Nongeometric heterotic strings and dual F-theory with enhanced gauge groups

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ABSTRACT: Eight-dimensional nongeometric heterotic strings were constructed as duals of F-theory on $\Lambda^{1,1} \oplus E_8 \oplus E_7$ lattice polarized K3 surfaces by Malmendier and Morrison. We study the structure of the moduli space of this construction. There are special points in this space at which the ranks of the non-Abelian gauge groups on the 7-branes in F-theory are enhanced to 18. We demonstrate that the enhanced rank-18 non-Abelian gauge groups arise as a consequence of the coincident 7-branes, which deform stable degenerations on the F-theory side. This observation suggests that the non-geometric heterotic strings include nonperturbative effects of the coincident 7-branes on the F-theory side. The gauge groups that arise at these special points in the moduli space do not allow for perturbative descriptions on the heterotic side.

We also construct a family of elliptically fibered Calabi-Yau 3-folds by fibering K3 surfaces with enhanced singularities over $\mathbb{P}^1$. Highly enhanced gauge groups arise in F-theory compactifications on the resulting Calabi-Yau 3-folds.

Contents

1 Introduction

F-theory [1–3] is a theoretical framework that extends type IIB superstring theory to a non-perturbative regime. The F-theory/heterotic duality [1–5] enables us to study non-perturbative aspects of heterotic strings. F-theory/heterotic duality states that heterotic string compactification[1] on an elliptically fibered Calabi-Yau n-fold and F-theory compactification on a K3 fibered Calabi-Yau $n + 1$-fold describe equivalent physical theories. This

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[1]See, e.g., [6–15] for recent studies of heterotic strings.

duality is strictly formulated when the stable degeneration [5, 16][2] where K3 fibers split into pairs of 1/2 K3s is considered on the F-theory side.

Recently, dual models of heterotic strings and F-theory, in which heterotic compactification does not allow for a geometric interpretation, have been discussed in [22]. The Narain space [23]

$$D_{2,18}/O(\Lambda^{2,18}) \tag{1.1}$$

is the moduli space of eight-dimensional heterotic strings. $D_{2,18}$ denotes the symmetric space of $O(2,18)$, i.e., $D_{2,18} := O(2) \times O(18) \backslash O(2,18)$. The double cover of this moduli space,

$$D_{2,18}/O^+(\Lambda^{2,18}), \tag{1.2}$$

coincides with the F-theory moduli on elliptically fibered K3 surfaces with a global section. ($O^+(\Lambda^{2,18})$ denotes an index two subgroup of $O(\Lambda^{2,18})$, defined as $O^+(\Lambda^{2,18}) := O(\Lambda^{2,18}) \cap O^+(2,18)$.) This defines the F-theory/heterotic duality. In [22], using the F-theory/heterotic duality Malmendier and Morrison constructed "non-geometric" eight-dimensional heterotic string compactifications on the 2-torus that possesses (a subgroup of) $O^+(\Lambda^{2,18})$ symmetry. The action of $O^+(\Lambda^{2,18})$ mixes the complex structure moduli, Kähler moduli, and the moduli of expectation values of the Wilson line. Consequently, heterotic strings that possesses $O^+(\Lambda^{2,18})$ symmetry do not yield a geometric interpretation.[3]

In [22], the moduli of F-theory on elliptically fibered K3 surfaces with a section with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra and the heterotic duals were considered for constructing non-geometric heterotic strings. Because the $\mathfrak{e}_8\mathfrak{e}_7$ algebra is unbroken for this specific case, we can take the orthogonal complement of the sum of $E_8 \oplus E_7$ and H (H is the lattice generated by the fiber and the zero-section) inside the K3 lattice to obtain the moduli space of eight-dimensional F-theory on an elliptically fibered K3 surface with an E_8E_7 singularity as

$$D_{2,3}/O^+(L^{2,3}). \tag{1.3}$$

$L^{2,3}$ is the orthogonal complement of the lattice $H \oplus E_8 \oplus E_7$ in the K3 lattice Λ_{K3}. The mathematical results of Kumar [25] and Clingher and Doran [26, 27] gave the Weierstrass equations of elliptically fibered K3 surfaces with a section with an E_8E_7 singularity, the coefficients of which are Siegel modular forms[4] of even weight. The periods $\underline{\tau}$ of this K3 surface give a point in the Siegel upper half-space of genus two, $\mathbb{H}_2$. Using this result, Malmendier and Morrison constructed heterotic strings with $O^+(L^{2,3})$ symmetry with the variable $\underline{\tau}$. The action of $O^+(L^{2,3})$ mixes complex structure moduli, Kähler moduli, and the moduli of Wilson line values. Therefore, the resulting heterotic strings are non-geometric. The Weierstrass equation of an elliptically fibered K3 surface with an E_8E_7 singularity determines a point in $D_{2,3}$, and the periods of this K3 surface determine the point $\underline{\tau}$ in $\mathbb{H}_2$. Because $D_{2,3}$ and the Siegel upper half-space of genus two, $\mathbb{H}_2$, are isomorphic, this construction defines a correspondence between F-theory on elliptic K3 surfaces with

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[2]See, for example, [17–21] for recent studies on the stable degenerations in F-theory/heterotic duality.

[3]Non-geometric type II theories were discussed in [24].

[4]The Siegel modular forms of genus two were discussed in the context of string compactifications in [28–30].

a section with an E_8E_7 singularity and non-geometric heterotic strings with $O^+(L^{2,3})$ symmetry.[5] We also refer the reader to [33–39] for recent studies on non-geometric strings.

In this study, we investigate the structure of the moduli space parameterizing F-theory on elliptic K3 surfaces with E_8E_7 singularity with a section

$$D_{2,3}/O^+(L^{2,3}), \qquad (1.4)$$

to analyze the special points of the moduli space at which the ranks of the non-Abelian gauge groups on the F-theory side are enhanced to 18. Because rank-18 non-Abelian gauge groups do not embed into $E_8 \times E_8 \times \mathrm{U}(1)^4$ or $\mathrm{SO}(32) \times \mathrm{U}(1)^4$, the non-geometric heterotic duals do not yield a perturbative interpretation of these gauge groups.

The divisors in a K3 surface form a lattice known as the Néron-Severi lattice. The rank of this lattice is referred to as the Picard number ρ, and for an elliptically fibered complex K3 surface with a section, the Picard number ρ takes values from two to 20. The elliptic K3 surfaces with Picard number 20 are called attractive K3 surfaces. The K3 surfaces with Picard number ρ in the complex structure moduli of K3 surfaces form a $(20 - \rho)$-dimensional subset. In particular, this means that the moduli of attractive K3 surfaces are discrete and zero-dimensional. Attractive K3 surfaces are known to be labelled by triplets of integers [40, 41]. An increase in the Picard number ρ implies that the rank of the gauge group increases. This equivalently means that the gauge group that arises in the F-theory compactification is enhanced. Of the Picard number $\rho = 20$ for an attractive K3 surface with a section, two of these degrees of freedom are used for the zero-section and the fiber. The non-Abelian gauge group generates a lattice, and the rank of this lattice comprise at most the remaining 18 degrees of freedom. An elliptic K3 surface with a section is referred to as the *extremal* K3 surface when the rank of the non-Abelian gauge group is the maximum 18. An extremal K3 has Picard number 20, which is the sum of the rank 18 of the non-Abelian gauge group and the rank two of the lattice generated by the zero-section and the fiber. Therefore, such a surface is always attractive.

The points in the complex structure moduli of K3 surfaces that correspond to the extremal K3s form a subset of the attractive K3 surfaces, and thus they are zero-dimensional and discrete. The heterotic strings dual to F-theory on the extremal K3 surfaces do not allow for a perturbative interpretation of the arising gauge groups. In this sense, the points that correspond to the extremal K3 surfaces in the F-theory moduli of elliptic K3 surfaces conspicuously demonstrate the non-geometric nature of dual heterotic strings.

In this work, we analyze the physics both on the F-theory and heterotic sides of compactifications at special points in the moduli that correspond to F-theory on extremal K3s.

In particular, we study the points in the moduli with an unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra at which elliptically fibered K3 surfaces become extremal. In addition, we study an extremal K3 surface obtained as the Jacobian fibration[6] of a genus-one fibered K3 surface that lacks

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[5]Connections of lattice polarized K3 surfaces, $O^+(\Lambda^{2,2})$-modular forms, and non-geometric heterotic strings were discussed in [31]. K3 surfaces with $\Lambda^{1,1} \oplus E_7 \oplus E_7$ lattice polarization and the construction of non-geometric heterotic strings were studied in [32].

[6]See [42] for the construction of the Jacobians of elliptic curves.

a global section[7] constructed as a (3,2) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$, as studied in [53]. Taking the Jacobian fibration of a genus-one fibered K3 surface yields an elliptic K3 surface with a global section. We analyze the physics of F-theory compactification on this extremal K3 and the heterotic dual.

It was found in [66] that some extremal K3 surfaces can be viewed as the deformations of K3 surfaces obtained as the reverse of the stable degeneration, in which a K3 surface splits into two rational elliptic surfaces. This can be understood as an effect of coincident 7-branes from the F-theory viewpoint. Under the stable degeneration limit, the dual of the F-theory model yields a geometric heterotic string. Thus, we deduce that non-geometric heterotic strings as duals of F-theory on extremal K3s are a consequence of the transitions from geometric heterotic strings when 7-branes are coincident. This suggests that non-geometric heterotic strings include the non-perturbative effects of coincident 7-branes on the F-theory side. In the heterotic language, these deformations from the stable degeneration are owing to the presence of 5-branes.

Furthermore, we construct elliptically fibered Calabi-Yau 3-folds with a section by fibering K3 Jacobian fibrations, as discussed in section 3.2.2 over $\mathbb{P}^1$. We analyze F-theory compactifications on these elliptically fibered Calabi-Yau 3-folds.[8] Recent studies have mainly considered local models in F-theory model building [84–87]. However, global F-theory models must be considered to discuss gravity and issues of the early universe and inflation. In this work, we study F-theory compactifications on elliptic Calabi-Yau 3-folds from the global perspective. Highly enhanced non-Abelian gauge groups arise in the compactifications on our constructions of the elliptic Calabi-Yau 3-folds. These Calabi-Yau 3-folds are constructed as the Jacobians of genus-one fibered Calabi-Yau 3-folds without a section, as hypersurfaces in the products of projective spaces. Similar constructions of genus-one fibered Calabi-Yau 4-folds can be found in [56].

In section 2, we briefly review the construction of non-geometric heterotic strings considered in [22]. We also review F-theory and extremal K3 surfaces. In section 3, we study special points of the 8-dimensional non-geometric heterotic string moduli at which the ranks of the non-Abelian gauge groups of F-theory duals reach 18. We find that eight-dimensional non-geometric heterotic strings at these points in the moduli are obtained as the transitions from geometric heterotic strings when 7-branes are coincident on the F-theory side. We also confirm that these points satisfy 5-brane solutions. Elliptic K3 surfaces generally admit distinct elliptic fibrations. Therefore, even when the complex structure of the compactification space is fixed, different gauge symmetries arise in F-theory compactification for distinct elliptic fibrations. An elliptically fibered K3 surface with a type II^* fiber and a type III^* fiber always admits another elliptic fibration with a type I_{10}^* fiber [27, 31]. Birational maps from an elliptic fibration with a type II^* fiber and a type III^* fiber to an elliptic fibration with a type I_{10}^* fiber were considered in [22]. These birational maps relate $E_8 \times E_8$ heterotic string to SO(32) heterotic string. Using these birational maps, we also study some applications to SO(32) heterotic strings in section 3.3.

