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Three-parameter integrable deformation of $\mathbb{Z}_4$ permutation supercosets

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Abstract: A three-parameter integrable deformation of $\mathbb{Z}_4$ permutation supercosets is constructed. These supercosets are of the form $\widehat{F}/F_0$ where F_0 is the bosonic diagonal subgroup of the product supergroup $\widehat{F} = \widehat{G} \times \widehat{G}$. They include the $AdS_3 \times S^3$ and $AdS_3 \times S^3 \times S^3$ supercosets. This deformation encompasses both the bi-Yang-Baxter deformation of the semi-symmetric space σ-model on $\mathbb{Z}_4$ permutation supercosets and the mixed flux model. Truncating the action at the bosonic level, we show that one recovers the bi-Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term.

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Contents

1 Introduction

In this article we investigate integrable deformations of the semi-symmetric space σ-model on $\mathbb{Z}_4$ permutation supercosets. These are supercosets that take the form

$$\frac{\widehat{F}}{F_0} = \frac{\widehat{G}\times\widehat{G}}{F_0}, \tag{1.1}$$

where F_0 is the bosonic diagonal subgroup of the superisometry group $\widehat{G}\times\widehat{G}$. The bosonic truncation of the undeformed theory yields the symmetric space σ-model on the $\mathbb{Z}_2$ permutation coset

$$\frac{F_0\times F_0}{F_0}, \tag{1.2}$$

which is equivalent to the principal chiral model on F_0. In [1] a three-parameter integrable deformation of this bosonic model was constructed, generalising the SU(2) case discussed in [2]. This deformed model can be understood as the bi-Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term. Our aim in this article is to construct the analogous three-parameter integrable deformation of the semi-symmetric space σ-model on the $\mathbb{Z}_4$ permutation supercoset (1.1).

The Yang-Baxter deformation of the principal chiral model was first introduced in [3], generalising the one-parameter anisotropic SU(2) principal chiral model of [4] to general Lie groups. The construction of a Lax pair for this model in [5] and the proof of Hamiltonian integrability in [6] rely on the deformation being governed by a solution of the modified classical Yang-Baxter equation. The two-parameter bi-Yang-Baxter deformation of the principal chiral model was introduced in [5] and shown to be integrable in [7, 8]. As demonstrated in [9], the bi-Yang-Baxter deformation of the SU(2) principal chiral model is equivalent to the two-parameter deformation of S^3 found in [10]. In [6] the Yang-Baxter deformation of the symmetric space σ-model was constructed and shown to be integrable. For symmetric spaces that take the form of $\mathbb{Z}_2$ permutation cosets (1.2), the deformed model is equivalent to a particular one-parameter model contained within the bi-Yang-Baxter deformation of the principal chiral model on F_0.

Adding the standard topological Wess-Zumino term [11–13] to the principal chiral model is well-known to preserve its classical integrability. Up to the expected quantization of the level, the resulting model interpolates between the principal chiral and conformal Wess-Zumino-Witten models. Generalising the SU(2) construction of [14, 15], the Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term for general Lie groups was derived in [16]. In order to construct this deformation it was assumed that the solution of the modified classical Yang-Baxter deformation governing the deformation cubes to its negative. This assumption is natural since it is satisfied by the standard Drinfel'd-Jimbo $\mathcal{R}$-matrix for a non-split real form of a semi-simple Lie algebra [17–19]. In this article we will continue to focus on deformations governed by such $\mathcal{R}$-matrices.

By allowing for an antisymmetric term in the action, the two-parameter deformation of [10] was generalised to an integrable four-parameter deformation of S^3 in [2]. Observing that this model also contains the TsT transformation of the SU(2) Wess-Zumino-Witten model [20, 21], it was proposed in [9] that the four-parameter model should be understood as the combined bi-Yang-Baxter deformation and TsT transformation of the σ-model on S^3 plus Wess-Zumino term. The bi-Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term was constructed in [1]. Applying a TsT transformation in the directions of the Cartan subalgebra, which is associated with adding a compatible abelian solution of the classical Yang-Baxter equation [22–29] to the Drinfel'd-Jimbo $\mathcal{R}$-matrix, and considering the SU(2) case it was shown that this model indeed generalises the four-parameter deformation of [2] to general Lie groups.

The semi-symmetric space σ-model of [30, 31] is integrable [32–34] and describes the Green-Schwarz superstring [35–39], and consistent truncations thereof, on various supergravity backgrounds supported by Ramond-Ramond flux [40]. Included amongst these are the $AdS_3\times S^3\times T^4$ [41–44] and $AdS_3\times S^3\times S^3\times S^1$ [45] backgrounds, each preserving 16 supersymmetries, which, in the context of this article, are of particular interest as the corresponding σ-models are on $\mathbb{Z}_4$ permutation supercosets (1.1) with $\widehat{F} = P\mathrm{SU}(1,1|2)$ and $\widehat{F} = D(2,1;\alpha)$ respectively.

Furthermore, it is well-known that both these two backgrounds can be supported by either Ramond-Ramond (R-R) flux, Neveu-Schwarz-Neveu-Schwarz (NS-NS) flux or a combination of the two. Classically, there is a one-parameter family of backgrounds interpolating between the pure R-R and pure NS-NS cases, whose bosonic truncation is the principal chiral model plus Wess-Zumino term. The corresponding one-parameter deformation of the semi-symmetric space σ-model, the mixed flux model, together with the Lax pair demonstrating its integrability, was derived in [46].

In [47, 48] the Yang-Baxter deformation of the semi-symmetric space σ-model for general $\mathbb{Z}_4$ supercosets was constructed. In the case the supercoset takes the form of a $\mathbb{Z}_4$ permutation supercoset (1.1) it is natural to expect that the model admits a two-parameter bi-Yang-Baxter deformation. This was indeed shown to be the case in [49]. In this article we generalise these constructions to find the bi-Yang-Baxter deformation of the mixed flux model of [46] giving a three-parameter integrable deformation of the semi-symmetric space σ-model on the $\mathbb{Z}_4$ permutation supercosets (1.1).

The plan of this article is as follows. In section 2 we present the action of the three-parameter deformation in two forms. The first involves additional gauge and auxiliary fields, the inclusion of which clarifies the underlying structure of the model. The additional fields are therefore particularly useful for constructing the Lax pair and demonstrating the classical integrability of the model, which is discussed in section 3. The second form is the action that follows from integrating out the additional fields, the explicit computation of which we give in section 4. In section 5 we demonstrate agreement with various known truncations and limits. First, in subsection 5.1, we consider the bosonic truncation and relate it to the model of [1]. We then study the limits that give the mixed flux model of [46] and the two-parameter bi-Yang-Baxter deformation of [49] in subsections 5.2 and 5.3 respectively. We conclude with some possible future directions and applications in section 6. There are three appendices. In appendix A we give the explicit form of coefficients of the linear maps used in section 3 to construct the Lax pair. In appendix B we start from the action of [1] and rewrite it in a form suitable for comparing with the bosonic truncation obtained in subsection 5.1. Finally, in appendix C, we rewrite the metric and B-field of the three-parameter deformation of S^3, first given in [2], in a particularly simple form.

2 Action of the three-parameter deformation

In this section, we give the definition of the three-parameter integrable deformation of $\mathbb{Z}_4$ permutation supercosets. As stated in the introduction and proved in subsection 5.1, the bosonic truncation of the integrable σ-model presented here is in agreement with the three-parameter integrable deformation of $\mathbb{Z}_2$ permutation cosets built in [1]. This model generalised the Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term constructed in [16]. To construct an integrable deformation one needs to demonstrate the existence of a Lax pair. On a technical level, this requires the inversion of some operator involving the $\mathcal{R}$-matrix, a skew-symmetric solution of the modified classical Yang-Baxter equation (mCYBE) used to define the Yang-Baxter deformation. As shown in [16], for the Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term this inversion is tractable when the $\mathcal{R}$-matrix takes the standard form.

To combine the bi-Yang-Baxter deformation and the WZ term the strategy developed in [1] is based on formulating the principal chiral model as a $\mathbb{Z}_2$ permutation coset (1.2). As shown in [1] introducing a gauge field makes the inversion of the relevant operators, which is necessary to prove the existence of a Lax pair, tractable. To go further and construct the action in the supercoset case we also introduce auxiliary fields. In this section we therefore give two formulations of the model. The first corresponds to the action including the gauge and auxiliary fields. It is this formulation that enables us to construct a Lax pair. The second is obtained after eliminating the gauge and auxiliary fields.

