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On the observer dependence of the Hilbert space near the horizon of black holes

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One of the pronounced characteristics of gravity, distinct from other interactions, is that there are no local observables which are independent of the choice of the spacetime coordinates. This property acquires crucial importance in the quantum domain in that the structure of the Hilbert space pertinent to different observers can be drastically different. Such intriguing phenomena as Hawking radiation and the Unruh effect are all rooted in this feature. As in these examples, the quantum effect due to such observer dependence is most conspicuous in the presence of an event horizon and there are still many questions to be clarified in such a situation. In this paper we perform a comprehensive and explicit study of the observer dependence of the quantum Hilbert space of a massless scalar field in the vicinity of the horizon of Schwarzschild black holes in four dimensions, both in the eternal (two-sided) case and in the physical (one-sided) case created by collapsing matter. Specifically, we compare and relate the Hilbert spaces of three types of observers, namely (i) the freely falling observer, (ii) the observer who stays at a fixed proper distance outside of the horizon, and (iii) the natural observer inside of the horizon analytically continued from outside. The concrete results we obtain have a number of important implications on black hole complementarity pertinent to the quantum equivalence principle and the related firewall phenomenon, on the number of degrees of freedom seen by each type of observer, and on the "thermal-type" spectrum of particles realized in a pure state.

Subject Index B22, B39

1. Introduction

A quantum black hole is a fascinating but as yet an abstruse object. Recent endeavors to identify it in a suitable class of conformal field theories (CFTs) in the AdS/CFT context [1–3] [4–7] or by an ingenious model such as the one proposed by Sachdev, Ye, and Kitaev [8–10] have seen only a glimpse of it, to say the most. Unfortunately, string theory, at the present stage of development, does not seem to give us a useful clue either. This difficulty is naturally expected since an object whose profile fluctuates by quantum self-interaction would be hard to capture. We must continue our struggle to find an effective means to characterize it more precisely.

Although the quantization of a black hole itself is still a formidable task, some analyses of quantum effects around a (semi-)classical black hole have been performed since a long time ago, and they have already uncovered various intriguing phenomena. Among them are the celebrated Hawking radiation [11] [12–14] and the closely related Unruh effect [15–17]. These effects revealed the non-trivial features of the quantization in curved spacetimes, in particular in those with event horizons.

At the same time, they brought out new puzzles of deep nature, such as the problem of information loss, the final fate of an evaporating black hole, and so on.

More recently, a further unexpected quantum effect in the black hole environment was argued to occur, namely that a freely falling observer encounters excitations of high-energy quanta, termed a "firewall," as he/she crosses the event horizon of a black hole [18,19] [20]. This is clearly at odds with the equivalence principle, which is one of the foundations of classical general relativity. An enormous number of papers have appeared since then, both for and against the assertion.[1] The various arguments presented have all been rather indirect, however, making use of the properties of the entanglement entropy, application of the no-cloning theorem, use of information-theoretic arguments, etc.

At the bottom of these phenomena lies the strong dependence of the quantization on the frame of observers, which is one of the most characteristic features of quantum gravity. This is particularly crucial when the spacetime of interest contains event horizons as seen by some observers, and leads to the notion of black hole complementarity [21].

The main aim of the present work is to investigate this observer dependence in some physically important situations as explicitly as possible to gain some firm and direct understanding of the phenomena rooted in this feature. For this purpose, we shall study the quantization of a massless scalar field in the vicinity of the horizon of the Schwarzschild black hole in four dimensions as perfomed by three typical observers. They are (i) the freely falling observer crossing the horizon, (ii) the stationary observer hovering at a fixed proper distance outside the horizon (i.e. one under constant acceleration), and (iii) the natural analytically continued observer inside the horizon.

Such an investigation, we believe, will be important for at least two reasons. One is that we will deal directly with the states of the scalar fields as seen by different observers and will not rely on any indirect arguments alluded to above. This makes the interpretation of the outcome of our study quite transparent (up to certain approximations that we must make for computation). Another role of our investigation is that the concrete result we obtain should serve as the properties of quantum fields in the background of a black hole, which should be compared, in the semi-classical regime, to the results to be obtained by other means of investigation, notably and hopefully by the AdS/CFT duality.[2] For some progress, and intriguing proposals in the related directions, see Refs. [22–29]. This is important since, as far as we are aware, there has not been a serious attempt to understand how the observer dependence is described in the context of AdS/CFT duality.

