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Joining spacetimes on fractal hypersurfaces

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Abstract

The theory of fractional calculus is attracting a lot of attention from mathematicians as well as physicists. The fractional generalisation of the well-known ordinary calculus is being used extensively in many fields, particularly in understanding stochastic process and fractal dynamics. In this paper, we apply the techniques of fractional calculus to study some specific modifications of the geometry of submanifolds. Our generalisation is applied to extend the Israel formalism which is used to glue together two spacetimes across a timelike, spacelike or a null hypersurface. In this context, we show that the fractional extrapolation leads to some striking new results. More precisely we demonstrate that, in contrast to the original Israel formalism, where many spacetimes can only be joined together through an intermediate thin hypersurface admitting effective matter fields violating standard energy conditions, the fractional generalisation allows these spacetimes to be smoothly sewed together without any such requirements on the stress tensor of the matter fields. We discuss the ramifications of these results for spacetime structure and the possible implications for gravitational physics.

1. Introduction

The theory of fractional calculus is considered a classical but obscure corner of mathematics [1–3]. It remained, until a few decades, a field by mathematicians, for mathematicians and of purely theoretical interest. Though it played a crucial role in the development of Abel's theory

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of integral equations and many mathematicians like Liouville, Riemann, Heaviside and Hilbert took an active interest in it, fractional calculus found limited applications and was referred to only occasionally, to simplify complicated solutions. For example, this formalism has been used quite often to simplify the solutions of both the diffusion as well as the wave equation (for example, see [4], and [5]).

During the last few decades, however, this theory has found important applications for large number of practical real life situations. Indeed, fractional calculus is providing excellent tools to develop models of polymers and materials [6,7]. In particular, it has been found that to understand properties of various materials which require long-range order to hold, fractional calculus provides a sound platform [8]. Fractional calculus has also been found to naturally incorporate some subtle effects in the dynamics of fluids, and these have found important applications in understanding mechanical, chemical and electrical properties of nano-fluids. However, possibly the most prominent application of these derivatives of non-integer order has been in the theory of fractals [9]. It has been found that for many stochastic processes, the phenomena progresses through increments which are not independent, but instead tend to retain some memory of previous increment, though not necessarily the immediately previous one [9–13]. In other words, these are random processes with long term memory. The theory of fractional Brownian motion, which provides a very natural explanation for these effects, incorporates these persistence effects (or anti-persistence effects) though the fractional modification of the usual Brownian formula relating displacement and time [14]. It has now been understood that statistically speaking, all the naturally occurring signals are of the Weierstrass type, *i.e.* continuous but non-differentiable [9, 15] (here differentiation is in the sense of the usual calculus) and indeed, such Weierstrass-like functions arise even in many quantum mechanical situations. For example, it has been shown that many quantum mechanical problems involving discontinuous potentials possesses energy spectrum of the Weierstrass type [16]. Furthermore, the Feynman paths in the path integral formulation of quantum mechanics are also examples of these kind [17]. However, the most significant discovery has been that, though the naturally occurring functions are of the Weierstrass type and are endowed with a fractal dimension, *they are fractionally differentiable* and that the maximal order of differentiability is related to the box-dimension of the function [18, 19]. Thus, fractional calculus has been highly advantageous in modelling dynamical processes in self-similar systems and for analysing processes which generate chaotic signals and are apparently irregular.

Fractal structures on spacetimes and fractal spacetimes have been the subject of many investigations in the last few decades [20–22]. These investigations predict that on sub-Planckian scales, the spacetime has a fractal structure which is effectively two dimensional [23–25]. It has also been argued that the structure of quantum spacetime should be characteristically different from the classical description of geometry: geometrical quantities like length, area and volume should be size, scale as well as observer dependent [26–28]. This scale dependence also extends to the dimension of spacetime and indeed, many of the quantum gravity theories including string theory and loop quantum gravity admit models where a *dimension flow* akin to those in multifractal sets take place [29–31]. Keeping in view of the wide variety of possible applications of fractal geometry in quantum gravity, [32,33] have developed a complete hierarchy of definitions pertaining to fractal, multi-fractal and multi-scale spacetimes along with their interrelations. Furthermore, the consequences and applications of these definitions in varied areas of quantum field theory, classical and quantum gravity as well as their observational consequences have been discussed extensively in [33].

In this paper, we apply the techniques of fractional calculus to study modifications of the spacetime structure. We show that fractional derivatives lead to modification of tensor functions as well as the Einstein equations (see Appendix C for the details). These equations then are expected to lead to modifications of the metric functions too. However, for our present region of interest, we have argued in Appendix C that the metric functions are not changed appreciably although the fractional modification of the connection have pronounced effect on the value of extrinsic curvature. Thus, we confine ourselves to construct fractal hypersurfaces on integral spacetimes and apply it to sew together two different spacetimes across a fractal hypersurface. As is well known, the issue of final state of gravitational collapse is a long standing open question in general relativity. The appearance of spacetime singularity reveals the domain of failure of the classical theory of general relativity [40]. Quite naturally, it is assumed that general relativity must be corrected to eliminate these failures. Both the string theories and effective field theories necessitate that one must add terms involving higher order as well as higher derivatives in the Riemann tensor to incorporate the effects of physics at small scales. It is a general hope that these higher order corrections will certainly get rid of the singularities [41]. Here, we propose to look at another alternative which may present itself at the small scales. As we shall see in the subsequent sections, fractional calculus, in any of its possible alternative forms, define differentiation through an integration. Hence, it naturally incorporates non-local spacetime correlations and long-range interactions, which are expected to be natural at high energy scales, into account. Thus, many subtle non-local effects may manifest itself if one replaces the ordinary differentiation by its fractional counterpart. One may immediately ask as to where should one look for such non-local terms to arise physically and envisage the regions of strong gravity where the classical theory of general relativity is known to require modifications.[1] An obvious candidate for the strong gravity regime is the black hole region since black holes are created due to gravitational collapse of matter fields in an intense gravitational field. That the interior of black hole may admit a fractal structure is quite old [35,36]. In this model, one envisages a framework where a fractal spacetime structure exists at small scales. They model this fractal spacetime as due to a particular distribution of (virtual) black holes. They show that at small length scales, where strong gravity may create such a distribution of small black holes, this *fractal foam* creates a regulatory mechanism. In fact it also arises that this fractal structure renders general relativity renormalisable (in the sense of an $1/N$ expansion), finite and stable without any higher derivative terms. There are two regions of the black hole spacetime which are ideally suited for the fractional non-local effects to manifest itself. First is the spacetime singularity, where, the effects of strong gravity though invisible to the asymptotic observer, are most spectacular. In fact, in this regime, one expects to generate curvature polynomials in the gravitational action, and many of these effective non-local models may lead to taming of the spacetime singularity. The second place of study is the horizon, which for small mass black holes are regions of intense gravitational field. The black hole horizon, being a surface of infinite redshift, is responsible for the characteristic Planckian spectrum of the Hawking radiation. As a result, the horizon allows existence of modes with frequency greater than the Planck energy. These modes can probe the spacetime at length scales much lower than the Planck length, where spacetime symmetries inherent in the continuum geometry are absent. In this microscopic regime all the local fields must be smeared over a region of Planck length and the macroscopic spacetime emerges only through a course-graining

[1] Most of these terms contribute non-local effects into the Green's function. It should not be surprising if many of the effects of the fractional generalisation arise naturally in the string theories or any other quantum theory.

procedure over a large number of quantum spacetime configurations. The question is whether these microstructures leave some imprint on the classical large scale description of geometry. In [37,38], it has been argued that if energy conservation is taken into account then, to have a correct thermodynamical description of classical black hole collapse, both the Hamiltonian of the physical system and the density of classical states must include modifications due to non-local field correlations. In fact, [37,38] constructs a class of model where the black holes may be treated as one-particle states of a non-local field theory. So, in either of these two high gravity regimes, presence of long range order should have interesting repercussions on modelling of spacetime.

Let us first discuss the implications for the region near a horizon. The black hole horizon, here taken to be an event horizon, is a null expansion-free hypersurface which lies in the region adjoining two spacetimes. So, the horizon may be thought of as a null hypersurface which glues the two spacetimes. Naturally, the joining of two spacetimes through the hypersurface requires that some conditions on the spacetime variables be satisfied on the hypersurface. The Israel–Darmois–Lanczos (IDL) junction condition demands that the metric on either side of the horizon, when pulled back to the hypersurface, must be continuous [39]. In contrast, the extrinsic curvature of the horizon is not required to be continuous. In fact, consistency requires that the Riemann tensor and hence the extrinsic curvature on the hypersurface admit delta function singularities. Using the Einstein equations, the Ricci part of this singularity is related to the stress tensor. Thus, the IDL condition only requires that the difference of the extrinsic curvatures of the hypersurface as embedded in these two spacetimes, must be proportional to the stress–energy tensor living on the horizon. In other words, due to the geometry itself, the hypersurface comes naturally equipped with a energy–momentum tensor. These junction conditions on the horizon has given rise to speculations of constructing a singularity free spacetimes, and in particular, non-singular black hole interiors, in the following way [42,43]: Take the exterior of the Schwarzschild horizon as the future spacetime region and the interior to de-Sitter horizon as the past spacetime region. The boundary between these two regions, the common hypersurface to these two regions, will be a thin null hypersurface endowed with some specific energy–momentum tensor derived from the IDL matching conditions. Thus, one may have well defined matching conditions to create a singularity free universe, with the exterior a Schwarzschild spacetime while the interior being a de-Sitter spacetime. However, in most of the cases, the matching conditions leads to energy–momentum tensors which violate some of the well known energy conditions. In [43], there have been attempts at constructing a singularity free universe by adjoining the de-Sitter interior with the inner horizon of a Reissner–Nordstrom black hole with particular values of charge and mass. In that particular case, the matching is smooth, with no requirement of any energy momentum tensor. In general situations however, these matching conditions require energy condition violating energy–momentum tensors on the matching hypersurface.

