



κ -deformed BMS symmetry

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ABSTRACT

We present the quantum κ -deformation of BMS symmetry, by generalizing the lightlike κ -Poincaré Hopf algebra. On the technical level our analysis relies on the fact that the lightlike κ -deformation of Poincaré algebra is given by a twist and the lightlike deformation of any algebra containing Poincaré as a subalgebra can be done with the help of the same twisting element. We briefly comment on the physical relevance of the obtained κ -BMS Hopf algebra as a possible asymptotic symmetry of quantum gravity.

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In the recent years there is a surge of interest in BMS symmetry at null infinity of asymptotically flat spacetimes [1], [2], [3]. This renewed interest in the seemingly exotic aspects of classical general relativity was fueled by the discovery of a surprising and close relationship between asymptotic symmetries and soft gravitons theorem [4] that, as it turned out, has its roots in Ward identities for supertranslations [5], [6]. Moreover, the gravitational memory effect [7] appears to be related to the two [8], so that there is a triangle of interrelationships, which vertices are BMS symmetry, Weinberg's soft graviton theorem, and the memory effect. The extensive discussion of these effects can be found in recent reviews [9] and [10].

It has also been argued recently [11] that the similarity of null infinity and black hole horizon suggests that the charges associated with the BMS symmetry might be present at the black hole horizon, so that the black hole might have an infinite number of hairs, so-called soft hairs, in addition to the three classical one: mass, charge, and angular momentum. It was argued in [11] and [12] that the presence of these charges may help solving the black hole information paradox.

In this letter we will investigate the properties of a κ -deformed generalization of the BMS symmetry. There are several reasons to be interested in such a generalization. The main one is that it is believed that investigations of κ -deformation might shed some light on the properties of elusive quantum gravity. The deformation parameter of κ -deformation of Poincaré algebra [13–17], see

[18] for a recent review and more references, has dimension of mass, and therefore it is natural to identify it with Planck mass, or inverse Planck length, which in turn suggests a possible close relationship between this deformation and quantum gravity. In fact, it was shown in [19] that in the case of gravity in 2 + 1 dimensions the κ -Poincaré algebra is the algebra of symmetries of quantum flat space in non-perturbatively quantized quantum gravity, and therefore it plays in quantum gravity the role analogous to the fundamental role played by the Poincaré algebra in quantum field theory.

There are three incarnations in which κ -deformation can appear. First, there is the mostly studied deformation in time direction, in which case the rotations subalgebra of the Poincaré algebra is kept undeformed. This deformation satisfies a natural requirement that rotations are not deformed, and in particular that the angular momentum and spin composition laws are not modified. It is for this same reason that space-like κ -deformation with a preferred spacial direction of deformation was never considered to be a phenomenologically viable possibility, and is treated as a mathematical curiosity with no real relevance for physics.

The third possibility is the lightlike κ -deformation [20–27]. In contrast to the previous two cases it is implemented by a triangular r -matrix, which satisfies the Classical Yang–Baxter Equation. A remarkable property of this so-called non-standard deformation is that it is given by a twist which makes its structure much simpler to construct [28]. Particularly, any embedding of the algebra into a larger Lie algebra preserves the triangular structure and allows to perform the deformation by the same twist element. Since the BMS algebra is ultimately tied in with null infinity and contains the Poincaré subalgebra, in this paper we deform it using

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the lightlike κ -deformation. Another characteristic property of the twist quantization is that it deforms the coalgebraic sector of the corresponding universal enveloping algebra, leaving the Lie algebra untouched.¹ Moreover, knowing the twist one can also obtain the underlying noncommutative quantum-deformed spacetime structure.

Let us start reviewing the construction of the lightlike deformed Poincaré algebra as a Hopf algebra following [24]. Then we will extend this procedure to the BMS algebra. We start with the Poincaré algebra in four dimensions with light-cone generators

$$M_{+-}, M_{+i}, M_{-i}, M_{ij}, P_+, P_-, P_i, \quad i, j = 1, 2 \quad (1)$$

satisfying

$$[M_{+i}, M_{-j}] = -i(M_{ij} + g_{ij}M_{+-}),$$

$$[M_{\pm i}, M_{\pm j}] = 0 \quad (2)$$

$$[M_{\pm i}, M_{jk}] = i(g_{ij}M_{\pm k} - g_{ik}M_{\pm j}),$$

$$[M_{+-}, M_{\pm i}] = \pm iM_{\pm i} \quad (3)$$

$$[M_{+-}, P_{\pm}] = \pm iP_{\pm},$$

$$[M_{\pm i}, P_j] = ig_{ij}P_{\pm} \quad (4)$$

$$[M_{\pm i}, P_{\pm}] = [M_{+-}, P_i] = 0,$$

$$[M_{\pm i}, P_{\mp}] = -iP_i \quad (5)$$

$$[M_{ij}, P_k] = i(g_{ik}P_j - g_{jk}P_i) \quad (6)$$

The metric used here has the nonzero entries $g_{+-} = g_{-+} = 1$, $g_{11} = g_{22} = -1$.