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[7]F-theory compactifications on genus-one fibrations without a section have recently been studied, e.g., in [43–65].

[8]See, for example, [67–81] for recent studies on F-theory compactifications on elliptic Calabi-Yau 3-folds. F-theory compactifications on Calabi-Yau 3-folds with a terminal singularity are discussed in [43, 82, 83].

In section 4, we discuss six-dimensional F-theory compactifications on elliptic Calabi-Yau 3-folds with a section built by fibering the Jacobian K3 surface discussed in section 3.2.2 over $\mathbb{P}^1$. We determine the Mordell-Weil groups for specific elliptic Calabi-Yau 3-folds. As a result, we determine the global structure of the gauge groups. F-theory on these specific elliptic Calabi-Yau 3-folds do not have a $U(1)$ gauge field. We also determine candidate matter spectra for some models. We deduce candidate matter spectra directly from the global defining equations of the Calabi-Yau 3-folds. We confirm that the anomaly cancels for these models. We state our concluding remarks in section 5.

2 Non-geometric heterotic strings, F-theory, and extremal K3

2.1 Review of F-theory

In this section, we briefly review F-theory compactification and the non-Abelian gauge groups that arise on 7-banes.

F-theory is compactified on Calabi-Yau spaces that admit genus-one fibrations. The modular parameter of an elliptic fiber is identified with the axio-dilaton $C_0 + ie^{-\phi}$. This F-theory formulation enables the axio-dilaton to have the monodromy property. A genus-one fibration does not necessarily have a global section: there are situations in which it admits a global section and those in which it does not. F-theory models discussed in this note admit a section,[9] namely we consider F-theory models on elliptically fibered Calabi-Yau manifolds that admit a global section. (However, we consider genus-one fibered Calabi-Yau spaces lacking a global section in sections 3.2.2 and 4 to construct elliptically fibered Calabi-Yau spaces with a global section. We take the Jacobian fibrations of Calabi-Yau genus-one fibrations to yield Calabi-Yau elliptic fibrations with a section.) F-theory compactified on a complex $(n+1)$-dimensional Calabi-Yau manifold yields a $(10-2n)$-dimensional theory. Fibers degenerate along the codimension one locus in the base space. This locus is referred to as the discriminant locus, and 7-branes are wrapped on the components of this locus. The fiber type over the component on which 7-branes are wrapped determines the non-Abelian gauge group that arises on the 7-branes [3, 112].

When an elliptic fibration admits a global section, it also admits a transformation to the Weierstrass form. In this case, the type of a singular fiber can be determined from the vanishing orders of the Weierstrass coefficients. The types of the singular fibers of an elliptic surface were classified, and their monodromies and j-invariants were computed, by Kodaira in [113, 114]. Techniques to determine the types of the singular fibers of an elliptic fibration are described in [115, 116]. The types of singular fibers of an elliptic surface, their monodromies and j-invariants, the vanishing orders of the Weierstrass coefficients, and the corresponding singularity types are listed in tables 1 and 2. The set of sections of an elliptic fibration forms a group, which is called the Mordell-Weil group. The rank of the Mordell-Weil group is equal to the number of $U(1)$ gauge fields that arise in F-theory compactification.

We also briefly review the geometries of elliptic K3 surfaces, because this is relevant to contents of section 3. The second integral cohomology group $H^2(S, \mathbb{Z})$ of a K3 surface S is

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[9]F-theory models with a global section have been intensively studied in the literature. See, for example, [20, 88–111].

fiber type	j-invariant	monodromy	order of monodromy	singularity type
I_0^*	regular	$-\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	2	D_4
I_b	∞	$\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$	infinite	A_{b-1}
I_b^*	∞	$-\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$	infinite	D_{b+4}
II	0	$\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	6	none
II^*	0	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$	6	E_8
III	1728	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	4	A_1
III^*	1728	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	4	E_7
IV	0	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$	3	A_2
IV^*	0	$\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	3	E_6

Table 1. J-invariants, monodromies, and the corresponding singularity types of fiber types. "Regular" for type I_0^* means that the j-invariant of this fiber type can take any finite value in $\mathbb{C}$.

fiber type	ord(f)	ord(g)	ord(Δ)
I_0	≥ 0	≥ 0	0
I_n $(n \geq 1)$	0	0	n
II	≥ 1	1	2
III	1	≥ 2	3
IV	≥ 2	2	4
I_0^*	≥ 2	3	6
	2	≥ 3	
I_m^* $(m \geq 1)$	2	3	$6+m$
IV^*	≥ 3	4	8
III^*	3	≥ 5	9
II^*	≥ 4	5	10

Table 2. The vanishing orders of the Weierstrass coefficients f, g of $y^2 = x^3 + f\,x + g$, and the vanishing order of the discriminant Δ, for the various fiber types.

called the K3 lattice, Λ_{K3}, and this lattice contains information on the geometry of a K3 surface, such as its divisors and their intersections. A K3 lattice is a rank-22 unimodular integral lattice with signature (3,19). From the structure theorem for an indefinite integral unimodular even lattice [117], the following isomorphism holds:

$$\Lambda_{K3} \cong H^3 \oplus E_8^2, \tag{2.1}$$

where H in this isomorphism denotes the "hyperbolic plane," which is a rank-two integral

lattice with the following intersection form:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

A sublattice of the K3 lattice Λ_{K3}, called the Néron-Severi lattice $NS(S)$, determines the information on the divisors of a K3 surface S. The intersection form of the Néron-Severi lattice $NS(S)$ corresponds to the intersections of the divisors. The rank of the Néron-Severi lattice $NS(S)$ is called the Picard number $\rho(S)$ of the K3 surface. This number varies depending on the complex structure of the K3 surface S. When a K3 surface is elliptically fibered and admits a global section, the fiber F and the zero-section generate the hyperbolic plane H. Thus, the property $H \subset NS(S)$ that the Néron-Severi lattice $NS(S)$ of a K3 surface S contains the hyperbolic plane H is equivalent [118] to the statement that the K3 surface S admits an elliptic fibration with a section.

When a genus-one fibered K3 surface has a section, inside the Néron-Severi lattice $NS(S)$, the orthogonal complement of the hyperbolic plane generated by the fiber and a section contains the information on the types of the singular fibers. A singular fiber is either $\mathbb{P}^1$ with a single singularity, or a sum of several smooth $\mathbb{P}^1$'s intersecting in a specific manner. The manner in which the $\mathbb{P}^1$'s intersect determines the type of a singular fiber [113, 114]. Each $\mathbb{P}^1$ component in a singular fiber is a divisor of the K3 surface S. Therefore, (except for $\mathbb{P}^1$'s that intersect with the zero-section) $\mathbb{P}^1$ components of a singular fiber belong to the orthogonal complement of the hyperbolic plane inside the K3 lattice Λ_{K3}. For this reason, the orthogonal complement captures the intersections of $\mathbb{P}^1$ components of singular fibers.

$D_{2,18}$ modded out by (a subgroup of) the automorphism group of the orthogonal complement of the hyperbolic plane H[10] inside the K3 lattice Λ_{K3},

$$D_{2,18}/O^+(\Lambda^{2,18}) \tag{2.3}$$

yields the parameter space for the elliptic K3 surfaces with a global section. Here, $\Lambda_{2,18}$ denotes the orthogonal complement of the hyperbolic plane H inside Λ_{K3}:

$$\Lambda_{2,18} := H^{\perp} \subset \Lambda_{K3}. \tag{2.4}$$

$\Lambda_{2,18}$ is an even integral unimodular lattice of signature (2,18), and this is unique up to isometry.

2.2 Review of non-geometric heterotic strings in Malmendier and Morrison

The moduli space of eight-dimensional heterotic strings compactified on the 2-torus T^2 is given by the Narain space

$$D_{2,18}/O(\Lambda^{2,18}). \tag{2.5}$$

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[10]The assumption that an elliptic K3 surface admits a global section cannot be dropped here. When a genus-one fibered K3 surface only has a multisection, the number of intersections between the multisection and the fiber is greater than one. Therefore, the multisection and the fiber do not generate the hyperbolic plane H.

In a suitable limit, this moduli space decomposes into the product of the complex structure moduli, Kähler moduli, and the moduli of the Wilson line expectation values [119]. This decomposition is preserved under the action of a parabolic subgroup Γ, but the action of $O(\Lambda^{2,18})$ does not preserve this decomposition. The action of the index-two subgroup $O^+(\Lambda^{2,18})$ still mixes the complex structure moduli, Kähler moduli, and the moduli of the Wilson line expectation values. Because of this, heterotic strings with $O^+(\Lambda^{2,18})$-symmetry do not yield a geometric interpretation. In this sense, they are non-geometric. This is the construction of the eight-dimensional non-geometric heterotic strings with the $O^+(\Lambda^{2,18})$-symmetry in [22].

The double cover (2.3) of the Narain space gives the moduli space of F-theory on elliptic K3 surfaces with a section. The moduli space that parameterizes F-theory on elliptic K3 surfaces with the $E_8 E_7$ singularity is

$$D_{2,3}/O^+(L^{2,3}), \tag{2.6}$$

where $L^{2,3}$ is the orthogonal complement of the lattice $H \oplus E_8 \oplus E_7$ in the K3 lattice Λ_{K3} (or equivalently, the orthogonal complement of the lattice $E_8 \oplus E_7$ in $\Lambda^{2,18}$).

An explicit construction of non-geometric heterotic strings with $O^+(L^{2,3})$-symmetry was discussed in [22]. Utilizing the F-theory/heterotic duality, the authors constructed such non-geometric heterotic strings with an unbroken $\mathfrak{e}_8 \mathfrak{e}_7$ gauge algebra as duals of F-theory on elliptic K3 surfaces with the $E_8 E_7$ singularity. Their construction proceeds as follows. There is a homomorphism from the Siegel modular group $Sp_4(\mathbb{R})$ to $O^+(2,3)$, and this induces an isomorphism of the Siegel upper half-space of genus two, $\mathbb{H}_2$, and $D_{2,3}$:

$$\mathbb{H}_2 \cong D_{2,3}. \tag{2.7}$$

Under this isomorphism, the ring of $O^+(L^{2,3})$-modular forms corresponds to the Siegel modular forms with genus two of even weight [120]. The Siegel modular forms with genus two of even weight form a polynomial ring generated by four generators, ψ_4, ψ_6, χ_{10}, and χ_{12}, of degrees 4, 6, 10, and 12, respectively [121].

The correspondence between the periods of elliptic K3 surfaces with a section with singularity type $E_8 E_7$ and the points in the Siegel upper half-space of genus two $\mathbb{H}_2$ was obtained by Kumar [25] and Clingher and Doran [26, 27]. The Weierstrass coefficients of such elliptic K3 surfaces were given in terms of Siegel modular forms with genus two of even weight in their work.