The second setting is related to the another such attempt where, the de-Sitter spacetime is glued to the Schwarzschild interior though a spacelike hypersurface. This attempt was made by [44], in their famous proposal of *limiting curvature*. They devised a model in which the Schwarzschild metric inside the black hole region is matched to a de-Sitter one at some spacelike junction surface which represent a thin transition layer. As a requirement of their proposal, this layer is placed at a region very close to the singularity where the curvature reaches its limiting value. However again, for general singularity free matching of the above kind, the junction layers admit energy momentum tensors which violate energy conditions. In particular, the effective stress–energy tensor of the model [44] violates the weak energy condition. In fact, in almost all similar attempts of creating singularity free models like that of [44], have energy condition violations. These kinds of violations are characteristic of quantum effects which become important in strong gravitational field.

As a remedy to these energy condition violations, we argue in this paper that the notion of fractional derivatives offers a possibility of creating singularity free universe through smooth matching of spacetimes. More precisely, we demonstrate the following: First, that the IDL junction conditions both for timelike/spacelike as well as for null hypersurfaces are modified due to the fractional generalisation of the spacetime connection. This fractional generalisation of the IDL conditions will in turn modify the energy–momentum tensor on the hypersurface.[2] Secondly, using specific examples, we show that this generalisation allows us to fix the conditions on the junction shell in such a way that the inner horizon of Reissner–Nordstrom spacetimes or the Schwarzschild interior can always be smoothly matched to the de-Sitter spacetime without any requirement either on the charge/mass of the spacetimes or on the energy condition violating energy–momentum tensors of the adjoining shell.

The paper is organised as follows: In the next section, section 2, we briefly discuss the mathematical formalism of fractional calculus and the relevant notations. In sections 3 and 4, we introduce the notations for timelike/spacelike and null hypersurfaces and discuss the generalisations of the IDL junction conditions for fractional exponents. We also argue that these generalised junction conditions lead to smooth joining of spacetimes, which otherwise are known to be joined only through a thin shell of matter. The implications of these results are discussed in the Discussion section.

2. Mathematical preliminaries

Let us discuss some notations useful for the mathematical formulation of geometry of hypersurfaces. Let us consider a 4 dimensional spacetime $(\mathcal{M}, g)$ with signature $(-,+,+,+)$. Let a hypersurface Δ be embedded in $\mathcal{M}$ and is given by $f : \Delta \to \mathcal{M}$. We shall assume that the embedding relation is such that the restriction of f to the image of Δ is C^∞. Let, $\{x^\mu\}$ be a local coordinate chart on $\mathcal{M}$ and $\{y^a\}$ be a local coordinate chart on Δ. The embedding relation implies $x^\mu = x^\mu(y^a)$. Let, $g_{\mu\nu}$ be the metric on the spacetime in terms of its local coordinates. The first fundamental form or the induced metric on Δ is the pull back of the metric g under the map f. In the local coordinates this can be written as h_{ab}.

$$h_{ab} \equiv g(\partial_a, \partial_b) = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}\, g(\partial_\mu, \partial_\nu) = e^\mu_a e^\nu_b\, g_{\mu\nu}, \tag{1}$$

where, $(\partial x^\mu/\partial y^a) = e^\mu_a$ and we have used that $e^\mu_a\,\partial_\mu$ is the push forward of the purely tangential vector field ∂_a onto the full spacetime $\mathcal{M}$. One may define a linear connection and hence a derivative operator on the spacetime. Let, $T\mathcal{M}$ denote the tangent bundle on $\mathcal{M}$ and let, X and Y are two arbitrary vector fields on it. The covariant derivative is a linear map

$$\begin{aligned}\nabla : T\mathcal{M} \otimes T\mathcal{M} &\to T\mathcal{M}\\ (X, Y) &\to \nabla_X Y.\end{aligned} \tag{2}$$

[2] The fractional generalisation developed in this paper assumes that fractal like structures are present at very high energy scales, just as they are present at low energy scales. There is no experimental basis for such assumptions. However, this assumption leads to some interesting consequences as developed in this paper.

The Riemannian theory assumes the covariant derivative to be metric compatible, $\nabla_Z\, g(X, Y) = g\,(\nabla_Z X, Y) + g\,(X, \nabla_Z Y)$. On the tangent bundle one may also define a covariant derivative (D) on Δ using the Gauss decomposition formula: For $X, Y \in \Delta$,

$$\begin{aligned} & D : T\Delta \otimes T\Delta \to T\Delta \\ & \quad \nabla_X Y = D_X Y + K(X, Y). \end{aligned} \tag{3}$$

$D_X Y$ is purely tangential and $K(X, Y)$ is an element of the normal bundle and refereed to as the extrinsic curvature. The Gauss equation also implies along with that metric compatibility of ∇ with g that the derivative operator D_a is metric compatible with the metric on the hypersurface h_{ab} (*i.e.* $D_a h_{bc} = 0$). In terms of the local coordinate charts, the Gauss equation gives the following expression for the derivative operator (for $X \equiv \partial_a$):

$$D_a Y_b = e^\mu_a\, e^\nu_b\, \nabla_\mu Y_\nu. \tag{4}$$

The extrinsic curvature can also be defined for the hypersurface in terms of the local coordinates. The normal bundle for the hypersurface is one dimensional. Let, n^μ be the normal. The extrinsic curvature is

$$K_{ab} = e^\mu_a\, e^\nu_b\, \nabla_\mu n_\nu = (1/2)\, \left(\pounds_n g_{\mu\nu}\right)\, e^\mu_a\, e^\nu_b. \tag{5}$$

For our later use, let us give the Gauss equation in terms of the local coordinates:

$$R_{\mu\nu\lambda\sigma}\, e^\mu_a\, e^\nu_b\, e^\lambda_c\, e^\sigma_d = R_{abcd} - \left(K_{ad} K_{bc} - K_{ac} K_{bd}\right). \tag{6}$$

The Codazzi equation in local coordinates is given by:

$$R_{\mu\nu\lambda\sigma}\, n^\mu\, e^\nu_b\, e^\lambda_c\, e^\sigma_d = K_{ab|c} - K_{ac|b}, \tag{7}$$

where $|$ denotes the covariant derivative with respect to the coordinates on the hypersurface.

Several of these spacetime functions have different values on either sides of a hypersurface. Then, it is required to express their continuity across the hypersurface. A useful and prominent example of this idea is that of the Israel–Darmois–Lanczos (IDL) junction condition [39]. Consider a hypersurface Δ which partitions the spacetime into two regions (M_+, g_+) with coordinates $\{x^\mu_+\}$ and (M_-, g_-) with coordinates $\{x^\mu_-\}$. The spacetime M_+ is assumed to be to the future of the spacetime M_-. Quite naturally, it is not generally true that the metrics on these two spacetimes could be continuously matched across the hypersurface Δ (the either side of the hypersurface Δ has been installed with coordinates $\{y^a\}$). The discontinuity in the metric would be reflected in the fact that Riemann tensor would have a delta-function singularity on the hypersurface. The Israel junction conditions provides a method to smoothly match these hypersurfaces by using the following trick: relate the Ricci part of the singular Riemann curvature tensor to the surface stress-tensor using the Einstein equations. For spacelike hypersurfaces, the Israel junction conditions for a smooth joining of hypersurfaces at Δ is given by

$$[h_{ab}] = 0 = [K_{ab}] \tag{8}$$

where $[A] \equiv A(M^+)|_\Delta - A(M^-)|_\Delta$. However, if the extrinsic curvature is not identical on both the sides on the hypersurface Δ, the surface stress tensor (S_{ab}) on the hypersurface is

$$8\pi\, S_{ab} = [K_{ab}] - [K] h_{ab}. \tag{9}$$

However, on the null surface, the standard extrinsic curvature corresponding the normal of the hypersurface (which is also the tangent to null hypersurface) is always continuous and hence, one needs to define a transverse curvature [39]. The metric induced on the null hypersurface is again continuous but the discontinuities in the components of the transverse curvature is related to the energy–momentum tensor induced on this hypersurface.