The primitive coproduct structure

$$\Delta_0(X) = \mathbb{1} \otimes X + X \otimes \mathbb{1} \quad (7)$$

and antipode

$$S_0(X) = -X, \quad (8)$$

where X denotes an arbitrary element of the algebra above, makes it a Hopf algebra. This Hopf algebra (established, in fact, on the enveloping algebra) can be deformed by performing a “twist” on the coproduct and the antipode (see below) while leaving the commutator structure unchanged.

Taking linear transformations

$$M^{+-} = i(l_0 + \bar{l}_0), \quad M^{12} = (l_0 - \bar{l}_0) \quad (9)$$

$$M^{+1} = \frac{i}{\sqrt{2}}(l_{-1} + \bar{l}_{-1}), \quad M^{+2} = \frac{1}{\sqrt{2}}(l_{-1} - \bar{l}_{-1})$$

$$M^{-1} = \frac{i}{\sqrt{2}}(l_1 + \bar{l}_1), \quad M^{-2} = \frac{-1}{\sqrt{2}}(l_1 - \bar{l}_1)$$

$$P_+ = \frac{i}{\sqrt{2}}T_{11}, \quad P_- = -\frac{i}{\sqrt{2}}T_{00}$$

$$P_1 = \frac{i}{2}(T_{10} + T_{01}), \quad P_2 = \frac{-1}{2}(T_{10} - T_{01})$$

one establishes that the Poincaré algebra is a subalgebra of the (extended) BMS algebra with generators l_m, \bar{l}_m, T_{pq} with $m, p, q \in \mathbb{Z}$, satisfying [29], [30]

$$[l_m, l_n] = (m - n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m - n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0 \quad (10)$$

$$[l_l, T_{m,n}] = \left(\frac{l+1}{2} - m\right)T_{m+l,n}, \quad [\bar{l}_l, T_{m,n}] = \left(\frac{l+1}{2} - n\right)T_{m,n+l}. \quad (11)$$

This is the infinite dimensional Lie algebra (10), (11) which we are going to deform as a Hopf algebra. It turns out however that this form of the algebra is not the one convenient for our purposes and therefore we will use another, obtained by linear transformations of the generators l_m, \bar{l}_m, T_{pq} . To this end we will use the original Poincaré generators M and P satisfying the algebra (2)–(6) along with the linear combinations of the higher BMS generators defined as follows

$$k_n = l_n + \bar{l}_n, \quad \bar{k}_n = -i(l_n - \bar{l}_n), \quad (12)$$

$$S_{mn} = \frac{1}{2}(T_{mn} + T_{nm}), \quad A_{mn} = -\frac{i}{2}(T_{mn} - T_{nm}). \quad (13)$$

Notice that we have $k_0 = -iM^{+-}$, $\bar{k}_0 = -iM^{12}$, $k_1 = -i\sqrt{2}M^{-1}$, $k_{-1} = -i\sqrt{2}M^{+2}$, $\bar{k}_1 = -i\sqrt{2}M^{-2}$, $\bar{k}_{-1} = -i\sqrt{2}M^{+1}$, $S_{00} = i\sqrt{2}P_-$, $S_{11} = -i\sqrt{2}P_+$, $S_{01} = -iP_1$, $A_{01} = -iP_2$. In terms of these generators the BMS algebra reads

$$[k_n, k_m] = (n - m)k_{n+m}, \quad [\bar{k}_n, \bar{k}_m] = -(n - m)k_{n+m}, \quad (14)$$

$$[k_n, \bar{k}_m] = (n - m)\bar{k}_{n+m}, \quad (15)$$

$$[k_n, S_{pq}] = \left(\frac{n+1}{2} - p\right)S_{p+n,q} + \left(\frac{n+1}{2} - q\right)S_{q+n,p}, \quad (16)$$

$$[\bar{k}_n, S_{pq}] = \left(\frac{n+1}{2} - p\right)A_{p+n,q} + \left(\frac{n+1}{2} - q\right)A_{q+n,p}, \quad (17)$$

$$[k_n, A_{pq}] = \left(\frac{n+1}{2} - p\right)A_{p+n,q} + \left(\frac{n+1}{2} - q\right)A_{p,q+n}, \quad (18)$$

$$[\bar{k}_n, A_{pq}] = \left(\frac{n+1}{2} - p\right)S_{p+n,q} + \left(\frac{n+1}{2} - q\right)S_{q+n,p}, \quad (19)$$

and one can check that it reproduces the Poincaré algebra (2)–(6) in the sector $|m|, |n|, \dots \leq 1$.

