2N^3} in the $O(N)^6$ symmetric Gurau-Witten (GW) model [1]. The results have shown an interesting structure. For example, for the $N = 2$ GW model the exact values of the 140 $SO(2)^6$ invariant energies were found [22]. Due to the discrete symmetries, there are only five distinct $E < 0$ eigenvalues and each one squares to an integer (the singlet spectrum also contains 50 zero-energy states).

The $O(N)^3$ model [2], has the Hamiltonian

$$H = \psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{1}{4}N^4, \tag{1.1}$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}, \quad a,b,c = 0,1,\ldots,N-1. \tag{1.2}$$

where, compared to [2,23], we have set the overall dimensionful normalization constant g to 4 in order to simplify the equations. For $N = 2$ there are only two gauge singlet states with $E = \pm 8$. For $N = 3$, as for any odd N, there are none [23]. While the complete spectra of Eq. (1.1) can be calculated for $N = 2$ and 3 using a laptop, this is no longer true for $N = 4$, where the total number of states is 2^{32}. However, they split into smaller sectors according to the charges (Q_0, Q_1) of the $U(1) \times U(1)$ Cartan subgroup of one of the SO(4) groups. The most complicated and interesting is the (0,0) sector; it is the part of the 32 qubit spectrum at the "half-half filling", i.e., where the first 16 qubits contain eight 0s and eight 1s, and the same applies to the remaining 16 qubits. In particular, all the $SO(4)^3$ invariant states are in this subsector; their number, 36, was found using the gauged version of the free fermion theory [23]. Since there are over 165 million states at half-half filling, the spectrum cannot be determined completely. However, using a Lanczos algorithm, we will be able to determine a number of low-lying eigenstates. We will also be able to find the complete spectrum of the 36 gauge singlet states, including their transformation properties under the residual discrete symmetries of the model where the $SO(4)^3$ symmetry is gauged. Thus, our work reveals the spectrum of a finite-N system without disorder, which is nearly conformal and solvable in the large-N limit, and identifies the discrete symmetries crucial for efficient numerical studies of such finite systems.

Using our numerical results we are able to infer the exact expressions for all the singlet eigenvalues. In particular, the ground state energy, which is numerically $E_0 \approx -160.140170$, agrees well with $E_0 = -\sqrt{32(447 + \sqrt{125601})}$ (for some results on the ground states in the SYK and related models, see Refs. [13,14,24]). Other gauge singlet energies either have similar expressions or are simply square roots of integers. This suggests that the Hamiltonian can be diagonalized exactly analytically.

Discrete symmetries acting on the gauge singlets.—For any even N, if we gauge the $SO(N)^3$ symmetry, there remain some gauge singlet states [23], which are annihilated by the symmetry charges

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}],$$
$$Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]. \tag{2.1}$$

These states may still have degeneracies due to the residual discrete symmetries. Indeed, each $O(N)$ group contains a Z_2 parity symmetry, which is an axis reflection. For example, inside $O(N)_1$ there is parity symmetry P_1, which sends $\psi^{0bc} \to -\psi^{0bc}$ for all b, c and leaves all other components invariant. The corresponding generator is

$$P_1 = P_1^\dagger = 2^{N^2/2} \prod_{bc} \psi^{0bc}. \tag{2.2}$$

One can indeed check that

$$P_1 \psi^{abc} P_1^\dagger = (-1)^{\delta_{a,0} + N^2} \psi^{abc}. \tag{2.3}$$

Similarly, there are Z_2 generators P_2 and P_3 inside $O(N)_2$ and $O(N)_3$.

It is also useful to introduce unitary operators P_{ij} associated with permutations of the $O(N)_i$ and $O(N)_j$ groups:

$$P_{23} = P_{23}^\dagger = i^{n(n-1)/2} \prod_a \prod_{b>c} (\psi^{abc} - \psi^{acb}),$$
$$P_{12} = P_{12}^\dagger = i^{n(n-1)/2} \prod_c \prod_{a>b} (\psi^{abc} - \psi^{bac}), \tag{2.4}$$

where $n = N^2(N-1)/2$ is the number of fields in the product. They satisfy

$$P_{23} \psi^{abc} P_{23}^\dagger = (-1)^{N^2(N-1)/2} \psi^{acb},$$
$$P_{12} \psi^{abc} P_{12}^\dagger = (-1)^{N^2(N-1)/2} \psi^{bac}. \tag{2.5}$$

These permutations flip the sign of H [23,25]:

$$P_{23} H P_{23}^\dagger = -H, \qquad P_{12} H P_{12}^\dagger = -H. \tag{2.6}$$

This explains why the spectrum is symmetric under $E \to -E$.