In deriving the above relations, we have implicitly made two crucial assumptions: First, that the point functions are continuous and differentiable in the region under consideration. However, it may happen that the scalar, vector or the tensor functions are only fractionally differentiable. In that case, the limiting values defined by our ordinary differential calculus become singular on the hypersurface. Thus, in addition to the IDL conditions, their fractional character must also be taken into account. Secondly, the spacetime connection is assumed to be a Levi-Civita connection. This arises since the spacetime is assumed to be a Riemannian spacetime and hence, the spacetime metric is compatible with the covariant derivative ($\nabla_{\gamma}\, g_{\alpha\beta} = 0$). The Gauss decomposition, eqn. (3), then implies that the connection on the hypersurface is also a Levi-Civita connection and that the extrinsic curvature is uniquely determined in terms of this connection. However, in the strong gravity regime we are interested in, the spacetime is modified from its Riemannian character and the connection is not a Levi-Civita connection derived from the metric. In fact, many perturbative field theoretic arguments leading to one loop or two loop effective actions imply that this restricted framework of Riemannian geometry may not hold true in strong gravity limits and many non-local and non-analytic contributions arise in the gravitational action [45,46]. In particular, it has been suggested that the low energy Einsteinian phase of gravity, where only non-degenerate metrics (and hence invertible metrics) are considered, must be replaced, in the strong gravity regime by a non-Einsteinian phase where the contributions from the degenerate metrics as well as non-symmetric connections also exist [47–49]. It has also been emphasised that contributions from degenerate metric configurations are essential in the Euclidean path integral approach to quantum gravity and lead to important effects like changes in the spacetime topology and introduction of non-symmetric spacetime connections like torsion fields [46,50]. Furthermore, in the first order formalism too, where the metric $g_{\mu\nu}$ is replaced by the tetrad and connection variables, e^{I}_{μ} and ω^{IJ}_{μ}, with I, J being the local Lorentz indices, this non-Einsteinian phase, $< g_{\mu\nu} >= 0$ or $\det e = 0$, has been shown to emerge at the level of one-loop effective action, leading not only to spacetimes with non-symmetric connections, but also providing possibilities for a quantum emergence of the Barbero–Immirzi parameter [49,51]. Thus, there are strong indications that the Riemannian structure of the spacetime and the Einstein–Hilbert action may not hold correct at the strong gravity regime and that the complete understanding of the gravitational degrees of freedom require other generalisations such as higher derivative and higher curvature generalisations. Such a point of view exists in almost all the proposals of quantum gravity, including string theory [52]. For example, supersymmetric generalisations of curvature squared terms present in the tree level actions of string theories have been found essential to understand the microscopic computation of entropy of extremal black holes in these theories [53]. As mentioned earlier, in loop quantum gravity too, where the Barbero–Immirzi parameter plays role at the kinematic level, the degenerate tetrads may play a crucial role in elucidating the quantum emergence of the Holst-like terms [49,51].

Keeping these points in view, we modify the Riemannian structure by considering connections which are not Levi-Civita connections but generalisations based on fractional derivatives. We have not used any quantum gravity framework, but have assumed that it may be incorporated, at least in principle, in the (multi)-fractal theories [32–34]. More precisely, at small distances or in the strong gravity regime, we expect the spacetime structure to attain a phase which satisfies at least some of the properties of a (multi)-fractal. In particular, the spacetime itself as well as the curves on this spacetime must become continuous but fractionally differentiable. Furthermore, all the scalar and tensor functions must also be only fractionally differentiable. Naturally, the metric is assumed to be *fractionally covariantly constant* in this regime. In other words the definition of the covariant derivative needs to be changed. We use a modification of the Levi-Civita connection obtained from a particular generalisation of the fractional Riemann–Liouville derivative [2,3]. In this situation, the Gauss decomposition implies that the connection on the hypersurface will also be modified (it will not remain to be a metric connection) and the expression of the extrinsic curvature will also change.

So, for a description of the classical fractal geometry of the hypersurface, we shall have to clearly specify a method to construct the metric and extrinsic curvature of the hypersurface. First, as may be seen from the eqn. (1) the induced metric on the hypersurface is a geometric projection of the metric of the full spacetime. Hence, the induced metric on a fractal hypersurface will remain identical to the one induced on the *integral* hypersurfaces. However, the Gauss decomposition eqn. (3), allows us to construct non-metric connections on the hypersurface provided we also modify the expression of the extrinsic curvature appropriately. In [54], a fractional generalisation of the Lie derivative has been proposed and utilised to generalise the definition of the extrinsic curvature for non-null hypersurfaces. The essential idea is to use the Einstein theory, but with the usual rules of differentiation modified to fractional ones. In a fractional generalisation, which is based on Caputo's modification of the Riemann–Liouville definition of fractional derivative (see the Appendix A), the usual definition of the extrinsic curvature $K_{ab} = (1/2)(\pounds_n g_{\alpha\beta})\, e^\alpha_a\, e^\beta_b$, is modified to give:

$$\begin{aligned} {}^qK_{ab} &= \frac{1}{2}\left({}^q\pounds_n g_{\alpha\beta}\right) e^\alpha_a\, e^\beta_b \\ &= \frac{1}{2}\left[n^\gamma\, \mathcal{D}^q_{r-\Delta,\gamma}\, g_{\alpha\beta} + g_{\gamma\beta}\mathcal{D}^q_{r-\Delta,r}n^\gamma + g_{\alpha\gamma}\mathcal{D}^q_{r-\Delta,r}n^\gamma\right] e^\alpha_a\, e^\beta_b. \end{aligned} \tag{10}$$

Here, the superscript q denotes the fractional parameter, $0 < q \le 1$ (see the Appendix A) and $\mathcal{D}^q_{r-\Delta,r}$ denotes the derivative:

$$\mathcal{D}^q_{r-\Delta,r}(g_{\alpha\beta}) = \frac{\Gamma(2-q)}{\Gamma(1-q)\,\Delta^{1-q}} \int_{r-\Delta}^{r} \frac{\partial g_{\alpha\beta}(w)}{\partial w}(r-w)^{-q}dw, \tag{11}$$

where the integration is carried out from a spacepoint $r - \Delta$ to r. In the context of matching of spacetimes across hypersurface, Δ is taken to be the thickness of the hypersurface. To understand this better, let us consider a spherically symmetric black hole horizon of radius R_0. Note that the high energy quantum behaviour of the horizon is expected to lead to some fuzziness Δ, at the location of the horizon and the location becomes $R_0 \pm \Delta$. If quantum effects are small, Δ/R_0 would be small whereas, $\Delta \simeq R_0$ for a pronounced quantum effect. Naturally, we take the Δ to be the thickness of the shell at the horizon and this is the region over which the non-local smearing of fields take place. The junction conditions will be modified from eqn. (9) to

$$8\pi\, {}^qS_{ab} = [{}^qK_{ab}] - [{}^qK]h_{ab}. \tag{12}$$

Naturally, because of the definition of the derivative, it has a non-local character imbedded into it.

In short, a hypersurface will be called a fractal hypersurface if it admits a metric induced from the full spacetime and has an extrinsic curvature given by eqn. (10). This fractal structure will definitely lead to modifications of the Einstein equations. In the Appendix C, we explicitly obtain corrections to the connection by using a particular kind of modification based on the Caputo derivative along with explicit expressions for the non-metric connections and the curvature tensors. We also derive the modified Einstein equation. As argued in this appendix itself, these corrections do not lead to substantial changes to our metric functions near the horizons since they are negligible at those spacetime regions. However, near the singularity, the fractal modifications should contribute to drastic modification of the spacetime structure. In this paper, we shall be dealing with only those regions of spacetime where the corrections to the metric is negligible and all the metric remain to be exactly identical to classical spacetimes associated with the uncorrected Einstein equations. On the other hand, we show that the contribution from the extrinsic curvatures are however not small and lead to changes in the effective hypersurface stress-tensors.

3. Junction conditions for non-null hypersurfaces

Let us consider a non-null hypersurface Δ. As discussed previously, the junction condition for the smooth joining of spacetimes along a timelike or spacelike hypersurface Δ is given by the following two conditions: $[h_{ab}] = 0$ and $[K_{ab}] = 0$. On the other hand, for joining spacetimes which contribute non-equal extrinsic curvatures on the hypersurface, a thin layer of matter is assumed to exist on the hypersurface with stress tensor $S_{ab} = (\epsilon/8\pi)([K_{ab}] - K h_{ab})$. The quantity $\epsilon = n \cdot n$, distinguishes spacelike hypersurfaces ($\epsilon = -1$) from timelike ones ($\epsilon = -1$).

As described in the previous subsection, the junction conditions differ if the fractional derivatives are used. For the non-null hypersurface, we elaborate on this method through two explicit examples. In the first example, we give a detail step by step calculation showing the matching of a slowly rotating Kerr metric to a Minkowski metric on a timelike hypersurface. We show that depending on the width of the shell, the energy momentum tensor of the shell changes. We utilize this observation in the second example, which deals with matching of a Schwarzschild spacetime with a de-Sitter spacetime on a spacelike hypersurface. Again the energy–momentum tensor residing on the thin shell differs substantially from the standard results.