Algebraic setting and related definitions. We consider supercosets of the type $\widehat{F}/F_0$ where F_0 is the bosonic diagonal subgroup of the product supergroup $\widehat{F} = \widehat{G} \times \widehat{G}$. The superalgebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{f}} = \hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ correspond to the supergroups $\widehat{G}$ and $\widehat{F}$ respectively. We shall make use of the standard block-diagonal matrix realisation of the product supergroup $\widehat{F}$. At the level of the superalgebra, for any element $\mathcal{X} = \mathrm{diag}(X^L, X^R) \in \hat{\mathfrak{f}}$ with $X^L, X^R \in \hat{\mathfrak{g}}$, we define the supertrace of $\mathcal{X}$ as $\mathrm{STr}(\mathcal{X}) = \mathrm{STr}(X^L) + \mathrm{STr}(X^R)$. The superalgebra $\hat{\mathfrak{f}}$ has a $\mathbb{Z}_4$ decomposition:

$$\hat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3,$$

with $\mathfrak{f}_0$ the Lie algebra associated with the Lie group F_0. Let us denote P_B and P_F the projections onto the even and odd parts of $\hat{\mathfrak{g}}$. The $\mathbb{Z}_4$ decomposition of $\mathcal{X}$ is defined as [45]

$$P_0\mathcal{X} = \mathcal{X}^{(0)} = \frac{1}{2}\begin{pmatrix} P_B(X^L + X^R) & 0 \\ 0 & P_B(X^L + X^R) \end{pmatrix}, \tag{2.1a}$$

$$P_1\mathcal{X} = \mathcal{X}^{(1)} = \frac{1}{2}\begin{pmatrix} P_F(X^L + iX^R) & 0 \\ 0 & -iP_F(X^L + iX^R) \end{pmatrix}, \tag{2.1b}$$

$$P_2\mathcal{X} = \mathcal{X}^{(2)} = \frac{1}{2}\begin{pmatrix} P_B(X^L - X^R) & 0 \\ 0 & -P_B(X^L - X^R) \end{pmatrix}, \tag{2.1c}$$

$$P_3\mathcal{X} = \mathcal{X}^{(3)} = \frac{1}{2}\begin{pmatrix} P_F(X^L - iX^R) & 0 \\ 0 & iP_F(X^L - iX^R) \end{pmatrix}. \tag{2.1d}$$

This decomposition is such that for any $\mathcal{X}, \mathcal{Y} \in \hat{\mathfrak{f}}$ we have $\mathrm{STr}(\mathcal{X}^{(a)}\mathcal{Y}^{(b)}) = 0$ if $a + b \neq 0 \bmod 4$. Finally, we introduce the matrix $W = \mathrm{diag}(1, -1)$, which satisfies the relations

$$P_0 W = W P_2, \qquad P_2 W = W P_0, \qquad P_1 W = W P_3, \qquad P_3 W = W P_1.$$

Action with gauge and auxiliary fields. The action depends on three parameters $\eta_{L,R}$ and k. The dynamical field of the model is

$$g = \mathrm{diag}(g_L, g_R) \quad \text{with} \quad g_{L,R}(\sigma^\pm) \in \widehat{G},$$

and where $\sigma^\pm$ are light-cone coordinates. The left-invariant current is defined as

$$\mathcal{J}_\pm = g^{-1}\partial_\pm g = \mathrm{diag}(g_L^{-1}\partial_\pm g_L, g_R^{-1}\partial_\pm g_R),$$

with $\partial_\pm = \partial_0 \pm \partial_1$. We also introduce $\mathcal{A}_\pm$ taking values in $\hat{\mathfrak{f}}$. As we shall see, the grade zero part, $\mathcal{A}_\pm^{(0)}$, plays the role of a gauge field while the other gradings, $\mathcal{A}_\pm^{(a)}$ with $a \neq 0$, are auxiliary fields. The definition of the action with gauge and auxiliary fields is

$$S_{\eta_{L,R},k}[g,\mathcal{A}_\pm] = \int d^2\sigma \ \mathrm{STr}\Big[(\mathcal{J}_+ - \mathcal{A}_+)\,\mathcal{O}_-\,(\mathcal{J}_- - \mathcal{A}_-) + \mathcal{A}_+\big(\mathcal{T}_- + kW\,(1+\mathcal{T}_-)\big)\mathcal{A}_-\Big] \\ + k\,\mathrm{STr}\Big[W\big(\mathcal{J}_+\mathcal{A}_- - \mathcal{A}_+\mathcal{J}_- + \mathcal{J}_-^{(2)}\mathcal{J}_+^{(0)} - \mathcal{J}_-^{(0)}\mathcal{J}_+^{(2)}\big)\Big] + S_{\mathrm{WZ},k}[g], \tag{2.2}$$

where the non-standard Wess-Zumino (WZ) term of [46], in the form already used in [49–51], is given by

$$S_{\mathrm{WZ},k}[g] = -4k \int d^2\sigma d\xi \ \epsilon^{\mu\nu\rho}\mathrm{STr}\left[\frac{2}{3}W\mathcal{J}_\mu^{(2)}\mathcal{J}_\nu^{(2)}\mathcal{J}_\rho^{(2)} + W[\mathcal{J}_\mu^{(1)},\mathcal{J}_\nu^{(3)}]\mathcal{J}_\rho^{(2)}\right], \tag{2.3}$$

with $\epsilon^{\mu\nu\rho}$ completely antisymmetric.

The operator $\mathcal{T}_-$ is defined as the following linear combination of projectors,

$$\mathcal{T}_- = -2P_2 + \frac{\lambda P_1 - kWP_3}{1-\lambda} - \frac{\lambda P_3 + kWP_1}{1+\lambda}, \tag{2.4}$$

with

$$\lambda = \sqrt{\frac{(1-k^2-\eta_L^2)(1-k^2-\eta_R^2)}{1-k^2}}. \tag{2.5}$$

As usual, we introduce a standard skew-symmetric solution $\mathcal{R}$ of the modified classical Yang-Baxter equation on $\hat{\mathfrak{f}}$. It therefore satisfies the three properties

$$[\mathcal{R}\mathcal{X},\mathcal{R}\mathcal{Y}] = \mathcal{R}\left([\mathcal{R}\mathcal{X},\mathcal{Y}] + [\mathcal{X},\mathcal{R}\mathcal{Y}]\right) + [\mathcal{X},\mathcal{Y}], \\ \mathrm{STr}[\mathcal{X}\,\mathcal{R}\mathcal{Y}] = -\mathrm{STr}[\mathcal{R}\mathcal{X}\,\mathcal{Y}], \qquad \mathcal{R}^3 = -\mathcal{R}, \tag{2.6}$$

for any $\mathcal{X},\mathcal{Y}\in\hat{\mathfrak{f}}$. Since $\mathcal{R}$ is a standard solution of the mCYBE, the operator

$$\Pi = 1 + \mathcal{R}^2$$

is the projector onto the Cartan subalgebra of $\hat{\mathfrak{f}}$ and satisfies (see for instance [16])

$$\Pi\mathcal{R} = \mathcal{R}\Pi = 0, \qquad \Pi[\mathcal{R}\mathcal{X},\mathcal{Y}] + \Pi[\mathcal{X},\mathcal{R}\mathcal{Y}] = 0, \qquad \mathcal{X},\mathcal{Y}\in\hat{\mathfrak{f}}. \tag{2.7}$$

As a consequence of the property $\mathcal{R}^3 = -\mathcal{R}$ we also have the following simple inversion formula

$$(\Pi + \beta\mathcal{R} + \gamma\mathcal{R}^2)^{-1} = \Pi + \frac{1}{\beta^2+\gamma^2}(-\beta\mathcal{R} + \gamma\mathcal{R}^2). \tag{2.8}$$

The operator $\mathcal{O}_-$ is defined in terms of the dressed operators

$$\mathcal{R}_g = \mathrm{Ad}_g^{-1}\circ\mathcal{R}\circ\mathrm{Ad}_g \quad \text{and} \quad \Pi_g = \mathrm{Ad}_g^{-1}\circ\Pi\circ\mathrm{Ad}_g,$$

such that the operator $\mathcal{R}_g$ is also a skew-symmetric solution of the mCYBE. We then introduce the operators

$$\Omega_\pm = \pm\frac{\sqrt{(\mu-1)(1-k^2\mu)}}{1\mp kW}\mathcal{R}_g \pm kW\left(\frac{\mu-1}{1\mp kW}\right)\mathcal{R}_g^2. \tag{2.9}$$

In this expression, μ is a block-diagonal matrix depending on the three parameters $\eta_{L,R}$ and k:

$$\mu = 1 + \frac{1}{\lambda^2 + k^2}\,\mathrm{diag}(\eta_L^2, \eta_R^2). \tag{2.10}$$

Finally, the operator $\mathcal{O}_-$ appearing in the action (2.2) is given by

$$\mathcal{O}_\pm = \frac{1 \pm kW\Omega_\pm}{1 + \Omega_\pm}, \tag{2.11}$$

where for later use we have also introduced $\mathcal{O}_+ = (\mathcal{O}_-)^t$.

Action after elimination of $\mathcal{A}_\pm$. In section 4 we compute the action obtained after elimination of the gauge and auxiliary fields $\mathcal{A}_\pm$. The result is given in equation (4.7). For convenience, we reproduce it here. It takes the form

$$S_{\eta_{L,R},k}[g] = \int d^2\sigma\ \mathrm{STr}\Big[\mathcal{J}_+\,\mathcal{S}_-\mathcal{J}_-\Big] + S_{\mathrm{WZ},k}[g]. \tag{2.12}$$

The definition of the operator $\mathcal{S}_-$ is

$$\mathcal{S}_- = \Big((1 - (2P_2 + P_F)\,kW\Omega_-)\,d_- + P_F\,kW\,(1 - d_-)\Big)\,(1 + \Omega_- d_-)^{-1}\,, \tag{2.13}$$

with

$$\begin{aligned} d_- = 2P_2 &+ \frac{1}{1-k^2}\Big(\big(\lambda - k^2\big)\,P_1 - (1+\lambda)\,kW P_3\Big) \\ &- \frac{1}{1-k^2}\Big(\big(\lambda + k^2\big)\,P_3 + (1-\lambda)\,kW P_1\Big). \end{aligned} \tag{2.14}$$

Gauge invariance. As we shall indicate in section 3 for the action (2.2) and prove in subsection 4 for the action (2.12), the field theory constructed in this article is on the supercoset $\widehat{F}/F_0$. This is the consequence of the existence of a gauge invariance under F_0. More precisely, the corresponding gauge transformations are

$$g \to g g_0, \qquad \mathcal{A}_\pm \to g_0^{-1}\partial_\pm g_0 + g_0^{-1}\mathcal{A}_\pm g_0, \tag{2.15}$$

with $g_0(\sigma^\pm)$ taking values in F_0. In particular, this means that $\mathcal{A}_\pm^{(0)}$ transforms as

$$\mathcal{A}_\pm^{(0)} \to g_0^{-1}\partial_\pm g_0 + g_0^{-1}\mathcal{A}_\pm^{(0)} g_0.$$

This shows that $\mathcal{A}_\pm^{(0)}$ is a gauge field. The other gradings $\mathcal{A}_\pm^{(a)}$ of $\mathcal{A}_\pm$ with $a \neq 0$, have the homogeneous gauge transformations

$$\mathcal{A}_\pm^{(a)} \to g_0^{-1}\mathcal{A}_\pm^{(a)} g_0.$$

Therefore, they correspond to auxiliary fields.