We now define the cyclic permutation operator $P = P_{12} P_{23}$ such that

$$P \psi^{abc} P^\dagger = \psi^{cab}, \quad PHP^\dagger = H, \quad P^3 = I. \tag{2.7}$$

Thus, P is the generator of the Z_3 symmetry of the Hamiltonian. Applying the Z_3 symmetry to the parity reflections P_i, we see that

$$P P_1 P^\dagger = P_2, \quad P P_2 P^\dagger = P_3, \quad P P_3 P^\dagger = P_1. \tag{2.8}$$

Forming all the possible products of I, P, P_1, P_2, P_3, we find that the full discrete symmetry group contains 24 elements. Using the explicit representation [Eq. (2.2)] for P_1, and the analogous ones for P_2 and P_3, we note that the three parity operators commute with each other. Furthermore,

$$[\Pi, P] = 0, \qquad \Pi = P_1 P_2 P_3, \qquad \Pi^2 = I. \tag{2.9}$$

Therefore, Π commutes with all the group elements, so that the group has a Z_2 factor with elements I and Π. The symmetry group turns out to be $A_4 \times Z_2$, and the 12 elements of the alternating group A_4 are

$$I, P_1, P_2, P_1P_2, P, P^2, P_1P, P_2P,$$
$$P_1P_2P, P_1P^2, P_2P^2, P_1P_2P^2. \tag{2.10}$$

Each of them can be associated with a sign preserving permutation of four ordered elements, and the action is

$$P_1(a_0, a_1, a_2, a_3) = (a_1, a_0, a_3, a_2),$$
$$P_2(a_0, a_1, a_2, a_3) = (a_2, a_3, a_0, a_1),$$
$$P_3(a_0, a_1, a_2, a_3) = (a_3, a_2, a_1, a_0),$$
$$P(a_0, a_1, a_2, a_3) = (a_0, a_3, a_1, a_2). \tag{2.11}$$

The degenerate $SO(N)^3$ invariant states of a given nonzero energy form irreducible representations of $A_4 \times Z_2$. For even N we can choose a basis where all the wave functions and matrix elements of the Hamiltonian are real. In this case we should study the representation of the symmetry group over the field $\mathbb{R}$. The degrees of the irreducible representations of A_4 over that field are 1,2,3. The Z_2 factor does not change the degrees since both irreducible representations of Z_2, the trivial one and the sign one, have a degree of 1.

Let us discuss the representations of A_4 in more detail. Using a reference eigenstate $|\psi_0\rangle$ not invariant under the Z_3 subgroup I, P, P^2, we can form a triplet of states

$$|\psi_0\rangle, \qquad P|\psi_0\rangle, \qquad P^2|\psi_0\rangle. \tag{2.12}$$

If the parities (P_1, P_2, P_3) of the state $|\psi_0\rangle$ are the same, then we can form a linear combination that transforms trivially under the Z_3,

$$|\psi\rangle = \frac{1}{\sqrt{3}}(1 + P + P^2)|\psi_0\rangle, \qquad P|\psi\rangle = |\psi\rangle, \tag{2.13}$$

while the remaining two linear combinations form the degree 2 representation of Z_3,

$$P|\psi_1\rangle = |\psi_2\rangle, \qquad P|\psi_2\rangle = -|\psi_1\rangle - |\psi_2\rangle, \tag{2.14}$$

where $|\psi_1\rangle = (1/\sqrt{3})|\psi_0\rangle - |\psi\rangle$. Because of this, some eigenstates have degeneracy of 2.