3.1. Joining Minkowski and slowly rotating Kerr metrics

Let us consider the metric of a Kerr spacetime in the slow-rotation approximation. We shall assume a shell of mass M and angular momentum J in the spacetime. The exterior spacetime ($\mathcal{M}^+$) has the following metric:

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,d\Omega^2 - \frac{4Ma}{r}\sin^2\theta\,dt\,d\phi, \tag{13}$$

where $f(r) = (1 - 2M/r)$ and $a = (J/M) \ll M$, is a parameter for the angular momentum which is usually used to replace the shell's angular momentum J. Let us assume that the shell is located at $r = R_0$. The induced metric on the shell becomes:

$$ds_\Sigma^2 = -f(R_0)\,dt^2 + R_0^2\,d\Omega^2 - \frac{4Ma}{R_0}\sin^2\theta\,dt\,d\phi. \tag{14}$$

Using the definitions, $\psi = (\phi - \omega t)$ with $\omega = (2Ma/r^3)$, and keeping terms upto first order of a, we get the induced metric to be

$$h_{ab}dy^a dy^b = -f(r)\,dt^2 + R_0^2\,(d\theta^2 + \sin^2\theta\,d\psi^2). \tag{15}$$

We shall use $y^a = (t, \theta, \psi)$ as the co-ordinates on the shell and the parametric equations for the hypersurface in the form $x^\alpha = x^\alpha(y^a)$ are $t = t$, $\theta = \theta$ and $\phi = (\psi + \omega t)$. The shell's unit normal is $n_\alpha = f(r)^{-1/2}\,\partial_\alpha r$, which in the coordinates is given by:

$$n^\alpha = \left(0, \sqrt{1 - 2M/r}, 0, 0\right). \tag{16}$$

Now let's calculate the non vanishing components of extrinsic curvature. The definition of transverse component of the fractional generalisation is:

$$\begin{aligned} {}^qK_{ab} &= \frac{1}{2}\left({}^q\pounds_n g_{\alpha\beta}\right) e^\alpha_a\, e^\beta_b \\ &= \frac{1}{2}\left[n^\gamma\, \mathcal{D}^q_{r-\Delta,\gamma}\, g_{\alpha\beta} + g_{\gamma\beta}\, \mathcal{D}^q_{r-\Delta,r} n^\gamma + g_{\alpha\gamma}\, \mathcal{D}^q_{r-\Delta,r} n^\gamma\right] e^\alpha_a\, e^\beta_b, \end{aligned} \tag{17}$$

where the projectors e^α_as are $e^\alpha_t\,\partial_\alpha = (\partial_t + \omega\,\partial_\phi)$, $e^\alpha_\theta\,\partial_\alpha = \partial_\theta$ and $e^\alpha_\phi\,\partial_\alpha = \partial_\phi$.

Let us first determine ${}^qK^+_{tt}$, where $+$ denotes that the variable is associated with the external spacetime. Note that since $e^\alpha_t \partial_\alpha = (\partial_t + \omega\partial_\phi)$, one gets the only contribution from ${}^qK^+_{tt} = (1/2)\,{}^q\pounds_n(g^+_{tt})$. The other contribution to ${}^qK^+_{tt}$ from $g^+_{t\phi}$ is neglected since $g^+_{t\phi} = -(2Ma\sin^2\theta/r)$, is directly proportional to a and further, together with $\omega = (2Ma/r^3)$ contributes an overall a^2 term. Note that due to the form of the normal vector, and the metric, only the first term in expansion in (17) contributes. Using the expression for $\mathcal{D}^q_{r-\Delta,r}(r^{-1})$ in the appendix, eqn. (B.15), we get

$${}^qK^+_{tt} = -\frac{M}{R_0^2}\left(1 - \frac{2M}{R_0}\right)^{1/2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \ldots\right],$$

and hence, using the metric, one easily determines that

$${}^qK^{+t}_t = \frac{M}{R_0^2}\left(1 - \frac{2M}{R_0}\right)^{-1/2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \ldots\right]. \tag{18}$$

Similarly, one finds the contribution from ${}^qK_{t\psi}$ as follows:

$${}^qK^+_{t\psi} = (1/2)\,{}^q\pounds_n(g^+_{ab})\,e^a_t\,e^b_\psi = (1/2)\,{}^q\pounds_n(g^+_{\phi t}) + (\omega/2)\,{}^q\pounds_n(g^+_{\phi\phi}). \tag{19}$$

Using $\mathcal{D}^q_{r-\Delta,r}(r^{-1})$ in the appendix, eqn. (B.15), we get that

$${}^q\pounds_n(g^+_{\phi t}) = n^r\mathcal{D}^q_{r-\Delta,r}(g^+_{\phi t}) = \frac{2Ma\sin^2\theta}{R_0^2}\left(1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \ldots\right). \tag{20}$$

Again, using $\mathcal{D}^q_{r-\Delta,r}(r^2)$ in the appendix, eqn. (B.9), we get

$$\begin{aligned} {}^q\pounds_n(g^+_{\phi\phi}) &= n^r\mathcal{D}^q_{r-\Delta,r}(r^2\sin^2\theta) \\ &= 2R_0\sin^2\theta\left(1 - \frac{2M}{R_0}\right)^{1/2}\left(1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \ldots\right). \end{aligned} \tag{21}$$

Putting $\omega = (2Ma/R_0^3)$ in eqn. (19), and using equations (20) and (21), we get

$${}^qK^+_{t\psi} = \frac{3Ma\sin^2\theta}{R_0^2}\left(1 - \frac{2M}{R_0}\right)^{1/2}\left[1 + \frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2 + \ldots\right].$$

The expression naturally leads to the following expressions for extrinsic curvatures:

$$ {}^{q}K_{\psi}^{+t} = g^{tt}\left({}^{q}K_{t\psi}^{+}\right) = \frac{-3Ma\sin^2\theta}{R_0^2}\left(1-\frac{2M}{R_0}\right)^{-1/2}\left[1+\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2+\dots\right], \tag{22} $$

$$ {}^{q}K_{t}^{+\psi} = g^{\psi\psi}\left({}^{q}K_{t\psi}^{+}\right) = \frac{3Ma}{R_0^4}\left(1-\frac{2M}{R_0}\right)^{1/2}\left[1+\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2+\dots\right]. \tag{23} $$

The angular components of the extrinsic curvatures are ${}^{q}K_{\theta\theta}^{+}$ and ${}^{q}K_{\psi\psi}^{+}$ and their expressions may be found in exactly the same method and we get:

$$ {}^{q}K_{\theta}^{+\theta} = {}^{q}K_{\psi}^{+\psi} = \frac{1}{R_0}\left(1-\frac{2M}{R_0}\right)^{1/2}\left(1-\frac{1-q}{2-q}\frac{\Delta}{R_0}+\dots\right). \tag{24} $$

For interior spacetime, we take it to be the flat Minkowski spacetime. So, to the past of the hypersurface at $r = R_0$, the spacetime $\mathcal{M}^-$ is given by the metric

$$ ds^2 = -\left(1-\frac{2M}{R_0}\right)dt^2 + d\rho^2 + \rho^2 d\Omega^2 \tag{25} $$

where ρ is a radial coordinate. The intrinsic metric on the hypersurface from the interior matches with the induced metric from the exterior region. The normal to the hypersurface is $n^\alpha = (\partial/\partial\rho)^\alpha$. The expressions for the extrinsic curvatures may be determined and the only non-vanishing components are ${}^{q}K_{\theta\theta}$ and ${}^{q}K_{\phi\phi}$:

$$ {}^{q}K_{\theta}^{-\theta} = {}^{q}K_{\psi}^{-\psi} = \frac{1}{R_0}\left(1-\frac{1-q}{2-q}\frac{\Delta}{R_0}+\dots\right). \tag{26} $$

Let us now determine the stress–energy tensor of the thin shell of matter forming the hypersurface joining the two spacetimes. The discontinuities in the extrinsic curvatures are related to the shell's surface stress–energy tensor S^{ab}.

$$ 8\pi S_t^t = \left[{}^{q}K_\theta^\theta\right] + \left[{}^{q}K_\psi^\psi\right], \tag{27} $$

$$ -8\pi S_\psi^t = \left[{}^{q}K_\psi^t\right], \tag{28} $$

$$ -8\pi S_t^\psi = \left[{}^{q}K_t^\psi\right], \tag{29} $$

$$ 8\pi S_\theta^\theta = \left[{}^{q}K_t^t\right] + \left[{}^{q}K_\psi^\psi\right]. \tag{30} $$

The shell's matter may be assumed to be made of perfect fluid, with density $\sigma = -S_t^t$, pressure $p = S_\theta^\theta$ and rotating with angular velocity $\omega = -S_\psi^t/(-S_t^t + S_\psi^\psi)$. The expressions for these components of the energy momentum tensor are:

$$ S_\psi^t = \frac{3Ma\sin^2\theta}{8\pi R_0^2}\left(1-\frac{2M}{r}\right)^{-1/2}\left[1+\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2+\dots\right], \tag{31} $$

$$ S_t^\psi = \frac{-3Ma}{8\pi R_0^4}\left(1-\frac{2M}{r}\right)^{1/2}\left[1+\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2+\dots\right], \tag{32} $$

$$ S_t^t = \frac{-1}{4\pi R_0}\left(1-\sqrt{1-2M/R_0}\right)\left[1-\frac{1-q}{2-q}\frac{\Delta}{R_0}+\dots\right], \tag{33} $$

$$S_\theta^\theta = \left[\frac{(1-2M/R_0)^{-1/2}}{8\pi R_0}\right]\left[1 - M/R_0 - \sqrt{1-2M/R_0}\right] - \left[\frac{(1-2M/R_0)^{-1/2}}{8\pi R_0}\right]$$
$$\left[1 - 4M/R_0 + \sqrt{1-2M/R_0}\right]\left(\frac{1-q}{2-q}\right)\left(\frac{\Delta}{R_0}\right) + \dots \quad (34)$$

Naturally, all the expressions of the energy momentum tensor are modified due to the improved notion of fractional differential. The modification takes the thickness of the shell into account. One very interesting notion is the determination of the angular velocity of the shell. The angular velocity is obtained from $\omega = S_\psi^t/(S_t^t - S_\psi^\psi)$. This gives for $R_0 \gg 2M$,

$$\omega_{shell} = \frac{3a}{2R_0^2} + \frac{3a}{2MR_0}\frac{1-q}{2-q}\frac{\Delta}{R_0} + \dots \quad (35)$$

This expression given above for the angular velocity is different from that obtained in the usual case [39] but reduces to it in the limit $\Delta/R_0 \to 0$.