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3 Equations of motion, the Maurer-Cartan equation and the Lax pair

In this section we will demonstrate the classical integrability of the bi-Yang-Baxter deformation of the mixed flux model as defined in section 2. To do so we first compute the equations of motion following from the action (2.2). These are of two types: two constraint equations arising from the variation with respect to $\mathcal{A}_\pm$ and a dynamical equation that comes from varying with respect to the supergroup-valued field g. Working on the constraint equations, we show that the dynamical equation in first-order form, i.e. in terms of currents, and the Maurer-Cartan equation for the currents follow from the zero curvature of a Lax pair.

Equations of motion for $\mathcal{A}_\pm$. Varying the action (2.2) with respect to $\mathcal{A}_\mp$, we find the constraint equations

$$(\mathcal{O}_\pm \mp kW)(\mathcal{J}_\pm - \mathcal{A}_\pm) = (1 \mp kW)\,\mathcal{T}_\pm \mathcal{A}_\pm, \tag{3.1}$$

where we have introduced a second sum of projectors, $\mathcal{T}_+$, defined as

$$\mathcal{T}_+ = (1 - kW)^{-1}\Big(\big(\mathcal{T}_- + kW(1+\mathcal{T}_-)\big)^t + kW\Big).$$

The linear combinations of projectors $\mathcal{T}_\pm$ are given by

$$\mathcal{T}_\pm = -2P_2 \mp \frac{\lambda P_1 - kWP_3}{1 \pm \lambda} \pm \frac{\lambda P_3 + kWP_1}{1 \mp \lambda}, \tag{3.2}$$

and satisfy the following relation

$$\mathcal{T}_-{}^t = \mathcal{T}_+ + \frac{4k}{\lambda^2 - 1} W(P_1 + P_3).$$

In order to show the existence of a Lax pair, it will be convenient to introduce a new current $\mathcal{Q}_\pm$ defined as

$$\mathcal{Q}_\pm = (1 + \Omega_\pm)^{-1}(\mathcal{J}_\pm - \mathcal{A}_\pm), \tag{3.3}$$

where we recall that $\Omega_\pm$ are defined in (2.9). In terms of this new current, the equations of motion for $\mathcal{A}_\mp$ (3.1) take the particularly simple form

$$\mathcal{Q}_\pm = \mathcal{T}_\pm \mathcal{A}_\pm, \tag{3.4}$$

from which it follows that

$$\mathcal{Q}_\pm^{(0)} = 0.$$

Equation of motion for g. Varying the action (2.2) with respect to g and eliminating $\mathcal{J}_\pm$ in favour of $\mathcal{Q}_\pm$ (3.3) we find the following equation of motion

$$\mathcal{E} \equiv \mathcal{D}_+(1 + kW)\mathcal{Q}_- + \mathcal{D}_-(1 - kW)\mathcal{Q}_+ + 2kW\mathcal{F}_{+-}(\mathcal{A}_\pm) = 0, \tag{3.5}$$

where

$$\mathcal{F}_{+-}(\mathcal{A}_\pm) = \partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-], \qquad \mathcal{D}_\pm = \partial_\pm + \mathrm{adj}_{\mathcal{A}_\pm},$$

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with $\mathrm{adj}_{\mathcal{A}_\pm} = [\mathcal{A}_\pm, \cdot]$. Decomposing $\mathcal{E}$ under the $\mathbb{Z}_4$ grading (2.1) we find that the grade 0 part is given by

$$\begin{aligned}\mathcal{E}^{(0)} = & kW(\partial_+ + \mathrm{adj}_{\mathcal{A}_+^{(0)}})(\mathcal{Q}_-^{(2)} + 2\mathcal{A}_-^{(2)}) - kW(\partial_- + \mathrm{adj}_{\mathcal{A}_-^{(0)}})(\mathcal{Q}_+^{(2)} + 2\mathcal{A}_+^{(2)}) \\ & + [\mathcal{A}_+^{(2)}, \mathcal{Q}_-^{(2)}] + [\mathcal{A}_-^{(2)}, \mathcal{Q}_+^{(2)}] + 2kW\big([\mathcal{A}_+^{(1)}, \mathcal{A}_-^{(1)}] + [\mathcal{A}_+^{(3)}, \mathcal{A}_-^{(3)}]\big) \\ & + [\mathcal{A}_+^{(1)}, \mathcal{Q}_-^{(3)}] + [\mathcal{A}_+^{(3)}, \mathcal{Q}_-^{(1)}] + [\mathcal{A}_-^{(1)}, \mathcal{Q}_+^{(3)}] + [\mathcal{A}_-^{(3)}, \mathcal{Q}_+^{(1)}] \\ & + kW\big([\mathcal{A}_+^{(1)}, \mathcal{Q}_-^{(1)}] + [\mathcal{A}_+^{(3)}, \mathcal{Q}_-^{(3)}] - [\mathcal{A}_-^{(1)}, \mathcal{Q}_+^{(1)}] - [\mathcal{A}_-^{(3)}, \mathcal{Q}_+^{(3)}]\big).\end{aligned}$$

Evaluating on the constraint equations (3.4) we find that $\mathcal{E}^{(0)}$ identically vanishes. This is a consequence of the gauge invariance (2.15) of the action (2.2).

Maurer-Cartan equation. We now turn to the Maurer-Cartan equation

$$\partial_+ \mathcal{J}_- - \partial_- \mathcal{J}_+ + [\mathcal{J}_+, \mathcal{J}_-] = 0, \tag{3.6}$$

which we rewrite in terms of $\mathcal{Q}_\pm$ and $\mathcal{A}_\pm$ using

$$\mathcal{J}_\pm = \mathcal{A}_\pm + (1 + \Omega_\pm)\mathcal{Q}_\pm,$$

which follows from (3.3). As outlined in section 2, in equation (2.9) the operators $\Omega_\pm$ are defined in terms of a standard skew-symmetric solution $\mathcal{R}$ of the mCYBE on $\hat{\mathfrak{f}}$. In particular, we have

$$1 + \Omega_\pm = \Pi_g + \beta_\pm \mathcal{R}_g + \gamma_\pm \mathcal{R}_g^2, \tag{3.7}$$

with the block diagonal matrix coefficients $\beta_\pm$ and $\gamma_\pm$ given by

$$\beta_\pm = \pm \frac{\sqrt{(\mu - 1)(1 - k^2\mu)}}{1 \mp kW}, \qquad \gamma_\pm = -\frac{1 \mp k\mu W}{1 \mp kW}. \tag{3.8}$$

The Maurer-Cartan equation (3.6) is then given by

$$\begin{aligned}\mathcal{F}_{+-}(\mathcal{A}_\pm) + (1 + \Omega_-)\mathcal{D}_+\mathcal{Q}_- - (1 + \Omega_+)\mathcal{D}_-\mathcal{Q}_+ - [(1 + \Omega_+)\mathcal{Q}_+, (1 + \Omega_-)\mathcal{Q}_-] \\ + (1 + \Omega_-)[(1 + \Omega_+)\mathcal{Q}_+, \mathcal{Q}_-] + (1 + \Omega_+)[\mathcal{Q}_+, (1 + \Omega_-)\mathcal{Q}_-] = 0.\end{aligned} \tag{3.9}$$

Using the equation of motion for g (3.5) to replace

$$\mathcal{D}_+\mathcal{Q}_- + \mathcal{D}_-\mathcal{Q}_+ \to -kW(\mathcal{D}_+\mathcal{Q}_- - \mathcal{D}_-\mathcal{Q}_+) - 2kW\mathcal{F}_{+-}(\mathcal{A}_\pm),$$

we substitute in the explicit expressions for $1 + \Omega_\pm$ (3.7) and simplify using the properties (2.6) and (2.7), to rewrite the Maurer-Cartan equation as

$$\begin{aligned}O_1\mathcal{F}_{+-}(\mathcal{A}_\pm) + O_2(\mathcal{D}_+\mathcal{Q}_- - \mathcal{D}_-\mathcal{Q}_+) + O_3[\mathcal{Q}_+, \mathcal{Q}_-] \\ - \beta_+(1 + \gamma_-)\Pi_g[\mathcal{Q}_+, \mathcal{R}_g\mathcal{Q}_-] - \beta_-(1 + \gamma_+)\Pi_g[\mathcal{R}_g\mathcal{Q}_+, \mathcal{Q}_-] = 0,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}O_1 &= \Pi_g + kW(\beta_+ - \beta_-)\mathcal{R}_g + (kW(\gamma_+ - \gamma_-) - 1)\mathcal{R}_g^2, \\ O_2 &= \Pi_g + \frac{1}{2}(\beta_+ + \beta_- + kW(\beta_+ - \beta_-))\mathcal{R}_g + \frac{1}{2}(\gamma_+ + \gamma_- + kW(\gamma_+ - \gamma_-))\mathcal{R}_g^2, \\ O_3 &= -(\beta_+\beta_- + \gamma_+\gamma_- + \gamma_+ + \gamma_-)\Pi_g - (\beta_+\gamma_- + \gamma_+\beta_-)\mathcal{R}_g + (\beta_+\beta_- - \gamma_+\gamma_-)\mathcal{R}_g^2.\end{aligned} \tag{3.11}$$