The Weierstrass equation of an elliptic K3 surface with a section and including the singularity type $E_8 E_7$ is given by

$$y^2 = x^3 + (a\,t^4 + c\,t^3)x + t^7 + b\,t^6 + d\,t^5. \tag{2.8}$$

Then (up to some scale factors), the coefficients a, b, c, d are given in terms of Igusa's generators as [25–27]:

$$a = -\frac{1}{48}\,\psi_4(\underline{\tau}), \quad b = -\frac{1}{864}\,\psi_6(\underline{\tau}), \quad c = -4\,\chi_{10}(\underline{\tau}), \quad d = \chi_{12}(\underline{\tau}). \tag{2.9}$$

The elliptic K3 surface determines a point in $D_{2,3}$, and this also determines a point in $\mathbb{H}_2$ under the isomorphism (2.7), which is denoted by $\underline{\tau}$.

Now, consider a manifold M with a line bundle $\mathcal{L}$, and choose sections a, b, c, d of $\mathcal{L}^{\otimes 4}$, $\mathcal{L}^{\otimes 6}$, $\mathcal{L}^{\otimes 10}$, and $\mathcal{L}^{\otimes 12}$, respectively. When the sections a, b, c, d are identified as (2.9), the compactification on M (which is the 2-torus T^2 when an eight-dimensional heterotic string is considered) yields a heterotic string with $O^+(L^{2,3})$-symmetry. This represents the construction of non-geometric strings discussed in [22].

In [22], the authors determined the locations of the 5-branes in the moduli of eight-dimensional non-geometric heterotic strings by computing the locus in the dual F-theory moduli on elliptic K3 surfaces with a section with the singularity type $E_8 E_7$, at which the singularity type is enhanced. The 5-brane solution is obtained from the conditions that either $c = 0$ (the type III^* fiber at $t = 0$ is enhanced to a type II^* fiber when this is satisfied) or the quartic polynomial in the discriminant Δ of the Weierstrass equation (2.8) degenerates, where the discriminant is

$$\Delta = t^9 \, s^9 \left(4s^2 \left(at + cs \right)^3 + 27t \left(t^2 + bts + ds^2 \right)^2 \right). \tag{2.10}$$

The resulting 5-brane solution is [22]

$$c = 0 \qquad \text{or} \qquad q = 0, \tag{2.11}$$

where q is a polynomial in the Weierstrass coefficients a, b, c, d, defined as [22]

$$\begin{aligned} q := {} & 11664 \, d^5 - 5832 \, b^2 d^4 + 864 \, a^3 d^4 + 16 \, a^6 d^3 \\ & + 216 \, a^3 b^2 d^3 + 16200 \, ac^2 d^3 - 2592 \, a^2 bcd^3 + 729 \, b^4 d^3 + 888 \, a^4 c^2 d^2 \\ & - 5670 \, ab^2 c^2 d^2 - 13500 \, bc^3 d^2 + 216 \, a^4 b^2 c^2 d + 16 \, a^7 c^2 d - 3420 \, a^3 bc^3 d \\ & + 4125 \, a^2 c^4 d + 729 \, ab^4 c^2 d + 6075 \, b^3 c^3 d - 16 \, a^6 bc^3 + 16 \, a^5 c^4 \\ & - 216 \, a^3 b^3 c^3 + 2700 \, a^2 b^2 c^4 - 5625 \, abc^5 - 729 \, b^5 c^3 + 3125 \, c^6. \end{aligned} \tag{2.12}$$

In section 3.2, we study points in the moduli of elliptic K3 surfaces with a section with the singularity type $E_8 E_7$ at which the singularity type is enhanced to rank 18. We confirm that either of the above conditions (2.11) are satisfied for non-geometric heterotic duals at such points.

2.3 Extremal K3 surfaces and enhanced gauge groups on the F-theory side

The Picard number ρ of a complex elliptic K3 surface with a section takes a value between two and 20, and when the K3 surface takes the maximum value 20 it is called an attractive K3 surface.[11] The types of the singular fibers of an elliptic K3 surface correspond to the singularity type of the surface. Because the $\mathbb{P}^1$ components of the singular fibers embed into the Néron-Severi lattice NS, the higher the rank of the singularity type of a K3 surface, the larger the Picard number ρ. The Néron-Severi lattice NS has rank 20 for an attractive K3 surface, with two of these degrees of freedom used for the fiber and the

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[11]We follow the convention for the term utilized in [122], referring to a K3 surface with the Picard number 20 as an attractive K3 surface.

zero-section. The $\mathbb{P}^1$ components of the singular fibers not intersecting the zero-section are orthogonal to the fiber and the zero-section. Therefore, the rank of the singularity type that corresponds to the diagram of the intersections of these $\mathbb{P}^1$ components is at most 18. When the singularity type of an attractive K3 surface attains the bound of 18, it is called an *extremal* K3 surface.[12] Physically, this means that a non-Abelian gauge group in an eight-dimensional F-theory compactification on an elliptic K3 surface that forms on the 7-branes has rank at most 18, and the rank is 18 when the K3 surface is extremal.

In section 3, we analyze points in the moduli of K3 surfaces (with some unbroken gauge algebras) at which K3 surfaces become extremal. As will be discussed in section 3.2, this enhancement of the gauge group can be seen as a consequence of coincident 7-branes.

It is known that the complex structure of an attractive K3 surface S is determined by its "transcendental lattice" $T(S)$, which is defined as the orthogonal complement of the Néron-Severi lattice $NS(S)$ inside the K3 lattice Λ_{K3} [40, 41]. For an attractive K3 surface, the transcendental lattice is a rank-two positive-definite integral even lattice. Under the action of $GL_2(\mathbb{Z})$, the intersection form of the transcendental lattice of an attractive K3 surface is as follows:

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \tag{2.13}$$

where $a, b, c \in \mathbb{Z}$ are such that

$$a \geq c \geq b \geq 0. \tag{2.14}$$

The relation between attractive K3 surfaces and the transcendental lattices is reviewed in the context of F-theory compactification in [53]. Based on this property of attractive K3 surfaces, when the intersection form of an attractive K3 surface is given by (2.13), we denote this attractive K3 surface by $S_{[2a\ b\ 2c]}$ in this paper.

All the types of the elliptic fibrations of an attractive K3 surface with the smallest discriminant three, $S_{[2\ 1\ 2]}$, and an attractive K3 with discriminant four, $S_{[2\ 0\ 2]}$, are determined [123].[13] Some types of these elliptic fibrations belong to extremal K3 surfaces in the moduli of elliptic K3 surfaces with a section, as discussed in section 3.2. Because some elliptic fibrations of these attractive K3 surfaces are relevant to results in this study, we summarize the elliptic fibrations of $S_{[2\ 1\ 2]}$ and $S_{[2\ 0\ 2]}$ in appendix A.

3   Special points in eight-dimensional non-geometric heterotic moduli and F-theory duals

3.1   Summary of discussion

Under the stable degeneration limit, an elliptic K3 surface splits into a pair of 1/2 K3s, i.e., a pair of rational elliptic surfaces. F-theory/heterotic duality is precisely formulated when

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[12]This condition is equivalent to an attractive K3 surface with a section having the Mordell-Weil rank zero.

[13]While in general a genus-one fibered K3 surface admits both elliptic fibrations with a section and genus-one fibrations without a section, these attractive K3 surfaces only have elliptic fibrations with a global section, as proved in [124]. F-theory compactifications on the attractive K3 surfaces $S_{[2\ 1\ 2]}$ and $S_{[2\ 0\ 2]}$ were discussed in relation to the appearances of U(1) gauge fields in [106].

this limit is taken. The Picard number of a rational elliptic surface is always 10, and the rank of the singularity type ranges from zero to eight. A rational elliptic surface with the singularity type of rank eight is called an extremal rational elliptic surface. Some extremal rational elliptic surfaces appear in section 3.2. The fiber types of the extremal rational elliptic surfaces were classified in [125]. We list the types of singular fibers for the extremal rational elliptic surfaces in table 6 in appendix B. The complex structure of an extremal rational elliptic surface is uniquely determined by the types of the singular fibers, except for those surfaces with two type I_0^* fibers [125]. The complex structures of the extremal rational elliptic surfaces with two type I_0^* fibers depend on the j-invariant j of the elliptic fibers. The fixed j determines the complex structure of an extremal rational elliptic surface with two type I_0^* fibers. Given these, we denote, for example, the extremal rational elliptic surfaces with singular fibers of type II^* and II as $X_{[II, \ II^*]}$. Here, n is simply employed to denote the type I_n fiber, and m^* represents the type I_m^* fiber. The extremal rational elliptic surface with type II^*, I_1, I_1 fibers is denoted by $X_{[II^*, \ 1,1]}$. An extremal rational elliptic surface with two type I_0^* fibers is denoted by $X_{[0^*, \ 0^*]}(j)$, because this surface depends on the j-invariant j of an elliptic fiber. The same notational convention is adopted in [66]. As discussed in [66], the singular fibers of an elliptic K3 surface obtained as the reverse of the stable degeneration is the sum of those of a pair of rational elliptic surfaces. Thus, the rank of the non-Abelian gauge group formed in F-theory compactification on an elliptic K3 surface obtained as the reverse of the stable degeneration can be up to 16. The heterotic dual of this compactification allows for a geometric interpretation by taking the large radius limit.

However, at some special points in the moduli of elliptically fibered K3 surfaces, K3 surfaces become extremal, and the non-Abelian gauge groups forming in F-theory compactifications on such extremal K3 surfaces have rank 18. Through dimensional reduction of 10-dimensional heterotic string to eight dimensions, $U(1)^2$ groups arise from each of the metric g_{MN} and the B field B_{MN}. Therefore, perturbative heterotic strings can have gauge groups up to $E_8 \times E_8 \times U(1)^4$ (or $SO(32) \times U(1)^4$). Rank 18 non-Abelian gauge groups, such as $E_8 \times E_8 \times SU(3)$, do not allow for perturbative descriptions on the heterotic side.

There is a special deformation of the stable degeneration in which 7-branes coincide and singular fibers collide, as studied in [66]. In this situation, the type of a singular fiber of the resulting K3 surface is enhanced, and the rank of the singularity type increases. Consequently, some extremal K3 surfaces are obtained in this special deformation of (the reverse of) the stable degeneration, as discussed in [66].

F-theory models on some extremal K3 surfaces are dual to non-geometric heterotic strings constructed by Malmendier and Morrison [22], as will be discussed in section 3.2. We demonstrate in section 3.2 that these F-theory models on extremal K3s that are dual to non-geometric strings constructed in [22] can be obtained as special deformations of F-theory models in the stable degeneration limit, as a consequence of coincident 7-branes. Because of this, the duality between eight-dimensional F-theory models with rank 18 gauge groups and non-geometric heterotic strings appears to suggest that non-geometric strings in [22] include the non-perturbative effect in which 7-branes coincide on the F-theory side. In addition, we confirm that the non-geometric heterotic duals of these F-theory models on

extremal K3s satisfy 5-brane solutions (2.11). The non-geometric heterotic duals of eight-dimensional F-theory models with rank 18 gauge groups satisfy multiple 5-brane solutions. Therefore, on the heterotic side, these deformations from the stable degeneration can be understood as a consequence of the insertion of multiple 5-branes. We also discuss some applications to SO(32) heterotic strings in section 3.3.

3.2 F-theory on extremal K3s and non-geometric heterotic duals

We discuss special points in the moduli of eight-dimensional non-geometric heterotic strings, at which the non-Abelian gauge groups that form on the F-theory side have rank 18. Eight-dimensional heterotic strings are compactified on the 2-torus T^2, and the dual F-theory models are compactified on elliptic K3 surfaces.