3.2. Matching the Schwarzschild and the de-Sitter spacetimes

The joining of exterior spacetime of the Schwarzschild black hole (taken as the exterior spacetime) with the de-Sitter spacetime has been the subject of many investigations, which were particularly directed to create singularity free models of black hole interior. One particularly interesting application was considered by Frolov, Markov and Mukhanov [44] to exemplify their *limiting curvature hypothesis*. They suggested that inside the Schwarzschild black hole, very close to the singularity, when the Planck scale is reached, there would be corrections to the Einstein theory of gravity. These corrections would not allow the curvature of the spacetime to dynamically grow to infinite values. Instead, the effective curvature of the spacetime would be bounded from below by ℓ_p^{-2}, where ℓ_p is the Planck length. Naturally, this hypothesis implies that there will be no curvature singularity. Instead, the model in [44] proposes that very close to the spacetime singularity, where the curvature reaches the ℓ_p^2, the spacetime makes a transition from the Schwarzschild to the de-Sitter spacetime by passing through a very thin transition layer (see Fig. 1). The spacetime passes through a deflation stage and instead of singularity, reaches a new inflating universe free of singularity. The matching of these two spacetimes require stress–energy tensors on the joining shell which violate energy conditions.

In the following, we recalculate the stress–energy tensor on the matching shell using fractional calculus. We show that contrary to the original Frolov, Markov and Mukhanov proposal, one may match the two spacetimes on any spacelike hypersurface of radius R_0 (and thickness Δ, with $\Delta < R_0$) such that $\ell_p << R_0 < 2M$. On this spacelike hypersurface, we also show that the stress-tensor is modified. The modified stress tensor will be shown to lead to smooth matching of the spacetimes. Let us match the de-Sitter spacetime with the interior Schwarzschild spacetime. The metric for the two spacetimes may be written in a combined form as:

$$ds^2 = f(r)\,dv^2 + 2dvdr + r^2\,d\Omega^2, \quad (36)$$

where $f(r) = (2M/r - 1)$ for the Schwarzschild metric and the $f(r) = \left[(r/l)^2 - 1\right]$ for the de-Sitter metric. For simplification, let us define a new set of coordinates: $v = \lambda/\sqrt{f}$. The induced metric on the spacelike surface becomes $ds^2 = d\lambda^2 + r^2 d\Omega^2$. The normal to this surface is given by:

$$n_\alpha = \left[0, -\frac{1}{\sqrt{f}}, 0, 0\right], \quad (37)$$

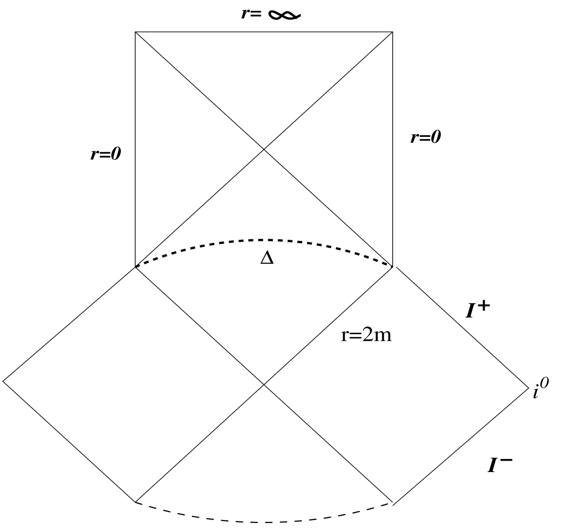

Fig. 1. The Frolov, Markov and Mukhanov model in which the Schwarzschild black hole has a de Sitter world in the interior. The spacelike hypersurface Δ represents the matching hypersurface joining the two spacetimes. The I^+, I^- and i^0 represent the future null, past null and the spatial infinities respectively.

and $n^\alpha = \left(1/\sqrt{f}, \sqrt{f}, 0, 0\right)$. The coordinates in the spacetime is taken to be $x^\alpha = (v, r, \theta, \phi)$ and that of the hypersurface to be $y^a = (\lambda, \theta, \phi)$. This implies that $e^\alpha_q \partial_\alpha = (1/\sqrt{f})(\partial/\partial v)^\alpha$, $e^\alpha_\theta \partial_\alpha = (\partial/\partial\theta)^\alpha$ and $e^\alpha_\phi \partial_\alpha = (\partial/\partial\phi)^\alpha$.

Let us evaluate the extrinsic curvatures. The general expressions for these quantities for either of these spacetimes are given by the following:

$$^qK_{qq} = \frac{1}{2\sqrt{f}}\,\mathcal{D}^q_{r-\Delta,r}(g_{vv}), \tag{38}$$

$$^qK_{\theta\theta} = (\sqrt{f}/2)\,\mathcal{D}^q_{r-\Delta,r}(g_{\theta\theta}), \tag{39}$$

$$^qK_{\phi\phi} = (\sqrt{f}/2)\,\mathcal{D}^q_{r-\Delta,r}(g_{\phi\phi}). \tag{40}$$

For the Schwarschild Metric, which is taken to be the interior spacetime, these expressions are obtained using the equations (B.15) and (B.9) are:

$$^qK^{q-}_q = -\frac{M}{R_0^2}\left(\frac{2M}{R_0^2} - 1\right)^{-1/2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right], \tag{41}$$

$$^qK^{\theta-}_\theta = {}^qK^{\phi-}_\phi = \frac{1}{R_0}\left(\frac{2M}{R_0} - 1\right)^{1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{42}$$

For the de-Sitter metric, taken to be the external or the future spacetime, the same expressions are given as, using (B.9) are:

$$^qK^{q+}_q = \frac{R_0}{l^2}\left[\left(\frac{R_0}{l}\right)^2 - 1\right]^{-1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right], \tag{43}$$

$$^qK^{\theta+}_\theta = = {}^qK^{\phi+}_\phi = \frac{1}{R_0}\left[\left(\frac{r}{l}\right)^2 - 1\right]^{1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{44}$$

The jump in the components of the extrinsic curvatures are given by:

$$\kappa = [{}^q K^q_q] = \frac{R_0}{l^2}\left[\left(\frac{R_0}{l}\right)^2 - 1\right]^{-1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]$$
$$+ \frac{M}{R_0^2}\left(\frac{2M}{R_0^2} - 1\right)^{-1/2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right], \tag{45}$$

$$\lambda = [{}^q K^\theta_\theta] = -\frac{1}{R_0^2}\left(\frac{2M}{R_0} - 1\right)^{1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]$$
$$+ \frac{1}{R_0}\left[\left(\frac{R_0}{l}\right)^2 - 1\right]^{1/2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{46}$$

The components of the stress–energy tensor is given by $S^q_q = \lambda/4\pi$ and $S^\theta_\theta = S^\phi_\phi = (\kappa + \lambda)/8\pi$. Notice that the values of the energy momentum tensors are markedly different from those obtained in [44]. The values differ by quantities which are proportional to the ratio $(1 - q/2 - q)(\Delta/R_0)$, and hence by choosing the value of this ratio judiciously, it can be easily seen that the energy momentum tensor can be made to vanish. Hence, one may match the two spacetimes smoothly across a spacelike hypersurface.

4. Junction conditions for null hypersurfaces

Let us consider a null hypersurface that partitions the 4-dimensional spacetime into two regions $\left[\mathcal{M}^+, g^+_{\mu\nu}(x^+)\right]$ and $\left[\mathcal{M}^-, g^-_{\mu\nu}(x^-)\right]$, which we shall conveniently call as the future and the past respectively. Let us denote the coordinates of the spacetime as x^α, $\alpha = 0, 1, 2, 3$, whereas the coordinates on either side of the hypersurface will be denoted by y^a, $a = 1, 2, 3$, which will mean the collective coordinates (λ, θ^A), where θ^A, $A = (2, 3)$ denotes the variables on the two-dimensional cross-sections of the hypersurface. On each side of the hypersurface, one may construct the tangents to the generators of the null hypersurface (ℓ^α) and the transverse spacelike vectors (e^α_A), which are tangents to the cross-sections (taken to be compact) of the hypersurface. These vectors shall be denoted by:

$$l^\alpha = e^\alpha_\lambda = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A}; \quad e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda, \tag{47}$$

with the following properties: $\ell^\alpha \ell_\alpha = 0$, $\ell_\alpha e^\alpha_A = 0$. These vectors may be constructed for both sides of the null hypersurface. Further, on each side, the basis needs four vectors and the fourth vector, will be taken to be a null vector. It will be denoted by n^α with the following properties: $\ell^\alpha n_\alpha = -1$, $n^\alpha n_\alpha = 0$, $n_\alpha e^\alpha_A = 0$.

