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Gapped gravitinos, isospin $\frac{1}{2}$ particles, and $\mathcal{N} = 2$ partial breaking

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Using results on topological band theory of phases of matter and discrete symmetries, we study topological properties of the band structure of physical systems involving spin $\frac{1}{2}$ and $\frac{3}{2}$ fermions. We apply this approach to study partial breaking in four-dimensional (4D) $\mathcal{N} = 2$ gauged supergravity in the rigid limit, and we describe the fermionic gapless mode in terms of the chiral anomaly. We also study the homologue of the usual spin–orbit coupling, $\vec{L}.\vec{S}$, that opens the vanishing band gap for free $s = \frac{1}{2}$ fermions; we show that it is precisely given by the central extension of the $\mathcal{N} = 2$ supercurrent algebra in 4D spacetime. We also comment on the rigid limit of Andrianopoli et al. [Phys. Lett. B **744**, 116 (2015)], and propose an interpretation of energy bands in terms of a chiral gapless isospin $\frac{1}{2}$ particle (iso-particle). Other features, such as discrete T-symmetry in the Fayet–Iliopoulos coupling space, the effect of quantum fluctuations, and the link with the Nielson–Ninomiya theorem, are also studied.

Subject Index B10, B11, B12, B31

1. Introduction

In the few years past there has been intensive interest in topological band theory in the Brillouin zone and in three-dimensional (3D) effective Chern–Simons field theories in connection with the phases of matter [1–9]. This interest has mainly concernd spin $s = \frac{1}{2}$ topological matter and spin 1 topological gauge theories in lower spacetime dimensions; this is because of their particular properties in dealing with condensed matter systems like topological insulators and superconductors, and also for the role they play in the quantum Hall effect as well as in the study of boundary states and anomalies [10–17]. By looking to extend some special features of these studies to systems with spacetime spins beyond $\frac{1}{2}$ or 1, we fall into supergravity-like models where fermionic modes of higher spins such as relativistic spin $\frac{3}{2}$ are also known to play a basic role. In this paper we would like to explore some topological aspects of the band theory of systems having spins less than or equal to 2, and to look for a physical model where the topological properties obtained for spins $s = \frac{1}{2}, 1$ can be extended to higher spins.

A priori, physical systems with spins $s \leq 2$ may exist in spacetime dimensions $D = d + 1 \geq 3$ where spin $\frac{3}{2}$ and 2 particle fields have non-trivial gauge degrees of freedom; to get started, it is then natural to begin by fixing the full-spin content of the physical system we are interested in here, and also to define the Hamiltonian model or the field equations describing the full dynamics. To find a physical system with higher spins where such kinds of studies may be relevant, and also to identify the appropriate approach to use as the starting point, we give here two motivations: the first concerns

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the choice of a particular system having fermions with different spins, say two types of fermionic spins $s = \frac{1}{2}$ and $\frac{3}{2}$, and the other regards the tools to use for approaching their band structure. First, by studying the constructions of Refs. [17–19], one comes to the conclusion that several topological condensed matter statements based on spin $\frac{1}{2}$ fermions may be approached by starting with the Dirac equation of $(1 + d)$ relativistic theory. From this theory one may engineer effective Hamiltonians breaking explicitly the $SO(1, d)$ Lorentz symmetry by allowing non-linear dispersion relations due to underlying lattice geometries and interactions. It follows from this description that quantities like fermionic gapless/gapped modes, chiral ones, and edge states have interpretations in terms of massless/massive states, quasi-particles with exotic statistics, and anomalies whose explanation requires the use of topological notions such as manifold boundaries, left/right windings, the Berry connection, and the Nielson–Ninomiya theorem.

To look to extend results on spin $\frac{1}{2}$ topological matter to spin $\frac{3}{2}$ gravitinos, we then have to go beyond the Dirac equation, for instance by considering the Rarita–Schwinger equation of gravitinos, and try to mimic the analysis done for spin $\frac{1}{2}$. Even though this is an interesting direction to take [20,21], we will not follow this path here because of the complicated $\mathcal{N} = 2$ supergravity interactions that make the field equations difficult to manage. Instead, we will rather use related equations given by extended supergravity Ward identities [22,23]. The use of these Ward identities has been motivated by the question of what kind of physical systems the specific properties of the gravitino band structure may serve. Recalling the role played by gravitinos in the spontaneous breaking of local supersymmetry, we immediately come to the point that gapless and gapped gravitinos can be applied to study the problem of partial supersymmetry breaking of $\mathcal{N}$-extended vector-like theories. Indeed, in the example of the effective $\mathcal{N} = 2$ gauged supergravity in 4D spacetime, one has, in addition to bosons (with $s = 0, 1, 2$) and spin $\frac{1}{2}$ fermions, two gravitinos $\left(\psi^1_{\alpha\mu}, \psi^2_{\alpha\mu}\right)$ forming an isospin $\frac{1}{2}$ particle; that is to say, a doublet under the SU(2) R-symmetry involving pairs of gapless gravitino modes. The breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ then requires a partial lifting of the degeneracies of mode doublets, which, as in the case of condensed matter with spin $\frac{1}{2}$ fermions, may be achieved by turning on a spin–orbit-like coupling $\vec{L}.\vec{S}$ [24]. The study of spin $\frac{3}{2}$ matter therefore offers a good opportunity to identify the iso-particle Hamiltonian including the homologue of $\vec{L}.\vec{S}$ that induces partial breaking of supersymmetry. This coupling will be denoted $\vec{\xi}.\vec{\mathcal{I}}$, where $\vec{\xi}$ plays the role of the angular momentum $\vec{L}$ and the isospin $\vec{\mathcal{I}}$ the role of the spin $\vec{S}$. In this regard, it interesting to recall that spontaneous partial breaking in $\mathcal{N} = 2$ supergravity may be done by the super-Higgs mechanism, in which, in $\mathcal{N} = 1$ supermultiplet language, a massive $\mathcal{N} = 1$ gravitino multiplet can be created by merging three multiplets: a massless $\mathcal{N} = 1$ gravitino eating a massless $\mathcal{N} = 1$ U(1) multiplet and an $\mathcal{N} = 1$ chiral multiplet [25]. But here, the partial breaking will be done by the isospin–orbit coupling that opens the gap energy between the two gravitinos. In this study, we will show that the $\vec{\xi}.\vec{\mathcal{I}}$ coupling is precisely given by the central anomaly of the $\mathcal{N} = 2$ supercurrent algebra in 4D spacetime [26,27].

The main purpose of this work is, then, to use results on topological band theory of fermionic matter and chiral anomalies as well as discrete symmetries to study partial breaking in $\mathcal{N} = 2$ gauged supergravity in four dimensions. The spacetime fields of our system are given by the field content of the standard $\mathcal{N} = 2$ supermultiplets; in particular, the field content of the gravity multiplet, n_V vector multiplets, and n_H matter multiplets. To perform this study we will use $\mathcal{N} = 2$ supergravity Ward identities in the rigid limit as considered in Ref. [28], and also study the partial breakings by using the topological approach along with the Nielson–Ninomiya theorem and the chiral anomaly. We also study the effect of quantum harmonic fluctuations in the Fayet–Iliopoulos (FI) coupling space, and show that the result of Ref. [28] is not affected by quantum corrections provided that a saturated condition holds.

The presentation is as follows: In Sect. 2, we describe some tools on partial breaking in the rigid limit of $\mathcal{N}=2$ supergravity theory and present the basic equations to start with. We also give some useful comments. In Sect. 3, we derive the free Hamiltonian of the iso-particles in $\mathcal{N}=2$ gauged supergravity, work out the isospin–orbit coupling that opens the zero gap between the two gravitino zero modes, and show how time reversing symmetry $\boldsymbol{T}$ and $\boldsymbol{PT}$ (combined $\boldsymbol{T}$ and parity $\boldsymbol{P}$) can be implemented. In Sect. 4 we study gapless and gapped gravitinos in $\mathcal{N}=2$ gauged supergravity, and describe the properties of partial supersymmetry breakings and their interpretation from the viewpoint of the Nielson–Ninomiya theorem and the chiral anomaly. We also discuss the effect of quantum fluctuations on partial breaking of $\mathcal{N}=2$ supersymmetry. Section 5 is devoted to our conclusions and comments.

2. Rigid limit of $\mathcal{N}=2$ Ward identity: the U(1) model

Following Ref. [28], partial breaking of rigid and local extended supersymmetries is highly constrained; it can occur in a certain class of supersymmetric field theories provided one evades some no-go theorems [29–32]; see also Refs. [33–38]. In global 4D $\mathcal{N}=2$ theories, this was first noticed in Refs. [26,39], and was explicitly realized in Refs. [40,41] for a model of a self-interacting $\mathcal{N}=2$ vector multiplet in the presence of $\mathcal{N}=2$ electric and magnetic FI terms. There, it was explicitly shown that the presence of electric $\vec{\nu}$ and magnetic $\vec{m}$ FI couplings is crucial to achieve partial breaking. The general conditions for $\mathcal{N}=2$ partial supersymmetry breaking have been recently elucidated by L. Andrianopoli et al. in Ref. [28], where it was also shown that $\vec{\nu}$ and $\vec{m}$ should be non-aligned ($\vec{\nu}\wedge\vec{m}\neq\vec{0}$). Their starting point for deriving the general conditions for partial supersymmetry breaking in the rigid limit[1] was the reduced $\mathcal{N}=2$ gauged supergravity Ward identity,

$$\mathcal{V}\delta_B^A + C_B^A = \sum_{i=1}^{n_V} \delta_B\lambda^{iC}\delta^A\lambda_{iC}, \tag{2.1}$$