In particular, we consider $E_8 \times E_8$ non-geometric heterotic strings with an unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra, as constructed in [22]. We also discuss non-geometric heterotic strings as duals of F-theory on the extremal K3 surface with the E_6^3 singularity. It is known [126] that an attractive K3 surface always admits a Shioda-Inose structure [41, 127], namely, it admits an elliptic fibration for which the singular fibers include two type II^* fibers. It follows from this fact that the extremal K3 surface with the E_6^3 singularity admits a birational transformation to some elliptic fibration with two type II^* fibers. (In fact, from table 2 in [128] we observe that the extremal K3 surface with the E_6^3 singularity is $S_{[2\ 1\ 2]}$, and all the elliptic fibrations of this are known [123]. These fibrations are listed in table 4 in appendix A. Fibration no. 5 corresponds to the E_6^3 singularity, and a certain birational transformation maps this fibration to fibration no. 1.) The singularity type of an attractive K3 surface with two type II^* fibers includes E_8^2, and thus the attractive K3 surface with a Shioda-Inose structure to which the extremal K3 surface with the E_6^3 singularity is mapped belongs to the moduli with an unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra. From this observation, we deduce that the Weierstrass coefficients of the extremal K3 surface with the E_6^3 singularity are given by Siegel modular forms. This observation implies that F-theory on the extremal K3 surface with the E_6^3 singularity is related to some non-geometric heterotic dual.

We determine the complex structures of these extremal K3 surfaces, and we find that some deformations from elliptic K3 surfaces, obtained as sums of pairs of identical 1/2 K3s as the reverse of the stable degenerations, yield these extremal K3 surfaces. These deformations can be understood as the effects of coincident 7-branes.

3.2.1 Unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra

We discuss the points in the moduli of elliptic K3 surfaces with a global section with the singularity type E_8E_7 at which K3 surfaces become extremal, namely those points at which the singularity types have rank 18.

A generic elliptic K3 surface in this moduli space has the singularity type E_8E_7, with singular fibers of types II^* and III^*. Identifying the Weierstrass coefficients of these K3 surfaces with sections of line bundles on the 2-torus T^2, as in (2.9), yields non-geometric heterotic duals, as reviewed in section 2.2. The singularity type is enhanced at special points in the moduli.

The singularity types of extremal K3 surfaces were classified in [128]. According to the result, there are four singularity types worse than $E_8 E_7$ that can occur for extremal K3 surfaces. These are as follows:

$$E_8 E_7 A_2 A_1, \quad E_8 E_7 A_3, \quad E_8^2 A_2, \quad E_8^2 A_1^2. \tag{3.1}$$

The complex structure of an extremal K3 surface for each of these singularity types is uniquely determined [128]. The extremal K3 surfaces with these singularity types have transcendental lattices that have the following intersection forms:

$$\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \tag{3.2}$$

respectively. Therefore, we denote these extremal K3 surfaces as $S_{[6\ 0\ 2]}$, $S_{[4\ 0\ 2]}$, $S_{[2\ 1\ 2]}$, $S_{[2\ 0\ 2]}$, respectively. These extremal K3 surfaces belong to the moduli of elliptic K3 surfaces with the singularity type $E_8 E_7$.

We discuss the Weierstrass equations of the four extremal K3 surfaces.

An attractive K3 surface $S_{[6\ 0\ 2]}$[14] admits an extremal fibration with the singularity type $E_8 E_7 A_2 A_1$ [128]. After some computation, we can determine the Weierstrass equation of this fibration. As reviewed in section 2.2, the Weierstrass equation with E_8 and E_7 singularities takes the form (2.8). We rewrite (2.8) in the homogeneous form as follows:

$$y^2 = x^3 + s^4 (a\, t^4 + c\, t^3 s) x + t^5 s^5 (t^2 + b\, ts + d\, s^2), \tag{3.3}$$

where $[t : s]$ is the homogeneous coordinate of the base $\mathbb{P}^1$. In the form (3.3), the type II^* fiber is at $[t : s] = [1 : 0]$ and the type III^* fiber is at $[t : s] = [0 : 1]$.

Using an automorphism of the base $\mathbb{P}^1$, we may assume that the type I_3 fiber is at $[t : s] = [-1 : 1]$. Furthermore, we require that this fibration also has an A_1 singularity. These two conditions impose the following equations to be satisfied:

$$4a^3 + 27b^2 + 54d = 27\left(b - \frac{3}{2}\right)^2 + 162\left(b - \frac{3}{2}\right) + 81 \tag{3.4}$$

$$12a^2 c + 54bd = 81\left(b - \frac{3}{2}\right)^2 + 162\left(b - \frac{3}{2}\right) + 27$$

$$27d^2 + 12ac^2 = 81\left(b - \frac{3}{2}\right)^2 + 54\left(b - \frac{3}{2}\right)$$

$$4c^3 = 27\left(b - \frac{3}{2}\right)^2.$$

The solution to these equations (3.4) is

$$a = -\frac{15}{2\sqrt[3]{2}}, \quad b = -\frac{5}{2}, \quad c = 3 \cdot \sqrt[3]{4}, \quad d = 10. \tag{3.5}$$

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[14]The types of elliptic fibrations of the attractive K3 surface $S_{[6\ 0\ 2]}$ were studied in [129].

This solution yields an elliptic fibration

$$y^2 = x^3 + \sqrt[3]{4}\, t^3 s^4 \left(-\frac{15}{4}t + 3s \right) x + t^5 s^5 \left(t^2 - \frac{5}{2}ts + 10s^2 \right). \tag{3.6}$$

The discriminant of this Weierstrass form is

$$\Delta \sim s^{10} t^9 (t - 4s)^2 (s + t)^3. \tag{3.7}$$

This confirms that the elliptic fibration (3.6) has a type II^* fiber at $[t : s] = [1 : 0]$, a type III^* fiber at $[t : s] = [0 : 1]$, a type I_3 fiber at $[t : s] = [-1 : 1]$, and a type I_2 fiber at $[t : s] = [4 : 1]$. Thus, the corresponding singularity type is $E_8 E_7 A_2 A_1$, and according to [128] an attractive K3 surface with this singularity type is unique, being $S_{[6\ 0\ 2]}$. We conclude from this that the Weierstrass equation (3.6) gives the desired extremal fibration of $S_{[6\ 0\ 2]}$. We confirm that the solution (3.5) in fact satisfies the 5-brane solution $q = 0$ in (2.11).

The F-theory compactification on the K3 extremal fibration (3.6) times a K3 surface yields a four-dimensional theory with $N = 2$ supersymmetry, and the anomaly cancellation condition requires that the total number of the 7-branes in this compactification is 24 [53]. The extremal fibration (3.6) with II^*, III^*, I_3, and I_2 fibers satisfies this requirement, and this provides a non-trivial consistency check.

According to table 2 in [128] (No. 324), the extremal fibration (3.6) has the trivial Mordell-Wei group. Therefore, the global structure of the gauge group that forms in the F-theory compactification on the extremal fibration (3.6) is

$$E_8 \times E_7 \times \mathrm{SU}(3) \times \mathrm{SU}(2). \tag{3.8}$$

Attractive K3 surface $S_{[4\ 0\ 2]}$ has an extremal fibration with the singularity type $E_8 E_7 A_3$. The elliptic fibrations with a section of the attractive K3 surface $S_{[4\ 0\ 2]}$ were obtained in [130]. The extremal fibration with the singularity type $E_8 E_7 A_3$ is particularly given by the following (general) Weierstrass form [130]:

$$y^2 - 2t\, xy - 2t^2(t - 1)\, y = x^3 + t^4(t - 1)^3. \tag{3.9}$$

A translation in y and completing the cube in x yields the following Weierstrass form:

$$y^2 = x^3 + t^3 s^4 \left(\frac{5}{3}t - 2s \right) x + t^5 s^5 \left(t^2 - \frac{70}{27}ts + \frac{5}{3}s^2 \right) \tag{3.10}$$

given in the homogeneous form. The discriminant is

$$\Delta \sim t^9 s^{10} (t - s)^4 (27t - 32s). \tag{3.11}$$

Thus, we confirm that the Weierstrass form (3.10) has a type III^* fiber at $[t : s] = [0 : 1]$, a type II^* fiber at $[t : s] = [1 : 0]$, a type I_4 fiber at $[t : s] = [1 : 1]$, and a type I_1 fiber at $[t : s] = [32 : 27]$.

We confirm that

$$a = \frac{5}{3}, \quad b = -\frac{70}{27}, \quad c = -2, \quad d = \frac{5}{3} \tag{3.12}$$

satisfy 5-brane solution $q = 0$ in (2.11).

According to table 2 in [128] (No. 325), the extremal fibration (3.10) has the trivial Mordell-Wei group. Therefore, the global structure of the gauge group that forms in F-theory compactification on the extremal fibration (3.10) is

$$E_8 \times E_7 \times \mathrm{SU}(4). \tag{3.13}$$

The attractive K3 surface $S_{[2\ 1\ 2]}$ has an extremal fibration with the singularity type $E_8^2 A_2$. The Weierstrass equation of this extremal fibration is given as follows [131]:

$$y^2 = x^3 + t^5 \left(t - \frac{1}{4}\right)^2, \tag{3.14}$$

or in the homogeneous form,

$$y^2 = x^3 + t^5 s^5 \left(t^2 - \frac{1}{2} ts + \frac{1}{16} s^2\right). \tag{3.15}$$

The discriminant is given by

$$\Delta \sim t^{10} s^{10} \left(t - \frac{1}{4} s\right)^4. \tag{3.16}$$

It can be confirmed from (3.15) and (3.16) that there are two type II^* fibers at $[t : s] = [0 : 1]$ and $[t : s] = [1 : 0]$, and there is a type IV fiber at $[t : s] = [1 : 4]$.

We find that

$$a = c = 0, \quad b = -\frac{1}{2}, \quad d = \frac{1}{16} \tag{3.17}$$

satisfy 5-brane solutions $c = 0$ and $q = 0$ in (2.11).

The global structure of the gauge group that forms in F-theory compactification on the extremal K3 surface (3.15) was discussed in [66]. It is

$$E_8^2 \times \mathrm{SU}(3). \tag{3.18}$$

An attractive K3 surface $S_{[2\ 0\ 2]}$ admits an extremal elliptic fibration with the singularity type $E_8^2 A_1^2$. The Weierstrass equation of this fibration is [131]

$$y^2 = x^3 - 3t^4 x + t^5 + t^7. \tag{3.19}$$

In the homogeneous form, this is given by

$$y^2 = x^3 - 3t^4 s^4 x + t^5 s^7 + t^7 s^5. \tag{3.20}$$

The discriminant is (in homogeneous form)

$$\Delta \sim t^{10} s^{10} (t - s)^2 (t + s)^2. \tag{3.21}$$

It can be seen from (3.20) and (3.21) that two type II^* fibers are at $[t : s] = [0 : 1]$ and $[t : s] = [1 : 0]$, and two type I_2 fibers are at $[t : s] = [-1 : 1]$ and $[t : s] = [1 : 1]$.

We find that

$$a = -3, \quad b = c = 0, \quad d = 1 \tag{3.22}$$

satisfy 5-brane solutions $c = 0$ and $q = 0$ in (2.11).

