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Entropy spectrum of charged BTZ black holes in massive gravity's rainbow

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Regarding the significant interests in massive gravity and combining it with gravity's rainbow and also BTZ black holes, we apply the formalism introduced by Jiang and Han in order to investigate the quantization of the entropy of black holes. We show that the entropy of BTZ black holes in massive gravity's rainbow is quantized with equally spaced spectra and it depends on the black holes' properties including massive parameters, electrical charge, the cosmological constant, and also rainbow functions. In addition, we show that quantization of the entropy results in the appearance of novel properties for this quantity, such as the existence of divergences, non-zero entropy in a vanishing horizon radius, and the possibility of tracing out the effects of the black holes' properties. Such properties are absent in the non-quantized version of the black hole entropy. Furthermore, we investigate the effects of quantization on the thermodynamical behavior of the solutions. We confirm that due to quantization, novel phase transition points are introduced and stable solutions are limited to only de Sitter black holes (anti-de Sitter and asymptotically flat solutions are unstable).

Subject Index E03, E05

1. Introduction

General relativity (GR) is a successful theory of gravity with certain shortcomings; for example, the accelerated expansion of the universe, massive gravitons, and the ultraviolet (UV) behavior cannot be explained with GR. To address these and other issues, GR needs to be modified. There are some modified theories, such as Horava–Lifshitz gravity [1,2], gravity's rainbow [3–6], and also massive gravity [7–13].

In order to understand the UV behavior of GR, various attempts have been made to obtain different models of UV completion of GR such that they should reduce to GR in the infrared (IR) limit. The first attempt in this field is related to Horava–Lifshitz gravity [1,2], in which space and time are made to have different Lifshitz scaling. Although this theory reduces to GR in the IR limit, its behavior is different from that of GR in the UV regime. It is notable that Horava–Lifshitz gravity is based on a deformation of the usual energy–momentum dispersion relation in the UV limit, in which it reduces to the usual energy–momentum dispersion relation in the IR limit. Another approach for extracting the UV completion of GR is called gravity's rainbow [3]. The theory is based on the deformation of the usual energy–momentum dispersion relation in the UV limit, and, similar to Horava–Lifshitz gravity, gravity's rainbow reduces to GR in the IR limit. It was shown that the quantum corrections in a gravitational system could be observed in the dependency of its spacetime on the energy of particles probing it, which is the gravity's rainbow point of view [14,15]. Also, by considering a

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suitable choice of the rainbow functions, Horava–Lifshitz gravity can be related to gravity's rainbow [16]. This is because both theories are based on modifying the usual energy–momentum dispersion relation in the UV limit. It is notable that such a modification of the usual energy–momentum relation has also been obtained in discrete spacetime [17], the spin-network in loop quantum gravity (LQG) [18], spacetime foam [19], ghost condensation [20], and non-commutative geometry [21]. Non-commutative geometry occurs due to background fluxes in string theory [22,23], and is used to derive one of the most important rainbow functions in gravity's rainbow [24,25].

In other words, the geometry of spacetime is modified to be energy dependent and this energy dependency of the metric is incorporated through the introduction of rainbow functions. The standard energy–momentum relation in gravity's rainbow is given as

$$E^2 f^2(E/E_{\mathrm{P}}) - p^2 g^2(E/E_{\mathrm{P}}) = m^2, \tag{1}$$

in which E and E_{P} are the energy of the test particle and the Planck energy, respectively. For the sake of simplicity we will use $\varepsilon = E/E_{\mathrm{P}}$. Also, $f(\varepsilon)$ and $g(\varepsilon)$ are energy functions which are restricted with $\lim\limits_{\varepsilon\to 0} f(\varepsilon) = 1$ and $\lim\limits_{\varepsilon\to 0} g(\varepsilon) = 1$ in the IR limit, and could be used to build an energy-dependent metric with the following recipe:

$$\hat{g}(\varepsilon) = \eta^{ab} e_a(\varepsilon) \otimes e_b(\varepsilon), \tag{2}$$

where

$$e_0(\varepsilon) = \frac{1}{f(\varepsilon)}\tilde{e}_0, \qquad e_i(\varepsilon) = \frac{1}{g(\varepsilon)}\tilde{e}_i, \tag{3}$$

in which $\tilde{e}_0$ and $\tilde{e}_i$ are related to the energy-independent frame fields. It is notable that E cannot exceed E_{P}, so $0 < \varepsilon \leq 1$. In other words, the gravity's rainbow produces a correction to the metric that becomes significant when the particle's energy approaches the Planck energy. It is notable that there are three models for these energy functions:

Case I is related to the hard spectra from gamma-ray bursts [24], with the following form:

$$f(\varepsilon) = \frac{e^{\beta\varepsilon} - 1}{\beta\varepsilon}, \qquad g(\varepsilon) = 1. \tag{4}$$

Case II is motivated by studies conducted in loop quantum gravity and non-commutative geometry [25], with

$$f(\varepsilon) = 1, \qquad g(\varepsilon) = \sqrt{1 - \eta\varepsilon^n}. \tag{5}$$

Case III is due to consideration of the constancy of the velocity of light [26],

$$f(\varepsilon) = g(\varepsilon) = \frac{1}{1 - \lambda\varepsilon}, \tag{6}$$

in which in the above models β, η, and λ are constants that could be determined by experiment.