where the spin $\frac{1}{2}$ fermions λ^{iA} and $\lambda_{iB} := \varepsilon_{BA} g_{i\bar{j}}\lambda^{\bar{j}A}$ refer to the chiral and antichiral projections of the gauginos respectively. Here, the SO(1, 3) spacetime spin index of the λ^{iA} fermions has been omitted for simplicity, while we have shown the other two indices, A and i. $A=1,2$ refers to the isospin $\frac{1}{2}$ representation of the $SU(2)_R$ symmetry of the $\mathcal{N}=2$ supersymmetric algebra, since $\mathcal{N}=2$ gauginos are iso-doublets under $SU(2)_R$; this index is lowered and raised by the antisymmetric tensor ε^{AB} and its inverse ε_{BA}. The index $i=1,\ldots,n_V$ designates the number of $\mathcal{N}=2$ vector multiplets in the Coulomb branch of the $\mathcal{N}=2$ gauged supergravity theory. Notice also that the quantity $\left(\delta_B\lambda^{iA}\right)$ is a convention notation for the $\mathcal{N}=2$ supersymmetric transformation of gauginos, which is given by $\delta_{\text{susy}}\lambda^{iA} = \left(\delta_B\lambda^{iA}\right)\epsilon^B$, with the two fermions $\epsilon^A = \left(\epsilon^1,\epsilon^2\right)$ standing for the supersymmetric transformation parameters. In Eq. (2.1), the right-hand side is restricted to the pure Coulomb branch and so corresponds to the rigid limit of the following local identities:

$$\sum_i \alpha_i\delta_B\lambda^{iC}\delta^A\lambda_{iC} = \tilde{\mathcal{V}}\delta_B^A - \sum_u \alpha_u\delta_B\zeta^u\delta^A\zeta_u - \sum_{\mu,\nu}\alpha_0\delta^A\psi_{\nu C}\Gamma^{\mu\nu}\delta_B\psi_\mu^C. \tag{2.2}$$

[1] The rigid limit is implemented through a rescaling of the field contents of the theory and the spacetime supercoordinates by using the dimensionless parameter $\mu=\frac{\Lambda}{M_{pl}}$. For explicit details, see Refs. [29,41].

The left-hand side of Eq. (2.1) contains two basic terms, namely the rigid limit of the scalar potential $\mathcal{V}\delta_B^A$ and an extra traceless constant matrix,

$$C_B^A = \vec{\xi}.\,(\vec{\tau})_B^A\,, \qquad \mathrm{Tr}\,C = 0. \tag{2.3}$$

This Hermitian traceless matrix can be interpreted as an anomalous central extension in the $\mathcal{N} = 2$ supersymmetric current algebra [27,29,39]; it only affects the commutator of two supersymmetry transformations of the gauge field [29,41] and contains data on hidden gravity and matter sectors. Recall that the basic anticommutator of the $\mathcal{N} = 2$ supercurrent algebra is

$$\left\{\mathcal{J}^{0A}(x)\,, \int d^3y \bar{\mathcal{J}}_B^0(y)\right\} = \delta_B^A \sigma_\mu \mathcal{T}^{\mu 0} + C_B^A, \tag{2.4}$$

where $\mathcal{J}_{\alpha A}^0(x)$, $\bar{\mathcal{J}}_{\dot{\alpha}A}^0(x)$, and $\mathcal{T}_\mu^0(x)$ are the time components of the supersymmetric current densities $\mathcal{J}_{\alpha A}^\nu$, $\bar{\mathcal{J}}_{\dot{\alpha}A}^\nu$, and $\mathcal{T}_\mu^\nu$, respectively. The time component densities in the current superalgebra, Eq. (2.4), are related to the Q_α^A, $\bar{Q}_{\dot{\alpha}B}$, and P_μ charges of the $\mathcal{N} = 2$ supersymmetric QFT$_4$ in the usual manner. For example,

$$Q_{\alpha A} = \int d^3x \mathcal{J}_{\alpha A}^0, \qquad P_\mu = \int d^3x \mathcal{T}_\mu^0, \tag{2.5}$$

obeying $Q_B\bar{Q}^A + \bar{Q}^A Q_B \sim \delta_B^A \sigma^\mu P_\mu$; the usual globally defined $\mathcal{N} = 2$ supersymmetric algebra with C_B^A constrained to vanish.

By comparing Eq. (2.1) with the general form of the Ward identities, Eq. (2.2), we deduce that the C_B^A term captures the contribution of the fermion shifts to the Ward identity coming from the rigid limit of the hidden gravity ($\delta^A \psi_{\nu C} \Gamma^{\mu\nu} \delta_B \psi_\mu^C$) and the matter ($\delta_B \zeta^u \delta^A \zeta_u$) branches. For the simple example of an Abelian U(1) gauge multiplet ($n_V = 1$), the anomaly iso-vector $\vec{\xi} = \mathrm{Tr}\,(\vec{\tau}C)$ has been realized in terms of the electric $\vec{\nu}$ and the magnetic $\vec{m}$ FI coupling constant iso-vectors of the Coulomb branch of the effective $\mathcal{N} = 2$ U(1) gauge theory as follows:[2]

$$\vec{\xi} = \vec{\nu} \wedge \vec{m}, \qquad \vec{m} \neq \mathbb{R}^* \vec{\nu}, \tag{2.6}$$

obeying the remarkable property $\vec{\xi}.\vec{\nu} = 0$ and $\vec{\xi}.\vec{m} = 0$; see Eq. (2.13). Moreover, partial breaking of supersymmetry takes place at [28,29]

$$\mathcal{V} = \left|\vec{\xi}\right| \geq 0. \tag{2.7}$$

This relation will be used later on when considering topological aspects of gapless fermions (Sect. 4.1) as well as harmonic fluctuations (Sect. 4.2), but before that let us add comments regarding Eqs. (2.6) and (2.7).

First, notice that in order to have a non-zero $\vec{\xi}$ it is sufficient to take the following particular and simple choice,

$$\vec{\nu} = \begin{pmatrix} \nu_x \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{m} = \begin{pmatrix} 0 \\ m_y \\ 0 \end{pmatrix}, \qquad \vec{\xi} = \begin{pmatrix} 0 \\ 0 \\ \xi_z \end{pmatrix}, \tag{2.8}$$

[2] The exact expression found in Ref. [28] is $\vec{\xi} = 2\vec{\nu} \wedge \vec{m}$. Here, the factor $2 = (\sqrt{2})^2$ has been absorbed by scaling the FI couplings.

satisfying $\vec{\nu}.\vec{m} = 0$ and $\vec{\nu} \wedge \vec{m} \neq \vec{0}$. This particular choice shows that a quadratic term of type

$$\frac{\gamma_\perp}{2} |\vec{m}| \times |\vec{\nu}|$$

like the one appearing in Eq. (2.17) becomes necessary for the contribution of the $\vec{\xi}$-direction normal to the $(\vec{\nu}, \vec{m})$ plane; that is to say:

$$\vec{\xi} = \vec{0} \qquad \Rightarrow \qquad \gamma_\perp = 0. \tag{2.9}$$

This implication is obviously not usually true since, for non-orthogonal $\vec{\nu}$ and $\vec{m}$, we have $\vec{\nu}.\vec{m} = |\vec{m}| \times |\vec{\nu}| \cos\theta \neq 0$ as long as $\theta \neq \pm\frac{\pi}{2} \bmod 2\pi$. The trick $\theta = \pm\frac{\pi}{2}$ will help us to detect the effect of $\vec{\xi}$, especially when studying quantum fluctuations around the $\mathcal{N} = 2$ supersymmetric ground states $\mathcal{V} = 0$ and $\mathcal{V} = \left|\vec{\xi}\right|$.

Second, observe that by setting $\tau_{[kl]} = \frac{1}{2}\varepsilon_{kln}\tau^n$, $\xi^{[lk]} = \varepsilon^{klm}\xi_n$, and $\varepsilon_{kln}\varepsilon^{klm} = 2\delta_n^m$, it follows that $\xi^{[kl]}\tau_{[kl]} = \xi_i\tau^i$, and then the central extension matrix in Eq. (2.3) can be also expressed as

$$C_B^A = \xi^{[kl]} \left(\tau_{[kl]}\right)_B^A, \qquad \tau_{[kl]} = \frac{1}{4i}\left[\tau_k, \tau_l\right]. \tag{2.10}$$

This way of expressing C_B^A is interesting since, supported by the dimensional argument, it gives an idea of how to realize the factor $\xi^{[kl]}$ in terms of the electrical ν^k and magnetic m^l couplings of FI. Antisymmetry implies the natural factorization

$$\xi^{[kl]} = \nu^k m^l - \nu^l m^k, \qquad \xi^{[xy]} = \nu^x m^y - \nu^y m^x, \tag{2.11}$$

which is nothing but the Andrianopoli et al. factorization of Eq. (2.6). We expect that this trick can also help to find the extension of the $\mathcal{N} = 2$ realization in Eq. (2.6) to higher supergravities, in particular to $\mathcal{N} = 4$ theory in the rigid limit where there is no matter branch; this generalization will not be considered here. Notice that for the simple choice of Eq. (2.8) we have the diagonal matrix

$$C_B^A = \begin{pmatrix} \nu_x m_y & 0 \\ 0 & -\nu_x m_y \end{pmatrix}, \tag{2.12}$$

showing that, in the rest frame, we have $\delta_B^A \sigma_\mu \mathcal{T}^{\mu 0} = \delta_B^A \mathcal{V}$, and then the $\mathcal{N} = 2$ current algebra of Eq. (2.4) splits into two $\mathcal{N} = 1$ copies with right-hand energy densities given by $\mathcal{V} \pm \nu_x m_y$.

Third, a non-vanishing $\vec{\xi}$ requires in general non-collinear $\vec{\nu}$ and $\vec{m}$ vectors, so the unit vectors $\vec{e}_\nu = \frac{\vec{\nu}}{|\vec{\nu}|}$, $\vec{e}_m = \frac{\vec{m}}{|\vec{m}|}$ generate a two-dimensional plane with a normal vector given by $\vec{e}_\xi = \frac{\vec{\xi}}{\left|\vec{\xi}\right|}$. These three vectors together form a 3D vector basis of $\tilde{\mathbb{R}}^3$ that we term the 3D iso-space:

$$\vec{e}_\nu \ ; \ \vec{e}_m \ ; \ \vec{e}_\xi, \qquad \vec{e}_\xi = \vec{e}_\nu \wedge \vec{e}_m. \tag{2.13}$$

By the terminology "3D iso-space" we intend to use its similarity with the usual Euclidean space $\mathbb{R}^3$ of the classical mechanics of point-like particles to propose a physical interpretation of Eq. (2.6) by using the notion of an isospin $I = \frac{1}{2}$ particle; this will be done in Sect. 3.