The global structure of the gauge group that forms in F-theory compactification on the extremal K3 surface (3.20) can be found in [66]. It is

$$E_8^2 \times \mathrm{SU}(2)^2. \tag{3.23}$$

Now, we state the main result of this section 3.2.1. We show that the four extremal K3 surfaces in the moduli of K3 surfaces with the $E_8 E_7$ singularity that we discussed previously can be viewed as special deformations of the stable degenerations in which singular fibers collide and 7-branes coincide. This shows that the non-geometric heterotic strings include the effect of coincident 7-branes and the enhancement of non-Abelian gauge groups on the F-theory side.

We saw that the extremal K3 surface $S_{[6\,0\,2]}$ has the singular fibers of type II^*, III^*, I_3, I_2. It was proved in [66] that for any pair of extremal rational elliptic surfaces with distinct complex structures (except for $X_{[II,\ II^*]}$, $X_{[III,\ III^*]}$, and $X_{[IV,\ IV^*]}$), there exists some K3 surface that stably degenerates into that pair. (For *any* pair of rational elliptic surfaces with an *identical* complex structure, there exists a K3 surface that stably degenerates into that pair [66].) There exists an elliptic K3 surface S_1 that splits into $X_{[II^*,\ 1,1]} \amalg X_{[III^*,\ 2,1]}$ under the stable degeneration. Because the singular fibers of the resulting K3 surface are given by the sum of the pair of rational elliptic surfaces [66], the elliptic K3 surface S_1 admits singular fibers of types $II^*, III^* I_2, I_1, I_1, I_1$. We consider the limit of this K3 surface S_1 at which the type I_2 fiber and a type I_1 fiber collide and two type I_1 fibers collide. In this limit, these fibers are enhanced to a type I_3 fiber and type I_2 fiber, respectively. Because the K3 surface obtained in the limit has the singularity type $E_8 E_7 A_2 A_1$, the classification result in [128] implies that the K3 surface obtained by taking this limit is the extremal K3 surface $S_{[6\,0\,2]}$. From this, we deduce that the extremal K3 surface $S_{[6\,0\,2]}$ is obtained when 7-branes coincide through the deformation of the K3 surface S_1 yielded as the reverse of the stable degeneration of the sum $X_{[II^*,\ 1,1]} \amalg X_{[III^*,\ 2,1]}$.

We can also consider another deformation of the elliptic K3 surface S_1. This time, let us consider the limit at which the type I_2 fiber and two type I_1 fibers collide. These are enhanced to a type I_4 fiber in this situation, and the resulting elliptic K3 surface obtained in this limit has singular fibers of types II^*, III^*, I_4, and I_1. The corresponding singularity type is $E_8 E_7 A_3$, and therefore from the classification result in [128] we conclude that the resulting surface is the extremal K3 surface $S_{[4\,0\,2]}$. This shows that the extremal K3 surface $S_{[4\,0\,2]}$ can also be obtained when 7-branes coincide as a deformation of the K3 surface S_1.

The reverse of the stable degeneration of the sum of two extremal rational elliptic surfaces $X_{[II,\ II^*]} \amalg X_{[II,\ II^*]}$ yields an elliptic K3 surface S_2, the singular fibers of which are two type II^* fibers and two type II fibers. It was deduced in [66] that the extremal K3 surface $S_{[2\,1\,2]}$ with the $E_8^2 A_2$ singularity is obtained as a deformation of S_2, in the limit at which two type II fibers collide. These are enhanced to a type IV fiber. This limit corresponds to the situation in which 7-branes over which type II fibers lie become coincident.

The reverse of the stable degeneration of the sum of two extremal rational elliptic surfaces $X_{[II^*,\ 1,1]} \amalg X_{[II^*,\ 1,1]}$ yields an elliptic K3 surface S_3, the singular fibers of which

are two type II^* fibers and four type I_1 fibers. As deduced in [66], the extremal K3 surface $S_{[2\,0\,2]}$ with the $E_8^2 A_1^2$ singularity is obtained as a deformation of S_3 in the limit at which two pairs of type I_1 fibers collide, and these are enhanced to two type I_2 fibers.

This argument shows that the four extremal K3 surfaces are deformations of some elliptic K3 surface that represent the stable degenerations of the sums of 1/2 K3s, and these deformations are the effect of coincident 7-branes.

3.2.2 Extremal K3 with the E_6^3 singularity

An attractive K3 surface $S_{[2\,1\,2]}$ has an extremal elliptic fibration with the singularity type E_6^3 (elliptic fibration no. 5 in table 4 in appendix A). This is given by taking the Jacobian fibration of the Fermat-type hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ studied in [53].

The Fermat-type hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ [53] is a genus-one fibered K3 surface defined by the equation of bidegree (3,2)

$$(t - \alpha_1)(t - \alpha_2)\,X^3 + (t - \alpha_3)(t - \alpha_4)\,Y^3 + (t - \alpha_5)(t - \alpha_6)\,X^3, \qquad (3.24)$$

where t denotes the inhomogeneous coordinate on $\mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1$, $[X : Y : Z]$ denotes the homogeneous coordinates on $\mathbb{P}^2$ in $\mathbb{P}^2 \times \mathbb{P}^1$, and α_i, $i = 1, \cdots, 6$ are parameters. Here, we particularly consider the most enhanced situation

$$\alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_4, \quad \alpha_5 = \alpha_6. \qquad (3.25)$$

In this situation, the genus-one fibration (3.24) becomes

$$(t - \alpha_1)^2\,X^3 + (t - \alpha_3)^2\,Y^3 + (t - \alpha_5)^2\,X^3. \qquad (3.26)$$

The Fermat-type genus-one fibration (3.24) does not have a global section [53], but its Jacobian fibration admits a global section. A genus-one fibration and its Jacobian fibration have the same types of the singular fibers, and their discriminant loci are identical. The Jacobian fibration of the K3 genus-one fibration (3.26) is [53]

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot (t - \alpha_1)^4 (t - \alpha_3)^4 (t - \alpha_5)^4. \qquad (3.27)$$

The singular fibers of the K3 genus-one fibration (3.26) and the Jacobian fibration (3.27) are three type IV^* fibers [53]. Thus, the Jacobian fibration (3.27) yields an extremal elliptic fibration of the attractive K3 surface $S_{[2\,1\,2]}$ with the singularity type E_6^3.

Being an attractive K3 surface, $S_{[2\,1\,2]}$ admits a Shioda-Inose structure, and thus it should admit another elliptic fibration, which includes the singularity type E_8^2. In fact, the elliptic fibration (3.14) that we discussed previously admits such a fibration. The Jacobian fibration (3.27) and the elliptic fibration (3.14) are two elliptic fibrations of the same attractive K3 surface $S_{[2\,1\,2]}$. Therefore, there exists a birational map that transforms the Jacobian fibration (3.27) into the elliptic fibration (3.14). Because the Jacobian fibration (3.27) and the elliptic fibration (3.14) are birational, the F-theory compactification on the Jacobian (3.27) relates to a non-geometric $E_8 \times E_8$ heterotic string.

The reverse of the stable degeneration of the sum $X_{[IV,\,IV^*]} \amalg X_{[IV,\,IV^*]}$ of two identical extremal rational elliptic surfaces $X_{[IV,\,IV^*]}$ yields an elliptic K3 surface, which we denote

as S_4. The singular fibers of this surface consist of two type IV^* fibers and two type IV fibers [66]. A deformation of S_4 obtained by taking the limit at which two type IV fibers collide yields an elliptic K3 surface $S_{[2\ 1\ 2]}$ with the extremal fibration (3.27) [66]. Thus, the Jacobian fibration (3.27) is obtained from the K3 surface S_4, given as the stable degeneration of the sum of two 1/2 K3s, as the effect of coincident 7-branes.

The Mordell-Weil group of the Jacobian fibration (3.27) is isomorphic to $\mathbb{Z}_3$ [123, 128], and the gauge group that arises for this fibration is [53]

$$E_6^3/\mathbb{Z}_3. \tag{3.28}$$

In section 4, we consider the fibering of the Jacobian fibrations of the genus-one fibrations (3.24) over $\mathbb{P}^1$, yielding an elliptic Calabi-Yau 3-fold.

3.3 Applications to SO(32) heterotic string

A birational map from K3 elliptic fibration the singularity type of which includes E_8E_7 to K3 surface with another elliptic fibration with a type I_{10}^* fiber was discussed in [22]. Using the birational transformation

$$x \to x^2 t, \quad y \to x^2 y, \quad t \to x, \tag{3.29}$$

the K3 elliptic fibration (2.8) was transformed to K3 elliptic fibration with a type I_{10}^* fiber (or worse) in [22], given by

$$y^2 = x^3 + (t^3 + a\, t + b)\, x^2 + (ct + d)\, x. \tag{3.30}$$

This fibration relates to the SO(32) heterotic string. The discriminant of the Weierstrass equation (3.30) is, in the homogeneous form, [22]

$$\Delta \sim s^{16} (ct + ds)^2 \left(t^6 + 2a\, t^4 s^2 + 2b\, t^3 s^3 + a^2\, t^2 s^4 + (2ab - 4c)\, ts^5 + (b^2 - 4d)\, s^6\right). \tag{3.31}$$

$[t:s]$ denotes the coordinate of the base $\mathbb{P}^1$.

We discuss the implications of the birational transformation (3.29) and the resulting Weierstrass form (3.30) with a type I_{10}^* fiber (or worse) for extremal K3 surfaces studied in section 3.2.

When the birational transformation is applied to the extremal fibration (3.6) of the K3 surface $S_{[6\ 0\ 2]}$, by plugging the values for a, b, c, d (3.5) for the Weierstrass form (3.30), we obtain the following equation:

$$y^2 = x^3 + \left(t^3 - \frac{15}{2\sqrt[3]{2}}\, t - \frac{5}{2}\right) x^2 + (3\sqrt[3]{4}\, t + 10)x. \tag{3.32}$$

The discriminant of the resulting Weierstrass form is

$$\Delta \sim s^{16} (3\sqrt[3]{4}\, t + 10s)^2\, (t - \sqrt[3]{2}s)^2\, (2t + 3\sqrt[3]{2}s)^3\, (2t - 5\sqrt[3]{2}s). \tag{3.33}$$

From the equations (3.32) and (3.33), we deduce that the transformed fibration (3.32) has the singular fibers of types $I_{10}^*, I_3, I_2, I_2, I_1$. Thus, the singularity type of the fibration (3.32)

is $D_{14}A_2A_1^2$. The rank of the singularity type is 18, therefore this fibration is also extremal. Birational map (3.29) does not alter the complex structure of a K3 surface, namely, the birational map (3.29) sends the attractive K3 surface $S_{[6\,0\,2]}$ to itself, but the fibration structure is not preserved under this map.

Therefore, the obtained results predict that the attractive K3 surface $S_{[6\,0\,2]}$ has the extremal elliptic fibration (3.32) with the singularity type $D_{14}A_2A_1^2$. This agrees with the classification of the singularity types of the extremal K3 surfaces in [128] (See No. 211 in table 2).

The extremal K3 surface with the singularity type $D_{14}A_2A_1^2$ has the Mordell-Weil group isomorphic to $\mathbb{Z}_2$ [128], therefore, the Mordell-Weil group of the elliptic fibration (3.32) is isomorphic to $\mathbb{Z}_2$. Thus, the gauge group that forms for this fibration is

$$\mathrm{SO}(28) \times \mathrm{SU}(3) \times \mathrm{SU}(2)^2/\mathbb{Z}_2. \tag{3.34}$$

The non-Abelian part of the gauge group has rank 18. This suggests that perturbative description of the gauge group is not possible for the SO(32) heterotic string (same as for heterotic string with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra).

