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The action of long strings in supersymmetric field theories

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ABSTRACT: Long strings emerge in many Quantum Field Theories, for example as vortices in Abelian Higgs theories, or flux tubes in Yang-Mills theories. The actions of such objects can be expanded in the number of derivatives, around a long straight string solution. This corresponds to the expansion of energy levels in powers of $1/L$, with L the length of the string. Doing so reveals that the first few terms in the expansions are universal, and only from a certain term do they become dependent on the originating field theory. Such classifications have been made before for bosonic strings. In this work we expand upon that and classify also strings with fermionic degrees of freedom, where the string breaks $D = 4$, $N = 1$ SUSY completely. An example is the confining string in $N = 1$ SYM theory. We find a general method for generating supersymmetric action terms from their bosonic counterparts, as well as new fermionic terms which do not exist in the non-supersymmetric case. These terms lead to energy corrections at a lower order in $1/L$ than in the bosonic case.

Contents

1 Introduction

String-like objects appear in many quantum field theories, such as flux tubes in quantum chromodynamics (QCD), vortices such as the Nielsen-Olesen strings in the 4d Abelian Higgs model [2], and domain walls in 3d theories such as the Ising model. Their appearance in QCD, as visible through the spectrum of mesons (and other hadrons), led to the development of the Veneziano model [1] and ultimately to the development of string theory.

A straight string is a 2d object which breaks the ISO($D-1,1$) symmetry of the D-dimensional bulk into an ISO($1,1$) × SO($D-2$) symmetry group, leading to ($D-2$) massless modes of excitation, known as the Nambu-Goldstone Bosons, or NGBs. These

massless excitations define the low-energy behavior of the string, and we can compute their energy levels expanded in powers of $1/L$, where L is the length of the string.

Naively, one might think that the actions computed for string-like objects in different QFTs are dependent on the underlying theory. However, as reviewed by Aharony and Komargodski [3], the first few terms in the expansion — up to and including order of $1/L^5$ — are universal, and only the higher order terms are dependent on the theory. This was shown in 3 different formalisms:

1. The general case, in which there is no gauge fixing, and allowed terms in the action must preserve both Lorentz symmetry and diffeomorphism.

2. The unitary ("static") gauge in which the parameterization of the world-sheet of the string ("diffeomorphism") is fixed and the Lorentz group is broken manifestly. In this formalism, the action can be expanded by the number of derivatives — corresponding to the $1/L$ expansion of the energy levels — constrained by Lorentz symmetries. In this formalism, it was shown that for $D > 3$ classical Lorentz invariance allows a six-derivative term, but its presence modifies the form of the generators (while higher-derivative allowed terms do not); and then quantum considerations show that its value is actually fixed.

3. The orthogonal ("conformal") gauge in which diffeomorphism is fixed up to conformal transformations and Lorentz symmetry is maintained. In this formalism, the action is constrained by conformal invariance.

This work aims at generalizing the results of Aharony and Komargodski to the case of Supersymmetry (SUSY), specifically $D = 4$, $N = 1$ SUSY. In a supersymmetric theory, a string may break $D = 4$, $N = 1$ SUSY either completely, or partially into $D = 2$, $N = (2, 0)$, as was shown by Hughes and Polchinski [4]. The breaking of SUSY generators adds massless fermionic modes of excitation, known as Goldstinos. The action can then be written as a functional of the NGBs and Goldstinos, and expanded as in the fully bosonic case by the number of derivatives. For the two cases of complete and partial breaking of SUSY, a complete classification of action terms has yet to be made. In the scope of this work we will only explore the case of complete SUSY breaking, which is relevant in particular for confining strings in supersymmetric Yang-Mills theory, and it is the main goal of this work to classify action terms for this case. As a final step, we will calculate the form of the energy level correction for a closed string on a circle, arising from the lowest order new term we find, so that our results can be verified by lattice simulations at some later point.

The outline of this paper is as follows. In the next section we review well established results, as well as notations and definitions we will use, and eventually a graphical approach, originally presented by Gliozzi and Meineri [9], to find invariant actions for bosonic effective strings. In section 3 we extend this approach to include Goldstinos, and in section 4 we use the extended approach to find invariant actions for SUSY breaking effective strings, including a new term at order $1/L^5$. In section 5 we formulate prohibition rules which show that our list of invariant actions is indeed exhaustive, and in section 6 we derive the

energy corrections that follow from our new term. Finally we discuss our results and draw some conclusions.

2 Review

2.1 Bosonic effective strings

Consider some gapped D-dimensional quantum field theory with a string-like field configuration, so that its width is much smaller than its length. Such a configuration could be either open, closed, infinite or semi-infinite. We define this configuration by the space-time coordinates of its worldsheet $X^\mu \left(\sigma^0, \sigma^1 \right)$, where σ^0, σ^1 are some parameterization of the worldsheet and $\mu = 0, \ldots, D-1$. The physics can't depend on the parameterization. The effective string action is the low energy action of the massless modes on the worldsheet

$$S = T \int d^2\sigma \mathcal{L} \left[X^\mu \left(\sigma^0, \sigma^1 \right) \right] \tag{2.1}$$

where T is the string tension. This general formalism is the first case referred to in the introduction.

The static gauge is where we fix $\sigma^0 = X^0$ and $\sigma^1 = X^1$. When working in this gauge we will denote these ξ^0, ξ^1 to avoid ambiguity. In this gauge the NGBs are given by the transverse coordinates X^i for $i = 2, \ldots, D-1$. In this formalism effective string action is

$$S = T \int d^2\xi \mathcal{L} \left(\partial_a X^i, \partial_a \partial_b X^i, \ldots \right) \tag{2.2}$$

where $a, b = 0, 1$. There is no X^i dependence with no derivatives due to translational invariance. For simplicity, we will work mainly in this formalism, and generalize our results whenever possible. We will use letters from the beginning of the Latin alphabet such as $a, b, c, d, \ldots$ to denote the worldsheet indices $0, 1$, and letters from the middle of the Latin alphabet such as $i, j, k, \ldots$ to denote the transverse indices $2, \ldots, D-1$.

The gauge choice (2.2) breaks the space-time symmetry $\mathrm{ISO}(D-1, 1)$ by choosing a preferred direction in space. $\mathrm{ISO}(D-1, 1)$ is the Poincaré group which is the group that preserves the Minkowski metric which we define as $\eta_{\mu\nu} = \mathrm{diag}\left(-1, 1, \ldots, 1 \right)$. It is generated by

$$J_{\mu\nu} = i \left(x_\mu \partial_\nu - x_\nu \partial_\mu \right) \tag{2.3}$$

$$P_\mu = i\partial_\mu \,. \tag{2.4}$$

It is broken into $\mathrm{ISO}(1, 1) \times \mathrm{SO}(D-2)$, where $\mathrm{ISO}(1, 1)$ is the symmetry on the worldsheet, which preserves the metric η_{ab}, and is generated by J_{ab} and P_a; and $\mathrm{SO}(D-2)$ is the symmetry of rotations around the string and is generated by J_{ij}. The remaining generators P_i and J_{ai} are broken. By acting with J_{ai} on the fields X^j we get

$$\delta X^j = i\epsilon^{ai} \left[J_{ai}, X^j \right] = -\epsilon^{ai}\delta^{ij}\xi_a - \epsilon^{ai} X^i \partial_a X^j \tag{2.5}$$

which is a non-linear realization of these generators.

When working in the static gauge, we will often work in light-cone coordinates

$$\xi^\pm = \xi^0 \pm \xi^1 . \tag{2.6}$$

A well known result in String theory is that the action of a string is proportional to the area of its worldsheet. This result can be expressed using the embedded metric on the string

$$g_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{2.7}$$

which in static gauge can be expressed as

$$g_{ab} = \eta_{ab} + \partial_a X^i \partial_b X^i \equiv \eta_{ab} + h_{ab}, \tag{2.8}$$

and the Nambu-Goto (NG) action equal to the area of the worldsheet

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{-\det\left(g_{ab}\right)}. \tag{2.9}$$

The NG action is highly non-linear. In the context of the effective string, we can work with it by expanding in terms of derivatives ∂ around the flat string solution of the static gauge. The determinant is given by

$$-\det\left(g_{ab}\right) = -\det\left(\eta_{ab} + h_{ab}\right) = 1 + \eta^{ab} h_{ab} - \det\left(h_{ab}\right) =$$
$$= 1 + \partial_a X^i \partial^a X^i - \frac{1}{2}\partial_a X^i \partial^b X^i \partial_b X^i \partial^a X^i + \frac{1}{2}\left(\partial_a X^i \partial^a X^i\right)^2 \tag{2.10}$$

and we get

$$S_{\text{NG}} = -T \int d^2\sigma \left(1 + \frac{1}{2}\partial_a X^i \partial^a X^i - \frac{1}{4}\partial_a X^i \partial^b X^i \partial_b X^i \partial^a X^i + \frac{1}{8}\left(\partial_a X^i \partial^a X^i\right)^2 + \mathcal{O}\left(\partial^6\right)\right). \tag{2.11}$$

This expansion is meaningful under the assumption of a long string of length scale L. We can then define a small dimensionless parameter $\left(\sqrt{T}L\right)^{-1}$ and expand the energy levels of the string in terms of this parameter. This expansion will take the form

$$E_n = TL + \frac{a_n^{(1)}}{L} + \frac{a_n^{(2)}}{TL^3} + \frac{a_n^{(3)}}{T^2 L^5} + \dots \tag{2.12}$$

Where the term at order L^{-k} corresponds to the terms in the action at order ∂^{k+1}. For the NG action, there is a known exact result for the energy levels of closed strings with no worldsheet momentum [10]

$$E_n = TL\sqrt{1 + \frac{8\pi}{TL^2}\left(n - \frac{D-2}{24}\right)}. \tag{2.13}$$

2.2 Supersymmetry

Supersymmetry (SUSY) is an extension of the Poincaré algebra to include fermionic generators. The simplest ($\mathcal{N} = 1$) super-Poincaré generators can be written in $D = 4$ as a single Majorana spinor

$$Q = \left(Q_1 \ Q_2 \ \overline{Q}_{\dot{2}} \ -\overline{Q}_{\dot{1}} \right)^T \tag{2.14}$$

with the following algebra

$$\begin{aligned}
\left\{ Q, \overline{Q} \right\} &= -2i\gamma^\mu P_\mu \\
[Q, P] &= 0 \\
[Q, J_{\mu\nu}] &= i\sigma_{\mu\nu} Q
\end{aligned} \tag{2.15}$$

where γ^μ are the Dirac gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\overline{Q}$ is obtained from Q using the charge conjugation operator $C = i\gamma^0\gamma^2$ such that

$$\overline{Q} = -Q^T C = \left(-Q_2 \ Q_1 \ \overline{Q}_{\dot{1}} \ \overline{Q}_{\dot{2}} \right) \tag{2.16}$$

and

$$\sigma_{\mu\nu} = \frac{i}{4} \left[\gamma_\mu, \gamma_\nu \right] . \tag{2.17}$$

We will generally use letters from the beginning of the Greek alphabet to denote the Majorana spinor indices such as in $Q_\alpha, \gamma^\mu_{\alpha\beta}$ where $\alpha, \beta, \cdots = 1, 2, 3, 4$. This symmetry can be realized by introducing a new anti-commuting space-time set of coordinates θ_α, such that

$$\{\partial_\alpha, \theta_\beta\} = \delta_{\alpha\beta}, \ \partial_\alpha = \frac{\partial}{\partial \theta_\alpha} . \tag{2.18}$$

We will take this to be a Majorana spinor, such that

$$\theta = \left(\theta_1 \ \theta_2 \ \overline{\theta}_{\dot{2}} \ -\overline{\theta}_{\dot{1}} \right)^T \tag{2.19}$$

$$\overline{\theta} = -\theta^T C = \left(-\theta_2 \ \theta_1 \ \overline{\theta}_{\dot{1}} \ \overline{\theta}_{\dot{2}} \right) . \tag{2.20}$$