Notice, moreover, the following features:

- The relation in Eq. (2.6) concerns an effective $\mathcal{N} = 2$ U(1) gauge theory with one gauge supermultiplet $n_V = 1$. The general expression of $\vec{\xi}$, extending Eq. (2.6), as well as the general form of the scalar potential energy $\mathcal{V}$ associated with generic $U(1)^{n_V}$ effective gauge theories reduced to FI couplings, have been shown to be functions of the characteristic data of the special geometry of the scalar manifold. They are given by the factorizations

$$\mathcal{V} = \frac{1}{2}\delta_{ab}\mathcal{P}^{aM}\mathcal{S}_{MN}\mathcal{P}^{bN}, \tag{2.14}$$

$$\xi_a = \frac{1}{2}\varepsilon_{abc}\mathcal{P}^{bM}\mathcal{C}_{MN}\mathcal{P}^{cN}, \tag{2.15}$$

where $\mathcal{P}^{aM} = \left(m^{aI}, \nu_I^a\right)^t$ are moment maps carrying quantum numbers of $SU(2)_R \times SP\,(2n_V, R)$, $\mathcal{C}_{MN}$ is the metric of $SP\,(2n_V, R)$, and $\mathcal{S}_{MN}$ is a symmetric matrix of the form

$$\mathcal{S}_{MN} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} \tag{2.16}$$

encoding data on the scalar manifold of the $\mathcal{N} = 2$ theory; see Refs. [28,29] for more details. For the example of an Abelian U(1) gauge model, ξ_a is as in Eq. (2.6) while the scalar potential of Eq. (2.14) has the following remarkable quadratic shape:

$$\mathcal{V} = \alpha\,|\vec{m}|^2 + \beta\,|\vec{\nu}|^2 + \frac{\gamma}{2}\,|\vec{m}| \times |\vec{\nu}|\,, \tag{2.17}$$

with $4\alpha\beta > \gamma^2 > 0$ and α, β assumed positive for later use. Notice that here γ should be viewed as the sum $\gamma_\parallel + \gamma_\perp$, with $\gamma_\parallel$ describing the coupling in the $(\vec{m}, \vec{\nu})$ plane and $\gamma_\perp$ in the normal $\vec{m} \wedge \vec{\nu}$ directions; see also Eqs. (3.9) and (3.10).

- By substituting Eq. (2.16) and $\mathcal{P}^a = (m^a, \nu^a)$ into Eq. (2.14) we learn that the real parameters α, β, and γ in the above scalar potential indeed have a geometric interpretation in terms of the effective prepotential $\mathcal{F}$ of the $\mathcal{N} = 2$ special geometry. For example, the parameters $\frac{\gamma}{2}$ in Eq. (2.17) depend on both the real $\mathcal{R}$ and imaginary $\mathcal{I}$ parts of the second derivative of $\mathcal{F}$.
- The scalar potential in Eq. (2.17) has a particular dependence on $|\vec{m}|$ and $|\vec{\nu}|$; it can be presented as a quadratic form $\mathcal{V} = P^i G_{ij} P^j$, with P^i and metric G_{ij} as follows:

$$\mathcal{V} = (|\vec{m}|\,, |\vec{\nu}|) \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix} \begin{pmatrix} |\vec{m}| \\ |\vec{\nu}| \end{pmatrix} \tag{2.18}$$

  with $\det G = \alpha\beta - \frac{\gamma^2}{4}$. This form will diagonalized later on for explicit calculations.
- The above $\mathcal{V}$ might be viewed as a special potential; a more general expression would involve more free parameters, such as

$$\begin{aligned} \mathcal{V} = V_0 + \varrho_a \nu^a + w_a m^a + B_{ab}\nu^a m^b + \\ A_{ab}\nu^a\nu^b + C_{ab} m^a m^b, \end{aligned} \tag{2.19}$$

  where V_0 is a number that depends on the vacuum expectation values (VEVs) of the scalar fields, and the parameters of the effective $\mathcal{N} = 2$ theory like masses and gauge coupling constants, ϱ_a and w_a are two iso-vectors scaling in same manner as the FI constants, and A_{ab}, B_{ab}, and C_{ab} are dimensionless real 3×3 matrices—A_{ab} and C_{ab} are symmetric, but B_{ab} is a general matrix. These moduli may also characterize the scalar manifold of the effective $\mathcal{N} = 2$ supergravity and likely external fields as suspected from Table 1; see also Eq. (3.25), where $\vec{w}$ of the $w_a m^a$ is interpreted in terms of an external iso-magnetic field.

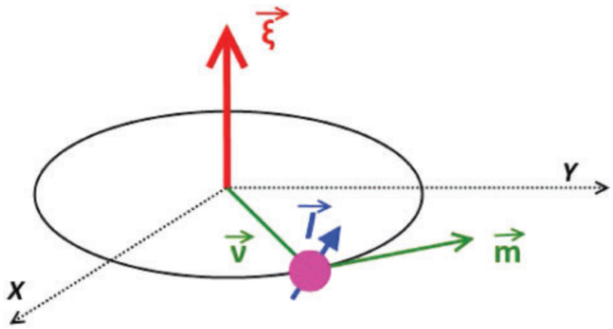

Fig. 1. A classical quasi-particle with angular momentum $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ in FI coupling space parameters. The electric FI coupling $\vec{\nu}$ is viewed as a position vector $\vec{r}$ and the magnetic coupling $\vec{m}$ as the momentum $\vec{p}$. In addition to $(\vec{\nu}, \vec{m})$, the quasi-particle also carries an intrinsic isospin charge $I = \frac{1}{2}$ as well as unit $U(1)_{elec} \times U(1)_{mag}$ charge due to the gauging of Abelian isometries of $\mathcal{N} = 2$ gauged supergravity. This image may be put in correspondence with an electron spinning around a nucleus.

Finally, notice that by giving these somehow explicit details on the $n_V = 1$ theory, we intend to use its simple properties to derive the iso-particle proposal and build the isospin–orbit coupling in $\mathcal{N} = 2$ supergravity mentioned in the introduction. We will also use these tools to study the isospin $\frac{1}{2}$ particle as well as hidden discrete symmetries that capture data on the topological phases of the right-hand side of the $\mathcal{N} = 2$ supersymmetry current algebra in Eq. (2.4).

3. Isospin $\frac{1}{2}$ particle proposal

The Andrianopoli et al. realization, Eq. (2.6), of the rigid anomaly iso-vector $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ in effective $\mathcal{N} = 2$ supersymmetric gauge theory is interesting and is very suggestive; see Fig. 1 for an illustration. This is because of the wedge product $\vec{\nu} \wedge \vec{m}$ that allows us to establish a correspondence between properties of the partial supersymmetry breaking and the electronic band theory with $\Delta_{soc}\vec{L}.\vec{S}$ spin–orbit coupling turned on ($\Delta_{soc} \neq 0$).

Indeed, the axial vector $\vec{\xi} = \vec{\nu} \wedge \vec{m}$, which we refer to below as the Andrianopoli et al. orbital vector, has the same form as the usual angular momentum vector,

$$\vec{L} = \vec{r} \wedge \vec{p}, \tag{3.1}$$

of a 3D classical particle with coordinate position $\vec{r}$ and momentum $\vec{p}$. By comparing the $\vec{\nu} \wedge \vec{m}$ formula of Eq. (2.6) with the above $\vec{r} \wedge \vec{p}$, it follows that the FI electric coupling $\vec{\nu}$ may be put in correspondence with the vector $\vec{r}$, and the magnetic $\vec{m}$ with the vector $\vec{p}$. Hence, we have the schematic picture shown in Table 1 linking the physics of classical particles (electrons) with the physics of iso-particles of $\mathcal{N} = 2$ gauged supergravity (gravitinos and gauginos).

In the left column of Table 1, the Euclidian $\mathbb{R}^3$ space is the usual 3D space with SO(3) isotropy symmetry. In this real space live bosons and fermions; in particular, fermions with intrinsic properties like spin $\frac{1}{2}$ particles with symmetry

$$SU(2)_{spin} \sim SO(3). \tag{3.2}$$

In the right column, the $\tilde{\mathbb{R}}^3$ is an iso-space with isotropy symmetry $SO(3)_R$ given by the R-symmetry $SU(2)_R$ of the $\mathcal{N} = 2$ supersymmetric algebra. This is a global symmetry group that will be imagined here as a global isospin group $SU(2)_{isospin}$ characterizing the quasi-particle of Fig. 1. Thus, the

Table 1. Comparison of classical particles and iso-particles.

Vectors in $\mathbb{R}^3$: electron	$\leftrightarrow$	Iso-vectors in $\tilde{\mathbb{R}}^3$: gravitinos
particle	:	iso-particle
$(\vec{r};\vec{p})$	:	$(\vec{\nu};\vec{m})$
$(\vec{r}+\delta\vec{r};\vec{p}+\delta\vec{p})$	:	$(\vec{\nu}+\delta\vec{\nu};\vec{m}+\delta\vec{m})$
isotropy SO(3)	:	R-symmetry SU(2)
orbital moment $\vec{L}=\vec{r}\wedge\vec{p}$	:	orbital moment $\vec{\xi}=\vec{\nu}\wedge\vec{m}$
Hamiltonian $h(\vec{r};\vec{p})$	:	Hamiltonian $h(\vec{\nu};\vec{m})$
spin $\vec{S}$	:	isospin $\vec{\mathcal{I}}$
gauge summetry $U(1)_{em}$	:	gauge symmetry $U(1)_{elec}\times U(1)_{mag}$

Table 2. Iso-space gravity and vector properties.