For an attractive K3 surface $S_{[2\,1\,2]}$ with the extremal fibration (3.15) with the singularity type $E_8^2A_2$, the birational transformation (3.29) maps the fibration (3.15) to the following fibration:

$$y^2 = x^3 + \left(t^3 - \frac{1}{2}\right)x^2 + \frac{1}{16}\,x. \tag{3.35}$$

The discriminant of the resulting fibration is

$$\Delta \sim s^{18}\,t^3\,(t^3 - s^3). \tag{3.36}$$

From the equations (3.35) and (3.36), we find that the resulting fibration (3.35) has the singular fibers of types $I_{12}^*, I_3, I_1, I_1, I_1$, and the singularity type of the fibration (3.35) is $D_{16}A_2$. The fibration (3.35) is also extremal.

This shows that the attractive K3 surface $S_{[2\,1\,2]}$ has an elliptic fibration with the singularity type $D_{16}A_2$. This agrees with the results in [123, 128]. (See fibration no.2 in table 4 in appendix A, and No.216 in table 2 in [128].) See also, e.g., [132] for studies of the fibration (3.35) of $S_{[2\,1\,2]}$.

The Mordell-Weil group of the fibration (3.35) is isomorphic to $\mathbb{Z}_2$ [123, 128], and the gauge group that forms for this fibration is

$$\mathrm{SO}(32) \times \mathrm{SU}(3)/\mathbb{Z}_2. \tag{3.37}$$

The rank of the non-Abelian part of the gauge group is 18, and the perturbative description of the gauge group for the SO(32) heterotic string is not possible either for this case.

The birational transformation (3.29) applied to an attractive K3 surface $S_{[2\,0\,2]}$ with the extremal fibration (3.19) with the singularity type $E_8^2A_1^2$ yields the following fibration:

$$y^2 = x^3 + (t^3 - 3\,t)\,x^2 + x. \tag{3.38}$$

The discriminant of this fibration is

$$\Delta \sim s^{18} \left(t^2 - 4s^2\right) (t-s)^2 (t+s)^2. \tag{3.39}$$

From the equations (3.38) and (3.39), we find that the fibration (3.38) has the singular fibers of types $I_{12}^*, I_2, I_2, I_1, I_1$, and the singularity type of the fibration (3.38) is $D_{16}A_1^2$. Therefore, the fibration (3.38) is also extremal.

These computations show that the attractive K3 surface $S_{[2\ 0\ 2]}$ has an elliptic fibration with the singularity type $D_{16}A_1^2$. This agrees with the results in [123, 128]. (See fibration no.3 in table 5 in appendix A, and No.215 in table 2 in [128].)

The Mordell-Weil group of the fibration (3.38) is isomorphic to $\mathbb{Z}_2$ [123, 128], and the gauge group that forms is

$$\mathrm{SO}(32) \times \mathrm{SU}(2)^2/\mathbb{Z}_2. \tag{3.40}$$

The rank of the non-Abelian part of the gauge group is 18, and the perturbative description of the gauge group for the SO(32) heterotic string is not possible either for this case.

4 Jacobian of Fermat-type Calabi-Yau 3-folds and F-theory

In this section, we consider the Jacobian fibrations of (3,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, in order to construct a family of elliptically fibered Calabi-Yau 3-folds.[15] This family admits a K3 fibration. Choosing a specific form of equations for elliptic Calabi-Yau 3-folds, this K3 fiber coincides with the Jacobian of the Fermat-type K3 hypersurface, as discussed in section 3.2.2. We discuss F-theory compactifications on the constructed elliptic Calabi-Yau 3-folds.

4.1 Fermat-type (3,2,2) hypersurface, the Jacobian fibration, and the discriminant locus

A (3,2,2) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ has the trivial canonical bundle $K = 0$, and therefore such a hypersurface yields a Calabi-Yau 3-fold. The projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ yields a genus-one fibration, and the projection onto $\mathbb{P}^1$ gives a K3 fibration.

In particular, we consider the (3,2,2) hypersurfaces given by the following form of the equations:

$$(t-\alpha_1)(t-\alpha_2)\, f_1\, X^3 + (t-\alpha_3)(t-\alpha_4)\, f_2\, Y^3 + (t-\alpha_5)(t-\alpha_6)\, f_3\, Z^3 = 0, \tag{4.1}$$

where $[X : Y : Z]$ denotes the homogeneous coordinates on $\mathbb{P}^2$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, t is the inhomogeneous coordinate on the first $\mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, and f_1, f_2, f_3 are degree 2 polynomials on the second $\mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, $\alpha_i, \ i = 1, \cdots, 6$ denote points in $\mathbb{P}^1$. We call the (3,2,2) hypersurfaces of this form *Fermat-type* hypersurfaces. (Similar naming conventions were employed for the K3 hypersurfaces and Calabi-Yau 4-folds constructed in [53, 56].) K3 fibers of these Calabi-Yau 3-folds are Fermat-type K3 (3,2) hypersurfaces [53], as discussed in section 3.2.2. In particular, K3 fibers are genus-one

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fibered, but they lack a section [53]. Splitting the degree 2 polynomials f_1, f_2, f_3 into linear factors, the equation (4.1) can be rewritten as

$$(t-\alpha_1)(t-\alpha_2)(u-\beta_1)(u-\beta_2)\,X^3 + (t-\alpha_3)(t-\alpha_4)(u-\beta_3)(u-\beta_4)\,Y^3$$
$$+(t-\alpha_5)(t-\alpha_6)(u-\beta_5)(u-\beta_6)\,Z^3 = 0. \tag{4.2}$$

By u we denote the inhomogeneous coordinate on the second $\mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, and β_i, $i = 1, \cdots, 6$ denote points in this $\mathbb{P}^1$.

Similar argument to that in [56] implies that the genus-one fibration (4.2) does not admit a global section. If the genus-one fibration (4.2) admits a section, this restricts to K3 fibers, giving a section to K3 fibers. K3 fibers are Fermat-type K3 (3,2) hypersurfaces (3.24), which do not admit a section, as explained previously. Therefore, this implies a contradiction. By an argument similar to those in [53, 56, 58], the genus-one fibration (4.2) admits a 3-section.[16]

By taking the Jacobian fibration of the genus-one fibrations (4.2), we construct a family of elliptically fibered Calabi-Yau 3-folds with a section. The Jacobian fibration of the genus-one fibration (4.2) is given by the following equation:

$$X^3 + Y^3 + \Pi_{i=1}^6(t-\alpha_i)\,\Pi_{j=1}^6(u-\beta_j)\,Z^3 = 0. \tag{4.3}$$

This equation transforms into the Weierstrass form as

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot \Pi_{i=1}^6(t-\alpha_i)^2\,\Pi_{j=1}^6(u-\beta_j)^2. \tag{4.4}$$

The Jacobian fibrations (4.4) yield elliptically fibered Calabi-Yau 3-folds. In section 4, we discuss F-theory compactifications on these Calabi-Yau 3-folds. These compactifications yield a six-dimensional theory with $N = 1$ supersymmetry.

The discriminant of the Weierstrass equation (4.4) is given by

$$\Delta \sim \Pi_{i=1}^6(t-\alpha_i)^4\,\Pi_{j=1}^6(u-\beta_j)^4. \tag{4.5}$$

Then, 7-branes are wrapped on the irreducible components of the discriminant locus $\Delta = 0$. There are 12 components, which we denote by A_i, $i = 1, \cdots, 6$, and B_j, $j = 1, \cdots, 6$. These are given by the following equations:

$$A_i = \{t = \alpha_i\} \quad (i = 1, \cdots, 6) \tag{4.6}$$
$$B_j = \{u = \beta_j\} \quad (j = 1, \cdots, 6).$$

Because A_i is isomorphic to $\{\text{pt}\} \times \mathbb{P}^1$ and B_j is isomorphic to $\mathbb{P}^1 \times \{\text{pt}\}$, these are isomorphic to $\mathbb{P}^1$.

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[16]Therefore, a discrete $\mathbb{Z}_3$ gauge symmetry [46, 51, 58] arises in F-theory compactifications on the genus-one fibered Calabi-Yau 3-folds (4.1).

Component	fiber type	non-Abel. Gauge Group
$A_{1,\cdots,6}$	IV	SU(3)
$A_{1,\cdots,6}$	IV^*	E_6
$B_{1,\cdots,6}$	IV	SU(3)
$B_{1,\cdots,6}$	IV^*	E_6

Table 3. Singular fiber types and the gauge groups on the discriminant components.

4.2 Gauge groups

4.2.1 Singular fibers and non-Abelian gauge groups

We determine the non-Abelian gauge groups that form on the 7-branes in F-theory compactifications on the Jacobian fibrations (4.4). In a generic case in which α_i and β_j are mutually distinct, all singular fibers are of type IV. When two α_i's become coincident, say $\alpha_1 = \alpha_2$, the corresponding 7-branes A_1 and A_2 coincide, and the singular fiber on the 7-brane A_1 is enhanced to type IV^*. Similarly, when two β_j's are coincident, say $\beta_1 = \beta_2$, the 7-branes B_1 and B_2 coincide, and the singular fiber on the 7-brane B_1 is enhanced to type IV^*. (See table 2 in section 2.1.) When three of the α_i's become coincident, the Calabi-Yau condition is broken [53, 56]. A similar statement holds for the β_j's.

From the equation (4.4), we find that both the type IV fiber and type IV^* fiber are split [112]. Therefore, the non-Abelian gauge group that forms is SU(3) for a type IV fiber and E_6 for a type IV^* fiber. The results are summarized in table 3.

Generically, there are 12 discriminant components with type IV fibers, and thus the non-Abelian gauge group that forms is

$$\text{SU}(3)^{12}. \tag{4.7}$$

When three pairs of α coincide,

$$\alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_4, \quad \alpha_5 = \alpha_6, \tag{4.8}$$

the Jacobian fibration (4.3) becomes

$$X^3 + Y^3 + (t - \alpha_1)^2 (t - \alpha_3)^2 (t - \alpha_5)^2 \, \Pi_{j=1}^6 (u - \beta_j) \, Z^3 = 0. \tag{4.9}$$

The Weierstrass form is then

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot (t - \alpha_1)^4 (t - \alpha_3)^4 (t - \alpha_5)^4 \, \Pi_{j=1}^6 (u - \beta_j)^2. \tag{4.10}$$

In this situation, the K3 fibers of the Jacobian (4.10) constitute the extremal K3 surface (3.27), as discussed in section 3.2.2. The 7-branes coincide as $A_1 = A_2$, $A_3 = A_4$, and $A_5 = A_6$, and the non-Abelian gauge group that forms in enhanced to

$$E_6^3 \times \text{SU}(3)^6. \tag{4.11}$$

We assumed that the β's are mutually distinct. When pairs of β are coincident, the corresponding 7-branes coincide. Then, $\text{SU}(3)^2$ is enhanced to E_6, and the gauge group (4.11) is further enhanced if this occurs.