From the definitions of $\beta_\pm$ and $\gamma_\pm$ (3.8) one can check that

$$\beta_+(1+\gamma_-) = \beta_-(1+\gamma_+). \tag{3.12}$$

This, together with the identity (2.7), implies that the left-hand side of the second line of (3.10) is identically zero. Furthermore, again using the definitions of $\beta_\pm$ and $\gamma_\pm$, the three operators (3.11) can be seen to be block proportional to each other with

$$O_1 = O_2 = \mu^{-1} O_3.$$

Together, this brings us to our final form of the Maurer-Cartan equation

$$\mathcal{Z} \equiv \mathcal{D}_+\mathcal{Q}_- - \mathcal{D}_-\mathcal{Q}_+ + \mu[\mathcal{Q}_+,\mathcal{Q}_-] + \mathcal{F}_{+-}(\mathcal{A}_\pm) = 0. \tag{3.13}$$

Lax pair. In order to construct a Lax pair we work on the constraint equations (3.4) and understand the equation of motion (3.5) and the Maurer-Cartan equation (3.13) as a set of two first-order equations for $\mathcal{A}_\pm$. We then attempt to construct two linear maps

$$\mathcal{A}_\pm = \sum_{i=0}^{3}\big(b^i_\pm P_i\mathcal{K}_\pm + c^i_\pm W P_i \mathcal{K}_\pm\big), \tag{3.14}$$

and (recalling that $\mathcal{E}^{(0)} = 0$) the two combinations

$$\begin{aligned}\widetilde{\mathcal{E}} &= \sum_{i=0}^{3}\big(e_i P_i\mathcal{E} + \tilde{e}_i W P_i\mathcal{E} + w_i P_i \mathcal{Z} + \tilde{w}_i W P_i \mathcal{Z}\big),\\ \widetilde{\mathcal{Z}} &= \sum_{i=0}^{3}\big(z_i P_i\mathcal{Z} + \tilde{z}_i W P_i\mathcal{Z} + f_i P_i \mathcal{E} + \tilde{f}_i W P_i \mathcal{E}\big),\end{aligned} \tag{3.15}$$

such that

$$\begin{aligned}
\widetilde{\mathcal{E}}^{(2)} &= D^{(0)}_+\mathcal{K}^{(2)}_- + D^{(0)}_-\mathcal{K}^{(2)}_+ + [\mathcal{K}^{(1)}_+,\mathcal{K}^{(1)}_-] - [\mathcal{K}^{(3)}_+,\mathcal{K}^{(3)}_-],\\
\widetilde{\mathcal{E}}^{(0)} &= 0, \qquad \widetilde{\mathcal{E}}^{(1)} = [\mathcal{K}^{(2)}_+,\mathcal{K}^{(3)}_-], \qquad \widetilde{\mathcal{E}}^{(3)} = [\mathcal{K}^{(1)}_+,\mathcal{K}^{(2)}_-],\\
\widetilde{\mathcal{Z}}^{(0)} &= F^{(0)}_{+-} + [\mathcal{K}^{(2)}_+,\mathcal{K}^{(2)}_-] + [\mathcal{K}^{(1)}_+,\mathcal{K}^{(3)}_-] + [\mathcal{K}^{(3)}_+,\mathcal{K}^{(1)}_-],\\
\widetilde{\mathcal{Z}}^{(2)} &= D^{(0)}_+\mathcal{K}^{(2)}_- - D^{(0)}_-\mathcal{K}^{(2)}_+ + [\mathcal{K}^{(1)}_+,\mathcal{K}^{(1)}_-] + [\mathcal{K}^{(3)}_+,\mathcal{K}^{(3)}_-],\\
\widetilde{\mathcal{Z}}^{(1)} &= D^{(0)}_+\mathcal{K}^{(1)}_- - D^{(0)}_-\mathcal{K}^{(1)}_+ + [\mathcal{K}^{(2)}_+,\mathcal{K}^{(3)}_-] + [\mathcal{K}^{(3)}_+,\mathcal{K}^{(2)}_-],\\
\widetilde{\mathcal{Z}}^{(3)} &= D^{(0)}_+\mathcal{K}^{(3)}_- - D^{(0)}_-\mathcal{K}^{(3)}_+ + [\mathcal{K}^{(2)}_+,\mathcal{K}^{(1)}_-] + [\mathcal{K}^{(1)}_+,\mathcal{K}^{(2)}_-],
\end{aligned} \tag{3.16}$$

where

$$D^{(0)}_\pm = \partial_\pm + \mathrm{adj}_{\mathcal{K}^{(0)}_\pm}, \qquad F^{(0)}_{+-} = \partial_+\mathcal{K}^{(0)}_- - \partial_-\mathcal{K}^{(0)}_+ + [\mathcal{K}^{(0)}_+,\mathcal{K}^{(0)}_-].$$

Such linear maps can indeed be found, with $b^i_\pm$ and $c^i_\pm$, along with the remaining parameters, given in appendix A. The equations

$$\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{Z}}^{(i)} = 0,$$

as defined in (3.16) then take the form of the familiar first-order equations of the semi-symmetric space σ-model [30]. It immediately follows that the Lax pair for the three-parameter deformation is given by [32]

$$\mathcal{L}_\pm(z) = \mathcal{K}^{(0)}_\pm + z^{-1}\,\mathcal{K}^{(1)}_\pm + z^{\pm 2}\,\mathcal{K}^{(2)}_\pm + z\,\mathcal{K}^{(3)}_\pm, \tag{3.17}$$

where $\mathcal{K}_\pm$ are defined in terms of $\mathcal{J}_\pm$ through (3.14), (3.4) and (3.3).

Comment. At this point we should emphasise that the existence of the Lax pair is highly non-trivial. Its existence stems from the peculiar form of the action (2.2). This includes, in particular, the very specific way in which the auxiliary fields appear as well as the tuning of the many coefficients that enter into its definition. Let us therefore stress that the action (2.2) does not come from nowhere! It is the result of a thorough and rather complicated investigation.

4 Elimination of gauge and auxiliary fields

The reason for introducing the gauge and auxiliary fields is that it enabled us to determine the Lax pair as demonstrated in the previous section. We now compute the action obtained after elimination of the gauge and auxiliary fields $\mathcal{A}_\pm$.

Equations of motion for $\mathcal{A}_\pm$. We first determine the on-shell values of $\mathcal{A}_\pm$ in terms of $\mathcal{J}_\pm$. For this we combine (3.3) and (3.4) to obtain

$$\mathcal{J}_\pm - \mathcal{A}_\pm = (1+\Omega_\pm)\mathcal{T}_\pm\mathcal{A}_\pm. \tag{4.1}$$

Defining

$$d_\pm = \frac{\mathcal{T}_\pm}{1+\mathcal{T}_\pm},$$

equation (4.1) may be rewritten as

$$\mathcal{J}_\pm - \mathcal{A}_\pm = \mathcal{T}_\pm\mathcal{A}_\pm + \Omega_\pm d_\pm(1+\mathcal{T}_\pm)\mathcal{A}_\pm.$$

We therefore have

$$\mathcal{A}_\pm = (1+\mathcal{T}_\pm)^{-1}(1+\Omega_\pm d_\pm)^{-1}\mathcal{J}_\pm, \tag{4.2}$$

which are the on-shell expressions for $\mathcal{A}_\pm$.

Action. One way to proceed is to rewrite the action (2.2) as

$$\begin{aligned} S_{\eta_{L,R},k}[g,\mathcal{A}_\pm] = &- \int d^2\sigma \mathrm{STr}\Big[\mathcal{A}_+\big((\mathcal{O}_- + kW)(\mathcal{J}_- - \mathcal{A}_-) - (1+kW)\mathcal{T}_-\mathcal{A}_-\big) + \mathcal{J}_+\mathcal{C}_-\Big] \\ &+ S_{\mathrm{WZ},k}[g], \end{aligned} \tag{4.3}$$

where

$$\mathcal{C}_- = (\mathcal{O}_- + kW)(\mathcal{J}_- - \mathcal{A}_-) + kW(2P_2 + P_F)\mathcal{A}_- - kW(2P_0+P_F)(\mathcal{J}_- - \mathcal{A}_-). \tag{4.4}$$

The first term in the action (4.3) vanishes upon imposing the equation of motion (3.1) for $\mathcal{A}_+$. It remains therefore to compute $\mathcal{C}_-$ on-shell. We find

$$\mathcal{C}_- = (1+kW)\mathcal{T}_-\mathcal{A}_- + kW(2P_2+P_F)\mathcal{A}_- - kW(2P_0+P_F)(1+\Omega_-)\mathcal{T}_-\mathcal{A}_- \tag{4.5}$$

$$= \big((1-(2P_2+P_F)kW\Omega_-)d_- + P_F kW(1-d_-)\big)(1+\mathcal{T}_-)\mathcal{A}_-, \tag{4.6}$$

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where we have first used the relations (3.1) and (4.1) to arrive at (4.5). We can now replace $\mathcal{A}_-$ in terms of $\mathcal{J}_-$ using its on-shell expression (4.2) to obtain

$$S_{\eta_{L,R},k}[g] = \int d^2\sigma \; \mathrm{STr}\Big[\mathcal{J}_+ \mathcal{S}_- \mathcal{J}_-\Big] + S_{\mathrm{WZ},k}[g], \tag{4.7}$$

where the operator $\mathcal{S}_-$ takes the form

$$\mathcal{S}_- = 2\,\mathcal{P}_-\,(1 + \Omega_- d_-)^{-1}\,, \tag{4.8}$$

with the quantity $\mathcal{P}_-$ defined as

$$\mathcal{P}_- = \frac{1}{2}\Big(\big(1 - (2P_2 + P_F)\,kW\Omega_-\big)\,d_- + P_F\,kW\,(1 - d_-)\Big). \tag{4.9}$$

The introduction of the operator $\mathcal{P}_-$ is useful to take the limits corresponding to both the mixed flux model and the bi-Yang-Baxter deformation in subsections 5.2 and 5.3 respectively.