Then we can express the super-Poincaré generators in the superspace $\{x^\mu, \theta_\alpha\}$

$$J_{\mu\nu} = i \left(X_\mu \partial_\nu - X_\nu \partial_\mu \right) + \theta_\alpha \left(\sigma_{\mu\nu} \right)_{\alpha\beta} \partial_\beta \tag{2.21}$$

$$Q_\alpha = -i\overline{\partial}_\alpha + \gamma^\mu_{\alpha\beta} \theta_\beta \partial_\mu \tag{2.22}$$

$$P_\mu = i\partial_\mu . \tag{2.23}$$

2.3 Fermionic effective strings

Much like the breaking of commuting symmetry operators results in the introduction of massless Nambu-Goldstone bosons, Akulov and Volkov showed [6] that the breaking of anti-commuting generators introduces massless fermions, which were later termed Goldstinos. In an $\mathcal{N} = 1$, $D = 4$ bulk, a string may break either all, or half of the 4 SUSY generators. Clearly the generators which square to translations transverse to the string must be broken. In this work we will focus on the case were all generators are broken. As in the bosonic case,

the coordinates which correspond to the broken generators become a field configuration which we will denote with the massless Majorana spinor ψ_α, and the effective string action is the action of the massless modes on the worldsheet

$$S = T \int d^2\sigma \mathcal{L} \left[X^\mu \left(\sigma^0, \sigma^1 \right), \psi_\alpha \left(\sigma^0, \sigma^1 \right) \right]. \tag{2.24}$$

The generalization of the Nambu-Goto action (2.9) to the supersymmetric case is obtained by replacing

$$\partial_a X^\mu \to \Pi_a^\mu \equiv \partial_a X^\mu - i\overline{\psi}\gamma^\mu\partial_a\psi \tag{2.25}$$

to get the Akulov-Volkov action

$$S_{\mathrm{AV}} = -T \int d^2\sigma \sqrt{-\det\left(\eta_{\mu\nu}\Pi_a^\mu\Pi_b^\nu\right)}. \tag{2.26}$$

When expanding this, dimensional analysis shows that terms of the form $\partial^k X^m \psi^{2n}$ contribute at order L^{-k-n+1}, so we will denote the free term $i\overline{\psi}\gamma^\mu\partial_a\psi \sim \mathcal{O}\left(\partial^2\right)$ and the rest of the terms accordingly. The AV action can then be expanded as

$$S_{\mathrm{AV}} = -T \int d^2\sigma \left(1 + \frac{1}{2}\partial_a X^i \partial^a X^i - \frac{1}{2}\left(\Delta_+^{2\dot{2}} + \Delta_-^{1\dot{1}}\right) + \mathcal{O}\left(\partial^3\right) \right) \tag{2.27}$$

where

$$\Delta_a^{\alpha\dot{\alpha}} \equiv i\overline{\psi}_{\dot{\alpha}}\partial_a\psi_\alpha - i\psi_\alpha\partial_a\overline{\psi}_{\dot{\alpha}}. \tag{2.28}$$

This implies the equations of motion

$$\partial_-\psi_1 + \mathcal{O}\left(\partial^3\right) = \partial_-\overline{\psi}_{\dot{1}} + \mathcal{O}\left(\partial^3\right) = \partial_+\psi_2 + \mathcal{O}\left(\partial^3\right) = \partial_+\overline{\psi}_{\dot{2}} + \mathcal{O}\left(\partial^3\right) = 0. \tag{2.29}$$

2.4 Classification of the action of bosonic strings

In their 2013 review of bosonic effective strings, Aharony and Komargodski (AK) classify the action terms by their scale (which they refer to as weight). The scale of a term is its dimension of *length*$^{-1}$, such that $\partial_a X^\mu$ has scale 0, $\partial_a\partial_b X^\mu$ has scale 1, and so on. Translational invariance guarantees that all terms in the action have non-negative scale. For bosonic strings, $\mathrm{ISO}(1,1)\times\mathrm{SO}(D-2)$ and parity invariance (that we assume) guarantee that all terms have even scale. AK then show that there is a unique invariant action at scale zero, which is the Nambu-Goto action (2.9). At scale 2, AK find a single term which is invariant up to a term proportional to the EOM, with 6 derivatives and 4 fields

$$\mathcal{L}_{6,4} = -32c_4 \left(\partial_+^2 X^i \partial_-^2 X^i\right)\left(\partial_+ X^j \partial_- X^j\right) + \ldots, \tag{2.30}$$
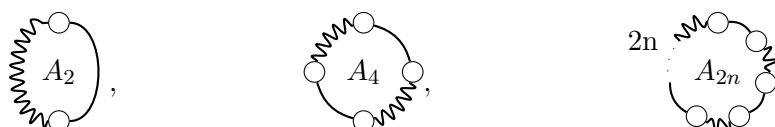
which can be shown to be forbidden quantum mechanically since it modifies the algebra of Lorentz transformations [7], which can lead to anomalies (terms whose variation is proportional to the EOM can be made invariant by changing the transformation rule, but this can modify the algebra). The next allowed terms are of scale 4, and have at least 8 derivatives. The existence of 8 derivatives implies that those terms contribute to the $1/L$ expansion of the energy levels at order of at least $1/L^7$, so that the coefficients up to and including order of $1/L^5$ are universal. Aharony and Klinghoffer [8] calculated how the first few terms of the NG action appear in the energy level expansion, as well as the effect of the $\mathcal{L}_{6,4}$ term.

2.5 Gliozzi-Meineri (GM) approach for classifying bosonic string action terms

In their 2013 Paper [9], Gliozzi and Meineri (GM) present a useful graphical approach to finding invariant terms for the action of a bosonic string. They associate terms with graphs, where the vertices are the fields X^i and their derivatives, and the edges represent contractions over indices. Since we have 2 types of indices — worldsheet indices denoted $a, b, c, \ldots$ and transverse indices denoted by $i, j, k, \ldots$ — we also have 2 types of edges. Worldsheet indices will be represented by solid lines, and transverse indices will be represented by wavy lines. The term $\partial_a X^i$ will be represented by a circular node (slightly changing GM notation) with 2 open edges

$$\partial_a X^i = \quad \text{ww}\!\circ\!\!-\!\!- \quad . \tag{2.31}$$

So that scale 0 terms can be represented as sums and products of ring graphs, so for example $\partial_a X^i \partial^a X^i$, $\partial_a X^i \partial^a X^j \partial_b X^j \partial^b X^i$ and a ring with 2n ∂X's will be represented and denoted as
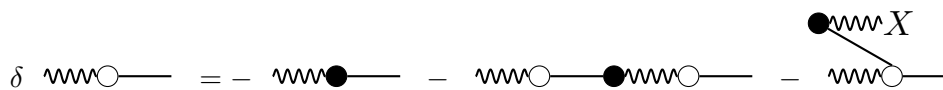

$$\tag{2.32}$$

correspondingly. General terms in the action are products of such rings. GM write the broken infinitesimal Lorentz transformations in a covariant form

$$\delta X^i = -\epsilon^{aj}\delta^{ij}\xi_a - \epsilon^{aj}X^j\partial_a X^i \tag{2.33}$$

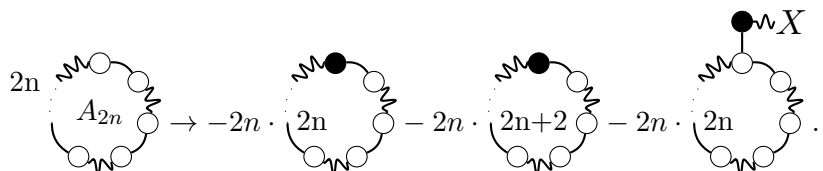$$\delta\left(\partial_b X^i\right) = -\epsilon^{aj}\delta^{ij}\eta_{ab} - \epsilon^{aj}\partial_b X^j \partial_a X^i - \epsilon^{aj}X^j\partial_a\partial_b X^i \,. \tag{2.34}$$

Eq. (2.34) can be expressed graphically as

$$\tag{2.35}$$

where the solid circles represent the transformation parameter ϵ^{aj}, the vertex X represents X^j, and the vertex which is connected to 3 edges is simply $\partial_a\partial_b X^i$. Using this transformation rule, one can transform the ring A_{2n} which has $2n$ vertices of the form $\partial_a X^i$ (we will refer to these as boson vertices) and express it graphically as

$$\tag{2.36}$$

We can cancel the first two terms in the variation by summing rings such that one variation from the ring A_{2n} will cancel the other from the ring A_{2n+2}. This gives a recursion relation for the coefficients of the rings

$$(2n+2)\,a_{2n+2} = -2na_{2n} \quad \Rightarrow \quad a_{2n} = (-1)^{n+1}\frac{1}{n}a_2 \,. \tag{2.37}$$
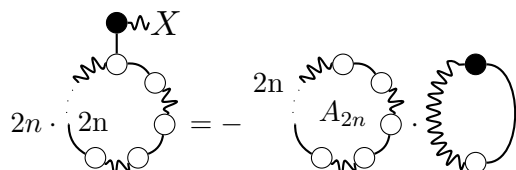
Summing this series we get

$$\sum_{n=1}^{\infty} a_{2n} A_{2n} = a_2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \mathrm{Tr} \left[\left(\partial_a X^i \partial_b X^i \right)^n \eta^{bc} \right] = a_2 \mathrm{Tr} \left[\log \left(\left(\eta_{ab} + \partial_a X \cdot \partial_b X \right) \eta^{bc} \right) \right] =$$
$$= a_2 \log \left[- \det \left(\eta_{ab} + h_{ab} \right) \right] = a_2 \log \left(-g \right) \tag{2.38}$$

where $g = \det(g_{ab})$. This summation cancels all variations which come out of those two terms except for the variation

$$\tag{2.39}$$

We can now consider a sum of terms of the form $b_n \left[\log(-g) \right]^n$. The nth order in this sum contains products of n rings, and in fact every n ring term is contained in it. Looking at the third term in the variation (2.36), we see that it has a "tumor" stemming from the ring. Such a tumor can be handled using integration by parts of the derivative from which the tumor stems. This will move the tumor around the ring, so that we get

$$\tag{2.40}$$

So for a product of n rings we can cancel this variation using the surviving A_2 variation from a product of $n+1$ rings. For this cancellation we require

$$b_{n+1} = \frac{1}{2(n+1)} b_n \quad \Rightarrow \quad b_n = \frac{1}{2^n n!} b_0 \tag{2.41}$$

and we get a unique invariant scale 0 Lagrangian

$$\mathcal{L}_0 = b_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{1}{2} \log(-g) \right]^n = b_0 \sqrt{-g}. \tag{2.42}$$

Which is exactly the NG Lagrangian. GM extend this approach for higher scaling. They obtain two scale 2 invariants

$$I_1 = \sqrt{-g} \partial_{ab}^2 X_i \partial_{cd}^2 X_j t^{ij} g^{ac} g^{bd} \tag{2.43}$$
$$I_2 = \sqrt{-g} \partial_{ab}^2 X_i \partial_{cd}^2 X_j t^{ij} g^{ab} g^{cd} \tag{2.44}$$

where

$$g^{ab} = \eta^{ab} - \eta^{ac} h_{cd} \eta^{db} + \eta^{ac} h_{cd} \eta^{de} h_{ef} \eta^{fb} - \ldots \tag{2.45}$$

is the matrix inverse of g_{ab}, and

$$t^{ij} = \delta^{ij} - \partial_a X^i \partial_b X^j g^{ab}. \tag{2.46}$$

However, looking at the invariants I_1, I_2 one may observe that $I_1 - I_2 = \sqrt{-g} R$ where R is the 2D Ricci scalar. This is a total derivative so it does not contribute to the action. Also,

the first terms of I_2, up to eight derivatives, are proportional to the free EOM and hence are vanishing at the six-derivative order. This shows there are no contributing invariants at the six-derivative order. This approach does not find the term (2.30) since this term is only invariant up to the EOM.

GM proceed to apply this method to find higher scale invariants which will be discussed in section 4.