4.2.2 Consistency check

By an argument similar to those given in [53, 56], smooth genus-one fibers of the genus-one fibration (4.2) are the Fermat curves. Thus, they have j-invariant 0. This means that generic fibers over the base surface have j-invariant 0, and this forces the singular fibers to have j-invariant 0. This condition strongly constrains possible fiber types of the singular fibers. Because a genus-one fibration and its Jacobian fibration have the identical types and configurations of the singular fibers, the same condition is imposed on the singular fibers of the Jacobian (4.3).

As in table 1 in section 2.1, the singular fibers with j-invariant 0 are: type II, II^*, IV, IV^*, and type I_0^* (type I_0^* fiber is also included, because type I_0^* fiber can have j-invariant 0.) We deduced in section 4.2.1 that the singular fibers have types IV and IV^*. We confirm that this result agrees with the constraint imposed. Monodromies of order 3 characterize the gauge groups.

4.3 Models without U(1) and their Mordell-Weil groups

Here, we discuss the particular Jacobian fibration (4.10). The K3 fiber of this Jacobian fibration is the extremal K3 surface $S_{[2\ 1\ 2]}$ with the singularity type E_6^3 (3.27), and therefore the Mordell-Weil group of the K3 fiber is isomorphic to $\mathbb{Z}_3$ [123, 128]. By an argument similar to those given in [56, 58], we can consider the specialization to the K3 fiber to find that the Mordell-Weil group of the Jacobian fibration (4.10) is isomorphic to the Mordell-Weil group of the K3 fiber. Thus, the Mordell-Weil group of the Jacobian fibration (4.10) is isomorphic to $\mathbb{Z}_3$:

$$MW \cong \mathbb{Z}_3. \tag{4.12}$$

The global structure [136–138] of the gauge group that forms on the 7-branes is

$$E_6^3 \times \mathrm{SU}(3)^6/\mathbb{Z}_3. \tag{4.13}$$

The F-theory compactifications on the Jacobians (4.10) do not have a U(1) gauge group.

4.4 Matter spectra

In this section, we deduce candidates for the matter spectra on F-theory compactifications on the Jacobian Calabi-Yau 3-folds. F-theory compactifications on the Jacobian Calabi-Yau 3-folds yield six-dimensional theories with $N = 1$ supersymmetry. We confirm that the deduced candidates for matter satisfy the anomaly cancellation condition [139–142]. Because the base of the Jacobian Calabi-Yau 3-folds (4.4) is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$,

$$T = h^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) - 1 = 2 - 1 = 1. \tag{4.14}$$

Thus, H and V are required to satisfy the following equation:

$$H - V = 273 - 29 = 244. \tag{4.15}$$

In particular, We consider the specific situations in which three pairs of the α's coincide:

$$\alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_4, \quad \alpha_5 = \alpha_6, \tag{4.16}$$

as described previously in section 4.2.1. As we saw in section 4.3, in this case the Mordell-Weil ranks of the Jacobian fibrations (4.10) are 0, and the Mordell-Weil groups are isomorphic to $\mathbb{Z}_3$.

First, we consider a generic case in which the β's are mutually distinct. The discriminant of the Jacobian (4.10) is given by

$$\Delta \sim (t - \alpha_1)^8 (t - \alpha_3)^8 (t - \alpha_5)^8 \Pi_{j=1}^6 (u - \beta_j)^4. \tag{4.17}$$

and the 7-branes intersect at 18 points:

$$(\alpha_i, \beta_j) \quad (i = 1, 3, 5, \; j = 1, \cdots, 6). \tag{4.18}$$

For all 18 intersections, the 7-branes over which type IV^* fibers lie intersect those over which type IV fibers lie. The corresponding singularity types are E_6 and A_2.

We saw in section 4.3 that the gauge group that arises is $E_6^3 \times \mathrm{SU}(3)^6/\mathbb{Z}_3$. Therefore,

$$V = 78 \times 3 + 8 \times 6 = 282. \tag{4.19}$$

Thus, from the anomaly cancellation condition (4.15) we must have

$$H = 282 + 244 = 526. \tag{4.20}$$

Although there appear to be three parameters of α in the equation (4.10), $\alpha_1, \alpha_3, \alpha_5$, using automorphism of $\mathbb{P}^1$ we may assume that they are $0, 1, \infty$. Thus, these three parameters are superfluous, and they do not count as the parameters for the deformation of the complex structure. By a similar reasoning, we may also assume that $\beta_1, \beta_2, \beta_3$ are equal to $0, 1, \infty$, respectively, using automorphism of $\mathbb{P}^1$. Therefore, the actual parameters of the deformation of the complex structure of the Jacobian Calabi-Yau 3-fold (4.10) are $\beta_4, \beta_5, \beta_6$, namely, the number of the parameters of the deformation of the complex structure is three. Thus, the number of the neutral hypermultiplets that arise from the deformation of the complex structure is

$$H^0 = 1 + 3 = 4. \tag{4.21}$$

We used H^0 to denote the number of the neutral hypermultiplets. It follows that the number of the hypermultiplets that arise from the 18 intersections of the 7-branes is

$$526 - 4 = 522. \tag{4.22}$$

When the following matter arise at each of the 18 intersections

$$\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{4.23}$$

the anomaly cancellation condition is satisfied:

$$(27 + 1 + 1) \times 18 = 522. \tag{4.24}$$

Therefore, we deduce that 18 $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ and 4 uncharged neutral hypermultiplets yield candidate matter spectrum in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.10).

Second, we consider the case in which a pair of β coincides:

$$\beta_1 = \beta_2. \tag{4.25}$$

Then, the Weierstrass equation of the Jacobian Calabi-Yau 3-fold is given by

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot (t - \alpha_1)^4 (t - \alpha_3)^4 (t - \alpha_5)^4 (u - \beta_1)^4 \Pi_{j=3}^6 (u - \beta_j)^2. \tag{4.26}$$

The discriminant of the Jacobian (4.26) is given by

$$\Delta \sim (t - \alpha_1)^8 (t - \alpha_3)^8 (t - \alpha_5)^8 (u - \beta_1)^8 \Pi_{j=3}^6 (u - \beta_j)^4. \tag{4.27}$$

and the 7-branes intersect at 15 points:

$$(\alpha_i, \beta_j) \quad (i = 1, 3, 5, \ j = 1, 3, 4, 5, 6). \tag{4.28}$$

For the 3 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1$, the 7-branes over which type IV^* fibers lie intersect with those over which type IV^* fibers lie. The corresponding singularity types are E_6 and E_6. The 7-branes over which type IV^* fibers lie intersect those over which type IV fibers lie for the remaining 12 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 3, 4, 5, 6$. The corresponding singularity types are E_6 and A_2.

The 7-branes B_1 and B_2 coincide, and the fiber type on B_1 is enhanced to type IV^*. Thus, the gauge group that arises in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.26) is

$$E_6^4 \times \mathrm{SU}(3)^4 / \mathbb{Z}_3. \tag{4.29}$$

Therefore, we have that

$$V = 78 \times 4 + 8 \times 4 = 344, \tag{4.30}$$

and the anomaly cancellation condition (4.15) requires that

$$H = 344 + 244 = 588. \tag{4.31}$$

By a reasoning similar to that stated previously, we may assume that $\beta_1, \beta_3, \beta_4$ are equal to $0, 1, \infty$, respectively, using automorphism of $\mathbb{P}^1$. Therefore, the parameters for the deformation of the complex structure are β_5, β_6, and the number of the neutral hypermultiplet that arises is

$$H^0 = 1 + 2 = 3. \tag{4.32}$$

It follows that the number of the hypermultiplets that arise from the 15 intersections of the 7-branes is

$$588 - 3 = 585. \tag{4.33}$$

When the following matter arise at each of the 3 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1$

$$\mathbf{78} \oplus \mathbf{1}, \tag{4.34}$$

and the following matter arise at each of the remaining 12 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 3, 4, 5, 6$

$$\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{4.35}$$

we find that the anomaly cancellation condition is satisfied:

$$(78 + 1) \times 3 + (27 + 1 + 1) \times 12 = 585. \tag{4.36}$$

Thus, we confirm that $3\ \mathbf{78} \oplus \mathbf{1}$, $12\ \mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ and 3 uncharged neutral hypermultiplets yield candidate matter spectrum in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.26).

Next, we discuss the case in which two pairs of β coincide:

$$\beta_1 = \beta_2, \quad \beta_3 = \beta_4. \tag{4.37}$$

In this case, the Weierstrass equation of the Jacobian Calabi-Yau 3-fold is given by the following equation

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot (t - \alpha_1)^4 (t - \alpha_3)^4 (t - \alpha_5)^4 (u - \beta_1)^4 (u - \beta_3)^4 (u - \beta_5)^2 (u - \beta_6)^2. \tag{4.38}$$

The discriminant is

$$\Delta \sim (t - \alpha_1)^8 (t - \alpha_3)^8 (t - \alpha_5)^8 (u - \beta_1)^8 (u - \beta_3)^8 (u - \beta_5)^4 (u - \beta_6)^4. \tag{4.39}$$

The 7-branes intersect at 12 points:

$$(\alpha_i, \beta_j) \quad (i = 1, 3, 5,\ j = 1, 3, 5, 6). \tag{4.40}$$

For the 6 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1, 3$, the 7-branes over which type IV^* fibers lie intersect with those over which type IV^* fibers lie. The corresponding singularity types are both E_6. The 7-branes over which type IV^* fibers lie intersect those over which type IV fibers lie for the remaining 6 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 5, 6$. The corresponding singularity types are E_6 and A_2.

The 7-branes B_1 and B_2 coincide, as do B_3 and B_4, and the singular fibers over these 7-branes are of type IV^*. Thus, the gauge group that arises in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.38) is

$$E_6^5 \times \mathrm{SU}(3)^2 / \mathbb{Z}_3. \tag{4.41}$$

We have that

$$V = 78 \times 5 + 8 \times 2 = 406, \tag{4.42}$$

and the anomaly cancellation condition (4.15) requires that the number of the hypermultiplets is

$$H = 406 + 244 = 650. \tag{4.43}$$

By a reasoning similar to that stated previously, using automorphism of $\mathbb{P}^1$ we may assume that $\beta_1, \beta_3, \beta_5$ are equal to $0, 1, \infty$, respectively. Therefore, the parameter for the

deformation of the complex structure is β_6, and the number of the neutral hypermultiplet that arises is

$$H^0 = 1 + 1 = 2. \tag{4.44}$$

It follows that the number of the hypermultiplets that arise from the 12 intersections of the 7-branes is

$$650 - 2 = 648. \tag{4.45}$$

When the following matter arise at each of the 6 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1, 3$

$$\mathbf{78} \oplus \mathbf{1}, \tag{4.46}$$

and the following matter arise at each of the remaining 6 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 5, 6$

$$\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{4.47}$$

the anomaly cancellation condition is satisfied:

$$(78 + 1) \times 6 + (27 + 1 + 1) \times 6 = 648. \tag{4.48}$$

Therefore, we find that 6 $\mathbf{78} \oplus \mathbf{1}$, 6 $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ and 2 uncharged neutral hypermultiplets yield candidate matter spectrum on F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.38).