$\mathcal{N}=2$ multiplets	Field content in spin			Spin s	Isospin I
	graviton	:	2	2	0
Gravity $\boldsymbol{G}_{\mathcal{N}=2}$	gravitinos ψ^A	:	$2\times\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{3}$
	graviphoton A^1_μ	:	1	1	0
	vector A^2_μ	:	1	1	0
Vector $\boldsymbol{V}_{\mathcal{N}=2}$	gauginos λ^A	:	$2\times\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	scalars	:	2×0	0	0

homologue of the real space symmetry of Eq. (3.2) is given by

$$SU(2)_{\text{isospin}} \sim SU(2)_R \sim SO(3)_R. \tag{3.3}$$

Matter in the iso-space $\tilde{\mathbb{R}}^3$ is then given by quasi-particles carrying isospin charges under $SU(2)_R$, in particular the isospin $I=\frac{1}{2}$ describing the two gravitinos and the n_V pairs of gauginos of the Coulomb branch of the $\mathcal{N}=2$ gauged supergravity. Recall that in this theory, the particle content belongs to three $\mathcal{N}=2$ supermultiplets, namely the gravity $\boldsymbol{G}_{\mathcal{N}=2}$, the vector $\boldsymbol{V}_{\mathcal{N}=2}$, and the matter $\boldsymbol{H}_{\mathcal{N}=2}$. The properties of the first two are summarized in Table 2.

The field content includes the fermions (gravitinos and gauginos) with a non-trivial isospin charge. It also contains two spin $s=1$ gauge fields A^M_μ (graviphoton A^1_μ and Coulomb A^2_μ) with

$$U(1)_{\text{elec}}\times U(1)_{\text{mag}} \tag{3.4}$$

gauge transformations given by Abelian isometries of the scalar manifold of the supergravity theory. The fermionic fields $F^A=\psi^A,\lambda^A$ carry a unit $U(1)_{elec}\times U(1)_{\text{mag}}$ charge, and interact with the gauge vector fields A^M_μ through the minimal coupling $D_\mu F^A$, where the covariant derivative $D_\mu=\partial_\mu+\vartheta_M A^M_\mu$ with the electric/magnetic coupling ϑ_M; see Refs. [31,32,40,41,44–46] for other features.

Moreover, in Table 1 we have an exotic variable τ playing the role of the real time t of the left column of the table. This τ may be imagined in terms of an energy scale variable, and hence one is left with running complings $\vec{\nu}=\vec{\nu}(\tau)$ and $\vec{m}=\vec{m}(\tau)$ with

$$\vec{m}(\tau)\sim\frac{d\vec{\nu}(\tau)}{d\tau} \qquad \leftrightarrow \qquad \vec{p}(t)\sim\frac{d\vec{r}(t)}{dt}. \tag{3.5}$$

In what follows, we assume that the classical correspondence in Table 1 is also valid at quantum level, and study the energy band properties of the isospin $\frac{1}{2}$ particles (gravitinos and gauginos) of the $\mathcal{N}=2$ gauged supergravity.

3.1. *Deriving the free Hamiltonian of the iso-particle*

Here, we use Table 1 to build the free Hamiltonian $\boldsymbol{h} = h(\nu, m)$ of the iso-particle and study its classical and quantum behaviors. We also comment on some interacting terms appearing in the scalar potential, Eq. (2.19).

3.1.1. *Classical description*

Using the proposal in Table 1, the free Hamiltonian $\boldsymbol{h}$ of the classical iso-particle is given by the scalar potential of the supergravity theory. It is just the energy density of the supergravity theory,

$$\boldsymbol{h} = \mathcal{V}(\vec{\nu}, \vec{m}). \tag{3.6}$$

Because this energy is quadratic in $\vec{m}$ and $\vec{\nu}$ as shown by the rigid limit of Ref. [28], $\boldsymbol{h}$ then describes the free dynamics of a classical iso-particle in the 6D phase space $\tilde{\mathbb{R}}^3 \times \hat{\mathbb{R}}^3$ parameterized by the FI coupling parameters. By using Eqs. (2.17) and (2.18), we have

$$\boldsymbol{h} = \alpha\, |\vec{m}|^2 + \beta\, |\vec{\nu}|^2 + \frac{\gamma_{\parallel}}{2}\, |\vec{m}| \times |\vec{\nu}|\,, \tag{3.7}$$

where α, β, and the planar $\gamma_{\parallel}$ are three real parameters that have an interpretation in the special geometry of the scalar manifold of the $\mathcal{N} = 2$ effective theory. Here, they will be given an interpretation in terms of an effective mass μ and a frequency ω with relationships as in Eqs. (3.15) and (3.20). Notice the following useful features:

- The above-described $\boldsymbol{h}$ has the form of a classical harmonic oscillator energy $\frac{p_x^2}{2M} + \frac{Mw^2}{2}x^2$, so one can take advantage of this feature to learn more about the properties of the iso-particle of the $\mathcal{N} = 2$ gauged supergravity.
- The notation $\gamma_{\parallel}$ in Eq. (3.7) is to distinguish it from another contribution $\gamma_{\perp}$ to be turned on later when switching on $\vec{\xi}.\vec{\mathcal{I}}$. By using the two types of vector products, a general quadratic term like $|\vec{m}| \times |\vec{\nu}|$ has the typical form

$$\frac{\gamma}{2}\, |\vec{m}| \times |\vec{\nu}| = \frac{\Lambda}{2}\vec{m}.\vec{\nu} + \frac{\Lambda'}{2}\, \|\vec{m} \wedge \vec{\nu}\|\,, \tag{3.8}$$

  showing that $\frac{\gamma}{2}$ may come from two sources: (1) from a scalar product like $\frac{\Lambda}{2}\vec{m}.\vec{\nu}$, and/or (2) from the norm of the wedge product of the two vectors as follows:

$$\frac{\Lambda}{2}\vec{m}.\vec{\nu} = \frac{\gamma_{\parallel}}{2}\, |\vec{m}| \times |\vec{\nu}|\,, \qquad \gamma_{\parallel} = \Lambda \cos\theta, \tag{3.9}$$

$$\frac{\Lambda'}{2}\, \|\vec{m} \wedge \vec{\nu}\| = \frac{\gamma_{\perp}}{2}\, |\vec{m}| \times |\vec{\nu}|\,, \qquad \gamma_{\perp} = \Lambda'\, |\sin\theta|\,. \tag{3.10}$$

- In order to fix a freedom in the signs of α, β, and $\gamma_{\parallel}$, we assume that the discriminant of the G_{ij} metric of Eq. (3.7) is positive definite,

$$\det G_{ij} = \alpha\beta - \frac{\gamma_{\parallel}^2}{4} > 0. \tag{3.11}$$

  As this discriminant is not sensitive to $(\alpha, \beta, \gamma_{\parallel}) \to (-\alpha, -\beta, -\gamma_{\parallel})$, we restrict α and β to both be positive; this constraint is also needed for $\boldsymbol{h}$ to be bounded from below, which is an important condition for the quantization of coupling fluctuation.

- The omission of the zero value in det G is because for $\alpha\beta - \frac{\gamma_{\|}^2}{4} = 0$, the Hamiltonian in Eq. (3.7) reduces to

$$\boldsymbol{h}_\eta = Z_\eta^2 \qquad \text{with} \qquad Z_\pm^2 = (|\vec{m}|\sqrt{\alpha} \pm |\vec{\nu}|\sqrt{\beta})^2, \tag{3.12}$$

ruling out the harmonic oscillations needed for quantum fluctuations; see Eq. (3.20). Nevertheless, the saturated limit also captures some interesting data; it will be discussed in Sect. 4.2.

With these features in mind, we are now in a position to deal with the Hamiltonian of Eq. (3.7). To do so, we perform a linear change of variables, $(|\vec{m}|, |\vec{\nu}|) \to (|\vec{m}'|, |\vec{\nu}'|)$, in order to put $\boldsymbol{h}$ into the following normal form:

$$\boldsymbol{h}_0 = \frac{1}{2\mu}\sum_{a=1}^{3}\left(m_a'\right)^2 + \frac{\kappa}{2}\sum_{a=1}^{3}\left(\nu_a'\right)^2, \tag{3.13}$$

where now $\frac{1}{2\mu}\vec{m}'^2$ stands for "kinetic energy" and $\frac{\kappa}{2}\vec{\nu}'^2$ for the "potential energy." The new $|\vec{m}'|$, $|\vec{\nu}'|$ are related to the old $|\vec{m}|$, $|\vec{\nu}|$ by

$$|\vec{m}'| = a\,|\vec{\nu}| + b\,|\vec{m}|, \qquad |\vec{\nu}'| = c\,|\vec{\nu}| + d\,|\vec{m}|, \tag{3.14}$$

with $ad - bc = 1$, which diagonalizes the metric in Eq. (2.18). The resulting positive mass μ and $\kappa = \mu\omega^2$ (oscillation frequency) are functions of the α, β, and $\gamma_{\|}$ parameters; their explicit expressions are

$$\frac{1}{\mu} = \alpha + \beta + \sqrt{(\alpha-\beta)^2 + \gamma_{\|}^2}, \qquad \kappa = \alpha + \beta - \sqrt{(\alpha-\beta)^2 + \gamma_{\|}^2}. \tag{3.15}$$

Notice that, using the condition $\gamma_{\|}^2 < 4\alpha\beta$ and the positivity of α and β, we have $(\alpha-\beta)^2 + \gamma_{\|}^2 < (\alpha+\beta)^2$ and then $\kappa > 0$. Notice also the following properties:

- The saturated value $(\gamma_{\|}^2)_{\max} = 4\alpha\beta$; then, $(\det G_{ij})_{\max} = 0$ and $(\kappa)_{\min} \to 2(\alpha+\beta)\,\omega_{\min}^2 = 0$.
- Classically, the Hamiltonian in Eq. (3.13) is positive and bounded from below,

$$\boldsymbol{h} \geq \boldsymbol{h}_0, \qquad \boldsymbol{h}_0 = 0. \tag{3.16}$$

This vanishing lower value $\boldsymbol{h}_0 = 0$ is important in the study of $\mathcal{N} = 2$ gauged supergravity in the rigid limit, since $\langle \mathcal{V}_{\text{class}} \rangle = \boldsymbol{h}_0 = 0$ corresponds to an exact $\mathcal{N} = 2$ rigid supersymmetric phase. This property requires $\vec{m} = \vec{\nu} = \vec{0}$.
- By restricting $\vec{m}$ and $\vec{\nu}$ to the particular choice in Eq. (2.8), the free Eq. (3.13) reduces to the Hamiltonian of a one-dimensional harmonic oscillator,

$$\boldsymbol{h}^{(1\mathrm{D})} = \frac{1}{2\mu}\left(m_y'\right)^2 + \frac{\kappa}{2}\left(\nu_x'\right)^2. \tag{3.17}$$

In what follows, we use this simple expression to study harmonic fluctuations of the FI couplings around the supersymmetric vacuum $m_y' = \nu_x' = 0$.