In the gravity's rainbow context, and by combining various gravities, the black hole and cosmological solutions have been studied in some works. For example, $F(R)$ gravity's rainbow [27], Gauss–Bonnet gravity's rainbow [28], dilatonic gravity's rainbow [29], and Galileon gravity's rainbow [30] have been investigated. Also, in Ref. [31] it was shown that by considering a special limit on rainbow functions, we encounter non-singular universes in gravity's rainbow. The remnant of the black objects and also the absence of black holes at LHC due to gravity's rainbow have been evaluated in Refs. [32–34]. The modified Tolman–Oppenheimer–Volkov (TOV) in gravity's rainbow, and investigations of the properties of magnetic neutron stars and dynamical stability conditions have been considered [35,36]. The heat engine and geometrothermodynamics of the black holes obtained in gravity's rainbow have been studied in Ref. [37].

On the other hand, and in order to have massive gravitons, GR must be modified, because the gravitons are massless particles in GR. Therefore, Fierz and Pauli were the first to study the theory describing massive gravitons (FP massive theory) [7,8]. Studies done by van Dam, Veltman, and Zakharov showed that FP massive theory encountered a discontinuity (the vDVZ discontinuity) [38–40]. In order to remove this problem, one has to generalize linear FP massive gravity to a non-linear one. Later, Boulware and Deser found that this theory of massive gravity suffers a ghost instability at the non-linear level [9,10]. Recently, there has been great interest in the modification of GR on the non-linear level to include massive gravitons. Among the studies done in this regard, de Rham, Gabadadze, and Tolley (dGRT) were able to introduce an interesting massive gravity without ghost in arbitrary dimensions [41,42], in which a stable non-linear massive gravity [11–13] was employed to conduct the investigations. Recently, several interesting black hole solutions have been obtained in various massive gravities [43–55]. Charged black holes and their thermodynamics in massive gravity have been evaluated in Refs. [56–59]. Van der Waals-like phase transition and geometrical thermodynamics of black holes in massive gravity have been investigated in Refs. [60–67]. Modified TOV in massive gravity and investigations of the properties of neutron stars and also dynamical stability conditions have been considered [68,69]. Black holes as heat engines in massive gravity have been investigated in Refs. [70,71]. Considering massive gravity, white dwarfs have been studied in Ref. [72].

The first three-dimensional black hole solution in the presence of the cosmological constant was obtained by Bañados, Teitelboim, and Zanelli, and is known as the BTZ black hole [73]. Later, it was shown that these solutions have central roles in understanding several issues such as black hole thermodynamics [74–76], quantum gravity, string theory, the anti-de Sitter spaces / conformal field theories (AdS/CFT) conjecture [77,78], and investigation of gravitational interaction in low-dimensional spacetime [79]. The charged BTZ black hole is the correspondence solution of Einstein–Maxwell gravity in three dimensions [74,80,81]. Recently, charged BTZ black holes with the two generalizations of massive gravity and gravity's rainbow have been studied [82–84].

As the first cornerstone, the concept of Hawking radiation of black holes improved our knowledge of the quantum theory of gravity. Then, Bekenstein showed that there is a lower bound for the event horizon area of black holes [85],

$$(\Delta A)_{\min} = 8\pi l_{\mathrm{p}}^2, \tag{7}$$

in which l_{p} is the Planck length. It is notable that this lower bound does not depend on the black hole parameters. On the other hand, quasinormal mode (QNM) frequencies are known as the characteristic sound of a black hole. These QNMs should have an adiabatic invariant quantity. Hod extracted the area and also the entropy spectrum of a black hole from QNMs [86,87]. Using the Bohr–Sommerfield quantization rule ($I_{\mathrm{adiabatic}} = n\hbar$), Hod showed that the area spectrum of a Schwarzschild black hole is equispaced. Using the well-known Bekenstein–Hawking area law, and considering the area spectrum, one can obtain the entropy spectrum of black holes as $\Delta S_{\mathrm{bh}} = \ln 3$. On the other hand, Kunstatter obtained the area spectrum of higher-dimensional spherical symmetric black holes by considering the adiabatic invariant quantity in the following form [88]:

$$I_{\text{adiabatic}} = \int \frac{dE}{\Delta\omega\,(E)}, \tag{8}$$

where $\Delta\omega = \omega_{n+1} - \omega_n$, and E and ω are the energy and frequency of the QNM, respectively. Later, Hod and Kunstatter calculated the area spectrum by considering the real part of the QNM frequency. Next, Maggiore [89] refined Hod's idea by proving that the physical frequency of the QNM is determined by its real and imaginary parts. A new method was proposed by Majhi and Vagenas in order to quantize the entropy without using QNM. They used the idea of relating an adiabatic invariant quantity to the Hamiltonian of the black hole, and then obtained an equally spaced entropy spectrum with its quantum equal to the one obtained by Bekenstein [90]. In the tunnelling picture, we can consider the horizon of a black hole to oscillate periodically when a particle tunnels in or out of the black hole. Therefore, we can use this viewpoint and consider an adiabatic invariant quantity (or an action of the oscillating horizon)

$$I = \int p_i dq_i, \tag{9}$$

in which p_i is the corresponding conjugate momentum of the coordinate of q_i ($i = 0, 1$, where $q_0 = \tau$ and $q_1 = r_{\mathrm{h}}$, in which τ and r_{h} are related to the Euclidean time and the horizon radius, respectively). By using Hamilton's equation ($\dot{q}_i = \frac{dH}{dp_i}$), one can rewrite Eq. (9) as

$$I = \int \int_0^H dH d\tau + \int \int_0^H \frac{dH}{\dot{r}_{\mathrm{h}}} dr_{\mathrm{h}} = 2 \int \int_0^H \frac{dH}{\dot{r}_{\mathrm{h}}} dr_{\mathrm{h}}, \tag{10}$$

where H is the Hamiltonian of the system and $\dot{r}_{\mathrm{h}} = \frac{dr_{\mathrm{h}}}{d\tau}$. Now, we want to calculate the above adiabatic invariant quantity, so we consider a static metric in gravity's rainbow as

$$ds^2 = -\frac{\psi\,(r,\varepsilon)}{f^2\,(\varepsilon)} dt^2 + \frac{1}{g^2\,(\varepsilon)} \left[\frac{dr^2}{\psi\,(r,\varepsilon)} + r^2 d\varphi^2 \right]. \tag{11}$$

It is notable that we can obtain r_{h} by using $\psi\,(r_{\mathrm{h}}, \varepsilon) = 0$. Finding the oscillating velocity of the black hole horizon, we can calculate Eq. (10). In the tunnelling picture, when a particle tunnels in or out, the horizon of the black hole will expand or shrink due to gaining or losing mass in the black hole. Since the tunnelling and oscillation happen simultaneously, the tunnelling velocity of the particle is equal and opposite to the oscillating velocity of the black hole horizon ($\dot{r}_{\mathrm{h}} = -\dot{r}$). Also, we have to Euclideanize the introduced metric of Eq. (11) by using the transformation $t \to -i\tau$. So, we have

$$ds^2 = \frac{\psi\,(r,\varepsilon)}{f^2\,(\varepsilon)} d\tau^2 + \frac{1}{g^2\,(\varepsilon)} \left[\frac{dr^2}{\psi\,(r,\varepsilon)} + r^2 d\varphi^2 \right]. \tag{12}$$

It is notable that, when a photon travels across the horizon of a black hole, the radial null path (or radial null geodesic) is given by

$$ds^2 = d\varphi^2 = 0 \;\to\; \dot{r} = \pm i \left(\frac{g\,(\varepsilon)\,\psi\,(r,\varepsilon)}{f\,(\varepsilon)} \right), \tag{13}$$

in which the negative sign denotes the incoming radial null paths, and the positive sign represents the outgoing ones. It is notable that we consider the outgoing paths—the positive sign of Eq. (13)—in order to calculate the area spectrum, because these paths are more related to the quantum behaviors under consideration. So, the shrinking velocity of the black hole horizon is given by

$$\dot{r_{\mathrm{h}}} = -\dot{r} = -i\left(\frac{g(\varepsilon)\,\psi(r,\varepsilon)}{f(\varepsilon)}\right). \tag{14}$$

Using the above equation and Eq. (10), we have

$$I = 2\int\int_0^H \frac{dH}{\dot{r_{\mathrm{h}}}}dr_{\mathrm{h}} = -2i\left[\int\int_0^H \frac{dH}{\left(\frac{g(\varepsilon)\psi(r,\varepsilon)}{f(\varepsilon)}\right)}dr\right]. \tag{15}$$

In order to solve the adiabatic invariant quantity of Eq. (15) we use the definition of Hawking's temperature, related to the surface gravity on the outer horizon (r_+) as $T_{\mathrm{bh}} = \frac{\hbar\kappa}{2\pi}$, in which κ is the surface gravity.