3 Extending the GM approach to include Goldstinos

To extend the GM approach to include Goldstinos, we need to look at the broken super-symmetry transformations on the string. The broken generators are

$$J_{ai} = i\left(X_a\partial_i - X_i\partial_a\right) + \psi^\alpha\left(\sigma_{ai}\right)_\alpha^\beta\partial_\beta \tag{3.1}$$

$$Q_\alpha = -i\overline{\partial}_\alpha + \gamma_{\alpha\beta}^\mu\psi^\beta\partial_\mu \tag{3.2}$$

so that the transformations can be written as

$$\delta X^j = -\epsilon^{ai}\delta^{ij}\xi_a - \epsilon^{ai}X^i\partial_a X^j + i\overline{\theta}^\alpha\gamma_{\alpha\beta}^j\psi^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\beta}^a\psi^\beta\partial_a X^j \tag{3.3}$$

$$\delta\psi^\beta = i\epsilon^{ai}\psi^\alpha\left(\sigma_{ai}\right)_\gamma^\beta - \epsilon^{ai}X_i\partial_a\psi^\beta + \overline{\theta}^\alpha C_\alpha^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\gamma}^a\psi^\gamma\partial_a\psi^\beta \tag{3.4}$$

$$\delta\overline{\psi}^\beta = \delta\psi^\delta C_\delta^\beta = -i\epsilon^{ai}\overline{\psi}^\alpha\left(\sigma_{ai}\right)_\alpha^\beta - \epsilon^{ai}X_i\partial_a\psi^\delta C_\delta^\beta + \overline{\theta}^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\gamma}^a\psi^\gamma\partial_a\psi^\delta C_\delta^\beta\,. \tag{3.5}$$

We can write any fermionic effective string action using the following vertices

$$\partial_a X^i = \quad\text{〰〰○——} \tag{3.6}$$

$$\partial_a\psi^\alpha = \quad\text{〰〰∂——} \tag{3.7}$$

$$\overline{\psi}^\alpha\gamma_{\alpha\beta}^b = \quad\text{——γ〰〰} \tag{3.8}$$

$$\overline{\psi}^\alpha\gamma_{\alpha\beta}^i = \quad\text{〰〰γ〰〰} \tag{3.9}$$

and their derivatives. In the above we used springs to express spinor indices. The transformation laws of these vertices can be written as

$$\delta\partial_b X^j = -\epsilon^{ai}\delta^{ij}\eta_{ab} - \epsilon^{ai}\partial_b X^i\partial_a X^j - \epsilon^{ai}X^i\partial_a\partial_b X^j + i\overline{\theta}^\alpha\gamma_{\alpha\beta}^j\partial_b\psi^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\beta}^a\partial_b\psi^\beta\partial_a X^j + i\overline{\theta}^\alpha\gamma_{\alpha\beta}^a\psi^\beta\partial_a\partial_b X^j \tag{3.10}$$

$$\delta\partial_b\psi^\beta = i\epsilon^{ai}\left(\sigma_{ai}\right)_\gamma^\beta\partial_b\psi^\alpha - \epsilon^{ai}\partial_b X_i\partial_a\psi^\beta - \epsilon^{ai}X_i\partial_a\partial_b\psi^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\gamma}^a\partial_b\psi^\gamma\partial_a\psi^\beta + i\overline{\theta}^\alpha\gamma_{\alpha\gamma}^a\psi^\gamma\partial_a\partial_b\psi^\beta \tag{3.11}$$

$$\delta\overline{\psi}^\beta\gamma_{\beta\gamma}^b = -i\epsilon^{ai}\overline{\psi}^\beta\gamma_{\alpha\beta}^b\left(\sigma_{ai}\right)_\gamma^\beta - \epsilon^{ai}\overline{\psi}^\alpha\gamma_{\alpha\gamma}^i\delta_a^b - \epsilon^{ai}X_i\partial_a\psi^\beta\left(C\gamma^b\right)_{\beta\gamma} + \overline{\theta}^\alpha\gamma_{\alpha\gamma}^b + i\overline{\theta}^\alpha\gamma_{\alpha\delta}^a\psi^\delta\partial_a\psi^\beta\left(C\gamma^b\right)_{\beta\gamma} \tag{3.12}$$

$$\delta\overline{\psi}^\beta\gamma_{\beta\gamma}^j = -i\epsilon^{ai}\overline{\psi}^\beta\gamma_{\alpha\beta}^j\left(\sigma_{ai}\right)_\gamma^\beta + \epsilon^{ai}\overline{\psi}^\alpha\gamma_{\alpha\gamma}^a\delta_i^j - \epsilon^{ai}X_i\partial_a\psi^\beta\left(C\gamma^j\right)_{\beta\gamma} + \overline{\theta}^\alpha\gamma_{\alpha\gamma}^j + i\overline{\theta}^\alpha\gamma_{\alpha\delta}^a\psi^\delta\partial_a\psi^\beta\left(C\gamma^b\right)_{\beta\gamma} \tag{3.13}$$

– 9 –

or in graphical representation

$$\delta\ \text{(diagram)} = -\ \text{(diagram)} \quad -\ \text{(diagram)}$$
$$+\, i\ \text{(diagram)}\,\partial\text{---} \quad +\, i\ \text{(diagram)}\,\partial\text{---} \tag{3.14}$$
$$+\, i\ \text{(diagram with }\psi\text{)} \quad -\ \text{(diagram with }X\text{)}$$

$$\delta\ \text{(diagram)}\,\partial\text{---} = i\ \text{(diagram)} \quad -\ \text{(diagram)}$$
$$+\, i\ \text{(diagram with }\psi\text{)} \quad -\ \text{(diagram with }X\text{)} \tag{3.15}$$
$$+\,\sigma\text{-term}$$

$$\delta\ \text{(diagram }\gamma\text{)} = \text{(diagram)} \quad -\ \text{(diagram)}$$
$$+\, i\ \text{(diagram with }\psi\text{)} \quad -\ \text{(diagram with }X\text{)} \tag{3.16}$$
$$+\,\sigma\text{-term}$$

$$\delta\ \text{(diagram }\gamma\text{)} = \text{(diagram)} \quad +\ \text{(diagram)}$$
$$i\ \text{(diagram with }\psi\text{)} \quad -\ \text{(diagram with }X\text{)} \tag{3.17}$$
$$+\,\sigma\text{-term}$$

where the σ-terms are different terms which involves σ_{ai} matrices, we used solid circles to represent transformation parameters, such that

$$\epsilon^{ai} = \ \text{(diagram)} \tag{3.18}$$
$$\overline{\theta}^{\alpha}\gamma^{a}_{\alpha\beta} = \ \text{(diagram)} \tag{3.19}$$
$$\overline{\theta}^{\alpha}\gamma^{i}_{\alpha\beta} = \ \text{(diagram)} \tag{3.20}$$

and we introduced 3-legged vertices and single legged vertices to represent double derivatives, the matrices $C\gamma$ and either X or ψ. We will refer to diagrams containing single legged vertices as "tumor diagrams".

4 Finding invariant terms in the unitary gauge

To find invariant terms, we will begin by eliminating the σ-terms and tumors from the transformations. To eliminate σ-terms, we note that we must only look at fermion bilinears. Noting that fermion bilinears with no derivatives will generate a Goldstino mass term which we know is forbidden, we can construct the following bilinears at scale zero

$$i\overline{\psi}^{\alpha}\gamma^{i}_{\alpha\beta}\partial_{b}\psi^{\beta} = i\ \text{(diagram)} \equiv \text{(diagram)} \tag{4.1}$$
$$i\overline{\psi}^{\alpha}\gamma^{a}_{\alpha\beta}\partial_{b}\psi^{\beta} = i\ \text{(diagram)} \equiv \text{(diagram)} \ . \tag{4.2}$$

These eliminate the σ-terms which appear both in the variations of the γ vertex and the ∂ vertex with opposite signs as can be seen from (3.11), (3.12) and (3.13). We also define transformation terms

$$i\bar{\theta}^\alpha \gamma^i_{\alpha\beta} \partial_b \psi^\beta = i \quad \text{WWW} \bullet \text{oo} \partial \text{---} \quad \equiv \quad \text{WWW} \blacktriangleright \text{---} \tag{4.3}$$

$$i\bar{\theta}^\alpha \gamma^a_{\alpha\beta} \partial_b \psi^\beta = i \quad \text{---} \bullet \text{oo} \partial \text{---} \quad \equiv \quad \text{---} \blacktriangleright \text{---} \tag{4.4}$$

so that the transformations laws become

$$\tag{4.5}$$

$$+ X - \text{tumor} + \psi - \text{tumor}$$

$$\tag{4.6}$$

$$+ X - \text{tumor} + \psi - \text{tumor}$$

$$\tag{4.7}$$

$$+ X - \text{tumor} + \psi - \text{tumor} \,.$$

Note that the variations with a solid boson vertex are due to Lorentz transformations, and the variations with a solid fermion vertex are due to SUSY transformations. As in the bosonic case, we can dispose of tumors through integration by parts, at the price of enlarging the number of disconnected pieces of a term by 1, where the added disconnected piece for the X and ψ -tumors are

$$\tag{4.8}$$

correspondingly. We will use this fact to examine fully connected terms, ignoring tumors, and then reinstate the tumors to sum up terms with multiple disconnected pieces.

4.1 Scale 0

Since the vertices defined above all have scale zero and two legs, we can build scale zero invariants from them using rings, similarly to what we have seen in the bosonic case. As in the bosonic case, we will start with a single ring, and for each (non-tumor) term in its variation find a new ring which can cancel it, and then repeat this process with any new rings we find, until all terms are canceled. We will consider a general ring which has n

worldsheet edges ($n \geq 1$), and cut all of them. The possible terms we could have between worldsheet edges and their variations are

$$\delta \ \text{(diagram)} = - \ \text{(diagram)} - \ \text{(diagram)} + \tag{4.9}$$
$$+ \ \text{(diagram)} + \ \text{(diagram)} + \text{transposed}$$

$$\delta \ \text{(diagram)} = \ \text{(diagram)} + \ \text{(diagram)} + \tag{4.10}$$
$$- \ \text{(diagram)} - \ \text{(diagram)}$$

$$\delta \ \text{(diagram)} = \ \text{(diagram)} - \ \text{(diagram)} + \tag{4.11}$$
$$+ \ \text{(diagram)} + \ \text{(diagram)} +$$
$$+ \ \text{(diagram)} + \ \text{(diagram)} +$$
$$- \ \text{(diagram)} - \ \text{(diagram)}$$

$$\delta \ \text{(diagram)} = \ \text{(diagram)} + \ \text{(diagram)} + \tag{4.12}$$
$$+ \ \text{(diagram)} - \ \text{(diagram)}$$
$$+ \text{transposed}$$

we will separate these to variations which preserve n, and variations which take $n \to n+1$. The variations which preserve n are

$$\delta_n \ \text{(diagram)} = - \ \text{(diagram)} + \ \text{(diagram)} + \text{transposed} \tag{4.13}$$

$$\delta_n \ \text{(diagram)} = \ \text{(diagram)} - \ \text{(diagram)} \tag{4.14}$$

$$\delta_n \ \text{(diagram)} = \ \text{(diagram)} - \ \text{(diagram)} + \tag{4.15}$$
$$+ \ \text{(diagram)}$$

$$\delta_n \ \text{(diagram)} = \ \text{(diagram)} + \text{transposed} \tag{4.16}$$

we can cancel most of these by looking at the combination vertex

$$\text{(diagram)} \equiv \ \text{(diagram)} - \ \text{(diagram)} - \ \text{(diagram)} \tag{4.17}$$
$$- \ \text{(diagram)} - \ \text{(diagram)} + \ \text{(diagram)}$$

and the ring A_n which is just n combination vertices connected to a ring. The combination vertex leaves only the variations

$$\delta_n \ \text{(diagram)} = - \ \text{(diagram)} - \ \text{(diagram)} + \text{transposed}. \tag{4.18}$$

Now, looking at the variations that take $n \to n+1$ we have

$$\delta_{n+1} \quad \text{[diagram]} \quad = - \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad + \tag{4.19}$$

$$+\text{transposed}$$

$$\delta_{n+1} \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad - \quad \text{[diagram]} \tag{4.20}$$

$$\delta_{n+1} \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad +$$

$$+ \quad \text{[diagram]} \quad - \quad \text{[diagram]} \quad +$$

$$- \quad \text{[diagram]} \tag{4.21}$$

$$\delta_{n+1} \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad +$$

$$- \quad \text{[diagram]} \quad +\text{transposed} \tag{4.22}$$

these include almost all combinations of terms from (4.17) and transformations from (4.18), which means we can cancel most of the $n \to n+1$ transformations of rings with n terms using n preserving transformations of rings with $n+1$ terms, exactly as we did in the boson case, taking

$$(2n+2)\, a_{n+1} = -2na_n \quad \Rightarrow \quad a_n = (-1)^{n+1}\,\frac{1}{n}a_1 \tag{4.23}$$

as the coefficient of the ring A_n. Note that as in the boson case there is no need to cancel the n preserving transformations for A_1 since it is a total derivative. This leaves us with 2 yet to be canceled terms:

- A single combination not represented in the $n \to n+1$ transformations $\quad$ [diagram]

- A single $n \to n+1$ transformation which cannot be expressed as such a combination $\quad$ [diagram]

To fix the first problem, we take note that the only term that can produce this transformation is $\quad$ [diagram] $\quad$. No other term can cancel it. To avoid this problem we will exclude it completely, by adding a canceling term into the definition of the combination vertex

$$\text{[diagram]} \quad = \quad \text{[diagram]} \quad - \quad \text{[diagram]} \quad - \quad \text{[diagram]}$$

$$- \quad \text{[diagram]} \quad - \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad +\alpha \quad \text{[diagram]} \quad . \tag{4.24}$$

One can check that in order to cancel the $\quad$ [diagram] $\quad$ transformation from the $\quad$ [diagram] $\quad$ in A_n with that transformation from the combinations of $\quad$ [diagram] $\quad$ and $\quad$ [diagram] $\quad$ in A_{n+1}, taking into account their respectable coefficients, we should take $\alpha = 1$. We now need to also include the variations of $\quad$ [diagram] $\quad$ which are

$$\delta \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad +$$

$$- \quad \text{[diagram]} \quad - \quad \text{[diagram]} \quad . \tag{4.25}$$

The first 3 transformations here are automatically dropped since all terms including ─────◁────▷───── were excluded. We are left with the last transformation, but this is exactly the last transformation we could not cancel before, and it is now canceled! This means we have constructed an invariant using A_n rings in exactly the same way we have in the boson case, with the switch

$$\text{───◯\text{\textasciitilde}◯───} \quad \rightarrow \quad \text{───□───} \tag{4.26}$$

switching back from the diagrammatic notation, this means

$$\partial_a X^i \partial_b X^i \rightarrow \partial_a X^i \partial_b X^i - i\overline{\psi}\gamma_a \partial_b \psi - i\overline{\psi}\gamma_b \partial_a \psi - i\overline{\psi}\gamma^i \partial_a \psi \partial_b X^i - i\overline{\psi}\gamma^i \partial_b \psi \partial_a X^i +$$
$$- \left(\overline{\psi}\gamma^i \partial_a \psi\right) \overline{\psi}\gamma^i \partial_b \psi - \eta_{cd}\left(\overline{\psi}\gamma^c \partial_a \psi\right)\overline{\psi}\gamma^d \partial_b \psi \tag{4.27}$$

(where the spinor indices are contracted between adjacent $\overline{\psi}'s$ and ψ's), or alternatively

$$\partial_a X^\mu \partial_b X_\mu = \eta_{ab} + \partial_a X^i \partial_b X^i \rightarrow$$
$$\rightarrow \eta_{ab} + \partial_a X^i \partial_b X^i - i\overline{\psi}\gamma_a \partial_b \psi - i\overline{\psi}\gamma_b \partial_a \psi - i\overline{\psi}\gamma^i \partial_a \psi \partial_b X^i - i\overline{\psi}\gamma^i \partial_b \psi \partial_a X^i +$$
$$- \left(\overline{\psi}\gamma^i \partial_a \psi\right)\overline{\psi}\gamma^i \partial_b \psi - \eta_{cd}\left(\overline{\psi}\gamma^c \partial_a \psi\right)\overline{\psi}\gamma^d \partial_b \psi = \tag{4.28}$$
$$= \partial_a X^\mu \partial_b X_\mu - \partial_a X^\mu i\overline{\psi}\gamma_\mu \partial_b \psi - i\overline{\psi}\gamma^\mu \partial_a \psi \partial_b X_\mu - \left(\overline{\psi}\gamma^\mu \partial_a \psi\right)\overline{\psi}\gamma_\mu \partial_b \psi =$$
$$= \left(\partial_a X^\mu - i\overline{\psi}\gamma^\mu \partial_a \psi\right)\left(\partial_b X_\mu - i\overline{\psi}\gamma_\mu \partial_b \psi\right)$$

and we get the invariant scale zero action

$$S_0 = -c_0 \int d^2\xi \sqrt{-\det\left[\left(\partial_a X^\mu - i\overline{\psi}\gamma^\mu \partial_a \psi\right)\left(\partial_b X_\mu - i\overline{\psi}\gamma_\mu \partial_b \psi\right)\right]} = -c_0 \int d^2\xi \sqrt{-g} \tag{4.29}$$

which is exactly the Akulov-Volkov action with

$$g = \det g_{ab} \tag{4.30}$$
$$g_{ab} = \left(\partial_a X^\mu - i\overline{\psi}\gamma^\mu \partial_a \psi\right)\left(\partial_b X_\mu - i\overline{\psi}\gamma_\mu \partial_b \psi\right) = \eta_{ab} + h_{ab} \tag{4.31}$$
$$h_{ab} = \partial_a X^i \partial_b X^i - i\overline{\psi}\gamma_a \partial_b \psi - i\overline{\psi}\gamma_b \partial_a \psi - i\overline{\psi}\gamma^i \partial_a \psi \partial_b X^i - i\overline{\psi}\gamma^i \partial_b \psi \partial_a X^i +$$
$$- \left(\overline{\psi}\gamma^i \partial_a \psi\right)\overline{\psi}\gamma^i \partial_b \psi - \eta_{cd}\left(\overline{\psi}\gamma^c \partial_a \psi\right)\overline{\psi}\gamma^d \partial_b \psi \tag{4.32}$$

and we have g^{ab} the matrix inverse of g_{ab}

$$g^{ab} = \eta^{ab} - \eta^{ac} h_{cd}\eta^{db} + \eta^{ac} h_{cd}\eta^{de} h_{ef}\eta^{fb} - \dots \tag{4.33}$$

We can see that this method is exhaustive since up to the overall c_0 it fixes the coefficients of all possible terms.

4.2 Scale 1

In order to find a scale one invariant action, we first list all possible independent scale one vertices, which are

$$\partial_a \partial_b X^i, \ \overline{\psi}\gamma^i \partial_a \partial_b \psi, \ \overline{\psi}\gamma^c \partial_a \partial_b \psi, \ \partial_a \overline{\psi}\partial_b \psi. \tag{4.34}$$

Since we can only include one such vertex in our action, all 3-legged vertices are excluded, and the only one we can use is $\partial_a \overline{\psi}\partial_b \psi$ in ring topology, where all other terms are scale zero. However this vertex is antisymmetric in the indices (a, b), while the rest of the ring is symmetric, and so the scale 1 action is dropped.

4.3 Scale 2

There are several ways to construct scale two invariants: either with two scale one vertices, or with a single scale two vertex. The scale one vertices are listed in (4.34). and we can either use 3-leeged vertices in "Θ" or "dumbbell" topologies as shown below, or two copies of the 2-legged vertex $\partial_a \overline{\psi} \partial_b \psi$ in a ring topology.

Θ topology   Dumbbell topology .

The possible independent scale two vertices are

$$\partial_a \partial_b \partial_c X^i,\ \partial_a \overline{\psi} \partial_b \partial_c \psi,\ \overline{\psi} \gamma^i \partial_a \partial_b \partial_c \psi,\ \overline{\psi} \gamma^d \partial_a \partial_b \partial_c \psi. \tag{4.35}$$

Excluding the 3-legged vertex $\partial_a \overline{\psi} \partial_b \partial_c \psi$ we are left with three 4-legged vertices which can be used in an "8" topology.

4.3.1 Ring topology

A ring topology invariant can be obtained by placing two $\partial_a \overline{\psi} \partial_b \psi$ vertices in a ring. We will first give this vertex a diagrammatic representation

$$i\partial_a \overline{\psi} \partial_b \psi = \quad \text{} \quad . \tag{4.36}$$

The variation for this vertex is

$$+ \text{ tumors}.$$

Looking back at our calculation for the scale zero term, we see immediately that this vertex has no n preserving variations, and that its $n \to n+1$ variations fit right into our cancellation scheme for vertices, without allowing for the excluded . Looking at the ring $B_{k\ell}$, $(k + \ell = n - 2 \geq 0)$ which is

$$B_{k\ell} = \partial^a \overline{\psi} \partial_b \psi \left(h^k \right)^b_{\ c} \partial^c \overline{\psi} \partial_d \psi \left(h^\ell \right)^d_{\ a} \tag{4.38}$$

where $\left(h^k \right)^b_{\ c}$ is the matrix $h^b_c = \eta^{ba} h_{ac}$ taken to the kth power, we see that its variation can be canceled by the variations of $B_{k+1,\ell}$ and $B_{k,\ell+1}$. Taking the sum so that the variations

cancel we have

$$\sum_{k,\ell=0}^{\infty} (-1)^{k+\ell} \, \partial^a \overline{\psi} \partial_b \psi \left(h^k \right)^b_{\ c} \partial^c \overline{\psi} \partial_d \psi \left(h^\ell \right)^d_{\ a} =$$
$$= \partial^a \overline{\psi} \partial_b \psi \sum_{k=0}^{\infty} \left((-h)^k \right)^b_{\ c} \partial^c \overline{\psi} \partial_d \psi \sum_{\ell=0}^{\infty} \left((-h)^\ell \right)^d_{\ a} = \qquad (4.39)$$
$$= \partial^a \overline{\psi} \partial_b \psi \left((\mathbb{1} + h)^{-1} \right)^b_{\ c} \partial^c \overline{\psi} \partial_d \psi \left((\mathbb{1} + h)^{-1} \right)^d_{\ a} =$$
$$= \partial_a \overline{\psi} \partial_b \psi g^{bc} \partial_c \overline{\psi} \partial_d \psi g^{da} \, .$$

Taking tumors into account means that this term must be multiplied by the scale zero invariant $\sqrt{-g}$, and we get the scale two ring invariant

$$\mathcal{L}_2^{\mathrm{ring}} = c_2 \sqrt{-g} \, \partial_a \overline{\psi} \partial_b \psi \partial_c \overline{\psi} \partial_d \psi \, g^{bc} g^{da} \qquad (4.40)$$

we can generalize this in a similar manner to what Gliozzi and Meineri did to obtain high scaling invariants. To do so, we define *seed* graphs, which are minimal connected graphs in the sense that they cannot be reduced to a non-trivial graph by erasing scale zero chains and connecting their edges together, and have no fermion vertices which can be reduced to boson vertices with the same scale and leg structure. The scale two ring topology seed graph is

$$(4.41)$$

Given a seed graph, for each worldsheet edge η^{ab}, if the vertices connected to it have a variation structure like the one in (4.37), we can replace $\eta^{ab} \to g^{ab}$ to eliminate the non-tumor variations, and multiply by $\sqrt{-g}$ to eliminate tumors.