Proof of gauge invariance. We can now present the postponed proof of the F_0-gauge invariance of the field theory we have constructed. Under the gauge transformation (2.15), the non-standard Wess-Zumino term $S_{\mathrm{WZ},k}[g]$ is invariant. The operator Ω_- defined by (2.9) transforms as

$$\Omega_- \to g_0^{-1}\Omega_- g_0.$$

Due to the presence of the projectors P_2, P_F and of d_-, defined by (2.14), in the expression of $\mathcal{S}_-$, the only gradings of $\mathcal{J}_\pm^{(a)}$ which contribute to the first term of the action (4.7) are the non zero ones. As these components of the currents have homogeneous gauge transformations, the action (4.7) is gauge invariant.

5 Bosonic truncation and limits

In this section we demonstrate agreement with three known results corresponding to certain truncations and limits. The first corresponds to the bosonic truncation, for which we recover the three-parameter deformation of $\mathbb{Z}_2$ permutation cosets worked out in [1]. The second and third cases are the mixed flux model and the bi-Yang-Baxter deformation respectively. These are obtained when $\eta_{L,R} = 0$ and $k = 0$ respectively. The mixed flux model has been constructed in [46] while the bi-Yang-Baxter deformation of $\mathbb{Z}_4$ permutation supercosets has been found in [49]. The corresponding actions have been constructed without a gauge or auxiliary fields. Therefore, here we make the comparison using the action (4.7).

5.1 Bosonic truncation

The bosonic model of [1] was constructed making use of a gauge field. Its form after eliminating the gauge field was also derived. For the three-parameter deformation of $\mathbb{Z}_4$ permutation supercosets, we not only have a gauge field but also auxiliary fields. This statement still holds for the bosonic truncation, where $\mathcal{A}_\pm^{(0)}$ plays the role of the gauge field and $\mathcal{A}^{(2)}_\pm$ is an auxiliary field. Therefore, the simplest way to show that the bosonic truncation coincides with the model of [1] is to compare the actions after eliminating the gauge and auxiliary fields.

An additional complication arises, however, as the action depending on the gauge field in [1] is not written explicitly in terms of projection operators associated with the $\mathbb{Z}_2$ grading. For this reason, in equation (B.1) of appendix B, we first rewrite this action in the language used in this article. We then eliminate the gauge field. The action after eliminating the gauge field is given in (B.15). The comparison between the bosonic truncation and (B.15) will then be immediate and leads to the map between the parameters used in this article for the supercoset case and those used in [1]. It is worth noting that the bosonic action after eliminating the gauge field (B.15) is written in a simpler form than in [1].

Action. When we consider the bosonic truncation the operator d_- in (2.14) becomes $d_- = 2P_2$. Therefore, the operator $\mathcal{S}_-$ in (4.8) is now

$$\begin{aligned}\mathcal{S}_- &= 2\left(1 - 2P_2 kW\Omega_-\right)P_2\left(1+2\Omega_- P_2\right)^{-1}\\ &= 2P_2\left(1-2kW\Omega_- P_2\right)\left(1+2\Omega_- P_2\right)^{-1}.\end{aligned} \tag{5.1}$$

The non-standard Wess-Zumino term (2.3) becomes the standard gauge invariant WZ term $S^B_{\mathrm{WZ},k}[g_L\, g_R^{-1}]$, which is written in terms of one copy of the supergroup $\widehat{G}$. Thus we have

$$\begin{aligned}S^B_{\eta_{L,R},k}[g_{L,R}] = &\int d^2\sigma\ \mathrm{STr}\Big[2\,\mathcal{J}_+P_2\left(1-2kW\Omega_-P_2\right)\left(1+2\Omega_-P_2\right)^{-1}\mathcal{J}_-\Big]\\ &+ S^B_{\mathrm{WZ},k}[g_L\, g_R^{-1}],\end{aligned} \tag{5.2}$$

with Ω_- given in equation (2.9). We need to compare this action to the action (B.15), which is reproduced here for convenience:

$$\begin{aligned}\tilde{S}^B_{\tilde{\eta}_{L,R},\tilde{k}}[g_{L,R}] = \frac{1}{2}\tilde{\mathcal{N}}\Bigg(&\int d^2\sigma\ \mathrm{STr}\Big[2\,\mathcal{J}_+P_2\big(1-2\tilde{\mathrm{b}}W\widetilde{\Omega}_-P_2\big)\big(1+2\widetilde{\Omega}_-P_2\big)^{-1}\mathcal{J}_-\Big]\\ &+ S^B_{\mathrm{WZ},\tilde{\mathrm{b}}}[g_L\, g_R^{-1}]\Bigg),\end{aligned} \tag{5.3}$$

where

$$\tilde{\mathcal{N}} = 2\tilde{k}\tilde{\mathrm{b}}^{-1}, \qquad \widetilde{\Omega}_- = \frac{1+\tilde{\eta}^2}{2\tilde{\alpha}^s_-(1+\tilde{\eta}^2+\tilde{k}W)^2}\left(-\tilde{A}\,\mathcal{R}_g - \frac{\tilde{k}W\tilde{\eta}^2}{1+\tilde{\eta}^2}\mathcal{R}_g^2\right). \tag{5.4}$$

The values of $\tilde{\mathrm{b}}$ and $\tilde{\alpha}^s_-$ in terms of the parameters $(\tilde{k}, \tilde{\eta}_{L,R})$ are given in equations (B.13) and (B.8) respectively, while $\tilde{A}$ and $\tilde{\eta}$ are defined through (B.7) and (B.2).

Map between the parameters. To determine the map between the parameters $(k, \eta_{L,R})$ and $(\tilde{k}, \tilde{\eta}_{L,R})$, we first focus on the WZ terms in the actions (5.2) and (5.3). This implies that we should identify k and $\tilde{\mathrm{b}}$, that is,

$$k = \tilde{\mathrm{b}} = \frac{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)\tilde{k}}{(1+\tilde{\eta}_L^2)(1+\tilde{\eta}_R^2)+\tilde{k}^2}. \tag{5.5}$$

It is then clear that we would have

$$\tilde{S}^B_{\tilde{\eta}_{L,R},\tilde{k}}[g] = \frac{1}{2}\tilde{\mathcal{N}} S^B_{\eta_{L,R},k}[g], \tag{5.6}$$

if $\widetilde{\Omega}_- = \Omega_-$. Comparing the coefficients of $\widetilde{\Omega}_-$ in (5.4) and of Ω_- in (2.9), we find the two conditions

$$\mu_L - 1 = \frac{\tilde{\eta}_L^2((1+\tilde{\eta}_R^2)^2 - \tilde{k}^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2},$$
$$\mu_R - 1 = \frac{\tilde{\eta}_R^2((1+\tilde{\eta}_L^2)^2 - \tilde{k}^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2}.$$

Using the form of μ given in equation (2.10), these relations may be rewritten as

$$\frac{(1-k^2)\eta_L^2}{(1-\eta_L^2)(1-\eta_R^2) - k^2(1-\eta_L^2-\eta_R^2)} = \frac{\tilde{\eta}_L^2((1+\tilde{\eta}_R^2)^2 - \tilde{k}^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2}, \tag{5.7a}$$
$$\frac{(1-k^2)\eta_R^2}{(1-\eta_L^2)(1-\eta_R^2) - k^2(1-\eta_L^2-\eta_R^2)} = \frac{\tilde{\eta}_R^2((1+\tilde{\eta}_L^2)^2 - \tilde{k}^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2}. \tag{5.7b}$$

Therefore, the map between the parameters used in the present article and those used in [1] is defined by (5.5) and (5.7). This demonstrates the agreement between the bosonic truncation of the three-parameter deformation of $\mathbb{Z}_4$ permutation supercosets and the model constructed in [1].

5.2 Mixed flux model

Now let us investigate the limit in which we expect to recover the mixed flux model of [46]. For this we take $\eta_{L,R} = 0$. It then immediately follows from (2.10) and (2.9) that $\mu = 1$ and $\Omega_- = 0$. As a consequence, the relation (4.8) becomes $\mathcal{S}_- = 2\,\mathcal{P}_-$. We also have from equation (2.5) that $\lambda = \sqrt{1-k^2}$. It remains to compute the values of d_- and $\mathcal{P}_-$, defined in (2.14) and (4.9) respectively, when $\eta_{L,R} = 0$. Doing so we find

$$\mathcal{P}_-(\eta_{L,R}=0,k) = \frac{1}{2}\big(d_- + P_F\, kW\,(1-d_-)\big) = P_2 + \frac{1}{2}\sqrt{1-k^2}(P_1 - P_3). \tag{5.8}$$

Therefore, we have

$$S_{\eta_{L,R}=0,k}[g] = 2\bigg(\int d^2\sigma\ \mathrm{STr}\Big[\mathcal{J}_+\,\mathcal{P}_-\mathcal{J}_-\Big] + \frac{1}{2}S_{\mathrm{WZ},k}[g]\bigg).$$

This action indeed corresponds to the mixed flux model of [46], written in the form given in [49–51].

5.3 Bi-Yang-Baxter deformation

The limit that should correspond to the bi-Yang-Baxter deformation of $\mathbb{Z}_4$ permutation supercosets is given by taking $k = 0$. We then have

$$\lambda = \sqrt{(1-\eta_L^2)(1-\eta_R^2)}, \qquad \mu_{L,R} = 1 + \frac{{\eta_{L,R}}^2}{\lambda^2}, \tag{5.9}$$

and

$$d_- = 2P_2 + \lambda(P_1 - P_3), \qquad \mathcal{P}_- = \frac{1}{2}d_-, \qquad \mathcal{S}_- = 2\mathcal{P}_-(1 + 2\Omega_-\mathcal{P}_-)^{-1}. \quad (5.10)$$

Therefore,

$$\mathcal{P}_-(\eta_{L,R}, k=0) = P_2 + \frac{\sqrt{(1-\eta_L^2)(1-\eta_R^2)}}{2}(P_1 - P_3). \quad (5.11)$$

It remains to compute Ω_- when $k = 0$. Starting from (2.9), we find

$$\Omega_-(\eta_{L,R}, k=0) = -\sqrt{\mu - 1}\,\mathcal{R}_g = -\frac{1}{2}\varkappa\mathcal{R}_g,$$

where we have used equation (5.9) and defined

$$\varkappa = \frac{2}{\sqrt{(1-\eta_L^2)(1-\eta_R^2)}}\mathrm{diag}(\eta_L, \eta_R).$$

Substituting this expression for Ω_- into $\mathcal{S}_-$ (5.10) we find that the action (4.7) becomes

$$S_{\eta_{L,R},k=0}[g] = 2\int d^2\sigma\ \mathrm{STr}\Big(\mathcal{J}_+\,\mathcal{P}_-\,(1 - \varkappa\mathcal{R}_g\,\mathcal{P}_-)^{-1}\,\mathcal{J}_-\Big),$$

with $\mathcal{P}_-$ given in equation (5.11). This action indeed coincides with the bi-Yang-Baxter deformation of $\mathbb{Z}_4$ permutation supercosets constructed in [49].