The area spectrum and also the entropy spectrum spacing change with respect to the change in coordinate transformation. In other words, the adiabatic invariant quantity ($\int p_i dq_i$) used in Majhi and Vagenas's method is not canonically invariant, so Jiang and Han modified this idea by considering the closed contour integral $\oint p_i dq_i$, which is invariant under coordinate transformations [91]. The closed contour integral can be considered as a path that goes from q_i^{out} to q_i^{in}, in which q_i^{out} and q_i^{in} are outside and inside the event horizon, respectively. So, the adiabatic invariant quantity is

$$I = \oint p_i dq_i = \int_{q_i^{\mathrm{in}}}^{q_i^{\mathrm{out}}} p_i^{\mathrm{out}} dq_i + \int_{q_i^{\mathrm{out}}}^{q_i^{\mathrm{in}}} p_i^{\mathrm{in}} dq_i, \tag{16}$$

where p_i^{in} and p_i^{out} are the conjugate momentums corresponding to the coordinates q_i^{in} and q_i^{out}, respectively, and $i = 0, 1, 2, \ldots$ It is notable that $q_1^{\mathrm{in}} = r_{\mathrm{h}}^{\mathrm{in}}$, $q_1^{\mathrm{out}} = r_{\mathrm{h}}^{\mathrm{out}}$, and $q_0^{\mathrm{in}} = q_0^{\mathrm{out}} = \tau$. Therefore, we can obtain the area spectrum of this black hole by using the tunnelling method and the covariant action of Eq. (16). The entropy spectra of various black holes have been studied in many works [92–104]. In the following, we obtain the entropy spectrum of BTZ black holes in massive gravity's rainbow. Then, we investigate the effects of such quantization on the properties of the black holes.

2. Entropy spectrum of BTZ black holes in massive gravity's rainbow

The metric of three-dimensional spacetime in the presence of the gravity's rainbow is given by

$$ds^2 = -\frac{\psi(r,\varepsilon)}{f(\varepsilon)^2}dt^2 + \frac{1}{g(\varepsilon)^2}\left(\frac{dr^2}{\psi(r,\varepsilon)} + r^2 d\varphi^2\right), \tag{17}$$

in which $\psi(r,\varepsilon)$ is the metric function of our black holes and the $f(\varepsilon)$ and $g(\varepsilon)$ functions are rainbow functions. The Lagrangian governing the three-dimensional form of massive gravity is given by

$$L_{\mathrm{massive}} = m(\varepsilon)^2 \sum_{i=1}^{3} c_i(\varepsilon)\mathcal{U}_i(g,f),$$

where the $c(\varepsilon)_i$ are some energy-dependent constants and the U_i are symmetric polynomials of the eigenvalues of the 3×3 matrix $\mathcal{K}_\nu^\mu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$ written as

$$\mathcal{U}_1 = [\mathcal{K}], \quad \mathcal{U}_2 = [\mathcal{K}]^2 - \left[\mathcal{K}^2\right], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}]\left[\mathcal{K}^2\right] + 2\left[\mathcal{K}^3\right],$$

which leads to the following field equation:

$$\chi_{\mu\nu} = -\frac{c_1(\varepsilon)}{2}\left(\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}\right) - \frac{c_2(\varepsilon)}{2}\left(\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1\mathcal{K}_{\mu\nu} + 2\mathcal{K}^2_{\mu\nu}\right)$$
$$- \frac{c_3(\varepsilon)}{2}(\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2\mathcal{K}_{\mu\nu} + 6\mathcal{U}_1\mathcal{K}^2_{\mu\nu} - 6\mathcal{K}^3_{\mu\nu}). \tag{18}$$

The only non-zero term of massive gravity is $\mathcal{U}_1$. Therefore, the action for three-dimensional Einstein–massive-rainbow gravity in the presence of a Maxwell field is given by

$$\mathcal{I} = -\frac{1}{16\pi G(\varepsilon)}\int d^3x\sqrt{-g}\left[\mathcal{R} - 2\Lambda(\varepsilon) - \mathcal{F} + m(\varepsilon)^2 c_1(\varepsilon)\mathcal{U}_1(g,f)\right], \tag{19}$$

in which R and F are the scalar curvature and the Lagrangian of Maxwell electrodynamics, respectively. $G(\varepsilon)$ is the gravitational constant, which is energy dependent. $\Lambda(\varepsilon)$ is the energy-dependent cosmological constant, and f and g are a fixed symmetric tensor and metric tensor, respectively. It is notable that $m(\varepsilon)$ is related to the energy-dependent mass of the graviton. In addition, $F = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor with A_μ as its gauge potential. It is a matter of calculation to show that the field equations are obtained as

$$R_{\mu\nu} - \left(\frac{R}{2} - \Lambda(\varepsilon)\right)g_{\mu\nu} + G(\varepsilon)\left(\frac{1}{2}g_{\mu\nu}\mathcal{F} - 2L_{\mathcal{F}}F_{\mu\rho}F^{\rho}_{\nu}\right) + m(\varepsilon)^2\chi_{\mu\nu} = 0, \tag{20}$$

$$\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0. \tag{21}$$

The metric function is obtained in this gravity as [105]

$$\psi(r,\varepsilon) = -\frac{\Lambda(\varepsilon)r^2}{g(\varepsilon)^2} - m_0(\varepsilon) - 2G(\varepsilon)f(\varepsilon)^2 q(\varepsilon)^2\ln\left(\frac{r}{l(\varepsilon)}\right) + \frac{m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)r}{g(\varepsilon)^2}, \tag{22}$$

where $m_0(\varepsilon)$ is an energy-dependent integration constant related to the total mass of the BTZ black hole.