4.3.2 Θ and dumbbell topologies

These invariants are created using two 3-legged scale one vertices, which are $\partial_a \partial_b X^i$, $i \overline{\psi} \gamma^i \partial_a \partial_b \psi$, $i \overline{\psi} \gamma^c \partial_a \partial_b \psi$. We will use the following graphical representations for them

$$(4.42)$$

and their variations

$$(4.43)$$

$$\delta \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} - \text{(diagram)} - \text{(diagram)} + $$

$$- \text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \text{tumors} \tag{4.44}$$

$$\delta \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} - \text{(diagram)} - \text{(diagram)} + $$

$$- \text{(diagram)} - \text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \text{tumors}. \tag{4.45}$$

Where in the last vertex it's important to note that we use an isosceles triangle, so that the two derivative legs behave differently than the γ^a leg. Looking at the first line in every variation, we see the exact same structure we saw for the ring topology, so these variations can be eliminated by replacing $\eta^{ab} \to g^{ab}$ on seed graphs in which worldsheet legs of scale one vertices are connected together. The seed graphs we can construct using these vertices are

- Θ topology

$$\tag{4.46}$$

- Dumbbell topology

$$\tag{4.47}$$
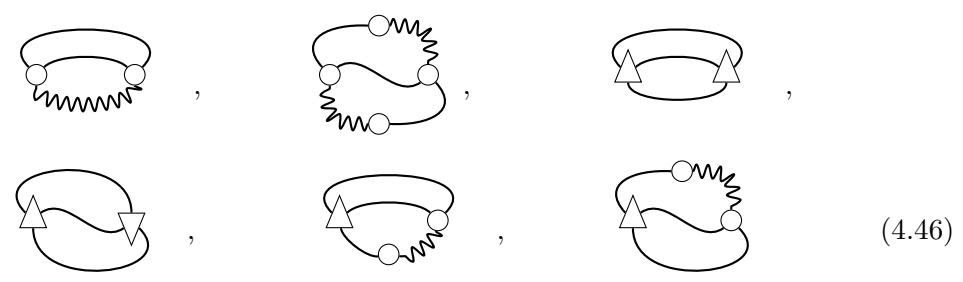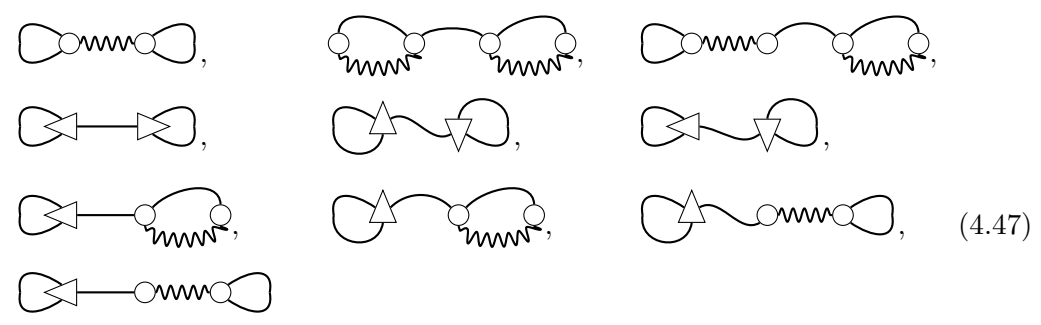
Most of these seed graphs can be easily eliminated as candidates for invariants, since they

have variations which cannot be canceled, such as

In fact, it is easy to check that each of these graphs with either a single scale-0 boson not connected to the γ leg of a scale-2 fermion, or a scale-2 fermion not connected by its γ leg to a scale-0 boson, will have such a variation. This leaves us with the following seed graphs

$$
\text{(4.48)}
$$
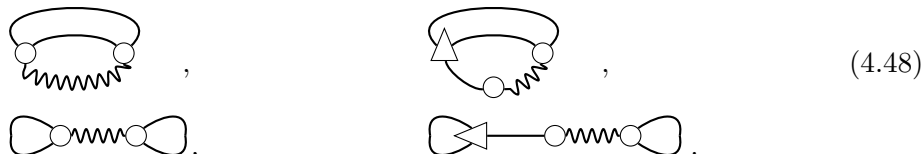
However, these seed graphs are not independent, since both graphs in each line of (4.48) appear in the same cancellation flow. We can define a new scale-2 fermion-boson vertex

$$
\text{(4.49)}
$$

and include such vertices as fermion vertices in our definition for seed graphs, leaving us with exactly 2 seed graphs we can create an invariant out of

$$
\text{(4.50)}
$$
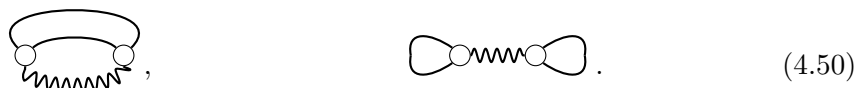
These are in fact exactly the same seed graphs GM used to create scale-2 invariants in the bosonic case. To make the invariants in our case, we need to eliminate all the variations. The variations on the derivative edges are eliminated by the exchange of the embedded metric. To eliminate the variations on the transverse edge, we look at all the legal chains between the scale one vertex and the first transverse edge. Those legal chains have to be made out of $\longrightarrow\!\triangleright\!\longrightarrow$ terms, which must be directed into the scale one vertex, since back-to-back fermions connected by a worldsheet vertex are forbidden. So we have the following chains

$$
n \geq 0 \,. \qquad \text{(4.51)}
$$
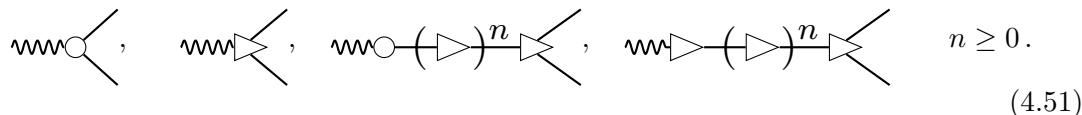
We can take a sum of these so that all variations which appear between the scale one vertex and the first transverse edge are canceled, by defining the scale one combination vertex
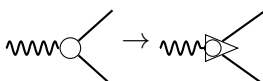
$$
= \partial_a \partial_b X^i - i\overline{\psi}\gamma^i \partial_a \partial_b \psi - \left(\partial_c X^i - i\overline{\psi}\gamma^i \partial_c \psi\right)\left(\delta_d^c + i\overline{\psi}\gamma^c \partial_d \psi\right)^{-1} i\overline{\psi}\gamma^d \partial_a \partial_b \psi \equiv C_{ab}^i . \qquad \text{(4.52)}
$$

Replacing the scale one boson vertices with these combination vertices, we get that there are still variations on the transverse edge. To eliminate these we sum up all the ways the two combination vertices can connect. This is either directly through the transverse edge, or the transverse edge can be terminated on both ends either by a boson vertex or a fermion vertex, and these connect through a scale zero chain with worldsheet edges on each side. This is equivalent to making a similar replacement to the one GM do for the boson case

$$\delta^{ij} \to t^{ij} = \delta^{ij} - \left(\partial_a X^i - i\overline{\psi}\gamma^i \partial_a \psi \right) g^{ab} \left(\partial_b X^j - i\overline{\psi}\gamma^j \partial_b \psi \right) . \tag{4.53}$$

Thus eliminating the rest of the variations. To conclude, scale two invariants are obtained by looking at scale two seed graphs and performing the following moves

1. Replacing $\eta^{ab} \to g^{ab}$ on worldsheet edges

2. Replacing the scale one vertices

3. Replacing $\delta^{ij} \to t^{ij}$ on transverse edges

Where in these topologies we get the invariants

$$I_1 = \sqrt{-g} C^i_{ab} t^{ij} C^j_{cd} g^{ac} g^{bd} \tag{4.54}$$

$$I_2 = \sqrt{-g} C^i_{ab} t^{ij} C^j_{cd} g^{ab} g^{cd} . \tag{4.55}$$

Here, as in the bosonic case, I_2 is proportional to the EOM up to $\mathcal{O}\left(\partial^6\right)$. We have not checked if $I_1 - I_2$ is a total derivative to any order, but at least up to $\mathcal{O}\left(\partial^6\right)$ it is. Thus, there are no new corrections to the fermionic string energy levels up to this order.

4.3.3   8 topology

Creating an 8 topology invariant requires using a single 4-legged scale two vertex, which can be either $\partial_a \partial_b \partial_c X^i$, $\overline{\psi}\gamma^i \partial_a \partial_b \partial_c \psi$ or $\overline{\psi}\gamma^d \partial_a \partial_b \partial_c \psi$. The available seed graphs are

where the third, with the vertex $\overline{\psi}\gamma^i \partial_a \partial_b \partial_c \psi$, is included in the chain obtained from $\partial_a \partial_b \partial_c X^i$. Both graphs have variations which cannot be canceled

so an 8 topology invariant is excluded.

4.4 Higher scaling

The number of invariants proliferates rapidly as the scaling increases, since the number of different vertices available and the number of different topologies both increase. We can generalize the vertices we have introduce into three types of higher scaling vertices

- $\partial^n X^i$ at scaling $n-1$ and with $n+1$ legs

- $\overline{\psi}\gamma^{a/i}\partial^n\psi$ at scaling $n-1$ and with $n+1$ legs

- $\partial^m\overline{\psi}\partial^n\psi$ at scaling $m+n-1$ and with $m+n$ legs

From their transformation laws, it is easy to see that the first two types are highly related. In fact, for any bosonic invariant that we can create using just the first type of vertices, we can generate a corresponding supersymmetric invariant by replacing the high scaling bosonic vertices with the appropriate combination vertices like we did in the last section

$$\partial^n_{a_1\cdots a_n}X^i \rightarrow \partial^n_{a_1\cdots a_n}X^i - i\overline{\psi}\gamma^i\partial^n_{a_1\cdots a_n}\psi - \left(\partial_c X^i - i\overline{\psi}\gamma^i\partial_c\psi\right)\left(\delta^c_d + i\overline{\psi}\gamma^c\partial_d\psi\right)^{-1}i\overline{\psi}\gamma^d\partial^n_{a_1\cdots a_n}\psi \equiv C^i_{a_1\cdots a_n}. \tag{4.58}$$

GM formulate the generation of higher scaling bosonic invariants by looking at the variation of the scaling $n-1$ vertex $(n>1)$

$$\delta\left(\partial^n_{a_1\cdots a_n}X^i\right) = -\epsilon^{bj}\left(\partial_b X^i\partial^n_{a_1\cdots a_n}X^j + \sum_k \partial_{a_k}X^j\partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}X^i + \right.$$
$$\left. + \sum_{k,l}\partial^2_{a_k a_l}X^j\partial^{n-1}_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_{l-1}a_{l+1}\cdots\cdots a_n}X^i + \ldots\right). \tag{4.59}$$

Where the first two terms add a scale zero vertex on each on the legs, and are canceled by the moves $\eta^{ab}\rightarrow g^{ab}, \delta^{ij}\rightarrow t^{ij}$ as we have seen in the previous section. The third term has a scale $n-2$ vertex connected to a scale 1 vertex so it can only be canceled by terms containing such vertices, the fourth has a scale $n-3$ vertex connected to a scale 3 vertex and so on. We can cancel these terms by defining a sort of covariant derivative. GM define this for the scale 2 term

$$\partial^3_{abc}X^i \rightarrow \nabla^3_{abc}X^i = \partial^3_{abc}X^i - \left(\partial^2_{ab}X^j\partial_d X^j\partial^2_{ec}X^i g^{de} + \text{cyclic permutations of } abc\right) \tag{4.60}$$

so that

$$\delta\left(\nabla^3_{abc}X^i\right) = -\epsilon^{bj}\left(\partial_b X^i\partial^n_{a_1\cdots a_n}X^j + \sum_k \partial_{a_k}X^j\partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}X^i\right) \tag{4.61}$$

which can be generalized to the n-th derivative with

$$\nabla^n_{a_1\cdots a_n}X^i = \partial^n_{a_1\cdots a_n}X^i - \left(\partial^{n-1}_{a_1\cdots a_{n-1}}X^j\partial_b X^j\partial^2_{ca_n}X^i g^{bc} + \text{cyclic permutations of } a_1\ldots a_n\right) +$$
$$- \left(\partial^{n-2}_{a_1\cdots a_{n-2}}X^j\partial_b X^j\partial^3_{ca_{n-1}a_n}X^i g^{bc} + \text{cyclic permutations of } a_1\ldots a_n\right) + \ldots \tag{4.62}$$