Knowing the undeformed BMS algebra, given by the commutators above with trivial Hopf structure, we can turn to the deformation. To this end we use the fact that the lightlike κ -deformation of Poincaré algebra corresponds to the triangular classical r -matrix $r_{LC} = M_{+-} \wedge P_+ + M_{+a} \wedge P^a$ found in [31], for which the twisting element \mathcal{F} is known as the so-called extended Jordanian twist proposed in [32]. It is based on the subalgebra containing six out of ten Poincaré generators: $(M_{+-}, P_+, M_{+1}, P_1, M_{+2}, P_2)$ and has the following form²

$$\begin{aligned} \mathcal{F} &= \exp\left(-i\frac{1}{\kappa}M_{+i} \otimes P^i\right) \exp(-iM_{+-} \otimes \log \Pi_+) \\ &= \exp\left(\frac{1}{\sqrt{2}\kappa}(ik_1 \otimes S_{01} + i\bar{k}_1 \otimes A_{01})\right) \\ &\quad \times \exp\left(-k_0 \otimes \sum_{j=0}^{\infty} \frac{1}{j+1} \left(\frac{-iS_{11}}{\sqrt{2}\kappa}\right)^{j+1}\right), \end{aligned} \quad (20)$$

where $\Pi_+ = 1 + \frac{P_+}{\kappa} = 1 + \frac{iS_{11}}{\sqrt{2}\kappa}$.

¹ However, it should be noticed that the first explicit formulas for the light-like κ -deformed coproducts have been provided in a non-classical basis [20–22] without using the twist.

² From this one also immediately gets the quantum R -matrix $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

We recall [28] that the twisting element $\mathcal{F} \equiv a_\alpha \otimes b^\alpha \in \mathcal{A} \otimes \mathcal{A}$ has to be an invertible, normalized 2-cocycle for the original (i.e. undeformed) algebra \mathcal{A} , i.e. it has to fulfill

$$\mathcal{F}_{12} \cdot (\Delta_0 \otimes \mathbb{1})(\mathcal{F}) = \mathcal{F}_{23} \cdot (\mathbb{1} \otimes \Delta_0)(\mathcal{F}), \quad \epsilon(a_\alpha)b^\alpha = 1 \quad (21)$$

with $\mathcal{F}_{12} = a_\alpha \otimes b^\alpha \otimes \mathbb{1}$, etc. It provides the deformed coproduct via the similarity transformation $\Delta(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}$.

The twist deformation can be extended from the Poincaré sub-algebra to the whole BMS algebra by making repeated use of the Hadamar formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \dots [A, B] \dots]]}_{n \text{ times}}$$

As a result, we get the expressions for the coproduct in the super-rotation sector³

$$\begin{aligned} \Delta(k_m) = & \mathbb{1} \otimes k_m + k_m \otimes \left(\mathbb{1} + m \frac{P_+}{\kappa} \right) + \frac{m-1}{\sqrt{2}\kappa} M^{+-} \otimes S_{1+m,1} \\ & + \frac{i}{\kappa} \left((M^{-1} \otimes \left(\frac{m+1}{2} S_{m1} + \frac{m-1}{2} S_{0,m+1} \right)) \right. \\ & \left. - M^{-2} \otimes \left(\frac{m+1}{2} A_{m1} - \frac{m-1}{2} A_{0,m+1} \right) \right) \\ & + \frac{i}{\sqrt{2}\kappa} \left(k_{m+1} \otimes (1-m)P_1 + \bar{k}_{m+1} \otimes (1-m)P_2 \right) \\ & + \mathcal{O}\left(\frac{1}{\kappa^2}\right) \end{aligned} \quad (22)$$

and

$$\begin{aligned} \Delta(\bar{k}_m) = & \mathbb{1} \otimes \bar{k}_m + \bar{k}_m \otimes \left(\mathbb{1} + m \frac{P_+}{\kappa} \right) + \frac{m-1}{\sqrt{2}\kappa} M^{+-} \otimes A_{1+m,1} \\ & + \frac{i}{\kappa} \left((M^{-1} \otimes \left(\frac{m+1}{2} A_{m1} + \frac{m-1}{2} A_{0,m+1} \right)) \right. \\ & \left. - M^{-2} \otimes \left(\frac{m+1}{2} S_{m1} - \frac{m-1}{2} S_{0,m+1} \right) \right) \\ & + \frac{i}{\sqrt{2}\kappa} \left(k_{m+1} \otimes (1-m)P_2 + \bar{k}_{m+1} \otimes (1-m)P_1 \right) \\ & + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \end{aligned} \quad (23)$$