The electric potential (U) and the total electric charge (Q) are calculated as [105]

$$U(\varepsilon) = -q(\varepsilon)\ln\left(\frac{r_+}{l(\varepsilon)}\right), \tag{23}$$

$$Q(\varepsilon) = \frac{1}{2}f(\varepsilon)G(\varepsilon)q(\varepsilon). \tag{24}$$

Using the standard definition of the Hawking temperature ($T = \frac{\hbar\kappa}{2\pi}$), the surface gravity is obtained by considering the metric in Eq. (17) as [105]

$$\kappa = \frac{1}{2\pi}\sqrt{\frac{-1}{2}\left(\nabla_\mu\chi_\nu\right)\left(\nabla^\mu\chi^\nu\right)} = \frac{1}{2}\left(\frac{g(\varepsilon)\psi'(r,\varepsilon)}{f(\varepsilon)}\right). \tag{25}$$

Therefore, the Hawking temperature of these black holes is [105]

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{4\pi}\left(\frac{g(\varepsilon)\psi'(r,\varepsilon)}{f(\varepsilon)}\right)\Bigg|_{r=r_+} = -\frac{\Lambda(\varepsilon)r_+}{2\pi f(\varepsilon)g(\varepsilon)} + \frac{m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)}{4\pi f(\varepsilon)g(\varepsilon)} - \frac{f(\varepsilon)g(\varepsilon)G(\varepsilon)q(\varepsilon)^2}{2\pi r_+}, \tag{26}$$

where r_+ is the outer horizon of the black hole. The entropy of black holes can be obtained by employing the area law as [105]

$$S = \frac{\pi r_+}{2g(\varepsilon)}. \tag{27}$$

The total mass of these solutions is given by [105]

$$M = \frac{m_0(\varepsilon)}{8f(\varepsilon)}. \tag{28}$$

Here, we want to quantize the entropy of this black hole using the adiabatic invariant quantity and Bohr–Sommerfeld quantization rule. Considering Eqs. (15) and (16), we have

$$I = \oint p_i dq_i = -4i\left[\int_{r_{\mathrm{out}}}^{r_{\mathrm{in}}}\int_0^H \frac{dH}{\psi(r,\varepsilon)}dr\right] \times \frac{f(\varepsilon)}{g(\varepsilon)}. \tag{29}$$

In order to solve the above equation we use the near-horizon approximation, so $\psi(r)$ can be Taylor expanded in the following form:

$$\psi(r,\varepsilon) = \psi(r,\varepsilon)_{r=r_+} + (r-r_+)\psi'(r,\varepsilon)_{r=r_+} + \cdots \tag{30}$$

The first term is zero ($\psi(r,\varepsilon)_{r=r_+} = 0$). Using the Cauchy integral theorem and the temperature in Eq. (26), Eq. (29) reduces to

$$I = \oint p_i dq_i = 4\pi\int_0^H \frac{dH}{\kappa} = 2\hbar\int_0^H \frac{dH}{T}. \tag{31}$$

The Smarr formula for a BTZ black hole in massive gravity's rainbow is

$$dM = dH = TdS - UdQ. \tag{32}$$

Therefore, Eq. (31) becomes

$$\oint p_i dq_i = 2\hbar S\left[1+\frac{U(\varepsilon)f(\varepsilon)G(\varepsilon)}{2Q(\varepsilon)}\ln\left(G(\varepsilon)\left[2\Lambda(\varepsilon)r_+ - m^2(\varepsilon)c(\varepsilon)c_1(\varepsilon)\right]r_+ + 8Q^2(\varepsilon)g^2(\varepsilon)\right)\right]. \tag{33}$$

On the other hand, the Bohr–Sommerfeld quantization rule is given by

$$\oint p_i dq_i = 2\pi n\hbar, \qquad n = 1,2,3,\ldots \tag{34}$$

Comparing Eq. (33) with Eq. (34), one can obtain the entropy spectrum as

$$S = \frac{n\pi}{1+\frac{U(\varepsilon)f(\varepsilon)G(\varepsilon)}{2Q(\varepsilon)}\ln\{G(\varepsilon)\left[2\Lambda(\varepsilon)r_+ - m^2(\varepsilon)c(\varepsilon)c_1(\varepsilon)\right]r_+ + 8Q^2(\varepsilon)g^2(\varepsilon)\}}. \tag{35}$$

Quantization of the entropy has the following specific physical results:

(1) The quantization results in the formation of a spectrum of the entropy characterized by n. The entropy is an increasing function of n, but the general behavior of the entropy is not determined by this parameter.
(2) The entropy spectrum is a function of the black hole's properties (the electric field, the massive parameters, the cosmological constant, and the gravity's rainbow generalizations and horizon radius).