We will perform our study both for the case of a two-sided eternal Schwarzschild black hole and for that of a one-sided physical black hole modeled by a simple Vaidya metric produced by collapsing matter or radiation at the speed of light[3] [30–32]. What makes such an investigation feasible explicitly is the well-known fact that near the horizon of the Schwarzschild black hole (roughly within the Schwarzschild radius from the horizon; see Sect. 3.1 for a more precise estimate) there exists a coordinate frame in which the metric takes the form of the flat four-dimensional Minkowski spacetime $M^{1,3}$. Thus, one can make use of the knowledge of the quantization in the flat space for observers

[1] It is practically impossible to list all such papers on this subject. We refer the reader to those citing the basic papers, Refs. [18,19].

[2] As far as the vicinity of the horizon is concerned, the Schwarzschild black hole and the AdS black hole have the same structure.

[3] Actually, we shall make an infinitesimal regularization to make the trajectory of the matter slightly timelike in order to avoid a certain singularity.

corresponding to the various Rindler frames. As this will serve as the platform upon which we develop our picture and computational methods for the black hole cases, we will give, in Sect. 2, a review of this knowledge together with some further new information about the relations between the quantizations by the three aforementioned observers.

In making use of this flat space approximation to the near-horizon region of a black hole, an important care must be taken, however. Although the scalar field and its canonical conjugate momentum are locally well-approximated by those in the flat space for the region of our interest, and hence the canonical quantization can be performed without any problem, as we try to extract the physical modes that create and annihilate the quantum states, such local knowledge is not enough in general. This is because the notion of *a quantum state* requires the global information of the wave function. Technically, this is reflected in the fact that the orthogonality relation needed for the extraction of the mode is expressed by an integral over the entire spacelike surface at equal time, and depending on the region of interest such a surface may not be totally contained within the region where the flat space approximation is valid.

One such problem, which, however, can be easily dealt with, stems from the simple fact that the approximation by the four-dimensional flat space includes that of the spherical surface of the horizon by a tangential plane around a point. Clearly, since the physical modes of the scalar field should better be classified by the angular momentum, not by the linear momentum, we shall use $\mathbb{R}^{1,1} \times \mathbb{S}^2$ instead of $M^{1,3}$ as the more accurately approximated spacetime, where $\mathbb{R}^{1,1}$ stands for a portion of two-dimensional flat spacetime realized near the horizon and $\mathbb{S}^2$ is the sphere at the Schwarzschild radius. Various formulas reviewed and/or developed in Sect. 2 for $M^{1,3}$ can be readily transplanted to this case by replacing the plane waves by spherical harmonics.

The problem pointed out above of the extraction of the modes within the flat region is much more non-trivial in the near-horizon region of $\mathbb{R}^{1,1}$, since the flat region which extends to infinity is only along the direction of the light cone. The problem with this situation is that the use of the trajectory along the light cone leads to the quantization of a *chiral* boson, which is known to be notoriously complicated. In addition, such a trajectory is not connected by a Lorentz transformation to the trajectory of a general observer, which is timelike. This problem is particularly severe when we deal with the one-sided black hole produced by a massless shock wave, the effect of which will be treated by the imposition of an effective Dirichlet boundary condition on the scalar field along the trajectory of the shock wave. To solve this problem, we have made a careful regularization of taking the trajectory of the shock wave to be *slightly timelike*.[4] Then we are able to treat the quantization for the observers freely falling with arbitrary velocity by making a suitable Lorentz transformation. Such a proper anaysis has not been performed in the literature and this allowed us to obtain firm results for the question of major interest.

Although we cannot summarize here all the results on how the different observers see their quanta and how they are related, let us list two that are of obvious interest:

- Under the assumption that the metric of the interior of a physical Schwarzschild black hole, in particular one large enough so that the curvature at the horizon is very small, can be described by a Vaidya-type solution, our results indicate that the equivalence principle still holds quantum mechanically near the horizon of the black hole, and the freely falling observer finds no surprise as he/she goes through the horizon.

[4] Evidently this corresponds to the case of slightly massive falling matter, which is physically reasonable.