In the supertranslation sector we get

$$\begin{aligned} \Delta(A_{pq}) = & \mathbb{1} \otimes A_{pq} + A_{pq} \otimes \left(\mathbb{1} + (p+q-1) \frac{P_+}{\kappa} \right) \\ & + \frac{i}{\sqrt{2}\kappa} (1-p) (A_{p+1,q} \otimes P_1 + S_{p+1,q} \otimes P_2) \\ & + \frac{i}{\sqrt{2}\kappa} (1-q) (A_{p,q+1} \otimes P_1 + S_{p,q+1} \otimes P_2) \\ & + \mathcal{O}\left(\frac{1}{\kappa^2}\right) \end{aligned} \quad (24)$$

and

$$\begin{aligned} \Delta(S_{pq}) = & \mathbb{1} \otimes S_{pq} + S_{pq} \otimes \left(\mathbb{1} + (p+q-1) \frac{P_+}{\kappa} \right) \\ & + \frac{i}{\sqrt{2}\kappa} (1-p) (A_{p+1,q} \otimes P_2 + S_{p+1,q} \otimes P_1) \\ & + \frac{i}{\sqrt{2}\kappa} (1-q) (A_{p,q+1} \otimes P_2 + S_{p,q+1} \otimes P_1) \\ & + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \end{aligned} \quad (25)$$

To obtain the antipodes of the deformed Hopf algebra it is often advantageous to make use of the defining property

$$m \circ (S \otimes \mathbb{1}) \circ \Delta(x) = \eta(x)\epsilon(x) = m \circ (\mathbb{1} \otimes S) \circ \Delta(x). \quad (26)$$

Thus, one obtains

$$\begin{aligned} S(k_m) = & -k_m \left(1 + m \frac{P_+}{\kappa} \right) - \frac{m-1}{\sqrt{2}\kappa} M^{+-} S_{1+m,1} \\ & - \frac{i}{\kappa} \left((M^{-1} \left(\frac{m+1}{2} S_{m1} + \frac{m-1}{2} S_{0,m+1} \right)) \right. \\ & \left. - M^{-2} \left(\frac{m+1}{2} A_{m1} - \frac{m-1}{2} A_{0,m+1} \right) \right) \\ & - \frac{i}{\sqrt{2}\kappa} \left(k_{m+1}(1-m)P_1 + \bar{k}_{m+1}(1-m)P_2 \right) \\ & + \mathcal{O}\left(\frac{1}{\kappa^2}\right), \end{aligned} \quad (27)$$

$$\begin{aligned} S(\bar{k}_m) = & -\bar{k}_m \left(1 + m \frac{P_+}{\kappa} \right) - \frac{m-1}{\sqrt{2}\kappa} M^{+-} A_{1+m,1} \\ & - \frac{i}{\kappa} \left((M^{-1} \left(\frac{m+1}{2} A_{m1} + \frac{m-1}{2} A_{0,m+1} \right)) \right. \\ & \left. - M^{-2} \left(\frac{m+1}{2} S_{m1} - \frac{m-1}{2} S_{0,m+1} \right) \right) \\ & - \frac{i}{\sqrt{2}\kappa} \left(k_{m+1}(1-m)P_2 + \bar{k}_{m+1}(1-m)P_1 \right) + \mathcal{O}\left(\frac{1}{\kappa^2}\right) \end{aligned} \quad (28)$$

and

$$\begin{aligned} S(A_{pq}) = & -A_{pq} \left(1 + (p+q-1) \frac{P_+}{\kappa} \right) \\ & - \frac{i}{\sqrt{2}\kappa} (1-p) (A_{p+1,q} P_1 + S_{p+1,q} P_2) \\ & - \frac{i}{\sqrt{2}\kappa} (1-q) (A_{p,q+1} P_1 + S_{p,q+1} P_2) + \mathcal{O}\left(\frac{1}{\kappa^2}\right), \end{aligned} \quad (29)$$

$$\begin{aligned} S(S_{pq}) = & -S_{pq} \left(1 + (p+q-1) \frac{P_+}{\kappa} \right) \\ & - \frac{i}{\sqrt{2}\kappa} (1-p) (A_{p+1,q} P_2 + S_{p+1,q} P_1) \\ & - \frac{i}{\sqrt{2}\kappa} (1-q) (A_{p,q+1} P_2 + S_{p,q+1} P_1) + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \end{aligned} \quad (30)$$