6 Conclusion

In this article we have constructed the bi-Yang-Baxter deformation of the mixed flux model of [46] giving a three-parameter integrable deformation of the semi-symmetric space σ-model on $\mathbb{Z}_4$ permutation supercosets. Furthermore, we demonstrated its classical integrability via the existence of a Lax pair and confirmed the agreement of various truncations and limits with known models.

For $\widehat{F} = P\mathrm{SU}(1,1|2)$ or $\widehat{F} = D(2,1;\alpha)$ the mixed flux model, together with the appropriate number of free compact bosons, is a κ-symmetry gauge-fixing of the Green-Schwarz superstring on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ or $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ supported by mixed R-R and NS-NS flux [45, 46, 52, 53]. Yang-Baxter deformations based on solutions of the modified classical Yang-Baxter equation do not typically describe strings on type II supergravity backgrounds [54, 55]. Instead the background fields satisfy a generalisation of the supergravity equations [56, 57]. This is indeed the case for the Yang-Baxter and bi-Yang-Baxter deformations of the $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ background supported by pure R-R flux [56, 58, 59]. It would be interesting to determine the R-R fluxes that support the metrics and B-fields of the three-parameter deformations of $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ and $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ (see [1, 2] and appendix C) and confirm that the generalised supergravity equations are satisfied. Furthermore, the generalised supergravity equations should imply that the model is scale invariant and UV finite on a flat two-dimensional worldsheet [56]. It would be important to confirm that this is indeed the case, for example, by checking the vanishing of the one-loop beta function for the σ-model coupling.

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While Yang-Baxter deformations based on solutions of the modified classical Yang-Baxter equation do not typically describe strings on supergravity backgrounds, their T-duals [56, 58] and Poisson-Lie duals [60] can. Poisson-Lie duality, introduced in [61–63], is a generalisation of (non-abelian) T-duality to models with Poisson-Lie symmetry. Poisson-Lie duals of the Yang-Baxter deformation of the principal chiral model plus Wess-Zumino term have been studied in [64, 65]. Extending this analysis to the bi-Yang-Baxter case, as well as to the R-R sector (see for instance [60, 66, 67]) would be necessary for investigating whether there exist duals of the three-parameter deformations of the $AdS_3\times S^3\times T^4$ and $AdS_3\times S^3\times S^3\times S^1$ backgrounds that are solutions of type II supergravity.

In order to understand the possible Poisson-Lie duals of the three-parameter integrable deformation it would also be helpful to study the Poisson-Lie symmetry of the model, together with the associated q-deformation of the global symmetry algebra [68]. An alternative route to exploring the q-deformed symmetry would be to compute the light-cone gauge dispersion relation and S-matrix as done for the Yang-Baxter deformation of the $AdS_5\times S^5$ superstring in [54, 69]. An initial proposal, based on symmetry considerations, for the deformed dispersion relation and S-matrix (up to overall phases) in the massive sector was given in [49, 70] following [71]. These are deformations of the undeformed dispersion relation and S-matrix constructed in [72–74].

Finally, it would also be interesting to perform the Hamiltonian analysis of this integrable σ-model and determine its twist function [34] (see [75] for a review). This would be the first step towards its reinterpretation as an affine Gaudin model, in the spirit of [76].

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A Coefficients of the linear maps used for the Lax pair

In this appendix we give the values of the various coefficients used in equations (3.14) and (3.15). We have

$$
\begin{aligned}
b^0_\pm &= 1, & b^2_\pm &= \frac{-(1-\eta_L^2)(1-\eta_R^2)+k^2(1-\eta_L^2-\eta_R^2)}{\sqrt{1-k^2}(1-k^2\eta_L^2\eta_R^2)},\\
c^0_\pm &= 0, & c^2_\pm &= \frac{-(1-k^2)(\eta_L^2-\eta_R^2)\pm k(1-k^2+\eta_L^2\eta_R^2)}{\sqrt{1-k^2}(1-k^2-\eta_L^2\eta_R^2)},\\
b^1_+ &= \mathbf{a}(k,\eta_L,\eta_R)+\mathbf{a}(-k,\eta_R,\eta_L), & b^1_- &= \mathbf{d}(k,\eta_L,\eta_R)+\mathbf{d}(-k,\eta_R,\eta_L),\\
b^3_+ &= \mathbf{d}(-k,\eta_L,\eta_R)+\mathbf{d}(k,\eta_R,\eta_L), & b^3_- &= \mathbf{a}(-k,\eta_L,\eta_R)+\mathbf{a}(k,\eta_R,\eta_L),\\
c^1_+ &= \mathbf{a}(k,\eta_L,\eta_R)-\mathbf{a}(-k,\eta_R,\eta_L), & c^1_- &= -\mathbf{d}(k,\eta_L,\eta_R)+\mathbf{d}(-k,\eta_R,\eta_L),\\
c^3_+ &= -\mathbf{d}(-k,\eta_L,\eta_R)+\mathbf{d}(k,\eta_R,\eta_L), & c^3_- &= \mathbf{a}(-k,\eta_L,\eta_R)-\mathbf{a}(k,\eta_R,\eta_L),
\end{aligned}
$$

$$\mathbf{a}(k,\eta_L,\eta_R) = -\left(\frac{1-k}{1+k}\right)^{\frac{3}{4}} \frac{(1-k^2-\eta_L^2)(\eta_R^2-2(1+k))+k(1-k^2)}{2\sqrt{1-k^2}\sqrt{1-k^2-\eta_L^2\eta_R^2}\sqrt{1-k^2-\eta_L^2}},$$
$$\mathbf{d}(k,\eta_L,\eta_R) = \left(\frac{1-k}{1+k}\right)^{\frac{1}{4}} \frac{(1-k^2-\eta_R^2)\eta_L^2+k(1-k^2)}{2\sqrt{1-k^2}\sqrt{1-k^2-\eta_L^2\eta_R^2}\sqrt{1-k^2-\eta_R^2}},$$

while for the remaining coefficients we find

$$e_2 = \frac{\sqrt{1-k^2}(1-k^2-\eta_L^2\eta_R^2)}{2\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)}, \qquad \tilde{f}_2 = \frac{k(1-k^2+\eta_L^2\eta_R^2)}{2\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)},$$
$$z_0 = \frac{(1-k^2)\big((1-\eta_L^2)(1-\eta_R^2)-k^2\big)}{(1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)}, \qquad \tilde{w}_0 = \frac{-k\sqrt{1-k^2}(1-k^2-\eta_L^2\eta_R^2)}{(1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)},$$
$$z_2 = \frac{\sqrt{1-k^2}(1-k^2-\eta_L^2\eta_R^2)}{(1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)}, \qquad \tilde{z}_2 = \frac{(1-k^2)(\eta_L^2-\eta_R^2)}{(1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)},$$
$$f_2 = w_0 = w_2 = 0, \qquad \tilde{e}_2 = \tilde{z}_0 = \tilde{w}_2 = 0,$$
$$e_1 = \mathbf{e}(k,\eta_L,\eta_R) + \mathbf{e}(-k,\eta_R,\eta_L), \qquad \tilde{e}_1 = \mathbf{e}(-k,\eta_L,\eta_R) - \mathbf{e}(k,\eta_R,\eta_L),$$
$$e_3 = -\mathbf{e}(-k,\eta_L,\eta_R) - \mathbf{e}(k,\eta_R,\eta_L), \qquad \tilde{e}_3 = -\mathbf{e}(k,\eta_L,\eta_R) + \mathbf{e}(-k,\eta_R,\eta_L),$$
$$w_1 = \mathbf{w}(k,\eta_L,\eta_R) + \mathbf{w}(-k,\eta_R,\eta_L), \qquad \tilde{w}_1 = -\mathbf{w}(-k,\eta_L,\eta_R) + \mathbf{w}(k,\eta_R,\eta_L),$$
$$w_3 = \mathbf{w}(-k,\eta_L,\eta_R) + \mathbf{w}(k,\eta_R,\eta_L), \qquad \tilde{w}_3 = -\mathbf{w}(k,\eta_L,\eta_R) + \mathbf{w}(-k,\eta_R,\eta_L),$$
$$z_1 = \mathbf{z}(k,\eta_L,\eta_R) + \mathbf{z}(-k,\eta_R,\eta_L), \qquad \tilde{z}_1 = \mathbf{z}(-k,\eta_L,\eta_R) - \mathbf{z}(k,\eta_R,\eta_L),$$
$$z_3 = \mathbf{z}(-k,\eta_L,\eta_R) + \mathbf{z}(k,\eta_R,\eta_L), \qquad \tilde{z}_3 = \mathbf{z}(k,\eta_L,\eta_R) - \mathbf{z}(-k,\eta_R,\eta_L),$$
$$f_1 = \mathbf{f}(k,\eta_L,\eta_R) + \mathbf{f}(-k,\eta_R,\eta_L), \qquad \tilde{f}_1 = -\mathbf{f}(-k,\eta_L,\eta_R) + \mathbf{f}(k,\eta_R,\eta_L),$$
$$f_3 = -\mathbf{f}(-k,\eta_L,\eta_R) - \mathbf{f}(k,\eta_R,\eta_L), \qquad \tilde{f}_3 = \mathbf{f}(k,\eta_L,\eta_R) - \mathbf{f}(-k,\eta_R,\eta_L),$$
$$\mathbf{e}(k,\eta_L,\eta_R) = -\left(\frac{1-k}{1+k}\right)^{\frac{1}{4}} \frac{(1-k^2)(1-k^2-\eta_L^2\eta_R^2)^{\frac{3}{2}}\big(1-k^2-(1+k)\eta_R^2\big)}{8\sqrt{1-k^2-\eta_R^2}\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)^2},$$
$$\mathbf{w}(k,\eta_L,\eta_R) = \left(\frac{1-k}{1+k}\right)^{\frac{1}{4}} \frac{(1-k^2)(1-k^2-\eta_L^2\eta_R^2)^{\frac{3}{2}}}{8\sqrt{1-k^2-\eta_R^2}\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)},$$
$$\mathbf{z}(k,\eta_L,\eta_R) = \left(\frac{1-k}{1+k}\right)^{\frac{1}{4}} \frac{\sqrt{1-k^2}\sqrt{1-k^2-\eta_L^2\eta_R^2}\big(2(1+k)(1-k^2-\eta_R^2)-k(1-k^2-\eta_L^2\eta_R^2)\big)}{4\sqrt{1-k^2-\eta_L^2}\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)},$$
$$\begin{aligned}\mathbf{f}(k,\eta_L,\eta_R) = \left(\frac{1-k}{1+k}\right)^{\frac{1}{4}} &\frac{\sqrt{1-k^2}\sqrt{1-k^2-\eta_L^2\eta_R^2}}{4\sqrt{1-k^2-\eta_L^2}\big((1-\eta_L^2)(1-\eta_R^2)-k^2(1-\eta_L^2-\eta_R^2)\big)^2} \\ &\times \big(k(1-k^2)^2 - k(1-k^2)(3+k)\eta_L^2 - 2(1-k^2)(1+k)\eta_R^2 \\ &+ (1+k)(4+k-k^2)\eta_L^2\eta_R^2 - (2+k(1+k))\eta_L^4\eta_R^2\big).\end{aligned}$$