- For a physical (one-sided) black hole, the vacuum[5] $|\hat{0}\rangle_-$ for the freely falling observer is a pure state which is not the same as the usual Minkowski vacuum $|0\rangle_{\rm M}$. Nevertheless, the expectation value of the number operator for the observer in the frame of the right Rindler wedge in $|\hat{0}\rangle_-$ has an Unruh-like distribution, which contains a "thermal" factor together with another portion depending on the assumed interaction between the scalar field and the collapsing matter, effectively expressed as a boundary condition. This is in contrast to the case of the two-sided eternal black hole, where tracing out of the modes of the left Rindler wedge must be performed and the resultant mixed state density matrix produces the usual purely thermal form of the Unruh distribution. The effect for the physical black hole occuring in the pure state described above is essentially of the same origin as the Hawking radiation seen by the asymptotic observer, who is a Rindler observer.[6]

The plan of the rest of the paper is as follows: In Sect. 2, we begin by describing the quantization of a massless scalar field in four-dimensional flat Minkowski space from the point of view of various observers, and provide explicit relations between them. Although this section is mostly a review, we also derive some useful relations that have not been discussed in the literature. This includes the construction of the explicit unitary transformation between the Minkowski mode operators and those of the future Rindler wedge, and how the Poincaré algebra is realized in various wedges. Next, in Sect. 3, this knowledge about the quantization in flat spacetime will be utilized to discuss how the scalar field is quantized by various observers in the vicinity of the event horizon of a two-sided Schwarzschild black hole, which by a suitable choice of coordinates can be approximated by part of $\mathbb{R}^{1,1} \times \mathbb{S}^2$. In Sect. 4, we study the similar problem in the case of a Vaidya model of the physical one-sided black hole that is produced by a collapse of matter with infinitesimal mass, introduced as a regularization. The effect of this collapse is treated as an effective boundary condition on the scalar field along a slightly timelike trajectory of such a shock wave. Even though we focus on the flat region near the horizon, the quantum states, which depend on the global situation, show different properties as compared with the two-sided case studied in Sect. 3. In Sect. 5, we disucss the implications of the results obtained in the previous sections on some important questions, such as the quantum equivalence principle, the firewall phenomenon, and the Unruh effect near the horizon. Finally, in Sect. 6, after summarizing the results, we re-emphasize that the effect of the observer dependence of quantization is one of the most crucial characteristics of any theory of quantum gravity, and it should be seriously investigated, in particular, in the framework of the AdS/CFT approach. Several appendices are provided to give further useful details of the formulas and calculations discussed in the main text.

2. Quantization of a scalar field in the Rindler wedges and the degenerate Kasner universes

We begin by describing the quantization of a massless[7] scalar field in the four-dimensional Minkowski space from the standpoint of uniformly accelerated Rindler observers for the right and the left wedges $\mathrm{W_R}$ and $\mathrm{W_L}$, and their appropriate analytic continuations for the future and the past wedges $\mathrm{W_F}$ and $\mathrm{W_P}$, which can be identified as degenerate Kasner universes. In Fig. 1, we draw the trajectories of the corresponding observers and the equal-time slices in each wedge.

[5] The vacuum referred to here will be explained in Sect. 4.2.3.
[6] For related work, though in a different setting, see Ref. [33].
[7] The massive case can be treated in an entirely similar manner.

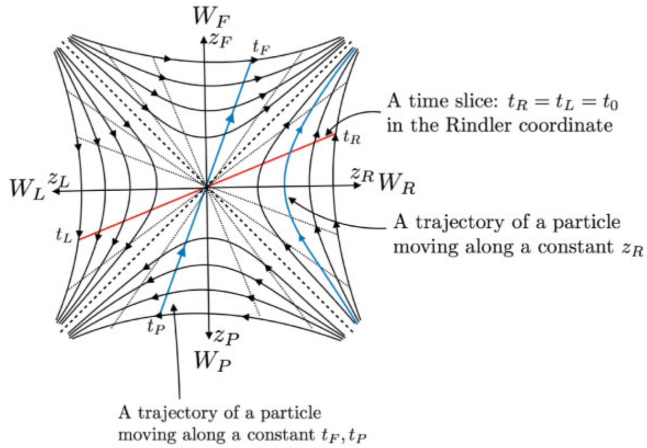

Fig. 1. Trajectories and equal-time slices of the Rindler observers in various wedges. The boundaries of the wedges W_R, W_F, W_L, and W_P are shown by dotted lines. The blue arrowed lines represent the trajectories of a particle, while the red line is a typical time slice at $t_R = t_L$ for W_R and W_L.

The subject of quantization by Rindler observers has a long history [17,34–36] and hence the content of this section is largely a review.[8] However, part of our exposition supplements the description in the existing literature by providing some clarifying details and new relations. The results of this section will serve as the foundation upon which to discuss the observer-dependent quantization around the horizon of Schwarzschild black holes, both eternal (two-sided) and physical (one-sided), as will be performed in Sect. 3.

2.1. Relation between Minkowski and Rindler coordinates

Before getting to the quantization of a scalar field, we need to describe the relationship between the Minkowski coordinates and the Rindler coordinates in various wedges.