If the parities (P_1, P_2, P_3) of the state $|\psi_0\rangle$ are not equal, then the triplet representation [Eq. (2.12)] of the full discrete group is irreducible. For example for $(P_1, P_2, P_3) = (+, +, -)$, i.e.,

$$P_1|\psi_0\rangle = |\psi_0\rangle, \quad P_2|\psi_0\rangle = |\psi_0\rangle, \quad P_3|\psi_0\rangle = -|\psi_0\rangle, \tag{2.15}$$

we find that the parities of the states $P|\psi_0\rangle$ and $P^2|\psi_0\rangle$ are given by the cyclic permutations of $(+, +, -)$. Indeed, using Eq. (2.8), we find that the parities of the state $P|\psi_0\rangle$ are

$$P_1 P|\psi_0\rangle = -P|\psi_0\rangle, \qquad P_2 P|\psi_0\rangle = P|\psi_0\rangle,$$
$$P_3 P|\psi_0\rangle = P|\psi_0\rangle. \tag{2.16}$$

Thus, each of the states in the triplet [Eq. (2.12)] has a distinct set of parities. Then it is impossible to form linear combinations, which are eigenstates of the parities, and we have an irreducible representation of A_4 of degree 3. In this situation we find that an energy eigenvalue has degeneracy of 3.

We also note the relations

$$P_{23} P_1 P_{23}^\dagger = (-1)^{N(N^2-1)/2} P_1,$$
$$P_{12} P_1 P_{12}^\dagger = (-1)^{N(N^2-1)/2} P_2,$$
$$P_{13} P_1 P_{12}^\dagger = (-1)^{N(N^2-1)/2} P_3 \tag{2.17}$$

and their cyclic permutations. Since an operator P_{ij} maps an eigenstate of energy E into an eigenstate of energy-E, we see that such mirror states have the same parities when $N/2$ is even, but opposite parities when $N/2$ is odd.

For the states at zero energy, the discrete symmetry group is enhanced to 48 elements because the permutation generators P_{ij} map them into themselves. Using the relations [Eq. (2.17)] we find

$$P_{12} \Pi P_{12}^\dagger = (-1)^{N(N^2-1)/2} \Pi, \tag{2.18}$$

which implies that $\Pi = P_1 P_2 P_3$ commutes or anticommutes with other elements depending on the value of N. Focusing on the case where $N(N^2-1)/2$ is even and the sign above is positive (this includes $N = 4$, which is our main interest in this Letter), we find that Π commutes with all other generators, so that the group has a Z_2 factor with elements I and Π. The symmetry group for $E = 0$ turns out to be $S_4 \times Z_2$, which is the full cube group. Its subgroup S_4 is formed out of the products of $I, P_1, P_2, P_{12}, P_{23}, P_{13}$. The parity generators are realized in the same way as in Eq. (2.11), while the permutations act by the natural embedding $S_3 \subset S_4$:

$$P_{12}(a_0, a_1, a_2, a_3) = (a_0, a_2, a_1, a_3),$$
$$P_{23}(a_0, a_1, a_2, a_3) = (a_0, a_1, a_3, a_2),$$
$$P_{13}(a_0, a_1, a_2, a_3) = (a_0, a_3, a_2, a_1). \tag{2.19}$$

The degrees of the irreducible representations of S_4 are 1,1,2,3,3.

Diagonalization of the Hamiltonian.—The Majorana fermions ψ^{abc} may be thought of as generators of the Clifford algebra in N^3-dimensional Euclidean space [26]. Restricting to the cases where N is even, the dimension of the Hilbert space is $2^{N^3/2}$, and the states may be represented by series of $N^3/2$ "qubits" $|s\rangle$, where $s = 0$ or 1. It is convenient to introduce operators [20,23]

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}[\psi^{ab(2k)} + i\psi^{ab(2k+1)}],$$
$$c_{abk} = \frac{1}{\sqrt{2}}[\psi^{ab(2k)} - i\psi^{ab(2k+1)}],$$
$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0,$$
$$\{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'}, \tag{3.1}$$

where $a, b = 0, 1, \ldots, N-1$, and $k = 0, \ldots, \frac{1}{2}N - 1$. In this basis, the $\mathrm{O}(N)^2 \times \mathrm{U}(N/2)$ symmetry is manifest, and the Hamiltonian is [20,23]

$$H = 2(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}). \tag{3.2}$$

If we number the qubits from 0 to $\frac{1}{2}N^3 - 1$, then operators $c_{abk}, \bar{c}_{abk}$ correspond to qubit number $N^2k + Nb + a$.