3.1.2. Quantum effect

The free iso-particle studied above is classical. However, like spin $s = \frac{1}{2}$ fermions in real 3D space the iso-particle also has intrinsic degrees of freedom, namely an isospin $I = \frac{1}{2}$, as shown in Table 2, and a unit electric/magnetic charge ϑ given by Eq. (3.4).

Assuming the classical correspondence of Eq. (1) to also hold at the quantum level in the iso-space $\hat{\mathbb{R}}^3$, it follows that the fluctuations of the FI couplings may also be governed by $\left|\Delta\vec{m}'\right| \times \left|\Delta\vec{\nu}'\right| \gtrsim \hbar$ in same manner as for the usual Heisenberg uncertainty $|\Delta x| \times |\Delta p_x| \gtrsim \hbar$, which is expressed in terms of the usual phase space coordinates $(\vec{r}, \vec{p})$. If one accepts this assumption, then we cannot have exactly $m'_y = \nu'_x = 0$ since $\left|\Delta\nu'_x\right| \times \left|\Delta m'_y\right| \gtrsim \hbar$, and so one expects $\mathcal{N} = 2$ supersymmetry in rigid limit to be broken by quantum effects since the ground state energy is now positive definite,

$$\left\langle \boldsymbol{h}_{\text{quant}}^{(1\text{D})} \right\rangle > 0. \tag{3.18}$$

In what follows, we restrict our study to exhibiting this quantum behavior and to checking the breaking $\mathcal{N} = 2 \to \mathcal{N} = 0$. We will return to study this feature in Sect. 4.2 when the isospin–orbit coupling is switched on. There, we will also give details of the condition for the partial breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$.

The quantum effect due to fluctuations of $\vec{m}$ and $\vec{\nu}$ around the supersymmetric ground state $\boldsymbol{h}_0 = \langle \mathcal{V} \rangle$ is induced by quantum isotropic oscillations with discrete energy $\hat{\epsilon}^{\parallel}_{(n_x,n_y,n_z)} = \epsilon^{\parallel}_{n_x} + \epsilon^{\parallel}_{n_y} + \epsilon^{\parallel}_{n_z}$ and fundamental oscillation frequency

$$\omega_{\parallel} = \sqrt{\frac{\kappa}{\mu}}. \tag{3.19}$$

By using Eq. (3.15), we have

$$\omega_{\parallel}^2 = 4\alpha\beta - \gamma_{\parallel}^2. \tag{3.20}$$

Observe that, because of the minus sign, this $\omega_{\parallel}$ vanishes for those parameters α, β, and $\gamma_{\parallel}$ satisfying the degenerate condition $\gamma_{\parallel}^2 = 4\alpha\beta$, which has been ruled out by the constraint in Eq. (3.11). To illustrate the quantum effect for $\omega_{\parallel} > 0$, we consider the particular choice of Eq. (2.8) bringing Eq. (3.13) to a one-dimensional quantum oscillator with Hamiltonian operator

$$\boldsymbol{H}_{\parallel}^{(1\text{D})} = \frac{\hbar\omega_{\parallel}}{2}\left[\left(\frac{\hat{\boldsymbol{m}}_y}{\sqrt{\mu\omega_{\parallel}}}\right)^2 + \left(\hat{\boldsymbol{\nu}}_x\sqrt{\mu\omega_{\parallel}}\right)^2\right]. \tag{3.21}$$

This has a diagonal form $\frac{\hbar\omega_{\parallel}}{2}\left(Y^2 + X^2\right)$, which by setting $A = \frac{X+iY}{\sqrt{2}}$ reads as usual like

$$\boldsymbol{H}_{\parallel}^{(1\text{D})} = \hbar\omega_{\parallel}\left(A^{\dagger}A + \frac{1}{2}\right), \tag{3.22}$$

with $AA^{\dagger} - A^{\dagger}A = I$. The energy spectrum $\hat{\epsilon}^{\parallel}_{(n_x,n_y,n_z)}$ reduces to

$$\epsilon_n^{\parallel} = \hbar\omega_{\parallel}\left(n + \frac{1}{2}\right) \geq \epsilon_0, \tag{3.23}$$

with the frequency $\omega_{\parallel}$ given by Eq. (3.20). The lowest energy value is given by $\epsilon_0^{\parallel} = \frac{\hbar\omega_{\parallel}}{2}$; it is non-zero for a non-vanishing frequency $\omega_{\parallel}$. Hence, the exact $\mathcal{N} = 2$ supersymmetry living at the classical vacuum $\langle \mathcal{V}_{\text{class}} \rangle = 0$ gets completely broken by the quantum effect

$$\left\langle \mathcal{V}_{\text{quant}} \right\rangle = \frac{\hbar\omega._{\parallel}}{2} > 0 \tag{3.24}$$

We end this subsection by giving two brief comments on interactions. The first interacting potential energy has a linear expression in $\vec{m}$,

$$h_{\text{int}}^{\left(\hat{R}^3\right)} = -q\vec{m}.\vec{\mathcal{A}}, \tag{3.25}$$

and concerns the electric $U(1)_{\text{elec}}$ gauge charge. This is a subgroup of the electric/magnetic $U(1)_{\text{elec}} \times U(1)_{\text{mag}}$ local symmetry of the $\mathcal{N} = 2$ gauged supergravity induced by gauging two Abelian isometries in the scalar manifold of the supergravity theory. The second interacting potential energy is given by the isospin–orbit coupling $h_{\text{ioc}}^{\left(\hat{R}^3\right)} = \vec{\xi}.\vec{\mathcal{I}}$ that we are particularly interested in here; it will be considered in detail in the next subsection.

Regarding Eq. (3.25), it is derived by taking the following two steps: First, start from the interaction energy $h_{\text{int}}^{\left(R^3\right)} = -e\vec{p}.\vec{A}$ of an electrically charged particle with momentum $\vec{p}$ moving in the presence of an external magnetic field $\vec{B}_{\text{ext}} = \vec{\nabla} \wedge \vec{A}$. Then, use the correspondence in Eq. (1) allowing us to imagine $-e\vec{p}$ in terms of the FI magnetic vector $-q\vec{m}$ and $\vec{A}$ as an iso-vector $\vec{\mathcal{A}}$. The obtained Eq. (3.25) describes just the term $w_a m^a$ in Eq. (2.19), from which we learn that $\vec{w} = -q\vec{\mathcal{A}}$.

3.2. *Isospin–orbit coupling*

The proposal in Table 1 has been useful for the physical interpretation of the rigid Ward identity in terms of an iso-particle Hamiltonian with phase space coordinates $(\vec{v}, \vec{m})$, thanks to the Andrianoploli et al. formula $\vec{\xi} = \vec{v} \wedge \vec{m}$ giving the orbital momentum of this iso-particle, and thanks also to the structure of the scalar potential $\mathcal{V}$, which turns out to be nothing but the free Hamiltonian $\boldsymbol{h}$ of Eq. (3.13). In this subsection, we derive the isospin–orbit coupling

$$\boldsymbol{h}_{\text{ioc}} = \vec{\xi}.\vec{\mathcal{I}}, \tag{3.26}$$

where $\vec{\mathcal{I}}$ stands for the isospin vector and $\vec{\xi} = \vec{v} \wedge \vec{m}$. For that purpose, recall that in Eq. (2.1) the rigid $\boldsymbol{C}$anomaly matrix appears in the form of a Hermitian traceless 2×2 matrix,

$$\boldsymbol{C} = \begin{pmatrix} \xi_z & \xi_x - i\xi_y \\ \xi_x + i\xi_y & -\xi_z \end{pmatrix} = \vec{\xi}.\vec{\tau}, \tag{3.27}$$

that reads in terms of the $\vec{\tau}$ Pauli matrices and the Andrianopoli et al. orbital vector as follows:

$$\boldsymbol{C} = (\vec{v} \wedge \vec{m}) .\vec{\tau}. \tag{3.28}$$

This factorized form of $\boldsymbol{C}$ teaches us that it can be imagined as describing the coupling of two things, namely the orbital isovector $\vec{\xi} = \vec{v} \wedge \vec{m}$ and the isospin vector

$$\vec{\mathcal{I}} = \frac{\vec{\tau}}{2}. \tag{3.29}$$

In what follows, we give two other different, but equivalent, ways to introduce $\vec{\xi}.\vec{\mathcal{I}}$. The first relies on comparing $\boldsymbol{h}_{\text{ioc}} = \vec{\xi}.\vec{\mathcal{I}}$ with the usual spin–orbit coupling $\boldsymbol{h}_{\text{soc}} = \vec{L}.\vec{S}$ of a particle with spin

$\vec{S} = \frac{\vec{\sigma}}{2}$ moving in real space $\mathbb{R}^3$ with coordinate vector $\vec{r}$. The second way extends the approach of the previous section for deriving the free Hamiltonian (3.13) by including the isospin effect.