The most important physical aspect of the co-product of an algebra is that it describes a rule of how to compose the associated charge for multiparticle systems. For example the co-product of P_+ , $\Delta(P_+)$ tells us what is the total energy of a system of many particles, each having the energy $P_+^{(i)}$; in the case of just two particles, we have from (24)

³ To simplify these expressions, in the main text we present the expressions to the leading order in $1/\kappa$ only. The complete expressions can be found in the appendix. We also use the standard notation for the Poincaré generators on the right hand side of the formulae below, so as to distinguish them from the superrotations and supertranslations.

$$\Delta P_+ = \mathbb{1} \otimes P_+ + P_+ \otimes \left(\mathbb{1} + \frac{P_+}{\kappa} \right) + O\left(\frac{1}{\kappa^2}\right), \quad (31)$$

so that for the two-particle system the total energy $P_+^{(1+2)}$ is

$$P_+^{(1+2)} \equiv P_+^{(1)} \oplus P_+^{(2)} = P_+^{(1)} + P_+^{(2)} + \frac{1}{\kappa} P_+^{(1)} P_+^{(2)} + O\left(\frac{1}{\kappa^2}\right). \quad (32)$$

Similarly the antipode tells you how to define the ‘inverse’, denoted sometimes by $\ominus X \equiv S(X)$, so that $X \oplus (\ominus X) = 0$. For example, the ‘negative momentum’ in direction 1 is given by (30)

$$\ominus P_1 = -P_1 \left(1 + \frac{P_+}{\kappa} \right). \quad (33)$$

The presence of a non-trivial Hopf structure, whose appearance is to be regarded as a quantum gravity effect, has important physical consequences. It may, for example, shed some new light on the ongoing discussion on black hole information paradox [33]. Recently, Hawking, Perry, and Strominger [11], [12] argued that the soft charges could dramatically change the standard analysis of black hole evaporation, because instead of being nearly bald (having – in the purely Einsteinian case – only mass and angular momentum as its charges) is in fact possessing a (potentially infinite) number of soft hair. Since the corresponding supertranslation charges are all conserved, their presence introduces previously overlooked correlations between early and late Hawking radiation, possibly rendering the black hole evaporation process unitary. In fact, the hard and soft part are not conserved separately, but the total charge is:

$$Q^-[f] = Q^+[f], \quad Q^\pm = Q_s^\pm + Q_h^\pm, \quad (34)$$

where \pm refers to past and future null infinity and f is an arbitrary function on the sphere. Exchange of charge between soft and hard part during the time evolution would introduce the correlations between early and late quanta.

This argument was refuted in a series of papers [34], [35], [36], [37] where it is argued that soft hair could not resolve the black hole unitarity problem, because their time evolution is trivial and their conservation laws are automatically satisfied. The soft modes carrying them effectively decouple from the hard ones and moreover there is a canonical transformation which strips the hard modes from all the supertranslational charges. As a result the supertranslational charges evolve trivially and each charge stays the same all the way through from \mathcal{I}^- to \mathcal{I}^+ . One can say that although a black hole has infinitely many soft hair, they are perfectly combed, so that they do not tangle with the hard ones.

The presence of deformation changes this qualitative picture considerably, the reason being the presence of the co-product structure and the associate modification of the composition laws. Indeed consider eqs. (25) and (24), describing the deformed co-product. It follows that in the associate conservation laws the total supertranslation charge will depend on the momenta carried by hard particles. Indeed consider the total supertranslation charge of the incoming system of a soft particle, having zero Poincaré momentum and some supertranslation charges T_{pq} and a hard one carrying only Poincaré momenta P_μ

$$\begin{aligned} T_{pq}^{Tot} = & T_{pq} + \frac{1}{\kappa} (p+q-1) T_{pq} P_+ \\ & + \frac{i}{\sqrt{2}\kappa} (1-p) (T_{p+1,q} P_1 + iT_{q,p+1} P_2) \\ & + \frac{i}{\sqrt{2}\kappa} (1-q) (T_{p,q+1} P_1 + iT_{q+1,i} P_2) + O\left(\frac{1}{\kappa^2}\right). \end{aligned} \quad (35)$$

If the coproduct was primitive the T_{pq}^{Tot} would be given purely by the eigenvalue T_{pq} of the soft particle and likewise for the late quanta. This corresponds to the argument of [34–37], because the supertranslation charges conservation would constrain only soft particles. But in the case of κ -deformed BMS the total supertranslation charges of early and late quanta are also functions of the Poincaré momenta of the hard particles. This makes the clear separation of hard and soft part impossible. Notice that this effect is negligible at the LHC energy scale, since it is suppressed by $1/\kappa$, which is expected to correspond to the Planck scale.