B Three-parameter deformation of $\mathbb{Z}_2$ permutation cosets

In this appendix we rewrite the three-parameter deformation of $\mathbb{Z}_2$ permutation cosets constructed in [1] in the language used in this article. We first give the action which includes a gauge field and then eliminate this gauge field. Note that, to be precise, we write the action of the $\mathbb{Z}_2$ permutation coset embedded in a $\mathbb{Z}_4$ permutation supercoset. This explains why the action is written using the supertrace as opposed to the negative trace, which would be the appropriate bilinear form for compact Lie groups.

Action with gauge field. The three parameters are denoted by $\tilde{k}$ and $\tilde{\eta}_{L,R}$. The action is

$$\begin{aligned}\tilde{S}^{B}_{\tilde{\eta}_{L,R},\tilde{k}}[g,\widetilde{\mathcal{A}}^{(0)}_{\pm}] = \int d^2\sigma \,\mathrm{STr}\Big[&(\mathcal{J}_+ - \widetilde{\mathcal{A}}^{(0)}_+)\widetilde{\mathcal{O}}_-(\mathcal{J}_- - \widetilde{\mathcal{A}}^{(0)}_-) \\ &+ (\mathcal{J}_+ - \widetilde{\mathcal{A}}^{(0)}_+)\tilde{k}W\mathcal{J}^{(2)}_- - \tilde{k}W\mathcal{J}^{(2)}_+(\mathcal{J}_- - \widetilde{\mathcal{A}}^{(0)}_-)\Big] \\ &+ S^{B}_{\mathrm{WZ},\tilde{k}}[g_L\, g_R^{-1}].\end{aligned} \tag{B.1}$$

The operator $\widetilde{\mathcal{O}}_-$ is defined as $\widetilde{\mathcal{O}}_- = \mathrm{diag}(\widetilde{O}^{L}_-, \widetilde{O}^{R}_-)$ with

$$\widetilde{O}^{L}_- = (1+\tilde{\eta}_L^2)\,\frac{1+\tilde{A}_L R_{g_L}}{1-\tilde{\eta}_L^2 R^2_{g_L}}, \qquad \widetilde{O}^{R}_- = (1+\tilde{\eta}_R^2)\,\frac{1+\tilde{A}_R R_{g_R}}{1-\tilde{\eta}_R^2 R^2_{g_R}}.$$

The coefficients $\tilde{A}_{L,R}$ take the values [1, 14–16]

$$\tilde{A}_L = \tilde{\eta}_L\sqrt{1-\frac{\tilde{k}^2}{1+\tilde{\eta}_L^2}}, \qquad \tilde{A}_R = \tilde{\eta}_R\sqrt{1-\frac{\tilde{k}^2}{1+\tilde{\eta}_R^2}}. \tag{B.2}$$

Action after eliminating the gauge field. The equation of motion for the gauge field $\widetilde{\mathcal{A}}^{(0)}_{\mp}$ is

$$P_0\,\widetilde{\mathcal{O}}_{\pm}(\mathcal{J}_{\pm} - \widetilde{\mathcal{A}}^{(0)}_{\pm}) \mp \tilde{k}W\mathcal{J}^{(2)}_{\pm} = 0, \tag{B.3}$$

with $\widetilde{\mathcal{O}}_+ = (\widetilde{\mathcal{O}}_-)^t$. Let us introduce the current $\widetilde{\mathcal{Q}}_{\pm}$ defined as

$$\widetilde{\mathcal{Q}}_{\pm} = (\widetilde{\mathcal{O}}_{\pm} \mp \tilde{k}W)(\mathcal{J}_{\pm} - \widetilde{\mathcal{A}}^{(0)}_{\pm}). \tag{B.4}$$

The equation of motion (B.3) implies that $P_0\,\widetilde{\mathcal{Q}}_{\pm} = 0$ and thus $P_2\,\widetilde{\mathcal{Q}}_{\pm} = \widetilde{\mathcal{Q}}_{\pm}$. Therefore,

$$\mathcal{J}^{(0)}_{\pm} - \widetilde{\mathcal{A}}^{(0)}_{\pm} = P_0(\widetilde{\mathcal{O}}_{\pm} \mp \tilde{k}W)^{-1}\widetilde{\mathcal{Q}}^{(2)}_{\pm}, \tag{B.5a}$$

$$\mathcal{J}^{(2)}_{\pm} = P_2(\widetilde{\mathcal{O}}_{\pm} \mp \tilde{k}W)^{-1}\widetilde{\mathcal{Q}}^{(2)}_{\pm}. \tag{B.5b}$$

It remains to compute the inverse operators appearing in these equations. This may be done by using the formula (2.8). Doing so we find

$$(\widetilde{\mathcal{O}}_{\pm} \mp \tilde{k}W)^{-1} = \frac{1}{1+\tilde{\eta}^2 \mp \tilde{k}W}\,\Pi_g + \frac{1+\tilde{\eta}^2}{(1+\tilde{\eta}^2 \mp \tilde{k}W)^2}\Big(\pm\tilde{A}\,\mathcal{R}_g - (1 \mp \tilde{k}W)\,\mathcal{R}_g^2\Big), \tag{B.6}$$

where

$$\tilde{\eta} = \mathrm{diag}(\tilde{\eta}_L, \tilde{\eta}_R) \quad \text{and} \quad \tilde{A} = \mathrm{diag}(\tilde{A}_L, \tilde{A}_R). \tag{B.7}$$

We then write $\Pi_g = 1+\mathcal{R}_g^2$ and decompose the coefficient multiplying Π_g into its symmetric and anti-symmetric part, that is

$$\frac{1}{1+\tilde{\eta}^2 \mp \tilde{k}W} = \tilde{\alpha}^s_{\pm}1 + \tilde{\alpha}^a_{\pm}W,$$

with

$$\tilde{\alpha}^s_\pm = \frac{2+\tilde{\eta}_L^2+\tilde{\eta}_R^2}{2(1+\tilde{\eta}_L^2\mp\tilde{k})(1+\tilde{\eta}_R^2\pm\tilde{k})}, \qquad \tilde{\alpha}^a_\pm = -\frac{\tilde{\eta}_L^2-\tilde{\eta}_R^2\mp 2\tilde{k}}{2(1+\tilde{\eta}_L^2\mp\tilde{k})(1+\tilde{\eta}_R^2\pm\tilde{k})}. \tag{B.8}$$

Finally we find

$$\big(\widetilde{\mathcal{O}}_\pm \mp \tilde{k}W\big)^{-1} = \tilde{\alpha}^s_\pm\Big(1+\frac{\tilde{\alpha}^a_\pm}{\tilde{\alpha}^s_\pm}W + 2\widetilde{\Omega}_\pm\Big), \tag{B.9}$$

where the operators $\widetilde{\Omega}_\pm$ are given by

$$2\tilde{\alpha}^s_\pm\widetilde{\Omega}_\pm = \frac{1+\tilde{\eta}^2}{(1+\tilde{\eta}^2\mp\tilde{k}W)^2}\Big(\pm\tilde{A}\,\mathcal{R}_g \pm \frac{\tilde{k}W\tilde{\eta}^2}{1+\tilde{\eta}^2}\mathcal{R}_g^2\Big). \tag{B.10}$$

Equations (B.5) may then be expressed as

$$\mathcal{J}^{(0)}_\pm - \widetilde{\mathcal{A}}^{(0)}_\pm = \tilde{\alpha}^s_\pm\Big(\frac{\tilde{\alpha}^a_\pm}{\tilde{\alpha}^s_\pm}W + 2P_0\widetilde{\Omega}_\pm\Big)\widetilde{\mathcal{Q}}^{(2)}_\pm, \tag{B.11a}$$

$$\mathcal{J}^{(2)}_\pm = \tilde{\alpha}^s_\pm\big(1+2P_2\widetilde{\Omega}_\pm\big)\widetilde{\mathcal{Q}}^{(2)}_\pm. \tag{B.11b}$$