By comparing the effect of the spin–orbit coupling $\vec{L}.\vec{S}$ in electronic systems and the effect of $\vec{\xi}.\vec{\mathcal{I}}$ in the partial breaking of $\mathcal{N} = 2$ supersymmetry, and by following Refs. [28,29], we learn that when the central extension matrix is turned off, i.e. $\boldsymbol{C} = 0$, then $\mathcal{N} = 2$ supersymmetry is preserved (two gapless gravitinos). However, it can be partially broken when it is turned on, i.e. $\boldsymbol{C} \neq 0$. This property can be viewed in terms of a non-zero gap energy $\mathrm{E_g}$ between the two fermionic iso-doublets, including the two charges Q_L, Q_R of $\mathcal{N} = 2$ supersymmetry with the expression

$$E_{\mathrm{g}} \propto \left|\vec{\xi}\right|. \tag{3.30}$$

This is exactly what happens for the case of two states of spin $\frac{1}{2}$ fermions in electronic condensed matter systems when the spin–orbit coupling $\vec{L}.\vec{S}$ is taken into account. This $\vec{L}.\vec{S}$ coupling is known to open the zero gap between the two states of free electrons. From this link with electronic properties, we deduce a correspondence between the central matrix $\boldsymbol{C}$ of the $\mathcal{N} = 2$ supercurrent algebra and the Hamiltonian $\boldsymbol{h}_{\mathrm{soc}} = \vec{L}.\vec{S}$. This link reads explicitly like

$$\vec{\xi}.\vec{\mathcal{I}} \quad \leftrightarrow \quad \vec{L}.\vec{S}, \tag{3.31}$$

where the isospin $\vec{\mathcal{I}}$ plays the role of the spin $\vec{S}$, and the Andrianopoli et al. vector $\vec{\xi}$ the role of the angular momentum $\vec{L}$. Adding the isospin–orbit coupling term to the free Hamiltonian in Eq. (3.13) we get $H = \mathcal{V} + \vec{\xi}.\vec{\mathcal{I}}$, which reads explicitly as

$$H = \frac{1}{2\mu}\vec{m}'^2 + \frac{\kappa}{2}\vec{\nu}'^2 + \vec{\xi}.\vec{\mathcal{I}}. \tag{3.32}$$

In matrix form, we have

$$H = \begin{pmatrix} \mathcal{V} + \xi_z & \xi_x - i\xi_y \\ \xi_x + i\xi_y & \mathcal{V} - \xi_z \end{pmatrix}, \tag{3.33}$$

with eigenvalues $E_\pm = \mathcal{V} \pm \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}$ and eigenstates

$$|\eta_\pm\rangle \sim \begin{pmatrix} \xi_z \pm \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} \\ \xi_x + i\xi_y \end{pmatrix}. \tag{3.34}$$

The second way to introduce Eq. (3.26) is a purely algebraic approach. The key idea relies on thinking of the free energy density term of the two $I_z = \pm\frac{1}{2}$ isospin states as

$$\boldsymbol{h}_B^A = \mathcal{V}\delta_B^A. \tag{3.35}$$

For each of the $I_z = \pm\frac{1}{2}$ states we have used Eq. (3.13) to derive its free Hamiltonian, but this result is just the diagonal term of a general Hamiltonian matrix H. The extension of $\mathcal{V}\delta_B^A$ to more interactions is then naturally given by the Ward identity, Eq. (2.1),

$$\boldsymbol{H}_B^A = \mathcal{V}\delta_B^A + C_B^A, \tag{3.36}$$

which is nothing but the right-hand side of the $\mathcal{N} = 2$ supersymmetric current algebra in Eq. (2.4), including the central matrix.

3.3. *Discrete symmetries*

From the rigid Ward identity of Andrianopoli et al., Eq. (2.1), we also learn that exact $\mathcal{N} = 2$ supersymmetry requires $\vec{\xi} = \vec{0}$: no isospin–orbit coupling in our modeling. But this vanishing value is just the fixed point of the $\boldsymbol{Z}_2$ discrete symmetry acting on the anomaly iso-vector as follows:

$$\boldsymbol{Z}_2: \quad \vec{\xi} \quad \rightarrow \quad -\vec{\xi}. \tag{3.37}$$

To figure out the meaning of this discrete transformation we use Eq. (2.6), from which we learn that the minus sign can be generated in two ways, either by the change $(\vec{\nu}, \vec{m}) \rightarrow (-\vec{\nu}, \vec{m})$ or by $(\vec{\nu}, \vec{m}) \rightarrow (\vec{\nu}, -\vec{m})$. To derive the physical interpretation of these two kinds of $\boldsymbol{Z}_2$ discrete symmetries, we use the analogy between the FI couplings $(\vec{\nu}, \vec{m})$ and the classical phase coordinates $(\vec{r}, \vec{p})$. Promoting this correspondence to dynamical (running) couplings, say

$$\begin{array}{c} \vec{r}\,(t) \\ \vec{p}\,(t) \end{array} \quad \leftrightarrow \quad \begin{array}{c} \vec{\nu}\,(\tau) \\ \vec{m}\,(\tau) \end{array}, \tag{3.38}$$

it follows that the transformation in Eq. (3.37) corresponds, for example, to the usual time-reversing symmetry $\boldsymbol{T}$ which maps the position $\vec{r}\,(t)$ and momentum $\vec{p}\,(t)$ respectively to $\vec{r}\,(-t)$ and $-\vec{p}\,(-t)$. On the side of the FI couplings, we then have the following action of the $\boldsymbol{T}$ analog of iso-time τ,

$$\boldsymbol{T}: \begin{array}{c} \vec{\nu}\,(\tau) \\ \vec{m}\,(\tau) \end{array} \quad \rightarrow \quad \begin{array}{c} \vec{\nu}\,(-\tau) \\ -\vec{m}\,(-\tau) \end{array}. \tag{3.39}$$

Notice that the usual space parity $\boldsymbol{P}$ which maps the $(\vec{r}, \vec{p})$ phase coordinates to $(-\vec{r}, -\vec{p})$ allows us, by using the $(\vec{r}, \vec{p}) \leftrightarrow (\vec{\nu}, \vec{m})$ correspondence, to write

$$\boldsymbol{P}: \begin{array}{c} \vec{\nu}\,(\tau) \\ \vec{m}\,(\tau) \end{array} \quad \rightarrow \quad \begin{array}{c} -\vec{\nu}\,(\tau) \\ -\vec{m}\,(\tau) \end{array}, \tag{3.40}$$

but this discrete $\boldsymbol{P}$ transformation leaves $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ invariant and so is not relevant for partial breaking. However, the combined $\boldsymbol{PT}$ transformation, which acts like

$$\boldsymbol{PT}: \begin{array}{c} \vec{\nu}\,(\tau) \\ \vec{m}\,(\tau) \end{array} \quad \rightarrow \quad \begin{array}{c} -\vec{\nu}\,(-\tau) \\ +\vec{m}\,(-\tau) \end{array}, \tag{3.41}$$

does affect the sign of $\vec{\xi}$. This combination can also be used to think about the $\boldsymbol{Z}_2$ transformation of Eq. (3.37). Actually, it corresponds to the second possibility of realizing $\vec{\xi} \rightarrow -\vec{\xi}$ from Eq. (2.6). Therefore, exact $\mathcal{N} = 2$ supersymmetry, which corresponds to $\vec{\xi} = \vec{0}$, lives at the fixed point of the $\boldsymbol{T}$ reversing time transformation of Eq. (3.37), or at the combined $\boldsymbol{PT}$ given by Eq. (3.41), or both.

4. Topological aspects and quantum effect

In this section we first study the topological behavior of gapless iso-particles of exact $\mathcal{N} = 2$ supersymmetry, as well as the gapless chiral ones that remain after partial breaking. We then study the effect of quantum fluctuations on partial supersymmetry breaking.

4.1. *Chiral anomaly*

Setting $H_B^A = \sum_i \delta_B \lambda^{iC} \delta^A \lambda_{iC}$, we can turn the rigid Ward identities of Eq. (2.1) into the matrix equation

$$H_B^A = \mathcal{V} \delta_B^A + C_B^A, \tag{4.1}$$

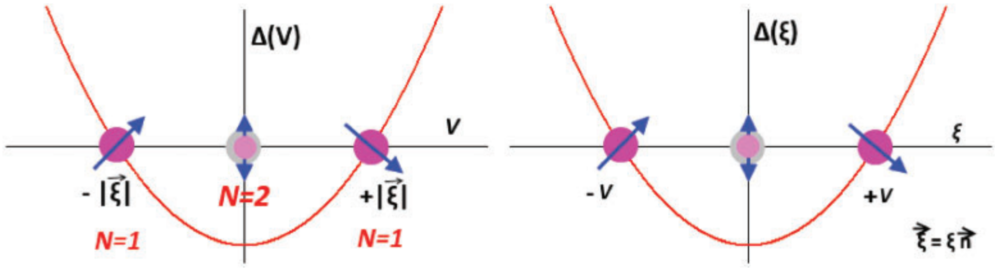

Fig. 2. On the left, the discriminant $\Delta_{\xi}(V)$ as a function of V and parameter $\left|\vec{\xi}\right|$. For $\vec{\xi} \neq \vec{0}$ there are two zeros, at $V = \pm\left\|\vec{\xi}\right\|$; one is visible in the rigid limit. At each zero, say $V = \left\|\vec{\xi}\right\|$, is a chiral gapless mode corresponding to a partially broken $\mathcal{N} = 2$ supersymmetric state. In the limit $\vec{\xi} \to \vec{0}$, the two chiral gapless modes at $V = \pm\left\|\vec{\xi}\right\|$ collide at the origin and form a gapless iso-doublet. On the right, $\Delta_V(\xi)$ as a function of $\vec{\xi}$. For non-zero V the chiral gapless mode lives at each $\vec{\xi}_{\pm} = \pm V\vec{n}$, merging for $V = 0$.