Thus the fundamental property of Hopf algebras, the presence of a non-trivial coproduct structure, translates into an irreducible entanglement between hard momenta and supertranslational charges. This opens a possibility that early and late Hawking radiation quanta are correlated after all. One may say that as a result of deformation, possibly being associated with quantum gravity effects, the combed hair of black hole become tangled again. Of course much more detailed calculations are required to claim that the level of deformation induced entanglement is sufficient to solve the black hole information paradox, and we hope to address this in future publication.

Apart from these physical investigations our work suggests a number of possible more technical developments. Firstly, it should be noticed that the mathematical studies of coboundary Lie bialgebra structures on infinite-dimensional Lie algebras and/or their supersymmetric extensions has extensive literature, e.g. [38–40]. Particularly, the case of Jordanian deformation of the (super) Witt and Virasoro algebras have been considered in the literature before, see [41,42] and references therein. This suggests the possibility of studying supersymmetric extensions of the BMS algebra and its deformation.

It was established recently [43] that the κ -Poincaré algebra with light-cone deformation is associated with the integrable deformation of principal sigma model. This indicates that similar construction might be possible in the case of the κ -deformed BMS algebra. We will address this issue in a future publication.

More systematic study of possible real forms of obtained deformation might be also of some interest. The one, implicitly assumed in the present paper, has been induced from the Poincaré reality condition: $l_m^\dagger = -\bar{l}_m$, $\bar{l}_m^\dagger = -l_m$, $T_{mm}^\dagger = -T_{nm}$. It manifests itself by the choice of Hermitian Poincaré generators making the twist (20) \dagger -unitary. More precisely, the BMS algebra treated as a complex Lie algebra can admit several real forms; some of them are induced by the reality condition on the embedded $\mathfrak{o}(4, \mathbb{C})$ Lie algebra.⁴

Another issue concerning the choice of possible physical observables in a deformed algebra which might be better adjusted for description of relativistic kinematics in a quantum gravity regime. Twist deformation can offer a canonical solution to this problem (see e.g. [47,48] and references therein).

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Appendix A

In this section we will list the all-orders expressions for the coproducts and antipodes. The leading order of these expressions was discussed in the main text.

⁴ See [44–46] for the discussion of reality conditions on a complex Lie bialgebra as well as quantized enveloping algebra level in the case of $\mathfrak{o}(4, \mathbb{C})$.

For the coproducts we have

$$\begin{aligned} \Delta(k_m) = & \mathbb{1} \otimes k_m + k_m \otimes \Pi_+^m + \frac{i(m-1)}{\sqrt{2\kappa}} k_0 \otimes \Pi_+^{-1} S_{1+m,1} \\ & - \frac{i}{\sqrt{2\kappa}} k_1 \otimes \left(\frac{m+1}{2} S_{m1} + \frac{m-1}{2} S_{0,m+1} \right) \\ & - \frac{i}{\sqrt{2\kappa}} \bar{k}_1 \otimes \left(\frac{m+1}{2} A_{m1} - \frac{m-1}{2} A_{0,m+1} \right) \\ & - \frac{m-1}{(\sqrt{2\kappa})^2} \left(k_1 \otimes \Pi_+^{-1} S_{01} S_{1,m+1} + \bar{k}_1 \otimes \Pi_+^{-1} A_{01} S_{1+m,1} \right) \\ & + \sum_{n=1}^{\infty} \left(\frac{-i}{\sqrt{2\kappa}} \right)^n \binom{n+m-2}{n} (k_{m+n} \otimes f_n \Pi_+^m \\ & + \bar{k}_{m+n} \otimes \bar{f}_n \Pi_+^m) \end{aligned} \quad (36)$$