We can now compute the action obtained after eliminating the gauge field $\widetilde{\mathcal{A}}^{(0)}_\pm$. We first eliminate $\widetilde{\mathcal{A}}^{(0)}_-$ by making use of the equation of motion of $\widetilde{\mathcal{A}}^{(0)}_+$. The Lagrangian defined by the first two lines of the action (B.1) becomes $\mathrm{STr}(\mathcal{J}_+\tilde{\mathcal{C}}_-)$ with

$$\tilde{\mathcal{C}}_- = P_2\Big(\big(\widetilde{\mathcal{O}}_- + \tilde{k}W\big)\big(\mathcal{J}_- - \widetilde{\mathcal{A}}^{(0)}_-\big) - 2\tilde{k}W\big(\mathcal{J}_- - \widetilde{\mathcal{A}}^{(0)}_-\big)\Big). \tag{B.12}$$

The quantity $\tilde{\mathcal{C}}_-$ can be written as

$$\begin{aligned}\tilde{\mathcal{C}}_- &= \widetilde{\mathcal{Q}}^{(2)}_- - 2\tilde{\alpha}^s_-\tilde{k}W\Big(\frac{\tilde{\alpha}^a_-}{\tilde{\alpha}^s_-}W + 2P_0\widetilde{\Omega}_-\Big)\widetilde{\mathcal{Q}}^{(2)}_- \\ &= \frac{(1+\tilde{\eta}_L^2)(1+\tilde{\eta}_R^2)+\tilde{k}^2}{(1+\tilde{\eta}_L^2+\tilde{k})(1+\tilde{\eta}_R^2-\tilde{k})}\Big(1-2\tilde{\mathrm{b}}WP_0\widetilde{\Omega}_-\Big)\widetilde{\mathcal{Q}}^{(2)}_-,\end{aligned}$$

where we have made use of the definition (B.4), the property $P_2W\mathcal{J}_- = P_2W\mathcal{J}^{(0)}_-$ and equations (B.11a). We have also defined

$$\tilde{\mathrm{b}} = \frac{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)\,\tilde{k}}{(1+\tilde{\eta}_L^2)(1+\tilde{\eta}_R^2)+\tilde{k}^2}. \tag{B.13}$$

The last step is to invert the relation (B.11b) and use the identities

$$\big(1+2P_2\widetilde{\Omega}_-\big)^{-1}P_2 = P_2\big(1+2\widetilde{\Omega}_-P_2\big)^{-1}, \tag{B.14a}$$

$$\big(1-2\tilde{\mathrm{b}}WP_0\widetilde{\Omega}_-\big)P_2 = P_2\big(1-2\tilde{\mathrm{b}}W\widetilde{\Omega}_-P_2\big), \tag{B.14b}$$

to write the action after elimination of the gauge field as

$$\begin{aligned}\tilde{S}^B_{\tilde{\eta}_{L,R},\tilde{k}}[g_{L,R}] = \frac{1}{2}\tilde{\mathcal{N}}\bigg(&\int d^2\sigma\ \mathrm{STr}\Big[2\,\mathcal{J}_+P_2\big(1-2\tilde{\mathrm{b}}W\widetilde{\Omega}_-P_2\big)\big(1+2\widetilde{\Omega}_-P_2\big)^{-1}\mathcal{J}_-\Big] \\ &+ S^B_{\mathrm{WZ},\tilde{b}}[g_L\,g_R^{-1}]\bigg),\end{aligned} \tag{B.15}$$

where

$$\tilde{\mathcal{N}} = 2\tilde{k}\tilde{\mathrm{b}}^{-1}, \qquad \widetilde{\Omega}_- = \frac{1+\tilde{\eta}^2}{2\tilde{\alpha}^s_-(1+\tilde{\eta}^2+\tilde{k}W)^2}\Big(-\tilde{A}\,\mathcal{R}_g - \frac{\tilde{k}W\tilde{\eta}^2}{1+\tilde{\eta}^2}\mathcal{R}_g^2\Big). \tag{B.16}$$

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C Metric and B-field for the three-parameter deformation of S^3

In this appendix we give simple expressions for the metric and B-field in the case of the three-parameter deformation of S^3. Other expressions have been previously obtained in [1, 2]. We start from the result (B.15) with the supertrace being replaced by the negative trace. In the case of $G = \mathrm{SU}(2)$, we take the familiar basis for the Lie algebra $\mathfrak{su}(2)$

$$T_a = \frac{i}{2}\sigma_a,$$

where σ_a are the Pauli matrices. The Drinfel'd-Jimbo [17–19] solution of the modified classical Yang-Baxter equation is then given by

$$RT_1 = -T_2, \qquad RT_2 = T_1, \qquad RT_3 = 0.$$

We use the gauge symmetry $g_{L,R} \to g_{L,R}g_0$ to fix $g_R = 1$ and parametrise g_L as

$$g_L = \exp[(-\varphi+\phi)T_3)].(r1 - 2\sqrt{1-r^2}T_1).\exp[(\varphi+\phi)T_3)].$$

We substitute into the action (5.2) and replace STr by $-$Tr to arrive at the three-parameter ($\tilde{\eta}_L$, $\tilde{\eta}_R$ and $\tilde{k}$) deformation of the S^3 σ-model. Comparing with the usual σ-model form in conformal gauge

$$-2\int d^2\sigma\ (\eta^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu) = 2\int d^2\sigma\ (G_{\mu\nu}+B_{\mu\nu})\partial_+X^\mu\partial_-X^\nu,$$

we find the following target space metric and B-field

$$\begin{aligned}
ds^2 = \tilde{F}^{-1}\bigg(&\frac{(1+\tilde{q}^2r^2(1-r^2))}{1-r^2}dr^2 - 2\tilde{q}\tilde{\kappa}_-(1-r^2)rdrd\varphi - 2\tilde{q}\tilde{\kappa}_+r^3drd\phi \\
&+ (1+\tilde{\kappa}_-^2(1-r^2))(1-r^2)d\varphi^2 + (1+\tilde{\kappa}_+^2r^2)r^2d\phi^2 + 2\tilde{\kappa}_+\tilde{\kappa}_-r^2(1-r^2)d\varphi d\phi\bigg), \\
B = \tilde{a}\tilde{F}^{-1}&\big(\tilde{\kappa}_+^2 - \tilde{\kappa}_-^2 + \tilde{q}^2(1-2r^2)\big)(-\tilde{\kappa}_+rdr\wedge d\varphi + \tilde{\kappa}_-rdr\wedge d\phi) \\
&- \tilde{a}\tilde{q}\tilde{F}^{-1}\big(1-2r^2-\tilde{\kappa}_+^2r^4+\tilde{\kappa}_-^2(1-r^2)^2\big)d\varphi\wedge d\phi, \\
\tilde{F} = 1 + \tilde{\kappa}_+^2r^2 &+ \tilde{\kappa}_-^2(1-r^2) + \tilde{q}^2r^2(1-r^2), \qquad \tilde{a} = \frac{1}{\sqrt{(\tilde{q}^2+\tilde{\kappa}_+^2+\tilde{\kappa}_-^2)^2 + 4(\tilde{q}^2-\tilde{\kappa}_+^2\tilde{\kappa}_-^2)}},
\end{aligned}$$

where we have introduced the new parameters

$$\tilde{\kappa}_\pm = \sqrt{\frac{(4+(\tilde{A}_L\pm\tilde{A}_R)^2)(1+\tilde{\eta}_L^2)(1+\tilde{\eta}_R^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2} - 1}, \qquad \tilde{q} = \frac{2\tilde{\eta}_L\tilde{\eta}_R\tilde{k}}{2+\tilde{\eta}_L^2+\tilde{\eta}_R^2}.$$

For convenience when relating the two sets of parameters we assume that $\tilde{\eta}_L$, $\tilde{\eta}_R$ and $\tilde{k}$ are all positive and all square roots are positive so that $\tilde{\kappa}_\pm$, $\tilde{q}$ and $\tilde{a}$ are also positive.

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Furthermore, we assume that we are in a region of parameter space such that the following relations hold

$$\tilde{\kappa}_+\tilde{\kappa}_- = \frac{(\tilde{\eta}_R^2-\tilde{\eta}_L^2)(1-\tilde{\eta}_L^2\tilde{\eta}_R^2-\tilde{k}^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2},$$
$$\tilde{q}\tilde{\kappa}_\pm = \frac{2\tilde{k}(\tilde{A}_L(1+\tilde{\eta}_L^2)\tilde{\eta}_R^2 \pm \tilde{A}_R(1+\tilde{\eta}_R^2)\tilde{\eta}_L^2)}{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2},$$
$$\tilde{a} = \frac{(2+\tilde{\eta}_L^2+\tilde{\eta}_R^2)^2}{4\tilde{\eta}_L\tilde{\eta}_R((1+\tilde{\eta}_L^2)(1+\tilde{\eta}_R^2)+\tilde{k}^2)}.$$

Let us conclude with some observations. First, we note that the first line of the B-field is closed and hence locally it can be set to zero by a gauge transformation. Second, the metric is invariant under the following formal transformation

$$r \to \sqrt{1-r^2}, \quad \varphi \leftrightarrow \phi, \qquad \tilde{\kappa}_+ \leftrightarrow \tilde{\kappa}_-, \qquad \tilde{q} \to -\tilde{q},$$

while the B-field changes just by an overall sign. Finally, the analogous deformation of AdS_3 can be found by analytically continuing

$$\rho \to ir, \qquad \varphi \to t, \qquad \phi \to \psi,$$

as well as flipping the overall sign of the metric and B-field.

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