We can generalize this for the supersymmetric case by noting that

$$\delta\partial^n_{a_1\cdots a_n}X^i = -\epsilon^{bj}\left(\partial_b X^i \partial^n_{a_1\cdots a_n}X^j + \sum_k \partial_{a_k}X^j \partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}X^i\right.$$

$$+\sum_{k,l}\partial^2_{a_k a_l}X^j\partial^{n-1}_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_{l-1}a_{l+1}\cdots\cdots a_n}X^i + \ldots\Bigg)$$

$$+ i\overline{\theta}\left(\gamma^i\partial^n_{a_1\cdots a_n}\psi + \gamma^b\sum_k \partial_{a_k}\psi\partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}X^i + \right. \qquad (4.63)$$

$$\left.+\gamma^b\sum_{k,l}\partial^2_{a_k a_l}\psi\partial^{n-1}_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_{l-1}a_{l+1}\cdots\cdots a_n}X^i + \ldots\right)$$

where the first two terms of the θ variation are canceled by the move $\partial^n_{a_1\cdots a_n}X^i \to C^i_{a_1\cdots a_n}$, and the following terms can be canceled by generalizing the above covariant derivative to the supersymmetric case such that

$$\nabla^n_{a_1\cdots a_n}X^i = \partial^n_{a_1\cdots a_n}X^i - \left(\partial^{n-1}_{a_1\cdots a_{n-1}}X^j\partial_b X^j\partial^2_{ca_n}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right) +$$

$$-\left(\partial^{n-2}_{a_1\cdots a_{n-2}}X^j\partial_b X^j\partial^3_{ca_{n-1}a_n}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right)+\ldots$$

$$-\left(i\overline{\psi}\gamma_b\partial^2_{ca_n}\psi\partial^{n-2}_{a_1\cdots a_{n-2}}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right) + \qquad (4.64)$$

$$-\left(i\overline{\psi}\gamma_b\partial^3_{ca_{n-1}a_n}\psi\partial^{n-3}_{a_1\cdots a_{n-3}}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right)+\ldots$$

and similarly for derivatives acting on fermions where

$$\delta\left(i\overline{\psi}\gamma^i\partial^n_{a_1\cdots a_n}\psi\right) = -i\epsilon^{bj}\left(\overline{\psi}\gamma^i\partial_b\psi\partial^n_{a_1\cdots a_n}X^j + \sum_k \partial_{a_k}X^j\overline{\psi}\gamma^i\partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}\psi + \right.$$

$$\left.+\sum_{k,l}\partial^2_{a_k a_l}X^j\overline{\psi}\gamma^i\partial^{n-1}_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_{l-1}a_{l+1}\cdots\cdots a_n}\psi + \ldots\right) +$$

$$+ i\overline{\theta}\left(\gamma^i\partial^n_{a_1\cdots a_n}\psi + \gamma^b\sum_k \partial_{a_k}\psi i\overline{\psi}\gamma^i\partial^n_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_n}\psi + \right.$$

$$\left.+\gamma^b\sum_{k,l}\partial^2_{a_k a_l}\psi i\overline{\psi}\gamma^i\partial^{n-1}_{ba_1\cdots a_{k-1}a_{k+1}\cdots a_{l-1}a_{l+1}\cdots\cdots a_n}\psi + \ldots\right) \qquad (4.65)$$

and a similar expression when replacing the transverse index i with a worldsheet index c. We get

$$\nabla^n_{a_1\cdots a_n}\psi = \partial^n_{a_1\cdots a_n}\psi - \left(\partial^{n-1}_{ba_1\cdots a_{n-2}}\psi\partial_c X^i\partial^2_{a_{n-1}a_n}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right) +$$

$$-\left(\partial^{n-2}_{ba_1\cdots a_{n-3}}\psi\partial_c X^i\partial^3_{a_{n-2}a_{n-1}a_n}X^i g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right)+\ldots$$

$$-\left(\partial^{n-1}_{ba_1\cdots a_{n-2}}\psi i\overline{\psi}\gamma_c\partial^2_{a_{n-1}a_n}\psi g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right) + \qquad (4.66)$$

$$-\left(\partial^{n-2}_{ba_1\cdots a_{n-3}}\psi i\overline{\psi}\gamma_c\partial^3_{a_{n-2}a_{n-1}a_n}\psi g^{bc}+\text{cyclic permutations of } a_1\ldots a_n\right)+\ldots$$

So we can get invariants by taking any bosonic seed graph, and acting on it with the following moves

1. Replacing $\eta^{ab} \to g^{ab}$ on worldsheet edges,

2. Replacing the bosonic vertices with combination vertices for $n \geq 2$, $\partial^n_{a_1 \cdots a_n} X^i \to D^i_{a_1 \cdots a_n}$,

3. Replacing $\delta^{ij} \to t^{ij}$ on transverse edges,

where $D^i_{a_1 \cdots a_n}$ is a combination vertex with higher derivatives replaced with covariant derivatives

$$D^i_{a_1 \cdots a_n} = \nabla^n_{a_1 \cdots a_n} X^i - i\overline{\psi}\gamma^i \nabla^n_{a_1 \cdots a_n} \psi - \left(\partial_c X^i - i\overline{\psi}\gamma^i \partial_c \psi\right)\left(\delta^c_d + i\overline{\psi}\gamma^c \partial_d \psi\right)^{-1} i\overline{\psi}\gamma^d \nabla^n_{a_1 \cdots a_n} \psi. \tag{4.67}$$

GM generate two scale 4 invariants

$$I_3 = \sqrt{-g}\, t^{ij} t^{kl} \partial^2_{ab} X^i \partial^2_{cd} X^j \partial^2_{ef} X^k \partial^2_{gh} X^l g^{ha} g^{bc} g^{de} g^{fg} \tag{4.68}$$

$$I_4 = \sqrt{-g}\, \nabla^3_{abc} X^i \nabla^3_{efg} X^j t^{ij} g^{ae} g^{bf} g^{cg} \tag{4.69}$$

which we can use to generate the supersymmetric invariants

$$I_3 = \sqrt{-g}\, t^{ij} t^{kl} D^i_{ab} D^j_{cd} D^k_{ef} D^l_{gh} g^{ha} g^{bc} g^{de} g^{fg} \tag{4.70}$$

$$I_4 = \sqrt{-g}\, D^i_{abc} D^j_{efg} t^{ij} g^{ae} g^{bf} g^{cg} . \tag{4.71}$$

We are left with the term $\partial^m \overline{\psi} \partial^n \psi$, which is the only vertex which gives us non-trivial supersymmetric invariants. First we note that it is antisymmetric, which means any invariant we generate must have an even number of these vertices. Second, we note that we can use the above argument for the variation of $\partial^n \psi$ to show that given a seed graph which contains such vertices, the above moves are sufficient to generate an invariant, along with replacing $\partial^m \overline{\psi} \partial^n \psi \to \nabla^m \overline{\psi} \nabla^n \psi$. We will define a seed graph at scaling higher than zero as a graph containing only boson and $\partial^m \overline{\psi} \partial^n \psi$ vertices, which does not contain scale zero vertices. The procedure for generating invariants at scale n will then be

1. Draw all seed graphs at this scale

2. Perform the above moves to generate an invariant

We can then generate two non-trivial scale 4 invariants

$$I_5 = \sqrt{-g}\, \partial_a \overline{\psi} \partial^2_{bc} \psi \partial_d \overline{\psi} \partial^2_{ef} \psi g^{ad} g^{be} g^{cf} \tag{4.72}$$

$$I_6 = \sqrt{-g}\, \partial_a \overline{\psi} \partial_b \psi \partial^2_{cd} \overline{\psi} \partial^2_{ef} \psi g^{ad} g^{be} g^{cf} \tag{4.73}$$

as well as many higher scaling invariants. As in the scale zero case, this method is exhaustive since up to the overall multiplicative constant it fixes the coefficients of all possible terms.

5 Exhaustiveness of the seed terms

To show that our list of invariants is exhaustive, we will formulate prohibition rules on the form the seed terms are allowed to take. To do so, we first define the lowering and raising variations under symmetry generators Q and J, such that

$$\delta_Q \mathcal{L}_f = \mathcal{L}_{f-1}^< + \mathcal{L}_{f+1}^> \tag{5.1}$$

$$\delta_J \mathcal{L}_d = \mathcal{L}_{d-1}^< + \mathcal{L}_{d,d+1}^> \tag{5.2}$$

where f is the number of fermions and d is the number of derivatives in the term.

In the case of supercharges the lowering part $\mathcal{L}_{f-1}^<$ is obtained by removal of one bare fermion (fermion without any derivatives acting on it); in the Lorentz transformation case to obtain the lowering part $\mathcal{L}_{d-1}^<$ we need to erase one ∂X term from the initial $\mathcal{L}_d$. Let's call the bare fermion ψ and ∂X — the lowering factor. If and only if a seed term contains at least one lowering factor, its lowering variation isn't zero.

5.1 First prohibition rule

Claim. If the seed term has two or more lowering factors it cannot be made invariant.

Note that we cannot add any terms with a higher number of fermions or derivatives to cancel the lowering variation. Hence, the lowering variation should be either zero or a total derivative.

Proof. Let's start from the reverse. Assume we have the term with two lowering factors. Now, let's try to make its lowering variation a total derivative. For simplicity of notations let's assume that the two lowering factors are two bare fermions (the other two cases: one bare ψ and one ∂X or two ∂X, are equivalent to this one).

Consider the most general form of the term with two bare fermions, where we have explicitly emphasized two derivatives we're going to use to make the lowering variation a total derivative

$$\mathcal{L}_{\psi\psi} = \psi^\alpha \psi^\beta \partial_a f \partial_b g h, \tag{5.3}$$

where f, g and h are any combinations of the fields and their derivatives. Consider its lowering variations under Q^α and Q^β

$$\delta_{Q^\alpha}^< \mathcal{L}_{\psi\psi} = \psi^\beta \partial_a f \partial_b g h,$$
$$\delta_{Q^\beta}^< \mathcal{L}_{\psi\psi} = \psi^\alpha \partial_a f \partial_b g h. \tag{5.4}$$

Let's make the first variation $\delta_{Q^\alpha}^< \mathcal{L}_{\psi\psi}$ a total derivative $\partial_a \left(\psi^\beta f \partial_b g h \right)$. To do so we need to add to the initial term $\mathcal{L}_{\psi\psi}$ other terms

$$\mathcal{L}_a^\alpha = \psi^\alpha (\partial_a \psi^\beta f \partial_b g h + \psi^\beta f \partial_{ab}^2 g h + \psi^\beta f \partial_b g \partial_a h). \tag{5.5}$$

Generalizing this, we can make the variation $\delta_{Q^\alpha}^< \mathcal{L}_{\psi\psi}$ or $\delta_{Q^\beta}^< \mathcal{L}_{\psi\psi}$ a total derivative with respect to ∂_a or ∂_b by adding one of the four terms $\mathcal{L}_{a(b)}^{\alpha(\beta)}$ to $\mathcal{L}_{\psi\psi}$, such that

$$\delta_{Q^\alpha}^< \left(\mathcal{L}_{\psi\psi} + \mathcal{L}_{a(b)}^\alpha \right) = \partial_{a(b)} N_{a(b)}^\alpha, \qquad \delta_{Q^\beta}^< \left(\mathcal{L}_{\psi\psi} + \mathcal{L}_{a(b)}^\beta \right) = \partial_{a(b)} N_{a(b)}^\beta \tag{5.6}$$

where $N_a^\alpha = \psi^\beta f \partial_b g h$ is $\mathcal{L}_{\psi\psi}$ without ψ^α and ∂_a.

It is crucial that we satisfy both of the variations simultaneously. Naively, one would add to $\mathcal{L}_{\psi\psi}$ one of the combinations $\mathcal{L}_a^\alpha + \mathcal{L}_a^\beta$ or $\mathcal{L}_a^\alpha + \mathcal{L}_b^\beta$ or $\mathcal{L}_b^\alpha + \mathcal{L}_a^\beta$ or $\mathcal{L}_b^\alpha + \mathcal{L}_b^\beta$. However, there is a problem that $\delta_{Q^\alpha}^< \mathcal{L}_{a(b)}^\beta \neq 0$ and $\delta_{Q^\beta}^< \mathcal{L}_{a(b)}^\alpha \neq 0$. Let us consider the combination $\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_a^\beta$. When we act on this term with $\delta_{Q^\alpha}^<$ we get

$$\delta_{Q^\alpha}^< (\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_a^\beta) = \partial_a N_a^\alpha + \delta_{Q^\alpha}^< \mathcal{L}_a^\beta \neq \partial_a F. \tag{5.7}$$

But we can notice that $\psi^\beta \delta_{Q^\beta}^< \mathcal{L}_a^\alpha = \psi^\alpha \delta_{Q^\alpha}^< \mathcal{L}_a^\beta$. So we can subtract it from the action and verify that the variation indeed gives a total derivative

$$\delta_{Q^\alpha}^< (\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_a^\beta - \psi^\alpha \delta_{Q^\alpha}^< \mathcal{L}_a^\beta) = \partial_a N_a^\alpha + \delta_{Q^\alpha}^< \mathcal{L}_a^\beta - \delta_{Q^\alpha}^< \mathcal{L}_a^\beta = \partial_a N_a^\alpha, \tag{5.8}$$

$$\delta_{Q^\beta}^< (\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_a^\beta - \psi^\beta \delta_{Q^\beta}^< \mathcal{L}_a^\alpha) = \partial_a N_a^\beta + \delta_{Q^\beta}^< \mathcal{L}_a^\alpha - \delta_{Q^\beta}^< \mathcal{L}_a^\alpha = \partial_a N_a^\beta. \tag{5.9}$$

So, both of the variations $\delta_{Q^\alpha}^<$ and $\delta_{Q^\beta}^<$ of $\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_a^\beta - \psi^\alpha \delta_{Q^\alpha}^< \mathcal{L}_a^\beta$ are total derivatives. However, this term by itself is a total derivative. Also, $\mathcal{L}_{\psi\psi} + \mathcal{L}_a^\alpha + \mathcal{L}_b^\beta$ and $\mathcal{L}_{\psi\psi} + \mathcal{L}_b^\alpha + \mathcal{L}_a^\beta$ don't work since $\psi^\beta \delta_{Q^\beta}^< \mathcal{L}_a^\alpha \neq \psi^\alpha \delta_{Q^\alpha}^< \mathcal{L}_b^\beta$. This concludes the proof.