In the basis [Eq. (3.1)], the parity operators P_i corresponding to ith group $\mathrm{O}(N)$ are

$$P_1 = \prod_{b=0}^{N-1}\prod_{k=0}^{N/2-1}[\bar{c}_{0bk}, c_{0bk}], \qquad P_2 = \prod_{a=0}^{N-1}\prod_{k=0}^{N/2-1}[\bar{c}_{a0k}, c_{a0k}],$$
$$P_3 = \prod_{a=0}^{N-1}\prod_{b=0}^{N-1}(\bar{c}_{ab0} + c_{ab0}). \tag{3.3}$$

The operator P_3 implements charge conjugation on the $k=0$ operators; i.e., it acts to interchange $\bar{c}_{ab0}$ and c_{ab0}. This conjugation is a symmetry of H. In fact, for each k the Hamiltonian is symmetric under the interchange of $\bar{c}_{abk}$ and c_{abk}.

The $U(1)^{N/2}$ subgroup of the $U(N/2)$ symmetry is realized simply. The corresponding charges,

$$Q_k = \sum_{a,b} \frac{1}{2}[\bar{c}_{abk}, c_{abk}], \qquad k = 0, \ldots, \frac{1}{2}N - 1, \quad (3.4)$$

are the Dynkin labels of a state of the third SO(N) group, and the spectrum separates into sectors according to their values. The oscillator vacuum state satisfies

$$c_{abk}|\mathrm{vac}\rangle = 0, \qquad Q_k|\mathrm{vac}\rangle = -\frac{N^2}{2}|\mathrm{vac}\rangle, \quad (3.5)$$

and other states are obtained by acting on it with some number of $\bar{c}_{abk}$.

For $N=4$ the total number of states is $2^{32} = 4\,294\,967\,296$, but they break up into $17^2 = 289$ smaller sectors due to the conservation of the $U(1) \times U(1)$ charges Q_0 and Q_1. The biggest sector is $(Q_0, Q_1) = (0,0)$; it consists of $[(16!)^2/(8!)^4] = 165\,636\,900$ states. The next biggest are the four sectors $(\pm 1, 0)$ and $(0, \pm 1)$; each of them contains 147 232 800 states. The smallest four sectors are $(\pm 8, \pm 8)$, and each one consists of just one state; each of these states has $E=0$. In general, the spectrum in the (q, q') sector is the same as in (q', q) due to the symmetry of H under interchange of the c_{ab0} and c_{ab1} oscillators.

Let us first study the (0,0) sector. These states are obtained by acting on $|vac\rangle$ with eight raising operators $\bar{c}_{ab0}$ and eight raising operators $\bar{c}_{ab1}$. In the qubit notation, both the first 16 qubits, and the second 16 qubits, have equal number, eight, of 0s and 1s. Clearly, all the $SO(4)^3$ invariant states are in this sector (there are additional constraints on the gauge singlet wave functions, but we will not discuss them explicitly here). While the numbers of such "half-half-filled" states is still very large, they turn out to be tractable numerically because the matrix we need to diagonalize is rather sparse. This has allowed us to study the low-lying eigenvalues of H, which occur in various representations of $SO(4)^3$. To find the gauge singlet energies, we study the operator proportional to $H + 100\sum_{i=1}^{3} C_2^i$, where the quadratic Casimir of the $SO(N)_1$ symmetry is $C_2^1 = \frac{1}{2} Q_1^{aa'} Q_1^{aa'}$, and analogously for $SO(N)_2$ and $SO(N)_3$. The Lanczos algorithm allows us to identify the lowest eigenvalues of this operator, which all correspond to $SO(4)^3$ invariant states; the nonsinglets receive large additive contributions due to the second term.

In Table I, we list the energies and parities of all 36 $SO(4)^3$ invariant states. In order to identify the values of P_i, we calculated the low-lying spectrum of operator

TABLE I. The list of all the $SO(4)^3$ invariant states, including their parities P_i.