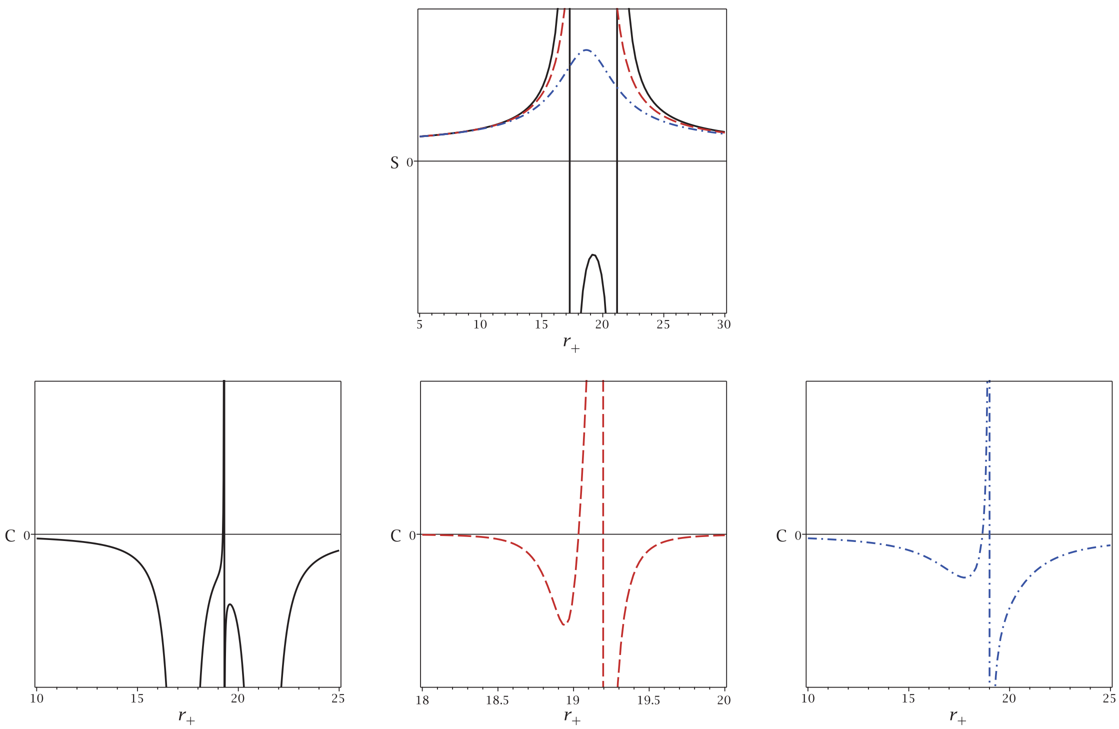

Fig. 1. S (upper panel) and C (lower panels) versus r_+ for $G(\varepsilon) = Q(\varepsilon) = U(\varepsilon) = c(\varepsilon) = c_1(\varepsilon) = m(\varepsilon) = n = 1, g(\varepsilon) = f(\varepsilon) = 1.1, \Lambda(\varepsilon) = 0.0130$ (continuous line), $\Lambda(\varepsilon) = 0.0131$ (dashed line), and $\Lambda(\varepsilon) = 0.0134$ (dash-dotted line).

(3) While the usual entropy of black holes, Eq. (27), is a divergence-free and smooth function of the horizon radius, the quantization results in the possibility of divergences for the entropy. The divergent points of entropy are obtained as

$$r_+|_{S\to\infty} = \frac{G(\varepsilon)c(\varepsilon)c_1(\varepsilon)m^2(\varepsilon) \pm\sqrt{G^2(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon) - 64G(\varepsilon)\Lambda(\varepsilon)Q^2(\varepsilon)g^2(\varepsilon) + 8G(\varepsilon)\Lambda(\varepsilon)e^{-\frac{2Q(\varepsilon)}{U(\varepsilon)f(\varepsilon)G(\varepsilon)}}}}{4G(\varepsilon)\Lambda(\varepsilon)}, \quad (36)$$

which shows that under certain conditions the quantized entropy could have up to two divergences (Fig. 1). It should be noted that only positive values of Eq. (36) are physically acceptable. In the absence of massive gravity, only for AdS solutions, divergent entropy could be obtained. In general, the major condition for the existence of the divergent quantized entropy is given by

$$\Lambda(\varepsilon) \leq \frac{G(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon)}{64Q^2(\varepsilon)g^2(\varepsilon) - e^{-\frac{2Q(\varepsilon)}{U(\varepsilon)f(\varepsilon)G(\varepsilon)}}}, \quad (37)$$

with positivity of Eq. (36). If one considers the absence of divergence in entropy as a requirement for having physical solutions, the mentioned condition gives us an upper (lower) limit on massive gravity's parameter (the cosmological constant).