This prohibition rule applied to two bare ψ's illuminates the fact that Goldstinos cannot have a mass term.

5.2 Second prohibition rule

Claim. Any seed term which contains a factor of $\psi_\alpha \partial_a \psi_\alpha \sim \psi \sigma^{bi} \partial_a \psi$ cannot generate an invariant chain.

Proof. The most general form of such terms is

$$\mathcal{L}_{\psi \partial \psi} = \psi_\alpha \partial_a \psi_\alpha \partial_b h f \tag{5.10}$$

where f should not contain bare ψ_α. Applying Q_α to this term we find in the leading order

$$\delta_{Q_\alpha}^< \mathcal{L}_{\psi \partial \psi} = \partial_a \psi_\alpha \partial_b h f. \tag{5.11}$$

We can not make this variation a total derivative with respect to ∂_a. Because then we should add $\psi_\alpha \psi_\alpha (\partial_{ab}^2 h f + \partial_a h \partial_b f)$ which is identically zero. We can try to make this variation a total derivative with respect to ∂_b, $\partial_b (\partial_a \psi_\alpha h f)$ by adding

$$\psi_\alpha \partial_a \psi_\alpha \partial_b h f + \psi_\alpha \partial_{ab}^2 \psi_\alpha h f + \psi_\alpha \partial_a \psi_\alpha h \partial_b f. \tag{5.12}$$

This expression can be rewritten after integration by parts of the middle term and then switching the order of fermions as:

$$\partial_a \psi_\alpha \partial_b \psi_\alpha h f. \tag{5.13}$$

This term is either zero, if $a = b$, or is proportional to the EOM (2.29), if $a \neq b$, since for any value of $\alpha = 1, \dot{1}, 2, \dot{2}$ one of the terms $\partial_a \psi_\alpha$ or $\partial_a \psi_\alpha$ will be proportional to the EOM. This concludes the proof.

5.3 Exhausting seeds up to scale 2

We first write all possible irreducible terms up to scale 2 with up to 3 indices

$$\text{Scale } -1: \ \bar{\psi}\psi, \ \bar{\psi}\gamma_\mu\psi, \ \bar{\psi}\sigma^{\mu\nu}\psi \tag{5.14}$$

$$\text{Scale } 0: \ \bar{\psi}\partial_a\psi, \ \bar{\psi}\gamma_\mu\partial_a\psi, \ \bar{\psi}\sigma^{\mu\nu}\partial_a\psi, \ \partial_a X^i \tag{5.15}$$

$$\text{Scale } 1: \ \partial_a\bar{\psi}\partial_b\psi, \ \partial_a\bar{\psi}\gamma^\mu\partial_b\psi, \ \partial_a\bar{\psi}\sigma^{\mu\nu}\partial_b\psi, \ \partial_{ab}X^i \tag{5.16}$$

$$\text{Scale } 2: \ \partial_a\bar{\psi}\partial_{bc}\psi \,. \tag{5.17}$$

Where we have ignored terms which can be eliminated using integration by parts. For example, on scale 1 we can write the term $\bar{\psi}\partial_a\partial_b\psi$, but when we consider $\psi\partial_a\partial_b\psi F$ for any F, we can integrate by parts to get $-\partial_a\bar{\psi}\partial_b\psi F - \bar{\psi}\partial_b\psi\partial_a F$, so it is enough to consider the scale 1 term $\partial_a\bar{\psi}\partial_b\psi$ and scale 0 term $\psi\partial_b\psi$.

Any possible term can be obtained by multiplying some combination of irreducible terms. Note that we are uninterested in purely bosonic terms, since they were considered in previous papers and it was shown that the first allowed term appears at higher orders, and that all seeds containing a scale -1 irreducible term are eliminated by the first prohibition rule.

Moreover, the second prohibition rule tells us that terms $\bar{\psi}\sigma^{ai}\partial_b\psi \sim \psi_\alpha\partial\psi_\alpha$ are prohibited. So we should use only $\bar{\psi}\sigma^{ab}\partial_c\psi$ or $\bar{\psi}\sigma^{ij}\partial_c\psi$ instead of $\bar{\psi}\sigma^{\mu\nu}\partial_c\psi$. Then immediately we can forget about $\bar{\psi}\sigma^{ij}\partial_c\psi$ because to contract transverse indices i and j we will need to go higher orders. Finally, we ignore terms which are proportional to the EOM and can be eliminated by field redefinitions

$$\gamma^a\partial_a\psi = 0 \quad \partial^2\psi = 0 \tag{5.18}$$

$$\partial^2 X_i = 0 \tag{5.19}$$

and for the fermions these can be written in light-cone coordinates as

$$\partial_-\psi_1 = \partial_-\bar{\psi}_1 = \partial_+\psi_2 = \partial_+\bar{\psi}_2 = 0 \,. \tag{5.20}$$

Scale 0 terms can be obtained by multiplying irreducible scale 0 terms or scale -1 and scale 1. However, as discussed above such terms with fermions are subjected to prohibition rules or proportional to the EOM. Scale 1 terms can be obtained by contracting the indices of a scale 1 irreducible term either with itself, or with the indices of scale 0 terms. This gives the following terms

$$(\bar{\psi}\partial_a\psi)(\partial^b\bar{\psi}\gamma^a\partial_b\psi), \qquad (\bar{\psi}\gamma^a\partial^b\psi)(\partial_a\bar{\psi}\partial_b\psi), \qquad (\bar{\psi}\gamma_a\partial_b\psi)(\partial^c\bar{\psi}\sigma^{ab}\partial_c\psi),$$
$$(\bar{\psi}\sigma^{ab}\partial^c\psi)(\partial_a\bar{\psi}\gamma_c\partial_b\psi), \qquad (\bar{\psi}\sigma^{ab}\partial^c\psi)(\partial_a\bar{\psi}\gamma_c\partial_b\psi) \,. \tag{5.21}$$

All of which are proportional to the EOM, as can be seen by writing them in light-cone coordinates.

Scale two terms can be obtained from contraction of scale 2 irreducible with scale 0 irreducible or as contraction of two scale 1 irreducible terms. We get

$$1 \times 1: (\partial_a \bar{\psi} \partial_b \psi)(\partial^a \bar{\psi} \partial^b \psi), \ (\partial_a \bar{\psi} \partial_b \psi)(\partial_c \bar{\psi} \sigma^{ab} \partial^c \psi), \ (\partial^a \bar{\psi} \gamma^b \partial^c \psi)(\partial_a \bar{\psi} \gamma_b \partial_c \psi),$$
$$(\partial^a \bar{\psi} \gamma^b \partial_a \psi)(\partial^c \bar{\psi} \gamma_b \partial_c \psi), \ (\partial^a \bar{\psi} \gamma^b \partial^c \psi)(\partial_b \bar{\psi} \gamma_c \partial_a \psi), \ (\partial^a \bar{\psi} \gamma^i \partial^b \psi)\partial_{ab} X_i,$$
$$(\partial^c \bar{\psi} \sigma^{ab} \partial_c \psi)(\partial^d \bar{\psi} \sigma_{ab} \partial_d \psi), \ (\partial_c \bar{\psi} \sigma^{ab} \partial_d \psi)(\partial_a \bar{\psi} \sigma^{cd} \partial_b \psi), \ (\partial^a \bar{\psi} \sigma^{bc} \partial^d \psi)(\partial_a \bar{\psi} \sigma^{bc} \partial_d \psi),$$
$$(\partial^a \bar{\psi} \sigma^{bc} \partial^d \psi)(\partial_b \bar{\psi} \sigma^{cd} \partial_a \psi) \tag{5.22}$$

$$2 \times 0: (\partial^a \bar{\psi} \partial_{ab} \psi)(\bar{\psi} \partial_b \psi), \ (\partial_a \bar{\psi} \partial_{bc} \psi)(\bar{\psi} \sigma^{ab} \partial_c \psi). \tag{5.23}$$

After some manipulation we see that among all of these terms only two are independent

$$(\partial_a \bar{\psi} \partial_b \psi)(\partial^a \bar{\psi} \partial^b \psi), \tag{5.24}$$

$$(\partial^a \bar{\psi} \gamma^i \partial^b \psi)\partial_{ab} X_i. \tag{5.25}$$

However using integration by parts, one can see that (5.25) is proportional to the EOM, leaving us with (5.24) as the only non-purely bosonic seed up to scale 2. This is the term we found in (4.40) above.

6 Energy correction of the $\partial\psi\partial\psi\partial\psi\partial\psi$ term

The most interesting lowest scale result we have arrived at in the analysis of the previous sections is the term

$$\mathcal{L}_2^{\text{ring}} = c_2 \sqrt{-g} \partial_a \overline{\psi} \partial_b \psi \partial_c \overline{\psi} \partial_d \psi g^{bc} g^{da}. \tag{6.1}$$

In order to make this result testable we would like to see how it affects the energy levels of the Akulov-Volkov string at large L. To do so we will consider this term in the static gauge, and in the lowest order in derivative expansion

$$\mathcal{L}_2^{\text{ring}} = c_2 \partial_a \overline{\psi} \partial_b \psi \partial_c \overline{\psi} \partial_d \psi \eta^{bc} \eta^{da} + \mathcal{O}\left(\partial^5\right) =$$
$$= 4 c_2 \partial_+ \overline{\psi}_{\dot{1}} \partial_- \overline{\psi}_{\dot{2}} \partial_+ \psi_1 \partial_- \psi_2 + \mathcal{O}\left(\partial^5\right). \tag{6.2}$$

We will treat this as a perturbation for the free part of the AV action

$$\mathcal{L}_{\text{AV,free}} = \frac{T}{2}\left(i\overline{\psi}_{\dot{2}}\partial_+\psi_2 + i\psi_2\partial_+\overline{\psi}_{\dot{2}} + i\overline{\psi}_{\dot{1}}\partial_-\psi_1 + i\psi_1\partial_-\overline{\psi}_{\dot{1}}\right) \tag{6.3}$$

$$\mathcal{L} = \mathcal{L}_{\text{AV,free}} + \mathcal{L}_2^{\text{ring}}. \tag{6.4}$$