Finally, we discuss the case in which three pairs of the β's coincide:

$$\beta_1 = \beta_2, \quad \beta_3 = \beta_4, \quad \beta_5 = \beta_6. \tag{4.49}$$

In this case, the Weierstrass form of the Jacobian Calabi-Yau 3-fold is given by

$$y^2 = x^3 - 3^3 \cdot 2^4 \cdot (t - \alpha_1)^4 (t - \alpha_3)^4 (t - \alpha_5)^4 (u - \beta_1)^4 (u - \beta_3)^4 (u - \beta_5)^4. \tag{4.50}$$

The discriminant is

$$\Delta \sim (t - \alpha_1)^8 (t - \alpha_3)^8 (t - \alpha_5)^8 (u - \beta_1)^8 (u - \beta_3)^8 (u - \beta_5)^8. \tag{4.51}$$

The 7-branes intersect in 9 points:

$$(\alpha_i, \beta_j) \quad (i = 1, 3, 5, \ j = 1, 3, 5). \tag{4.52}$$

For all 9 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1, 3, 5$, the 7-branes over which type IV^* fibers lie intersect with those over which type IV^* fibers lie. The corresponding singularity types are both E_6.

The 7-branes B_1 and B_2, B_3 and B_4, B_5 and B_6 coincide, and the singular fibers over these 7-branes are of type IV^*. Thus, the gauge group that arises in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.50) is

$$E_6^6/\mathbb{Z}_3. \tag{4.53}$$

We have

$$V = 78 \times 6 = 468. \tag{4.54}$$

The anomaly cancellation condition (4.15) requires that the number of the hypermultiplets is

$$H = 468 + 244 = 712. \tag{4.55}$$

By a reasoning similar to that stated previously, using automorphism of $\mathbb{P}^1$ we may assume that $\beta_1, \beta_3, \beta_5$ are equal to $0, 1, \infty$, respectively. Therefore, the complex structure of the Jacobian Calabi-Yau 3-fold for this case is fixed. Thus, the number of the neutral hypermultiplet that arises is

$$H^0 = 1 + 0 = 1. \tag{4.56}$$

It follows that the number of the hypermultiplets that arise from the 12 intersections of the 7-branes is

$$712 - 1 = 711. \tag{4.57}$$

When the matter

$$\mathbf{78} \oplus \mathbf{1} \tag{4.58}$$

arise at each of the 9 intersections (α_i, β_j), $i = 1, 3, 5$, $j = 1, 3, 5$ the anomaly cancellation condition is satisfied:

$$(78 + 1) \times 9 = 711. \tag{4.59}$$

Therefore, we deduce that 9 $\mathbf{78} \oplus \mathbf{1}$ and 1 uncharged neutral hypermultiplets yield candidate matter spectrum in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.50).

We expect that each of the first and the last matter spectra ($18 \times (\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1})$ and $9 \times (\mathbf{78} \oplus \mathbf{1})$) is the unique choice that is free of anomaly. Concerning the first matter spectrum, we saw that 7-branes intersect at 18 points. A symmetry argument suggests that the identical matter arise from these 18 points. Thus, the dimension of the matter representation at each of the 18 intersections of the 7-branes should be 29, for the anomaly to be cancelled. An E_6 singularity and an A_2 singularity collide at each of these intersections. Therefore, it is natural to expect that $\mathbf{27}$ of E_6 arises at the intersections. It follows that the matter representation $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ at each of the 18 intersections of the 7-branes is the unique choice.

As to the last matter spectrum, the 7-branes intersect at 9 points. A symmetry argument again suggests that the identical matter arise from these 9 points, and the dimension of the matter representation at each of these points should be 79. Then the representation $\mathbf{78} \oplus \mathbf{1}$ is the unique natural choice. (The remaining two choices, $\mathbf{27} + 52 \times \mathbf{1}$, or $2 \times \mathbf{27} + 25 \times \mathbf{1}$, do not seem natural.)

The second and the third matter spectra, $3 \times (\mathbf{78} \oplus \mathbf{1}) + 12 \times (\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1})$, and $6 \times (\mathbf{78} \oplus \mathbf{1}) + 6 \times (\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1})$, are not the unique choices. For example, $3 \times (\mathbf{27} \oplus \mathbf{27} \oplus \mathbf{1}) + 12 \times (\mathbf{27} \oplus \mathbf{8})$ is also a potentially viable matter spectrum that is free of anomaly in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.26). $6 \times \mathbf{27} + 6 \times (\mathbf{27}, \mathbf{3})$, and $6 \times \mathbf{78} + 6 \times (\mathbf{27} \oplus \mathbf{3})$ can also be viable matter spectra that are free of anomaly in F-theory compactification on the Jacobian Calabi-Yau 3-fold (4.38).

If we assume that matter representation $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises from the collision of E_6 and A_2 singularities of the Jacobian Calabi-Yau 3-folds, and $\mathbf{78} \oplus \mathbf{1}$ arises from the collision of two

E_6 singularities,[17] the anomaly cancels in F-theory compactifications on the four families of the Jacobian Calabi-Yau 3-folds (4.10), (4.26), (4.38), (4.50). Determining whether these interpretations indeed give the matter spectra by resolving the singularities of the Jacobian Calabi-Yau 3-folds constructed in this note can be a direction of future studies.

5 Conclusions

We analyzed some points in the eight-dimensional F-theory moduli at which the ranks of the gauge groups are enhanced to 18, and their non-geometric heterotic duals. On the F-theory side, these models are understood as deformations of the stable degenerations, and non-Abelian gauge groups are enhanced as the effect of coincident 7-branes. This suggests that non-geometric heterotic strings include the effect of coincident 7-branes. In the heterotic language, this corresponds to the effect of the insertion of 5-branes.

We particularly considered $E_8 \times E_8$ heterotic strings, but by applying a birational transformation from a K3 surface with type II^* and III^* fibers to a K3 surface with a type I_{10}^* fiber [22, 27, 31], we also discussed SO(32) heterotic string. Considering F-theory dual of SO(32) heterotic string, we deduced the Weierstrass equation of an extremal K3 surface.

Non-geometric heterotic duals of F-theory on K3 surfaces with the Picard number 19 with the singularity rank 17 also do not allow for perturbative interpretations of the gauge groups that arise on the heterotic side. This case is not studied in this note. It can be interesting to investigate F-theory on such K3 surfaces and the non-geometric heterotic duals.

In section 4, by taking the Jacobian fibration of the Fermat-type (3,2,2) hypersurfaces we constructed elliptically fibered Calabi-Yau 3-folds, and we studied F-theory compactifications on these Calabi-Yau 3-folds. Highly enhanced gauge groups arise in these compactifications. We deduced candidate matter spectra directly from the global defining equation of the Calabi-Yau 3-folds. These Calabi-Yau 3-folds admit a K3 fibration. They include special cases in which K3 fibrations become extremal. In this situation, F-theory compactifications on the K3 fibers have non-geometric heterotic duals (by applying a birational map if necessary). Determining whether six-dimensional F-theory on Calabi-Yau 3-folds have heterotic duals is a likely direction of future studies.

A Elliptic fibrations of attractive K3 $S_{[2\ 1\ 2]}$ and $S_{[2\ 0\ 2]}$

The types of the elliptic fibrations of attractive K3 surfaces with the discriminant three and four, $S_{[2\ 1\ 2]}$ and $S_{[2\ 0\ 2]}$, were classified, and the Mordell-Weil groups of the fibrations

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[17]The collision of two E_6 singularities in the context of four-dimensional conformal matter in F-theory was discussed in [143]. See also, e.g., [16, 74] for discussion of the collision of two E_6 singularities in F-theory.

Elliptic fibrations of $S_{[2\ 1\ 2]}$	singularity type	MW group
no.1	$E_8^2 A_2$	0
no.2	$D_{16} A_2$	$\mathbb{Z}_2$
no.3	$E_7 D_{10}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
no.4	A_{17}	$\mathbb{Z} \oplus \mathbb{Z}_3$
no.5	E_6^3	$\mathbb{Z}_3$
no.6	$D_7 A_{11}$	$\mathbb{Z}_4$

Table 4. Elliptic fibrations of the attractive K3 surface $S_{[2\ 1\ 2]}$, singularity types of the fibrations, and the Mordell-Weil groups.

Elliptic fibrations of $S_{[2\ 0\ 2]}$	singularity type	MW group
no.1	$E_8^2 A_1^2$	0
no.2	$E_8 D_{10}$	0
no.3	$D_{16} A_1^2$	$\mathbb{Z}_2$
no.4	$E_7^2 D_4$	$\mathbb{Z}_2$
no.5	$E_7 D_{10} A_1$	$\mathbb{Z}_2$
no.6	$A_{17} A_1$	$\mathbb{Z}_3$
no.7	D_{18}	0
no.8	$D_{12} D_6$	$\mathbb{Z}_2$
no.9	$D_8^2 A_1^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
no.10	$A_{15} A_3$	$\mathbb{Z}_4$
no.11	$E_6 A_{11}$	$\mathbb{Z} \oplus \mathbb{Z}_3$
no.12	D_6^3	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
no.13	A_9^2	$\mathbb{Z}_5$

Table 5. Elliptic fibrations of the attractive K3 surface $S_{[2\ 0\ 2]}$, singularity types of the fibrations, and the Mordell-Weil groups.

were determined in [123]. We list the types of the elliptic fibrations and the Mordell-Weil groups of $S_{[2\ 1\ 2]}$ and $S_{[2\ 0\ 2]}$ deduced in [123] in table 4 and table 5.

B Fiber types of the extremal rational elliptic surfaces

We list the types of the singular fibers for the extremal rational elliptic surfaces [125] in table 6 below. The complex structure of an extremal rational elliptic surface is uniquely determined by the fiber type except the surface $X_{[0^*,\ 0^*]}(j)$, and the complex structure of the extremal rational elliptic surface $X_{[0^*,\ 0^*]}(j)$ depends on the j-invariant j of the elliptic fiber [125].

Extremal rational elliptic surface	fiber type	singularity type
$X_{[II,\ II^*]}$	$II^*,\ II$	E_8
$X_{[III,\ III^*]}$	$III^*,\ III$	$E_7 A_1$
$X_{[IV,\ IV^*]}$	$IV^*,\ IV$	$E_6 A_2$
$X_{[0^*,\ 0^*]}(j)$	$I_0^*,\ I_0^*$	D_4^2
$X_{[II^*,\ 1,1]}$	$II^*\ I_1\ I_1$	E_8
$X_{[III^*,\ 2,1]}$	$III^*\ I_2\ I_1$	$E_7 A_1$
$X_{[IV^*,\ 3,1]}$	$IV^*\ I_3\ I_1$	$E_6 A_2$
$X_{[4^*,\ 1,1]}$	$I_4^*\ I_1\ I_1$	D_8
$X_{[2^*,\ 2,2]}$	$I_2^*\ I_2\ I_2$	$D_6 A_1^2$
$X_{[1^*,\ 4,1]}$	$I_1^*\ I_4\ I_1$	$D_5 A_3$
$X_{[9,1,1,1]}$	$I_9\ I_1\ I_1\ I_1$	A_8
$X_{[8,2,1,1]}$	$I_8\ I_2\ I_1\ I_1$	$A_7 A_1$
$X_{[6,3,2,1]}$	$I_6\ I_3\ I_2\ I_1$	$A_5 A_2 A_1$
$X_{[5,5,1,1]}$	$I_5\ I_5\ I_1\ I_1$	A_4^2
$X_{[4,4,2,2]}$	$I_4\ I_4\ I_2\ I_2$	$A_3^2 A_1^2$
$X_{[3,3,3,3]}$	$I_3\ I_3\ I_3\ I_3$	A_2^4

Table 6. Fiber types of the extremal rational elliptic surfaces are listed.

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