(4) Quantization of the entropy could also provide the possibility of the formation of a root for this quantity. The root of the entropy is given by

$$r_+|_{S=0} = \frac{G(\varepsilon)c(\varepsilon)c_1(\varepsilon)m^2(\varepsilon) \pm \sqrt{G^2(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon) - 64G(\varepsilon)\Lambda(\varepsilon)Q^2(\varepsilon)g^2(\varepsilon)}}{4G(\varepsilon)\Lambda(\varepsilon)}, \tag{38}$$

which shows that the existence of the entropy root is restricted to satisfaction of the condition

$$\Lambda(\varepsilon) \leq \frac{G(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon)}{64Q^2(\varepsilon)g^2(\varepsilon)}, \tag{39}$$

and positivity of Eq. (38). It should be noted that for the usual entropy of Eq. (27), the only root for the entropy exists at $r_+ = 0$, whereas for the quantized entropy, the root is modified to a non-zero horizon radius.

(5) If the relation

$$r_+|_{\text{free}} = \frac{G(\varepsilon)c(\varepsilon)c_1(\varepsilon)m^2(\varepsilon) \pm \sqrt{G^2(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon) - 64G(\varepsilon)\Lambda(\varepsilon)Q^2(\varepsilon)g^2(\varepsilon) + 8G(\varepsilon)\Lambda(\varepsilon)}}{4G(\varepsilon)\Lambda(\varepsilon)} \tag{40}$$

holds, the quantized entropy will be independent of the black hole's properties. In other words, the quantized entropy will have a fixed value irrespective of variations in the black hole's electric charge, the massive parameters, the cosmological constant, and the horizon radius. This is one of the consequences of the quantization of entropy which says that the entropy will be independent of the size and electric charge of the black hole (if Eq. (40) holds).

(6) One of the most important results of the quantization is non-zero entropy for $r_+ = 0$, and is given by

$$S = \frac{n\pi}{1 + \frac{U(\varepsilon)f(\varepsilon)G(\varepsilon)}{2Q(\varepsilon)} \ln\{8Q^2(\varepsilon)\, g^2(\varepsilon)\}}. \tag{41}$$

The limit $r_+ \to 0$ is known as the high-energy limit. Evidently, in this limit the quantized entropy is non-zero (in contrast to usual entropy) and it is governed by the electric part of black holes, gravity's rainbow generalization, and n. Another interpretation of this limit is that for black hole evaporation, despite the vanishing internal energy of the black holes, the entropy remains non-zero. Interestingly, in this limit, a non-zero temperature could also be observed, but while the entropy in this limit is independent of massive gravity, the temperature only depends on massive gravity [105].

The asymptotic behavior of quantized entropy is given by

$$\lim_{r_+\to\infty} S = \frac{n\pi}{1 + \frac{U(\varepsilon)f(\varepsilon)G(\varepsilon)}{2Q(\varepsilon)} \ln\{2G(\varepsilon)\, \Lambda(\varepsilon)\, r_+^2\}} + O\left(\frac{1}{r_+}\right), \tag{42}$$

which shows that in this limit, the only non-contributing factor on the behavior of entropy is the massive gravity.

To further clarify the effects of quantization on the thermodynamics of black holes, we investigate the heat capacity. The heat capacity gives a detailed picture regarding thermal/thermodynamical behavior of the solutions. In general, for black holes with quantized entropy, this quantity is given by

$$
\begin{aligned}
C &= T\frac{\left(\frac{\partial S}{\partial r_+}\right)_{Q,U}}{\left(\frac{\partial T}{\partial r_+}\right)_{Q,U}} \\
&= \frac{n\pi\, Q(\varepsilon)\, U(\varepsilon) f(\varepsilon)\, G(\varepsilon)^2 \left(m^2(\varepsilon)\, c(\varepsilon)\, c_1(\varepsilon) - 4\,\Lambda(\varepsilon)\, r_+\right) r_+}{\left[2\, Q(\varepsilon) + UfG \ln\{G(\varepsilon)\left[2\Lambda(\varepsilon)\, r_+ - m^2(\varepsilon)\, c(\varepsilon)\, c_1(\varepsilon)\right] r_+ + 8Q^2(\varepsilon)\, g^2(\varepsilon)\}\right]^2 Z},
\end{aligned}
\tag{43}
$$

where $Z = G(\varepsilon)\, r_+^2 \Lambda(\varepsilon) - 4\, Q(\varepsilon)^2 g^2(\varepsilon)$.