which is nothing but the Hamiltonian matrix of Eq. (3.32). Multiplying both sides of this 2×2 matrix relation by $\eta_A = (\eta_1, \eta_2)^{\mathrm{T}}$, describing the two states of the iso-particle, we end up with the eigenvalue equation $H.\eta = E\eta$ whose two eigenvalues are given by $E_{\pm} = \mathcal{V} \pm \left|\vec{\xi}\right|$; the eigenstates $\hat{\eta}_{\pm}$ associated with these $E_{\pm}$ are linear combinations of η_1 and η_2, and read like $\hat{\eta}_{\pm} = A_{\pm}\eta_1 + B_{\pm}\eta_2$ with amplitudes $A_{\pm}$ and $B_{\pm}$ as follows:

$$A_{\pm} = \frac{\xi_z \pm \left|\vec{\xi}\right|}{\sqrt{2\left(\xi_z - \left|\vec{\xi}\right|\right)}}, \qquad B_{\pm} = \frac{\xi_x + i\xi_y}{\sqrt{2\left(\xi_z - \left|\vec{\xi}\right|\right)}}. \tag{4.2}$$

The determinant $\det H = \Delta$ that captures data on the singular points in the $\left(\mathcal{V}, \left|\vec{\xi}\right|\right)$ plane is given by the product of the eigenvalues $E_{\pm}$, and reads as

$$\Delta = \left(\mathcal{V} + \left|\vec{\xi}\right|\right)\left(\mathcal{V} - \left|\vec{\xi}\right|\right). \tag{4.3}$$

It is a function of two real quantities, namely $\mathcal{V}$ and $\left|\vec{\xi}\right|$, but here we will treat it as a parametric function of one variable like $\Delta_{\zeta}(x)$. The choice of the variable x depends on the property we are interested in exhibiting; see Fig. 2. From the viewpoint of the scalar potential energy, the variable is given by $x = \mathcal{V}$, while $\zeta = \left|\vec{\xi}\right|$ is seen as a free parameter.

From the viewpoint of the $\vec{\xi}$ vector we have the reverse picture: $x = \left|\vec{\xi}\right|$ is the variable while $\zeta = \mathcal{V}$ stands for a free parameter. In the first image, $\det H$ has two zeros at $\mathcal{V}_{\pm} = \pm\left|\vec{\xi}\right|$, one positive, $\mathcal{V}_+$, that is visible in the global supersymmetry sector, and a hidden negative, $\mathcal{V}_-$. In the second picture, the discriminant $\det H$ has zeros at $\left|\vec{\xi}_+\right| = +\mathcal{V}$ for positive $\mathcal{V}$ and $\left|\vec{\xi}_-\right| = -\mathcal{V}$ for negative $\mathcal{V}$. Let us express these two zeros in $\tilde{\mathbb{R}}^3$ as $\vec{\xi}_{\pm} = \pm\mathcal{V}\vec{n}$ with unit vector $\vec{n} = \frac{\vec{\xi}}{\left|\vec{\xi}\right|}$. The effective gap energy $E_{\mathrm{g}} = E_+ - E_-$ between the two $E_{\pm}$ energy density bands is given by

$$E_{\mathrm{g}} = 2\left|\vec{\xi}\right|; \tag{4.4}$$

it vanishes for $\left|\vec{\xi}\right| = 0$ and then for $\vec{\xi} = \vec{0}$. Because of the property $\left|\vec{\xi}\right| \geq 0$, the zeros of $\det H$ are of two kinds: simple for $\left|\vec{\xi}\right| > 0$ and double for $\left|\vec{\xi}\right| = 0$. At each simple zero lives a gapless fermionic mode (gravitino and gaugino) and a gapped one. For $\left|\vec{\xi}_+\right| = +\mathcal{V}$, with positive energy density $\mathcal{V}$, we have the conducting band, and for $\left|\vec{\xi}_-\right| = -\mathcal{V}$ with negative $\mathcal{V}$ we have the valence band. Notice that $\det H = \mathcal{V}^2 - \vec{\xi}^2$ is conserved under[3] the discrete change

$$\boldsymbol{Z}_2 : \mathcal{V} \to -\mathcal{V} \quad \Rightarrow \quad \vec{\xi}_+ \to \vec{\xi}_- = -\vec{\xi}_+. \tag{4.5}$$

Its two zeros $\mathcal{V}_\pm = \pm\left|\vec{\xi}\right|$ are not fixed points of $\boldsymbol{Z}_2$ except for the origin; they are interchanged as shown by Eq. (4.5)—for instance, properties at $\vec{\xi}_-$ may be deduced from those at $\vec{\xi}_+$.

Now let us approach $\det H$ from the viewpoint of the iso-space vector $\vec{\xi}$ and consider the two-spheres $\mathbb{S}^2_{+\upsilon}$ and $\mathbb{S}^2_{-\upsilon}$, with a surface normal to $\vec{n}$, surrounding respectively the zeros

$$\vec{\xi}_\pm = \pm\mathcal{V}\vec{n}. \tag{4.6}$$

The two-sphere $\mathbb{S}^2_{+\upsilon}$ is described by the vector $\vec{p} = \vec{\xi} - \vec{\xi}_+$, and $\mathbb{S}^2_{-\upsilon}$ by $\vec{q} = \vec{\xi} - \vec{\xi}_-$. These two-spheres should not be confused with the unit two-sphere

$$\mathbb{S}^2_{\vec{n}} : n_x^2 + n_y^2 + n_z^2 = 1 \tag{4.7}$$

associated with the unit vectors of Eq. (2.13), but all three of $\mathbb{S}^2_{+\upsilon}$, $\mathbb{S}^2_{-\upsilon}$, $\mathbb{S}^2_{\vec{n}}$ live in the iso-space $\tilde{\mathbb{R}}^3$ and are related to each other by continuous mappings like

$$\pi_+ : \mathbb{S}^2_{+\upsilon} \to \mathbb{S}^2_{\vec{n}}, \qquad \pi_- : \mathbb{S}^2_{-\upsilon} \to \mathbb{S}^2_{\vec{n}}. \tag{4.8}$$

Focusing, for instance, on $\mathbb{S}^2_{+\upsilon}$, the continuity of π_+ shows that it has a winding $w(\mathbb{S}^2_{+\upsilon})$ describing the net number of times $\mathbb{S}^2_{+\upsilon}$ wraps the unit sphere $\mathbb{S}^2_{\vec{n}}$; the integer number $w(\mathbb{S}^2_{+\upsilon})$ just reflects the mathematical property $\pi_2\left(\mathbb{S}^2\right) \cong \mathbb{Z}$. A similar thing can be said about $\mathbb{S}^2_{-\upsilon}$ thanks to the $\boldsymbol{Z}_2$ parity of Eq. (4.5), under which the gauge curvature $\mathcal{F}$ of the underlying Berry connection $\mathcal{A}$ is odd; see Eq. (4.11) below.

Moreover, each gapless state at the two zeros $\vec{\xi}_\pm = \pm\mathcal{V}\vec{n}$ is anomalous in the sense that it has one gapless chiral mode and then violates the Nielson–Ninomiya theorem [17,42,43]. Recall that in theories that are free from chiral anomalies the usual Nielson–Ninomiya theorem [17,42] states that the sum of winding numbers $w\left(\mathbb{S}^2_i\right)$ around two-spheres $\mathbb{S}^2_i$ surrounding the $\vec{\xi}_{*i}$ zeros where gapless modes live vanishes identically. Here, this statement reads explicitly as

$$\sum_i w\left(S_i^2\right) = \sum_i \int_{\mathbb{S}^2_i} \frac{\text{Tr}\left(\mathcal{F}\right)}{2\pi} = 0, \tag{4.9}$$

where $\mathcal{F}$ is a gauge curvature whose explicit expression will be given below. For positive $\mathcal{V}$, Eq. (4.3) has one zero given by an outgoing $\vec{\xi}_+ = +\mathcal{V}\vec{n}$ with positive sense in the normal $\vec{n}$ direction; then, a two-sphere $\mathbb{S}^2_{+\upsilon}$ surrounding the point $\vec{\xi}_+ = \left(\xi_{+x}, \xi_{+y}, \xi_{+z}\right)$ has a positive winding number

$$w\left(S_+^2\right) = \int_{\mathbb{S}^2_{+\upsilon}} \frac{\text{Tr}\left(\mathcal{F}\right)}{2\pi} = 1. \tag{4.10}$$

Here, the curvature $\mathcal{F}$ is given by the following rank-2 antisymmetric tensor,

$$\mathcal{F}_{ab} = \frac{1}{2}\vec{n}.\left(\frac{\partial\vec{n}}{\partial\xi^a} \wedge \frac{\partial\vec{n}}{\partial\xi^b}\right). \tag{4.11}$$

[3] From the charged particle's viewpoint, this mapping from valence- to conducting-like bands and vice versa may be imagined as a ***CT*** transformation combining time reversing ***T*** and charge conjugation ***C***.

Table 3. Properties of zero modes of H.

Zeros of det H	Multiplicity of zeros	Winding number	Conserved SUSY charges
$\left\|\vec{\xi}_+\right\| = +\mathcal{V}$	1	+1	$\tilde{Q}^+$
$\left\|\vec{\xi}_-\right\| = -\mathcal{V}$	1	−1	$\tilde{Q}^-$
$\left\|\vec{\xi}_\pm\right\| = 0$	2	0	$\begin{pmatrix} \tilde{Q}^+ \\ \tilde{Q}^- \end{pmatrix}$

The Nielson–Ninomiya theorem is then violated due to the existence of one gapless chiral moving mode, and so the partially broken theory has a chiral anomaly: only one of the two supersymmetric charges $\left(\hat{Q}_L, \hat{Q}_R\right)$, say the right, $\hat{Q}_R$, is preserved; the left, $\tilde{Q}_L$, is broken. For the incoming $\vec{\xi}_- = -\mathcal{V}\vec{n}$ we have the negative winding number

$$w\left(S_-^2\right) = \int_{\mathbb{S}^2_{-\upsilon}} \frac{\mathrm{Tr}\left(\mathcal{F}\right)}{2\pi} = -1. \tag{4.12}$$

This negative value follows from the mapping $\vec{n} \to -\vec{n}$, due to Eq. (4.5) and using Eq. (4.11). For the special case where the VEV of the scalar potential vanishes, $\mathcal{V} = 0$, the discriminant of the matrix in Eq. (4.3) reduces to $\det \boldsymbol{H} = -\left|\vec{\xi}\right|^2$ and its zero, $\left|\vec{\xi}\right| = 0$, has a multiplicity 2. In this case, the Nielson–Ninomiya theorem reads as

$$w\left(S_+^2\right) + w\left(S_-^2\right) = 1 - 1 = 0. \tag{4.13}$$

At the fixed point of the transformation in Eq. (4.5), the two zeros collide at $\left|\vec{\xi}_\pm\right| = 0$. Then, the two effective gravitino zero modes with opposite chiralities form a massless doublet (a massless iso-particle) and $\mathcal{N} = 2$ supersymmetry gets restored.