The typical situation with a null surface is that the usual extrinsic curvature, $K_{ab} = (1/2)\left(\pounds_\ell\, g_{\alpha\beta}\right) e^\alpha_a\, e^\beta_b$, corresponding to the normal to the hypersurface is continuous, since the normal is also the tangent ℓ^α. So, one usually defines the transverse component of the extrinsic curvature corresponding to the null vector field normal to the transverse cross-sections of the hypersurface. This vector is n^a, such that $\ell.n = -1$. The transverse extrinsic curvature may be defined as $C_{ab} = (1/2)\left(\pounds_n\, g_{\alpha\beta}\right) e^\alpha_a\, e^\beta_b$. The stress–energy tensor of the shell is given by: $S^{\alpha\beta} = \mu\, \ell^\alpha \ell^\beta + p\, \sigma^{AB}\, e^\alpha_A\, e^\beta_B$, where $\mu = (-1/8\pi)\sigma^{AB}[C_{AB}]$ is the shell's surface density and $p = (-1/8\pi)[C_{\lambda\lambda}]$ is the surface pressure.

4.1. Null charged shell collapsing on a charged black hole

Let us consider a spherically symmetric charged black hole of mass M and charge Q on which a null charged shell of mass E and charge q collapses. Outside the shell, the spacetime outside the total configuration may be viewed as a spherically symmetric Reissner–Nordstrom type geometry with mass $(M + E)$ and charge $(Q + q)$. As before, the spacetimes outside and inside the shell shall be denoted by $+$ and $-$ respectively.

$$ds^2 = -f_{\pm}(r) + dv^2 + 2dvdr + r^2 d\Omega^2, \tag{48}$$

$$f_+(r) = 1 - \frac{2(M+E)}{r} + \frac{(Q+q)^2}{r^2}, \tag{49}$$

$$f_-(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{50}$$

The coordinates of the spacetime is given by $x^\alpha = (v, r, \theta, \phi)$. The surface of the shell is given by $v = v_0$ with coordinates of the shell being $y^a = (r, \theta, \phi)$. The vector fields are given by $e^\alpha_r \partial_\alpha \equiv \ell^\alpha = -(\partial/\partial r)^\alpha$, $e^\alpha_\theta \partial_\alpha = (\partial/\partial\theta)^\alpha$, $e^\alpha_\phi \partial_\alpha = (\partial/\partial\phi)^\alpha$. Note that the vector ℓ^α is the generator of the null surface. The transverse null vector required to complete the basis is $n^\alpha = \{f_\pm(r)/2\}\,(\partial/\partial r)^\alpha$.

The metric is continuous across the shell. Let us check that the extrinsic curvature of the null surface corresponding to the null normal ℓ^α is also continuous on either side of the surface. This is always true for the non-fractional case and precisely for this reason, the concept of the transverse curvatures have been introduced. We show that for the fractional case too, the extrinsic curvatures corresponding to the null normal of the surface is continuous. For the interior solution, we get that $\ell^{-\alpha} = (0, 1, 0, 0)$ and hence, the components of the fractional extrinsic curvature on the cross-sections are: ${}^qK^-_{AB} = (1/2)^q £_\ell\, g_{AB}$. Using the formulae from the appendix, equation (B.9), we get

$$ {}^qK^-_{\theta\theta} = R_0\Big[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\Big].$$

$$ {}^qK^-_{\phi\phi} = R_0 \sin^2\theta\Big[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\Big].$$

These two equations may be combined to the following form:

$$ {}^qK^-_{AB} = R_0^2\Big[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\Big]\sigma_{AB}. \tag{51}$$

For the exterior solution too, the null normal is given by $\ell^{+\alpha} = (0, 1, 0, 0)$ and the extrinsic curvature corresponding to this null normal is ${}^qK^+_{AB} = \frac{1}{2}{}^q£_\ell(g_{AB})$ is given by:

$$ {}^qK^+_{AB} = R_0^2\Big[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\Big]\sigma_{AB}. \tag{52}$$

This implies that the extrinsic curvatures are also continuous ${}^qK^+_{AB} ={}^q K^-_{AB}$.

The transverse extrinsic curvature is not continuous for this metric. The expression for ${}^qC^+_{\theta\theta}$ is given by

$$ \begin{aligned} {}^qC^+_{\theta\theta} &= (1/2)\left[n^r \mathcal{D}^q_{r-\Delta,r}(g_{\theta\theta})\right], \\ &= f_+(r)\, R_0\Big[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\Big]. \end{aligned} \tag{53}$$

Similarly for ${}^{q}C^{+}_{\phi\phi}$, we get:

$$ {}^{q}C^{+}_{\phi\phi} = f_{+}(r) R_0 \sin^2\theta \left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{54} $$

So these two expressions may be combined to give:

$$ {}^{q}C^{+}_{AB} = \frac{f_{+}(r)}{R_0}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]\sigma_{AB}, \tag{55} $$

where $\sigma_{AB} = R_0^2 + R_0^2 \sin^2\theta$. Similarly, for the interior spacetime, the transverse component of the extrinsic curvature is given by:

$$ {}^{q}C^{-}_{AB} = \frac{f_{-}(r)}{R_0}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]\sigma_{AB} \tag{56} $$

These equations immediately imply that the shell's surface pressure is zero for this case. The shell's surface density is

$$ \mu = -\frac{1}{4\pi R}\Big(\frac{2Qq + q^2}{R^2} - \frac{2E}{R}\Big)\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R} + \cdots\right]. \tag{57} $$

This relation clearly implies that to satisfy the weak energy condition, we must have

$$ 2E \geq \frac{2Qq + q^2}{M + \sqrt{M^2 - Q^2}}. \tag{58} $$

As a simple application, let us study if the charged black hole may be overcharged, so that the total charge $(Q + q)$ exceed the total mass $(M + E)$. It is a simple matter to check that the condition for overcharging violates the weak energy condition. So, even in the fractional modification, a charged black hole cannot be overcharged.

4.2. Matching Schwarschild and de-Sitter spacetimes across horizons

Let's start with a general form of the metric and then we shall specialize to the individual cases. The general form for a spherical symmetric metric in the advanced Eddington–Finkelstein coordinates is given by:

$$ ds^2 = -f(r)\, dv^2 + 2dv\, dr + r^2\, d\Omega^2. \tag{59} $$

The coordinates of the spacetime is given by $x^\alpha = (v, r, \theta, \phi)$. Let us assume a null hypersurface (a shell) given by $r = r_0$ with coordinates of the shell being $y^a = (v, \theta, \phi)$. The null surface is foliated by compact surface S^2. The vector fields tangent to the sphere are given by, $e^\alpha_\theta = (\partial/\partial\theta)^\alpha$ $e^\alpha_\phi = (\partial/\partial\phi)^\alpha$. Let us now determine the set of null vectors tangent to the null surface which is given by the relation $f(r_0) = 0$. The generator of the null surface is $\ell^\alpha = (\partial/\partial v)^\alpha$ and the transverse null vector is $n^\alpha = -(\partial/\partial r)^\alpha$.

Let us consider the interior metric $(\mathcal{M}^-)$ to be the de-Sitter spacetime:

$$ ds_-^2 = -\left[1 - \left(\frac{r}{l}\right)^2\right] dv^2 + 2dvdr + r^2 d\Omega^2. \tag{60} $$

As usual, the standard extrinsic curvatures associated to the null normals of the are continuous and hence let us calculate the transverse extrinsic curvatures ${}^{q}C^{-}_{\theta\theta}$ and ${}^{q}C^{-}_{\phi\phi}$:

$$
{}^{q}C^{-}_{\theta\theta} = R_0\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right],
$$

and similarly for ${}^{q}C^{+}_{\phi\phi}$, we get

$$
{}^{q}C^{-}_{\phi\phi} = R_0 \sin^2\theta\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right].
$$

So combining them together, we get:

$$
{}^{q}C^{-}_{AB} = \frac{1}{R_0}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]\sigma_{AB}, \tag{61}
$$

where $\sigma_{AB} = R_0^2 + R_0^2\sin^2\theta$. The quantity ${}^{q}C^{-}_{vv}$ gives:

$$
{}^{q}C^{-}_{vv} = \frac{R_0}{l^2}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{62}
$$

The exterior spacetime is taken to be the Schwarzschild spacetime (M^{+}), with the metric

$$
ds_-^2 = -\Big(1 - 2m/r\Big)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{63}
$$

Again, the co-ordinate on null shell are (v, θ, ϕ). Let us calculate transverse curvatures. Just as in the previous case, the result is

$$
{}^{q}C^{+}_{AB} = \frac{1}{R_0}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]\sigma_{AB}. \tag{64}
$$

The ${}^{q}C^{+}_{vv}$ is given by:

$$
{}^{q}C^{+}_{vv} = -\frac{m}{R_0^2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{65}
$$

Let us calculate the quantities associated with the shell. The surface density $\mu = 0$. The pressure is given by:

$$
p = \frac{-1}{8\pi}\left[\left(\frac{R_0}{l^2} + \frac{m}{R_0^2}\right) - \left(\frac{R_0}{l^2} - \frac{2m}{R_0^2}\right)\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{66}
$$

On the horizon, $R_0 = 2m = l$ and hence the pressure must be non vanishing. For $q = 1$, the result match with the known results of general relativity that it is not possible to directly match the de-Sitter spacetime to the Schwarzschild exterior on the null hypersurface of the Schwarzschild horizon, and that one requires a layer of noninflationary layer at the interface [42,43,55]. This layer of non-inflationary layer of non-zero pressure but vanishing energy is not easy to motivate in the general relativistic framework. Quite surprisingly, the result is identical in the fractional modification too. On the horizon, the term proportional to (Δ/R_0) vanishes identically and hence again, one obtains a pressure term at the interface of the two spacetimes. Although the matching is not smooth since the horizon is now thick, the pressure term may easily be associated with this layer.