E	P_1	P_2	P_3	E	P_1	P_2	P_3
−160.140 170	1	1	1	160.140 170	1	1	1
−97.019 491	1	1	−1	97.019 491	1	1	−1
−97.019 491	−1	1	1	97.019 491	−1	1	1
−97.019 491	1	−1	1	97.019 491	1	−1	1
−88.724 292	−1	−1	−1	88.724 292	−1	−1	−1
−54.434 603	1	1	1	54.434 603	1	1	1
−50.549 167	1	1	−1	50.549 167	1	1	−1
−50.549 167	−1	1	1	50.549 167	−1	1	1
−50.549 167	1	−1	1	50.549 167	1	−1	1
−39.191 836	1	1	1	39.191 836	1	1	1
−39.191 836	1	1	1	39.191 836	1	1	1
−38.366 652	1	−1	−1	38.366 652	1	−1	−1
−38.366 652	−1	1	−1	38.366 652	−1	1	−1
−38.366 652	−1	−1	1	38.366 652	−1	−1	1
0.000 000	1	1	1	0.000 000	−1	−1	−1
0.000 000	−1	1	1	0.000 000	1	−1	−1
0.000 000	1	−1	1	0.000 000	−1	1	−1
0.000 000	1	1	−1	0.000 000	−1	−1	1

$$H + 100\sum_{i=1}^{3} C_2^i + \sum_{i=1}^{3} a_i P_i, \quad (3.6)$$

where a_i are unequal small coefficients. (The states at ± 39.191836 are doubly degenerate and have identical parities; these states form the degree 2 representation of the Z_3 subgroup of A_4. To split such double degeneracies we added a small amount of noise to the Hamiltonian.) The biggest degeneracy is found for the $E=0$ states; it corresponds to the 2^3 independent choices of the three parities. Since the discrete group acting on the $E=0$ states is $S_4 \times Z_2$, which is the full cube group, we find two different irreducible representations of S_4: the trivial one of degree 1 and the standard one of degree 3. We may associate the eight $E=0$ states with the vertices of a cube. The energies of the gauge singlet states and their degeneracies are plotted in Fig. 1.

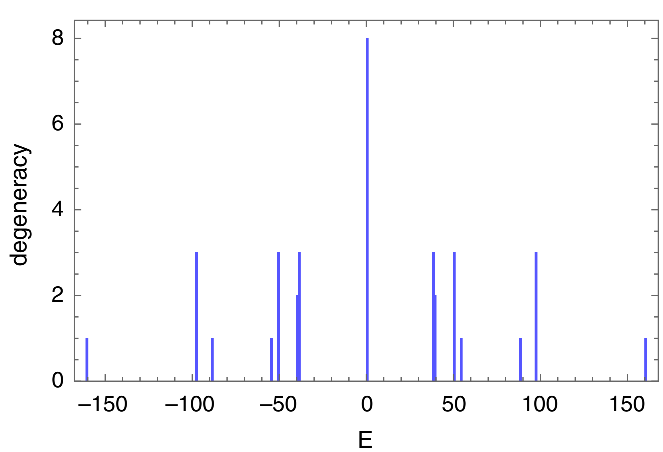

FIG. 1. Spectrum of gauge singlets in the $O(4)^3$ model.

TABLE II. The low-lying energies in the (0,0) sector, i.e., at half-half filling, including the values of the quadratic Casimir invariants of each $SO(N)$ group. When the C_2^i are not all equal, there are additional states of the same energy with their values obtained by a cyclic permutation.

C_2^1	C_2^2	C_2^3	E
0	0	0	−160.140 170
0	4	8	−136.559 039
0	0	12	−136.417 554
0	0	24	−128.490 197
4	4	4	−122.553 686
0	0	12	−121.606 040
4	8	8	−121.552 284
4	8	8	−120.699 077
4	8	8	−119.685 636
0	8	12	−119.659 802
0	12	8	−119.204 505
0	8	4	−118.699 780
0	4	16	−118.541 049
4	4	4	−116.774 758