Evidently, by quantizing the entropy, the heat capacity is consequently quantized. But here, we should be a little bit cautious. The reason is that quantization is only done for the entropy while the temperature is not quantized. In addition, we have considered an ensemble where the electric charge and the potential are both fixed (canonical ensemble). The obtained heat capacity highlights several important contributions of quantized entropy:

(1) Here as well, due to quantization, a spectrum is formed by the heat capacity characterized by n. But overall, the behavior of heat capacity is not determined by n.

(2) The positivity/negativity of the heat capacity determines the thermal stability/instability of the solutions. Therefore, the stability conditions are given by the following set of conditions:

$$
\begin{cases}
\frac{4\,Q(\varepsilon)^2 g^2(\varepsilon)}{G(\varepsilon) r_+^2} < \Lambda(\varepsilon) < \frac{m^2(\varepsilon)c(\varepsilon)c_1(\varepsilon)}{4\, r_+}, \\[2ex]
\frac{4\,Q(\varepsilon)^2 g^2(\varepsilon)}{G(\varepsilon) r_+^2} > \Lambda(\varepsilon) > \frac{m^2(\varepsilon)c(\varepsilon)c_1(\varepsilon)}{4\, r_+},
\end{cases}
\tag{44}
$$

which confirms that for asymptotical flat and AdS solutions, the heat capacity will be negative and solutions will always be thermally unstable. The only possible thermally stable solution exists for the dS branch and with the satisfaction of certain conditions.

(3) The root of the heat capacity is given by

$$
r_+|_{C=0} = \begin{cases}
\frac{m^2(\varepsilon)c(\varepsilon)c_1(\varepsilon)}{4\,\Lambda(\varepsilon)}, \\[2ex]
\frac{G(\varepsilon)c(\varepsilon)c_1(\varepsilon)m^2(\varepsilon) \pm \sqrt{G^2(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon) - 64G(\varepsilon)\Lambda(\varepsilon)Q^2(\varepsilon)g^2(\varepsilon)}}{4G(\varepsilon)\Lambda(\varepsilon)},
\end{cases}
\tag{45}
$$

which confirms two important points: first of all, one of the possible roots originates only from the contribution of the massive gravity. The second point is that some of the roots of the heat capacity are also entropy roots—please compare Eqs. (38) and (45).

(4) The divergences in heat capacity are where thermal phase transitions take place. The divergences of heat capacity are given by

$$
r_+|_{C\to\infty} = \begin{cases}
\frac{2Q(\varepsilon)g(\varepsilon)}{\sqrt{G(\varepsilon)\Lambda(\varepsilon)}}, \\[2ex]
\frac{G(\varepsilon)c(\varepsilon)c_1(\varepsilon)m^2(\varepsilon) \pm \sqrt{G^2(\varepsilon)c^2(\varepsilon)c_1^2(\varepsilon)m^4(\varepsilon) - 64G(\varepsilon)\Lambda(\varepsilon)Q^2(\varepsilon)g^2(\varepsilon) + 8G(\varepsilon)\Lambda(\varepsilon)e^{-\frac{2Q(\varepsilon)}{U(\varepsilon)f(\varepsilon)G(\varepsilon)}}}}{4G(\varepsilon)\Lambda(\varepsilon)}.
\end{cases}
\tag{46}
$$

The quantization has also resulted in a modification in the place and number of the divergences in heat capacity compared to the non-quantized case. Evidently, it is possible for the heat capacity to have up to three divergences (see the right panel of Fig. 1). One of these divergences is due to the contributions and interactions of the cosmological constant and the electric charge. The other divergences are same as those obtained for the entropy—please compare Eqs. (36) and (46)—indicating that these phase transitions are due to the quantization. In other words, through the quantization of entropy, novel thermal phase transitions are introduced in the thermodynamical structure of the black holes.

3. Conclusions

In this paper, we considered BTZ black holes in the presence of massive gravity's rainbow. We studied the quantization of entropy of these black holes using an adiabatic invariant integral method put forwarded by Majhi and Vagenas with modifications proposed by Jiang and Han, and the Bohr–Sommerfeld quantization rule.

It was shown that quantization of the entropy results in the formation of a spectrum of entropy. In addition, the quantized entropy leads to the existence of divergences and roots for the entropy that were absent in the usual entropy. It was also shown that in the high-energy limit and/or for a vanishing horizon radius the entropy has a non-zero value, which again was in contrast to the usual entropy. In general, the behavior of quantized entropy depends on the parameter of horizon radius, massive gravity, the rainbow functions, the cosmological and Newton constants, and also the electric charge. But it was shown that for specific choices of different parameters, the effects of the black holes' properties (both gravitational and matter field contributions) could be cancelled, resulting in a fixed entropy. In other words, for this specific case, the quantized entropy of black holes was independent of the size, electric field, and other characteristics of the black holes. In addition, the heat capacity of solutions was investigated. It was shown that quantization resulted in the appearance of novel phase transition points in the structure of the black holes. Also, it was shown that due to quantization, only dS black holes could be thermally stable while asymptotically flat and AdS solutions were unstable.

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