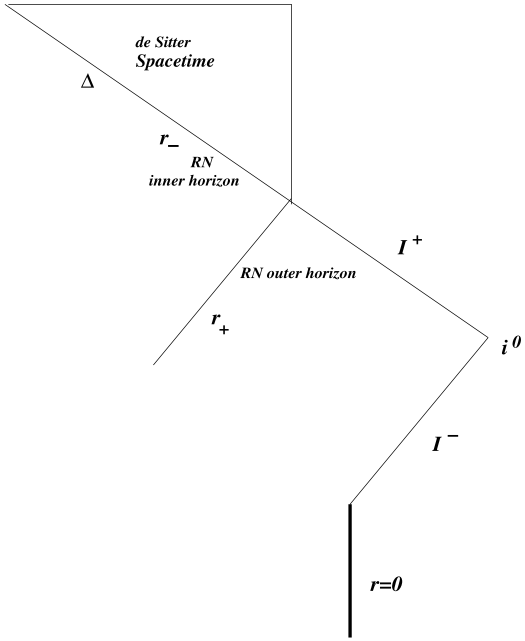

Fig. 2. The Barrabes–Israel model in which the inner horizon of the non-extremal Reissner–Nordstrom black hole is joined to a de Sitter world in the interior. The null surface Δ represents the matching hypersurface joining the two spacetimes. The I^+, I^- and i^0 represent the future null, past null and the spatial infinities.

4.3. Matching the Reissner–Nordstrom and the de-Sitter spacetimes

Let us determine the criteria for matching the de-Sitter spacetimes on the inner horizon of the non-extremal charged Reissner–Nordstrom spacetime black hole. Interestingly the matching is to be carried out on the inner horizon as was first proposed in [43]. The Penrose diagram is given in Fig. 2.

The external spacetime is the Reissner–Nordstrom spacetime (M^+) with the following metric:

$$ds_-^2 = -\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\Big)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{67}$$

The co-ordinates on null shell are (v, θ, ϕ) and $R_0 = m - \sqrt{m^2 - Q^2}$. The transverse extrinsic curvatures ${}^qC^+_{\theta\theta}$ and ${}^qC^+_{\phi\phi}$ may be written as

$${}^qC^+_{AB} = \frac{1}{R_0}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]\sigma_{AB}. \tag{68}$$

$${}^qC^+_{vv} = \left[\left(\frac{Q^2}{R_0^3} - \frac{m}{R_0^2}\right) - \frac{1-q}{2-q}\frac{\Delta}{R_0}\left(\frac{2m}{R_0^2} - \frac{3Q^2}{R^3}\right) + \cdots\right]. \tag{69}$$

The interior spacetime is the de-Sitter spacetime with matching at $R_0 = l$. The curvatures have already been found out in the previous subsection. The properties of the shell may be immediately obtained. The surface density $\mu = 0$ but the pressure is

$$p = \frac{-1}{8\pi}\left[\left(\frac{R_0}{l^2}+\frac{m}{R_0^2}-\frac{Q^2}{R_0^3}\right)-\left(\frac{R_0}{l^2}-\frac{2m}{R_0^2}+\frac{3Q^2}{R_0^2}\right)\frac{1-q}{2-q}\frac{\Delta}{R_0}+\cdots\right]. \tag{70}$$

So, again, in the standard case, when $(\Delta/R_0)=0$ or $q=1$, the spacetimes matching requires a shell which shall hold this pressure and hence the matching is not smooth. Incidentally, in [42, 43], the authors noted that (in general relativity), for special case of $3l^2 = Q^2$, there is a smooth matching of the two spacetimes. This matching is however a special case. On the other hand, the fractional generalisation, shows that it is possible to adjust the parameter q and (Δ/R_0) to get a vanishing pressure and hence, a smooth matching of the spacetimes on the hypersurface without any such requirement on the charge or mass.

5. Discussions

In this paper, we have developed the fractional generalisation of the Israel–Darmois–Lanczos junction conditions for spacelike/timelike as well as for null hypersurfaces. We have observed that there is a significant modification due to the fractional generalisation. First, due to the definition of the fractional differentiation through an integral, it automatically incorporates the non-local spacetime correlations into itself. As a manifestation of this, the thickness of the shell gets incorporated into to the values of the shell's properties like the energy and pressure. We have taken several examples and have demonstrated that by choosing this thickness parameter Δ/R_0 judiciously, it is possible to join many spacetimes smoothly across spacelike timelike or null hypersurfaces. It is also possible to interpret the factor (Δ/R_0), as a manifestation of quantum effects. Indeed, quantum fluctuations at the horizon (at R_0) would lead to a fuzziness Δ of the horizon. Naturally, $\Delta \simeq R_0$ would imply large quantum effects. At large scales, where Δ (the quantum fluctuations) is small and R_0 large, these corrections to the metric are negligible, of the order of Δ/R_0^3 (see Appendix C). However, on the horizon, where the quantum fluctuations and contributions from non-local effects are not quite negligible (Δ/R_0^3 is still small), there exists a fractional contribution to the stress-tensor on the horizon of the order of Δ/R_0. So, although there is no fundamental scale, implicitly, there is a scale, dictated by the fact whether Δ/R_0 is close to unity or small enough to be neglected. Note that very close to the singularity, the quantum fluctuations shall contribute overwhelmingly (since $\Delta/R_0 \simeq 1$) and alter the spacetime structure. As we have argued in this paper, it is possible to utilise the combination $(1-q)\Delta/R_0$ to obtain smooth matching of many spacetimes including the Schwarzschild and the de-Sitter on a spacelike hypersurface as well as the Reissner–Nordstrom and the de-Sitter spacetime on a null hypersurface. Both of these examples generate singularity free spacetimes.

Note that this approach is different from that adopted for the *multifractal spacetimes* [32–34], where one postulates that our universe is naturally multifractal with the spacetime geometry being fundamentally scale dependent. This formalism modifies the differential and integral structure by incorporating scale dependence directly into them. For example the measure $d^n x$ is replaced by $d^n x\, q(x)$, where $q(x)$ contain the information of multifractal geometry and may vary with each scale. Hence, this method may be more fundamental since the fractal effects arise directly from geometry and, would be immensely useful to construct the entire hierarchy of theories connecting the classical regime to the quantum one. On the other hand, in our case, the large scale spacetime is classical 4-dimensional and the fractional effects arise due to quantum fluctuations $(1-q)\Delta/R_0$ and, given a scale one needs to see if it is negligible or large.

A point of crucial importance is that the dimension of the spacetime has been taken to be integral. Fractal dimensions may also be possibly included. In fact, general relativity may also be suitably adapted for *fractal spacetimes*, which would also require revising our notions of coordinate transformations and covariance. However, we have not attempted this path, of altering the theory of general relativity to recast it for all spacetime dimensions, integral or non-integral. Instead, we have looked for alternate avenues by generalising the notion of Lie-derivative which is intrinsically attached to the differentiable structure of the spacetime. Unlike the usual Levi-Civita connection, there is no requirement of the metric and hence the Lie derivative is much primitive and turns out to be most useful. This generalisation has been used here to construct the extrinsic curvatures and hence the surface properties of the shell.

In the appendix, we have developed the reasons as to why we should expect that there should be some modification in the dynamics as well. We show that the Einstein equations modify significantly. The ramifications of the modified dynamics and further issues related to higher dimensional fractal structures shall be dealt with in future papers.

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Appendix A. Fractional derivative

The Riemann–Liouville definition of fractional calculus is usually given in the form of a integral transform of a specialised type, as given below [1–3]:

$$D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} f(y)\, dy, \tag{A.1}$$

where $\nu > 0$. This definition is the foundation of the theory of fractional differentiation (and integration), but breaks down at the integral points, $\nu = 0, -1, -2, \cdots$. At those points the integration may however be replaced by the ordinary integration formula.

The Caputo derivative is a modification of the Riemann–Liouville derivative where suitable modification have been applied so that it satisfies all the rules of a derivative. The Caputo derivative is defined as follows [1–3]:

$$D_x^q f(x) = \frac{1}{\Gamma(1-q)} \int_a^x (x-y)^{-q} \frac{\partial f(y)}{\partial y}\, dy, \tag{A.2}$$

where the superscript q denotes the fractional parameter, $0 < q \le 1$. To take into account of the tensor indices, a further modification is added in [54] as follows:

$$\mathcal{D}_{x,k}^q (x'^i) = \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int_x^{x'} \frac{\partial y^i}{\partial y^k} (x'-y)^{-q} dy. \tag{A.3}$$

For example, if the integration of the metric variable $g_{ij}(r)$ is to be carried out from one end of the shell (of thickness Δ) to the other, the above definition gives:

$$\mathcal{D}^{q}_{r-\Delta,r}(g_{ij}) = \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int\limits_{r-\Delta}^{r} \frac{\partial g_{ij}(w)}{\partial w}(r-w)^{-q}dw, \tag{A.4}$$

where the integration limits have been chosen appropriately. This definition has been utilised in this paper.