and

$$\begin{aligned} \Delta(\bar{k}_m) = & \mathbb{1} \otimes \bar{k}_m + \bar{k}_m \otimes \Pi_+^m + \frac{i(m-1)}{\sqrt{2\kappa}} k_0 \otimes \Pi_+^{-1} A_{1+m,1} \\ & - \frac{i}{\sqrt{2\kappa}} k_1 \otimes \left(\frac{m+1}{2} A_{m1} + \frac{m-1}{2} A_{m+1,0} \right) \\ & - \frac{i}{\sqrt{2\kappa}} \bar{k}_1 \otimes \left(\frac{m+1}{2} S_{m1} - \frac{m-1}{2} S_{m+1,0} \right) \\ & - \frac{m-1}{(\sqrt{2\kappa})^2} \left(k_1 \otimes \Pi_+^{-1} S_{01} A_{1,m+1} + \bar{k}_1 \otimes \Pi_+^{-1} A_{01} A_{1+m,1} \right) \\ & + \sum_{n=1}^{\infty} \left(\frac{-i}{\sqrt{2\kappa}} \right)^n \binom{n+m-2}{n} (k_{m+n} \otimes g_n \Pi_+^m \\ & + \bar{k}_{m+n} \otimes \bar{g}_n \Pi_+^m), \end{aligned} \quad (37)$$

where the elements $f_n, g_n, \bar{f}_n, \bar{g}_n$ are defined by the following recurrent relations

$$f_1 = \bar{g}_1 = S_{01}, \quad g_1 = \bar{f}_1 = A_{01}, \quad (38)$$

$$f_{n+1} = f_n S_{01} - \bar{f}_n A_{01}, \quad \bar{f}_{n+1} = f_n A_{01} + \bar{f}_n S_{01}, \quad (39)$$

$$g_{n+1} = g_n S_{01} - \bar{g}_n A_{01}, \quad \bar{g}_{n+1} = g_n A_{01} + \bar{g}_n S_{01}. \quad (40)$$

Here the usual binomial notation $\binom{n+m-2}{n} = \frac{(n+m-2)(n+m-1)\dots(m-1)}{n!}$ is in use and the infinite sums in the equations (34), (35) become finite whenever $m \leq 1$.

In the supertranslation sector we get

$$\begin{aligned} \Delta(A_{pq}) = & \mathbb{1} \otimes A_{pq} + A_{pq} \otimes \Pi_+^{(p+q-1)} \\ & + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r,s}^{n \geq r+s > 0} \left(\frac{i}{\sqrt{2\kappa}} \right)^{r+s} \left(A_{p+r,q+s} \otimes g_{rs}^{(pq)} \Pi_+^{(p+q-1)} \right. \\ & \left. + S_{p+r,q+s} \otimes f_{rs}^{(pq)} \Pi_+^{(p+q-1)} \right) \end{aligned} \quad (41)$$

and

$$\begin{aligned} \Delta(S_{pq}) = & \mathbb{1} \otimes S_{pq} + S_{pq} \otimes \Pi_+^{(p+q-1)} \\ & + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r,s}^{n \geq r+s > 0} \left(\frac{i}{\sqrt{2\kappa}} \right)^{r+s} \left(A_{p+r,q+s} \otimes h_{rs}^{(pq)} \Pi_+^{(p+q-1)} \right. \\ & \left. + S_{p+r,q+s} \otimes j_{rs}^{(pq)} \Pi_+^{(p+q-1)} \right), \end{aligned} \quad (42)$$

where we abbreviate

$$g_{10}^{(pq)} = (1-p)S_{01}, \quad g_{01}^{(pq)} = (1-q)S_{01}, \quad (43)$$

$$f_{10}^{(pq)} = (1-p)A_{01}, \quad f_{01}^{(pq)} = (1-q)A_{01}, \quad (44)$$

$$g_{r+1,s}^{(pq)} = (1-(p+r))(g_{rs}^{(pq)} S_{01} + f_{rs}^{(pq)} A_{01}), \quad (45)$$

$$g_{r,s+1}^{(pq)} = (1-(q+s))(g_{rs}^{(pq)} S_{01} - f_{rs}^{(pq)} A_{01}), \quad (46)$$

$$f_{r+1,s}^{(pq)} = (1-(p+r))(g_{rs}^{(pq)} A_{01} + f_{rs}^{(pq)} S_{01}), \quad (47)$$

$$f_{r,s+1}^{(pq)} = (1-(q+s))(g_{rs}^{(pq)} A_{01} + f_{rs}^{(pq)} S_{01}), \quad (48)$$

and

$$j_{10}^{(pq)} = S_{01}, \quad j_{01}^{(pq)} = (1-q)S_{01}, \quad (49)$$

$$h_{10}^{(pq)} = (1-p)A_{01}, \quad h_{01}^{(pq)} = (1-q)A_{01}, \quad (50)$$

$$j_{r+1,s}^{(pq)} = (1-(p+r))(j_{rs}^{(pq)} S_{01} + h_{rs}^{(pq)} A_{01}), \quad (51)$$

$$j_{r,s+1}^{(pq)} = (1-(q+s))(j_{rs}^{(pq)} S_{01} + h_{rs}^{(pq)} A_{01}), \quad (52)$$

$$h_{r+1,s}^{(pq)} = (1-(p+r))(j_{rs}^{(pq)} A_{01} + h_{rs}^{(pq)} S_{01}), \quad (53)$$

$$h_{r,s+1}^{(pq)} = (1-(q+s))(-j_{rs}^{(pq)} A_{01} + h_{rs}^{(pq)} S_{01}). \quad (54)$$