Since the leading order perturbation is purely fermionic, the boson field is completely free and we can ignore it for this derivation. We define the conjugate momenta

$$\overline{\Pi} = \frac{\delta \mathcal{L}}{\delta\left(\partial_0 \psi\right)} = \begin{pmatrix} \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi_1)} \\ \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi_2)} \\ \frac{\delta \mathcal{L}}{\delta(\partial_0 \overline{\psi}_{\dot{2}})} \\ \frac{\delta \mathcal{L}}{\delta(-\partial_0 \overline{\psi}_{\dot{1}})} \end{pmatrix}^T = \begin{pmatrix} -\frac{1}{2}Ti\overline{\psi}_{\dot{1}} - 2c_2\partial_+\overline{\psi}_{\dot{1}}\partial_-\overline{\psi}_{\dot{2}}\partial_-\psi_2 \\ -\frac{1}{2}Ti\overline{\psi}_{\dot{2}} + 2c_2\partial_+\overline{\psi}_{\dot{1}}\partial_-\overline{\psi}_{\dot{2}}\partial_+\psi_1 \\ \frac{1}{2}Ti\psi_2 + 2c_2\partial_+\overline{\psi}_{\dot{1}}\partial_+\psi_1\partial_-\psi_2 \\ -\frac{1}{2}Ti\psi_1 + 2c_2\partial_-\overline{\psi}_{\dot{2}}\partial_+\psi_1\partial_-\psi_2 \end{pmatrix} \tag{6.5}$$

so that the Hamiltonian is

$$
\mathcal{H} = \int\limits_0^{2\pi R} d\sigma \left(\overline{\Pi} \partial_0 \psi - \mathcal{L} \right) =
$$

$$
= \int\limits_0^{2\pi R} d\sigma \left(\overline{\Pi} \partial_1 \psi \right) - \frac{1}{4} c_2 \int\limits_0^{2\pi R} d\sigma \left(\partial_1 \overline{\psi}_{\dot{1}} \partial_1 \overline{\psi}_{\dot{2}} \partial_1 \psi_1 \partial_1 \psi_2 + \mathcal{O}\left(\partial^5 \right) \right) =
$$

$$
= \mathcal{H}_{\text{free}} + \mathcal{H}_4 \tag{6.6}
$$

where we use $L = 2\pi R$ and have plugged in the equations of motion (2.29). We look at the Fourier expansion of the fields on a closed string in the NS sector at $\tau = 0$ (for the R sector take $n \in \mathbb{Z}$ instead of $r \in \mathbb{Z} + \frac{1}{2}$)

$$
\psi_1 \left(\sigma \right) = \sqrt{\frac{2}{\pi R T}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^1 e^{ir\frac{\sigma}{R}}, \qquad \overline{\Pi}_1 \left(\sigma \right) = -i \sqrt{\frac{T}{2\pi R}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^1 e^{ir\frac{\sigma}{R}} \tag{6.7}
$$

$$
\psi_2 \left(\sigma \right) = \sqrt{\frac{2}{\pi R T}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^2 e^{ir\frac{\sigma}{R}}, \qquad \overline{\Pi}_2 \left(\sigma \right) = -i \sqrt{\frac{T}{2\pi R}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^2 e^{ir\frac{\sigma}{R}} \tag{6.8}
$$

$$
\overline{\psi}_{\dot{1}} \left(\sigma \right) = \sqrt{\frac{2}{\pi R T}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\dot{1}} e^{ir\frac{\sigma}{R}}, \qquad \Pi_1 \left(\sigma \right) = -i \sqrt{\frac{T}{2\pi R}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\dot{1}} e^{ir\frac{\sigma}{R}} \tag{6.9}
$$

$$
\overline{\psi}_{\dot{2}} \left(\sigma \right) = \sqrt{\frac{2}{\pi R T}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\dot{2}} e^{ir\frac{\sigma}{R}}, \qquad \Pi_2 \left(\sigma \right) = -i \sqrt{\frac{T}{2\pi R}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\dot{2}} e^{ir\frac{\sigma}{R}} \tag{6.10}
$$

where $\sigma \in [0, 2\pi R]$ and $\left\{ b_r^s, b_{r'}^{s'} \right\} = \delta_{r+r'} \delta_{ss'}$, $s = 1, 2, \dot{1}, \dot{2}$. Note that σ is really periodic only under $\sigma \to \sigma + 4\pi R$ in the NS-NS sector since

$$
\psi \left(\sigma \right) = -\psi \left(\sigma + 2\pi R \right). \tag{6.11}
$$

The commutator is

$$
\left\{ \psi_s \left(\sigma \right), \overline{\Pi}_{s'} \left(\sigma' \right) \right\} = -i \delta_{ss'} \left[2\delta \left(\frac{\sigma - \sigma'}{4\pi R} \right) - \delta \left(\frac{\sigma - \sigma'}{2\pi R} \right) \right] \tag{6.12}
$$

where $\delta \left(x \right)$ is non-zero for all $x \in \mathbb{Z}$. Plugging this into the free (fermion) Hamiltonian we get

$$
\mathcal{H}_{\text{free}} = \frac{2}{R} \sum_{s=1,2,\dot{1},\dot{2}} \left(\sum_{r \in \mathbb{N} + \frac{1}{2}} r \left(b_{-r}^s b_r^s - b_r^s b_{-r}^s \right) \right). \tag{6.13}
$$

This Hamiltonian is Weyl ordered, meaning that products of fermionic operators appear in the form

$$
\frac{1}{k!} \sum_{(p_1, \dots p_k) \in \text{perms}(k)} (-1)^{s(p_1, \dots p_k)} b_{r_{p_1}} \cdots b_{r_{p_k}} \tag{6.14}
$$

where $s(p_1, \ldots p_k)$ is the parity of the permutation $(p_1, \ldots p_k)$. We now take the normal ordering to get

$$\mathcal{H}_{\text{free}} = \frac{4}{R} \sum_{s=1,2,\dot{1},\dot{2}} \left(\sum_{r \in \mathbb{N}+\frac{1}{2}} r b^s_{-r} b^s_r - \frac{1}{48} \right) \tag{6.15}$$

where we used zeta function regularization to take sums of the form

$$\sum_{r=\frac{1}{2}}^{\infty} r^k = \sum_{n=1}^{\infty} \left(\frac{2n+1}{2} \right)^k = -\frac{2^k-1}{2^k} \zeta(-k) = \begin{cases} \frac{1}{24} & k=1 \\ 0 & k=2,4 \\ -\frac{7}{8*120} & k=3. \end{cases} \tag{6.16}$$

The perturbation Hamiltonian is

$$\mathcal{H}_4 = \frac{c_2}{4} \int_0^{2\pi R} d\sigma \partial_1 \overline{\psi}_{\dot{1}} \partial_1 \overline{\psi}_{\dot{2}} \partial_1 \psi_2 \partial_1 \psi_1 =$$

$$= \frac{c_2}{\pi^2 T^2 R^6} \int_0^{2\pi R} d\sigma \sum_{r_1,r_2,r_3,r_4} b^{\dot{1}}_{r_1} r_1 e^{ir_1 \frac{\sigma}{R}} b^{\dot{2}}_{r_2} r_2 e^{ir_2 \frac{\sigma}{R}} b^2_{r_3} r_3 e^{ir_3 \frac{\sigma}{R}} b^1_{r_4} r_4 e^{ir_4 \frac{\sigma}{R}} =$$

$$= \frac{c_2}{\pi^2 T^2 R^6} \sum_{r_1,r_2,r_3,r_4} r_1 r_2 r_3 r_4 b^{\dot{1}}_{r_1} b^{\dot{2}}_{r_2} b^2_{r_3} b^1_{r_4} \int_0^{2\pi R} d\sigma e^{-2\pi i(r_1+r_3+r_2+r_4)\frac{\sigma}{2\pi R}} =$$

$$= \frac{2c_2}{\pi T^2 R^5} \sum_{n \in \mathbb{Z}} \sum_{r,r'} r(r+n) r'(r'-n) b^{\dot{1}}_{-r} b^1_{r+n} b^{\dot{2}}_{-r'} b^2_{r'-n}. \tag{6.17}$$

This clearly annihilates the ground state. The simplest of its eigenstates with non-zero eigenvalues are

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(\left| 1^1_{\frac{1}{2}}, 1^2_{\frac{1}{2}} \right\rangle \pm \left| 1^{\dot{1}}_{\frac{1}{2}}, 1^{\dot{2}}_{\frac{1}{2}} \right\rangle \right) \tag{6.18}$$

since

$$\mathcal{H}_4 |\psi\rangle = \frac{2c_2}{\pi T^2 R^5} \sum_{n \in \mathbb{Z}} \sum_{r,r'} r(r+n) r'(r'-n) b^{\dot{1}}_{-r} b^1_{r+n} b^{\dot{2}}_{-r'} b^2_{r'-n} \frac{1}{\sqrt{2}} \left(\left| 1^1_{\frac{1}{2}}, 1^2_{\frac{1}{2}} \right\rangle \pm \left| 1^{\dot{1}}_{\frac{1}{2}}, 1^{\dot{2}}_{\frac{1}{2}} \right\rangle \right) =$$

$$= \frac{2c_2}{\pi T^2 R^5} \sum_{n \in \mathbb{Z}} \sum_{r,r'} r(r+n) r'(r'-n) b^{\dot{1}}_{-r} b^1_{r+n} b^{\dot{2}}_{-r'} b^2_{r'-n} \frac{1}{\sqrt{2}} \left(b^1_{-\frac{1}{2}} b^2_{-\frac{1}{2}} \pm b^{\dot{1}}_{-\frac{1}{2}} b^{\dot{2}}_{-\frac{1}{2}} \right) |0\rangle =$$

$$= \frac{c_2}{8\pi T^2 R^5} \frac{1}{\sqrt{2}} \left(b^{\dot{1}}_{-\frac{1}{2}} b^1_{\frac{1}{2}} b^{\dot{2}}_{-\frac{1}{2}} b^2_{\frac{1}{2}} b^1_{-\frac{1}{2}} b^2_{-\frac{1}{2}} \pm b^{\dot{1}}_{\frac{1}{2}} b^1_{-\frac{1}{2}} b^{\dot{2}}_{\frac{1}{2}} b^2_{-\frac{1}{2}} b^{\dot{1}}_{-\frac{1}{2}} b^{\dot{2}}_{-\frac{1}{2}} \right) |0\rangle =$$

$$= \frac{c_2}{8\pi T^2 R^5} \frac{1}{\sqrt{2}} \left(b^{\dot{1}}_{-\frac{1}{2}} b^{\dot{2}}_{-\frac{1}{2}} \pm b^1_{-\frac{1}{2}} b^2_{-\frac{1}{2}} \right) |0\rangle = \pm \frac{c_2}{8\pi T^2 R^5} |\psi\rangle$$

which gives rise to the energy correction

$$\Delta E_{\text{NS}} = \pm \frac{c_2}{8\pi T^2 R^5}. \tag{6.19}$$

Where other eigenstates will give rise to more complicated energy corrections at this order in $1/R$. We can repeat this analysis for the Ramond sector with $|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|1_1^1, 1_1^2\right\rangle \pm \left|1_1^{\dot{1}}, 1_1^{\dot{2}}\right\rangle\right)$ to get

$$\mathcal{H}_{\text{free}} = \frac{4}{R} \sum_{s=1,2,\dot{1},\dot{2}} \left(\sum_{n\in\mathbb{N}} n b_{-n}^s b_n^s + \frac{1}{24}\right) \tag{6.20}$$

$$\Delta E_R = \pm \frac{2c_2}{\pi T^2 R^5} \,. \tag{6.21}$$

7 Discussion and conclusions

In this work we present a general method to generate invariant actions for effective strings which break $D = 4$, $N = 1$ SUSY. We have shown that this method recreates known results, as well as producing new ones. Our method does not generate terms which are only invariant up to the equations of motion, which may be related to anomalies as in the bosonic case, but seems to be exhaustive otherwise. We can summarize our method as taking a seed term — a minimal term of Goldstone bosons and Goldstinos which is invariant under the non-broken $\text{ISO}(1,1) \times \text{SO}(D-2)$, and performing 4 simple moves: (a) replacing the Minkowski metric η^{ab} with the worldsheet metric g^{ab} as defined in (4.33), (b) replacing $n \geq 2$ scaling boson vertices with the combination vertex $C_{a_1 \cdots a_n}^i$ as defined in (4.58), (c) replacing $n \geq 3$ derivatives with covariant derivatives as defined in (4.63), (4.66) and (d) replacing the Euclidean transverse metric δ_{ij} with the transverse metric t_{ij} as defined in (4.53). This method clearly shows that every known bosonic invariant has a supersymmetric counterpart, as well as the existence of new supersymmetric invariants with no bosonic counterparts, the simplest of which we have termed $\mathcal{L}_2^{\text{ring}}$ and for which we have calculated its energy corrections. As directions for future research, we can consider repeating this analysis for the case in which only half of the SUSY generators are broken, such that the worldsheet theory has $\mathcal{N} = (0,2)$ supersymmetry, analyzing $\mathcal{L}_2^{\text{ring}}$ in the conformal gauge and verifying it does not contribute to the conformal anomaly and that no other terms are possible also in that approach, and generalizing this work to other dimensions and $\mathcal{N} > 1$.

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