Appendix B. Incomplete Beta functions, Hypergeometric functions and fractional differentiation

In the paper, we have frequently made use of the Beta function, defined as:

$$B_x(a,b) = \int\limits_0^x t^{a-1}(1-t)^{b-1}dt, \tag{B.1}$$

where $a > 0, b > 0$. In general, we may also write it as

$$B_x(a,b) = x^a\left(\frac{1}{a} + \frac{1-b}{1+a}x + \cdots\right), \tag{B.2}$$

and hence the above equation implies naturally that:

$$B_{\frac{\Delta}{R_0}}(1-q,2) = \left(\frac{\Delta}{R_0}\right)^{1-q}\frac{1}{1-q}\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{B.3}$$

If, $(\Delta/R) \neq 1$, we may also use the relation between Beta function and Hypergeometric function:

$$B_x(a,b) = \left(x^a/a\right){}_2F_1(a,b,c;x) = \left(x^a/a\right){}_2F_1(a,b-1,a+1;x), \tag{B.4}$$

where

$${}_2F_1(a,b,c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{1} + \cdots. \tag{B.5}$$

This gives the following two useful forms:

$$B_{\frac{\Delta}{R_0}}(1-q,-1) = \frac{(\Delta/R_0)^{1-q}}{1-q}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + 6\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2 + \cdots\right] \tag{B.6}$$

$$B_{\frac{\Delta}{R_0}}(1-q,-2) = \frac{(\Delta/R_0)^{1-q}}{1-q}\left[1 + 3\frac{1-q}{2-q}\frac{\Delta}{R_0} + 12\frac{1-q}{3-q}\left(\frac{\Delta}{R_0}\right)^2 + \cdots\right]. \tag{B.7}$$

These equations have been used below to derive the results used in the main text. Let us first evaluate $\mathcal{D}^{q}_{r-\Delta,r}(r^2)$. Using eqn. (A.4), we get:

$$\mathcal{D}^{q}_{r-\Delta,r}(r^2) = \frac{2\,\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int\limits_{r-\Delta}^{r} w(r-w)^{-q}dw, \tag{B.8}$$

where the integration limit is chosen to take the thickness of the hypersurface into account. Using the change of variables, $r - w = t$, the limit also changes from $r - \Delta$ to Δ and r to 0. This gives us:

$$\mathcal{D}^q_{r-\Delta,r}(r^2) = \frac{2\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} \int_0^{\Delta} (r-t)t^{-q}dt. \tag{B.9}$$

Again, make a change of variables $t/r = y$ and also put $r = R_0$ as we match on the hypersurface placed at $r = R_0$.

$$\mathcal{D}^q_{r-\Delta,r}(r^2) = \frac{2\,\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{2-q} \int_0^{\frac{\Delta}{R_0}} (1-y)y^{-q}dy \tag{B.10}$$

$$= \frac{2\,\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{2-q} B_{\frac{\Delta}{R_0}}(1-q,2). \tag{B.11}$$

Using the form of eqn. (B.3) and property of Gamma function $(1-q)\Gamma(1-q) = \Gamma(2-q)$, we get,

$$\mathcal{D}^q_{r-\Delta,r}(r^2) = 2R_0\left[1 - \frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{B.12}$$

For $\mathcal{D}^q_{r-\Delta,r}(r^{-1})$, a similar calculation yields the following result:

$$\mathcal{D}^q_{r-\Delta,r}(r^{-1}) = \frac{\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-1-q} \int_0^{\frac{\Delta}{R_0}} (1-y)^{-2}y^{-q}dy, \tag{B.13}$$

$$= -\frac{\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-1-q} B_{\frac{\Delta}{R_0}}(1-q,-1). \tag{B.14}$$

Using eqn. (B.6) and property of Gamma function *i.e.* $(1-q)\Gamma(1-q) = \Gamma(2-q)$ we get:

$$\mathcal{D}^q_{r-\Delta,r}(r^{-1}) = -\frac{1}{R_0^2}\left[1 + 2\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{B.15}$$

The computation for $\mathcal{D}^q_{r-\Delta,r}(r^{-2})$ proceeds along similar lines and gives:

$$\mathcal{D}^q_{r-\Delta,r}(r^{-2}) = \frac{-2\,\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-2-q} \int_0^{\frac{\Delta}{R_0}} (1-y)^{-3}y^{-q}dy \tag{B.16}$$

$$= \frac{-2\,\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-2-q} B_{\frac{\Delta}{R_0}}(1-q,-2), \tag{B.17}$$

which using the equation (B.7) and $(1-q)\Gamma(1-q) = \Gamma(2-q)$ gives us:

$$\mathcal{D}^q_{r-\Delta,r}(r^{-2}) = -\frac{2}{R_0^3}\left[1 + 3\frac{1-q}{2-q}\frac{\Delta}{R_0} + \cdots\right]. \tag{B.18}$$

Appendix C. Fractional modification of the Einstein equations and the spacetime metric

Throughout the text, we have assumed that fractional derivatives lead to modification of the connection. Naturally, this would lead to corrections of the Einstein equations itself. Furthermore, the black hole uniqueness theorems also may not hold for this corrected theory of gravity.

Consequently, one also expects the unique solutions of the Einstein theory to receive corrections. In the following, we shall however argue that these corrections are small on the horizon although, they would contribute substantially near the spacetime singularity.

Note that the fractional derivative leads to a modification of the partial derivative. From the previous sections, we note that the Caputo derivative modifies the derivative through a factor $(1-q)\Delta/R_0$. Let us use this form to write for any function g, a modification of the derivative operator as:

$$\mathcal{D}g = \partial \mathbf{g}\left[1 \pm \beta(1-q)\frac{\Delta}{R_0} \pm \cdots\right] \tag{C.1}$$

here β is some constant, and q denotes the fractional parameter. Using this definition of the derivative, the relation between new Christoffel symbol (for non-Levi-Civita connection) and old Christoffel symbol (for Levi-Civita connection) becomes:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} \pm \tilde{\alpha}(1-q)\frac{\Delta}{R_0} + \cdots \tag{C.2}$$

where $\tilde{\alpha}$ is some constant. This gives a relation between old Riemann tensor and new Riemann tensor. The usual definition

$$R^{\alpha}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\nu}_{\beta\delta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma}\Gamma^{\alpha}_{\nu\delta} \tag{C.3}$$

is modified to a new definition

$$\tilde{R}^{\alpha}_{\beta\gamma\delta} = \mathcal{D}_{\gamma}\tilde{\Gamma}^{\alpha}_{\beta\delta} - \mathcal{D}_{\delta}\tilde{\Gamma}^{\alpha}_{\beta\gamma} + \tilde{\Gamma}^{\nu}_{\beta\delta}\tilde{\Gamma}^{\alpha}_{\nu\gamma} - \tilde{\Gamma}^{\nu}_{\beta\gamma}\tilde{\Gamma}^{\alpha}_{\nu\delta}. \tag{C.4}$$

The relation between the two Riemann tensors is given by:

$$\tilde{R}^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} \pm (1-q)\frac{\Delta}{R_0}[\mu] \pm \cdots \tag{C.5}$$

where $\mu = \left[\beta\partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} + \partial_{\gamma}\tilde{\alpha} - \beta\partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\delta}\tilde{\alpha} \pm \tilde{\alpha}(\Gamma^{\nu}_{\beta\delta} \pm \Gamma^{\alpha}_{\nu\gamma} \mp \Gamma^{\nu}_{\beta\gamma} \mp \Gamma^{\alpha}_{\nu\delta})\right]$. The Ricci tensor is

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} \pm (1-q)\frac{\Delta}{R_0}[\eta] \pm \cdots \tag{C.6}$$

Similarly, the Ricci scalar is given by

$$\tilde{R} = R \pm (1-q)\frac{\Delta}{R_0}[\tau] \pm \cdots \tag{C.7}$$

where η and τ are some functions of spacetime obtained from complicated combination of the Christoffel symbols. The Einstein field equations get modified as well:

$$\tilde{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{R} = 8\pi G T_{\alpha\beta} \pm (1-q)\frac{\Delta}{2R_0}[\eta-\tau]\,g_{\alpha\beta} \pm \cdots. \tag{C.8}$$

So, the dynamics of the gravitational fields get modified for the fractional generalisation. The question which now arises is this: how much correction would the metric receive? Note that at large distance from the system, the Einstein theory continues to hold and the asymptotic observer would continue to find the spherically symmetric spacetime to be the unique Schwarzschild geometry. The modifications would only become important near the horizon where the fractional corrections arise. In view of the eqn. (C.8), modification of the Einstein theory may alternatively be viewed as adding a term $(1-q)(\Delta/2R_0)\,(\eta-\tau)\,g_{\alpha\beta}$ to the energy momentum tensor. Such terms are perturbations to the original Schwarzschild metric since the expression is of the order of Δ/R_0^3. Additionally, since Δ/R_0 is small, and R_0 is large (being the horizon radius), these terms have negligible contribution to the metric near the horizon. The crucial point is that the contribution to the metric correction may be small (of the order of Δ/R_0^3), but the fractional correction to the Israel–Darmois–Lanczos junction condition is non-trivial of the order of Δ/R_0). This is because one further correction arises due to the fractional modification of the connection itself which is of the order of Δ/R_0. Note that, near the black hole singularity, R_0 is very small and hence, Δ/R_0^3 will lead to a substantial variation of the spacetime.

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