Let us note that for the important Poincaré subalgebra these results coincide with those found in [25]. One can also notice that the first lines in the formulas (34), (35), (42), (43) represent the coproduct deformed by the purely Jordanian twist $\mathcal{J} = \exp\left(-k_0 \otimes \sum_{j=0}^{\infty} \frac{1}{j+1} \left(\frac{-iS_{11}}{\sqrt{2\kappa}}\right)^{j+1}\right)$ while the remaining lines come from the twist extension (cf. (20)).

For the antipodes it is in general only possible to write down implicit formulas like

$$\begin{aligned} S(A_{pq}) = & -A_{pq} S(\Pi_+^{-(1-(p+q))}) \\ & - \sum_{n=1}^{\infty} \sum_{r,s}^{n \geq r+s > 0} \frac{1}{n!} \left(A_{p+r,q+s} S(g_{rs}^{(pq)}) S(\Pi_+^{-(1-(p+q))}) \right. \\ & \left. + S_{p+r,q+s} S(f_{rs}^{(pq)}) S(\Pi_+^{-(1-(p+q))}) \right), \end{aligned} \quad (55)$$

$$\begin{aligned} S(S_{pq}) = & -S_{pq} S(\Pi_+^{-(1-(p+q))}) \\ & - \sum_{n=1}^{\infty} \sum_{r,s}^{n \geq r+s > 0} \frac{1}{n!} \left(A_{p+r,q+s} S(h_{rs}^{(pq)}) S(\Pi_+^{-(1-(p+q))}) \right. \\ & \left. + S_{p+r,q+s} S(j_{rs}^{(pq)}) S(\Pi_+^{-(1-(p+q))}) \right), \end{aligned} \quad (56)$$

and similar for the superrotations.

To define the q -analog of the Hopf algebra at hand (see [25]) one also needs to make sure that all structures including the coproduct are at most polynomial in the generators. In the Poincaré sub-Hopf algebra this is achieved with the help of Π_+ and Π_+^{-1} . One has to define all the Hopf algebra structures including these auxiliary objects which replace $S_{11} = -i\sqrt{2}P_+$. The only nontrivial commutators are

$$[k_m, \Pi_+] = \frac{(m-1)}{\sqrt{2\kappa}} S_{1+m,1}, \quad (57)$$

$$[\bar{k}_m, \Pi_+] = \frac{(m-1)}{\sqrt{2\kappa}} A_{1+m,1} \quad (58)$$

and

$$0 = [k_m, \Pi_+ \Pi_+^{-1}] = \Pi_+ [k_m, \Pi_+^{-1}] + [k_m, \Pi_+] \Pi_+^{-1} \quad (59)$$

$$\Rightarrow [k_m, \Pi_+^{-1}] = \frac{-(m-1)}{\sqrt{2\kappa}} S_{1+m,1} (\Pi_+^{-1})^2, \quad (60)$$

$$[\bar{k}_m, \Pi_+^{-1}] = \frac{-(m-1)}{\sqrt{2\kappa}} A_{1+m,1} (\Pi_+^{-1})^2. \quad (61)$$

The coproducts then take the form

$$\Delta(\Pi_+) = \mathbb{1} \otimes \mathbb{1} + \frac{i}{\sqrt{2\kappa}} \Delta(S_{11}) \quad (62)$$

$$= \Pi_+ \otimes \Pi_+, \quad (63)$$

$$\Delta(\Pi_+^{-1}) = \Pi_+^{-1} \otimes \Pi_+^{-1} \quad (64)$$

and for the counits and antipodes one has

$$\epsilon(\Pi_+) = \epsilon(\Pi_+^{-1}) = 1, \quad (65)$$

$$S(\Pi_+) = \Pi_+^{-1}, \quad S(\Pi_+^{-1}) = \Pi_+. \quad (66)$$

Note that for general generators of the BMS algebra (apart from the purely Jordanian case) this is not fulfilled in the current notation, e.g. the coproducts for k_m, \bar{k}_m contain infinite power series (in $1/\kappa$) for $m > 1$. Finding a reformulation to remedy the situation, if possible, will be the goal of future research.

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