These results are summarized in Table 3.

4.2. *Quantum fluctuation*

Here, we study quantum fluctuations in the FI couplings around the partial breaking vacuum $\langle\mathcal{V}\rangle = \left|\vec{\xi}\right|$ and comment on their effect by using the special choice in Eq. (2.8). For that purpose we use $\left|\Delta\vec{m}'\right| \times \left|\Delta\vec{v}'\right| \sim \hbar$ to promote the matrix equation in Eq. (4.1) into an effective quantum eigenvalue matrix equation $\boldsymbol{H}\left|\eta\right\rangle = E\left|\eta\right\rangle$ that we split into two eigenvalue equations:

$$\begin{aligned} \boldsymbol{H}_+\left|\eta_+\right\rangle &= E_+\left|\eta_+\right\rangle, \\ \boldsymbol{H}_-\left|\eta_-\right\rangle &= E_-\left|\eta_-\right\rangle. \end{aligned} \tag{4.14}$$

In these relations we have $\boldsymbol{H}_\pm = \hat{\mathcal{V}} \pm \hat{\boldsymbol{\xi}}$, where the hatted $\hat{\mathcal{V}}$ and $\hat{\boldsymbol{\xi}}$ refer to the quantized operators associated with $\mathcal{V}$ and $\left|\vec{\xi}\right|$ expressed in terms of the phase space vectors $\vec{m}$ and $\vec{v}$. For the particular FI coupling choice in Eq. (2.8) we have $\left|\Delta m'_y\right| \times \left|\Delta v'_x\right| \sim \hbar$, and find, after repeating the steps between Eqs. (3.7) and (3.22), the two quantum 1D Hamiltonians

$$\boldsymbol{H}_\pm^{(1\mathrm{D})} = \hbar\omega_\pm\left(A^\dagger A + \frac{1}{2}\right) \tag{4.15}$$

describing two oscillators with different frequencies $\omega_{\pm}$. Their energies are given by $\epsilon_n^{\pm} = \hbar\omega_{\pm}\left(n+\frac{1}{2}\right)$, with

$$\omega_{\pm}^2 = 4\alpha\beta - \left(\gamma_{\parallel} \pm \gamma_{\perp}\right)^2 \tag{4.16}$$

with the remarkable minus sign. Notice that imposing the constraint in Eq. (3.11) on both $|\eta_{\pm}\rangle$ eigenstates, we have

$$\alpha\beta - \frac{\left(\gamma_{\parallel} - \gamma_{\perp}\right)^2}{4} \geq 0, \quad \alpha\beta - \frac{\left(\gamma_{\parallel} + \gamma_{\perp}\right)^2}{4} \geq 0, \tag{4.17}$$

leading to

$$0 \leq \left(\gamma_{\parallel} - \gamma_{\perp}\right)^2 \leq 4\alpha\beta, \quad 0 \leq \left(\gamma_{\parallel} + \gamma_{\perp}\right)^2 \leq 4\alpha\beta, \tag{4.18}$$

and then to

$$-\alpha\beta \leq \gamma_{\parallel}\gamma_{\perp} \leq \alpha\beta. \tag{4.19}$$

For the case where one of the bounds of the constraints in Eq. (4.18) is saturated, for example if the upper bound of the squared deviation $(\gamma_{\parallel} - \gamma_{\perp})^2 \leq 4\alpha\beta$ is saturated, we can fix one of the four parameters in terms of the three others like

$$\left(\gamma_{\parallel} - \gamma_{\perp}\right)^2 = 4\alpha\beta \quad \Rightarrow \quad \gamma_{\perp} = \gamma_{\parallel} \pm 2\sqrt{\alpha\beta}. \tag{4.20}$$

By substituting back into Eq. (4.16), we end up with two energy spectrums. First, $\epsilon_n^{-} = \hbar\omega_{-}^{\text{sat}}\left(n+\frac{1}{2}\right)$ with

$$\omega_{-}^{\text{sat}} = 4\alpha\beta - \left(\gamma_{\parallel} - \gamma_{\perp}\right)^2 = 0, \tag{4.21}$$

describing gapless iso-particles (gravitinos/gauginos) with $E_{-} = 0$, which corresponds to the ground state $\langle\mathcal{V}\rangle = \left|\vec{\xi}\right|$ where partial breaking takes place, and second, $\epsilon_n^{+} = \hbar\omega_{+}^{\text{sat}}\left(n+\frac{1}{2}\right)$ with

$$\left(\omega_{+}^{\text{sat}}\right)^2 = 4\alpha\beta - \left(\gamma_{\parallel} + \gamma_{\perp}\right)^2 = -4\gamma_{\parallel}\gamma_{\perp} > 0, \tag{4.22}$$

describing a gapped iso-particle. Thus, along with the gapless modes ($\omega_{-}^{\text{sat}} = 0$), we have gapped states with harmonics $n\omega_{+}^{\text{sat}}$. The ϵ_n^{+} energies are bounded as

$$\epsilon_n^{+} \geq \epsilon_0^{+} = \frac{1}{2}\hbar\omega_{+}^{\text{sat}} > 0, \tag{4.23}$$

with the ground state energy ϵ_0^{+} corresponding to the classical $E_{+} = 2\left|\vec{\xi}\right|$, which is also the gap energy between the two polarizations of the iso-particle.

As a conclusion of this subsection, quantum fluctuations in the FI coupling space with $\gamma_{\perp} = \gamma_{\parallel} \pm 2\sqrt{\alpha\beta}$ do not destroy the partial breaking supersymmetry of the Andrianopoli et al. rigid limit; this property holds for the saturated condition of Eq. (4.20), otherwise quantum corrections also break the residual $\mathcal{N} = 1$ supersymmetry.

5. Conclusion

In this paper we have used results on topological band theory of usual $s = \frac{1}{2}$ matter to study partial breaking of $\mathcal{N} = 2$ gauged supergravity in the rigid limit. By using supergravity Ward identities and results from Refs. [28] and [17–19,31], we have derived a set of interesting conclusions on the band structure of gravitinos and gauginos in $\mathcal{N} = 2$ theory. Some of these conclusions have been obtained from the proposal in Table 1 and its quantum extension, and we rephrase them below:

(1) The interpretation of the Andrianopoli realization $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ as an angular momentum vector of a quasi-particle with phase space coordinates $(\vec{\nu}, \vec{m})$ allowed us to think of the two gravitinos and the two gauginos in terms of classical isospin $\frac{1}{2}$ particles (iso-particles) charged under $U(1)_{\text{elec}} \times U(1)_{\text{mag}}$ gauge symmetry. As a consequence of this observation, the scalar potential $\mathcal{V}$ has been interpreted as the Hamiltonian, Eqs. (3.6) and (refhp), of a free iso-particle, and the central extension of the $\mathcal{N} = 2$ supercurrent algebra in Eq. (2.4) as describing the isospin–orbit coupling $\vec{\xi}.\vec{\mathcal{I}}$. This isospin–orbit interaction is the homologue of the usual spin–orbit coupling $\vec{L}.\vec{S}$ in electronic systems of condensed matter. The proposal in Table 1 also allowed us to derive two discrete symmetries, T and TP, capturing data on partial breaking of $\mathcal{N} = 2$ supersymmetry; see Sect. 3.3 for details. Exact $\mathcal{N} = 2$ lives at the fixed point of these symmetries. In summary, we can say that the classical properties of the iso-particle are given by the $\mathcal{N} = 2$ supersymmetric current algebra in Eq. (2.3).

(2) By using the Nielson–Ninomiya theorem, we have studied the topological properties of the fermionic gapless states given by zeros of the discriminant in Eq. (4.3). The two bands of the rigid Ward operator H are gapped except at isolated points in the phase space of the electric and magnetic coupling constants, where supersymmetry is partially broken and where there is a gapless chiral state with a chiral anomaly violating the Nielson–Ninomiya theorem. From the study of the properties of H, it follows that the gap energy is given by $E_g = 2\left|\vec{\xi}\right|$ and vanishes for $\left|\vec{\xi}\right| = 0$; that is, for a vanishing central extension in the $\mathcal{N} = 2$ supercurrent algebra. The zero modes of H and their properties like windings and conserved supersymmetric charges are as reported in Table 3. At the particular point $\mathcal{V} = 0$, the discriminant $\det H$ reduces to $-\left|\vec{\xi}\right|^2$ and has an $SU(2)$ singularity at the origin $\vec{\xi} = \vec{0}$. There, the Nielson–Ninomiya theorem $\sum_i w\left(S_i^2\right) = 0$ is trivially satisfied, as shown in Table 3, and $\mathcal{N} = 2$ supersymmetry is exact with compensating chiral anomalies.

(3) We have used the proposal in Table 1 to study the effect of quantum corrections induced by fluctuations of FI coupling constants (running couplings). We have found that the quantum effect in the iso-space of FI couplings may break supersymmetry completely except for the saturated bounds in Eq. (4.21), where half of the oscillating modes disappear.

Finally, we would like to add that this approach might be helpful to explore the picture in higher supergravities, in particular for $\mathcal{N} = 4$; progress in this direction will be reported in a future publication.

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