Some of the energies agree within the available precision with square roots of integers: $8\sqrt{23} \approx 38.366\,652$, $8\sqrt{24} \approx 39.191\,836$, and $8\sqrt{123} \approx 88.724\,292$. Furthermore, the four eigenvalues with parities (1,1,1), $\pm 160.140\,170$ and $\pm 54.434\,603$, are approximations to the analytic expressions $\pm\sqrt{32(447 \pm \sqrt{125\,601})}$, while the triplet eigenvalues, $\pm 97.019\,491$ and $\pm 50.549\,167$, are approximations to $\pm\sqrt{32(187 \pm \sqrt{11\,481})}$. To demystify these exact results, we note that there are only two $SO(4)^3$ invariant states with $P_1 = P_2 = P_3 = -1$ (see Table I). Since the Hamiltonian has symmetry under $E \to -E$, the eigenvalue equation in this subsector must have the form of the second order even polynomial: $E^2 - A_1 = 0$. This explains why some of the eigenvalues are simply square roots. On the other hand, there are four $SO(4)^3$ invariant states with $P_1 = P_2 = P_3 = 1$. Thus, the eigenvalue equation in this subsector must have the form

$$E^4 + 2A_2E^2 + A_3 = 0, \tag{3.7}$$

and this explains why some of the energies satisfy $E^2 = -A_2 \pm \sqrt{A_2^2 - A_3}$. Similar symmetry considerations explain the form of all the gauge singlet energies in terms of the square roots. We leave exact derivation of the parameters A_i for future work.

The list of all the low-lying energy levels in the (0,0) sector, singlets and nonsinglets, and the corresponding values of quadratic Casimir invariants C_2^i, is shown in Table II. In order to identify the values of C_2^i, we have calculated the low-lying spectrum of $H + \sum_{i=1}^3 a_i C_2^i$ where a_i are unequal small coefficients. When the C_2^i are not all equal, there are also states of the same energy with their values obtained by a cyclic permutation. For example, at $E = -136.559\,039$ we find states with $(C_2^1, C_2^2, C_2^3) = (0,4,8), (4,8,0), (8,0,4)$.

TABLE III. The low-lying states in the sectors $(\pm 1, 0)$ and $(0, \pm 1)$, i.e., with one extra hole (h) or particle (p) added to half-half filling The energies are the same within the accuracy shown, which is a good test of our diagonalization procedure. When the C_2^i are not all equal, there are additional states of the same energy with their values obtained by a cyclic permutation.

C_2^1	C_2^2	C_2^3	$E_h = E_p$
3	3	3	−140.743 885
3	3	9	−128.059 272
3	3	15	−124.547 555
3	9	9	−118.371 087
3	3	9	−117.798 571
3	3	19	−115.861 910
3	9	9	−114.885 221
3	3	15	−114.660 576
3	3	9	−114.539 928

Absent from the list in Table II is the lowest possible value of the quadratic Casimir, $C_2 = 3$, which corresponds to the $(1/2, 0) + (0, 1/2)$ i.e., fundamental representation 4 of SO (4). Let us proceed to the sectors adjacent to one-particle and one-hole sectors, $(\pm 1, 0)$ and $(0, \pm 1)$, which contain some of the additional representations, including the (4,4,4) of $SO(4)^3$. The refined bound [23] for this representation gives $|E_{(4,4,4)}| < 72\sqrt{5} \approx 160.997$, while the actual lowest state in this representation has $E \approx -140.743885$. The low-lying states in the sectors $(\pm 1, 0)$ and $(0, \pm 1)$ are given in Table III. We have also calculated the energies in other charge sectors. We find that the absolute ground state lies in the (0,0) sector: as the magnitudes of charges increase, the energies tend to get closer to 0.

The simulations presented in this Letter were performed on computational resources managed and supported by Princeton's Institute for Computational Science & Engineering and OIT Research Computing. We are grateful to Yakov Kononov and Douglas Stanford for useful discussions. K. P. was supported by the Swiss National Science Foundation through the Early Postdoc.Mobility Grant No. P2EZP2_172168. The work of I. R. K. and F. P. was supported in part by the US NSF under Grant No. PHY-1620059. The work of G. T. was supported in part by the MURI Grant No. W911NF-14-1-0003 from ARO and by DOE Grant No. de-sc0007870.

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