The d-dimensional Minkowski metric is described in the usual Cartesian coordinate as

$$ds^2 = -(dt_M)^2 + (dx^1)^2 + \sum_{i=2}^{d-1}(dx^i)^2. \tag{2.1}$$

Since we will be mostly concerned with the first two coordinates, and the roles of the rest of the $d-2$ coordinates are essentially the same, hereafter we will deal with the four-dimensional case, i.e. $d = 4$.

As for the Rindler coordinates, we begin with the one in the right wedge W_R shown in Fig. 1. As is well known, it is related to the coordinates of the observer who is acclerated in the positive x^1 direction with a uniform acceleration. The trajectory of the observer in the (t_M, x^1) Minkowski plane with a value of acceleration κ (> 0) is given by

$$(x^1)^2 - (t_M)^2 = (1/\kappa)^2 = z_R^2. \tag{2.2}$$

Here, the symbol z_R is introduced as a variable, meaning that different values of z_R describe different trajectories. Thus the Rindler coordinate system is spanned by the proper time τ_R of the observer

[8] For a review article closely related to this section, see Ref. [37].

and the spatial coordinate z_R. The relation to the Minkowski coordinate is given by

$$t_M = z_R \sinh t_R, \qquad x^1 = z_R \cosh t_R \qquad (z_R > 0), \tag{2.3}$$

where we introduced for convenience the rescaled time t_R defined by

$$t_R \equiv \kappa \tau_R. \tag{2.4}$$

The metric in terms of these variables is

$$ds^2 = -z_R^2 dt_R^2 + dz_R^2 + \sum_{i=2}^{3} (dx^i)^2. \tag{2.5}$$

Note that $z_R = 0$ corresponds to the (Rindler) horizon, which consists of two-dimensional planes along the lightlike lines bounding the region W_R. It will often be convenient to use the following light cone variables:

$$x^{\pm} \equiv x^1 \pm t_M = z_R e^{\pm t_R}. \tag{2.6}$$

This shows that t_R is nothing but the *rapidity-like variable* and gets simply translated by the Lorentz boost in the x^1 direction.

The coordinates (t_L, z_L) in the left wedge W_L can be obtained in an entirely similar manner and are related to the Minkowski coordinates by

$$t_M = -z_L \sinh t_L, \qquad x^1 = -z_L \cosh t_L \qquad (z_L > 0). \tag{2.7}$$

The metric takes exactly the same form as Eq. (2.5), with the subscript R replaced by L. Note that as t_R increases from $-\infty$ to ∞, the Minkowski time t_M also increases, while when t_L increases from $-\infty$ to ∞, t_M decreases, as indicated by the arrows in Fig. 1.

Next, consider the future and the past wedges, W_F and W_P, which describe the interior of the Rindler horizon. The relation to the Minkowski coordinate for W_F is

$$t_M = z_F \cosh t_F, \qquad x^1 = z_F \sinh t_F \qquad (z_F > 0), \tag{2.8}$$

and the metric takes the form

$$ds^2 = -dz_F^2 + z_F^2 dt_F^2 + \sum_{i=2}^{3} (dx^i)^2. \tag{2.9}$$

This means that in W_F, z_F is the timelike and t_F is the spacelike coordinate. As in the case of W_R, the following light cone combinations are often useful:

$$x^{\pm} \equiv x^1 \pm t_M = \pm z_F e^{\pm t_F}. \tag{2.10}$$

Just like t_R, under a Lorentz transformation the variable t_F undergoes a simple shift.

This interchange of the timelike and the spacelike natures also occurs in the past wedge W_P. In an entirely similar manner, we have

$$t_M = -z_P \cosh t_P, \qquad x^1 = -z_P \sinh t_P \qquad (z_P > 0), \tag{2.11}$$

with the form of the metric identical to Eq. (2.9) with the subscript F $\to$ P.

In Sect. 3, where we discuss how the similar Rindler wedges for a flat space appear in the vicinity of the horizon of a Schwarzschild black hole, we will see that the z_R variable expresses the proper distance from the horizon in the outside region and is related to the radial variable r and the Schwarzschild radius $2M$ (where M is the mass of the black hole) by $z_R \simeq \sqrt{8M(r-2M)}$. Hence, as we go through the horizon from W_R into W_F, we must make an analytic continuation by choosing a branch for the square-root cut. Similarly, an analytic continuation connects W_F and W_L, and so on. This continuation process must be such that as we go once around all the wedges, we should come back to the same branch for W_R. A simple analysis for this consistency yields the following continuation rules, with a sign $\eta = \pm 1$ that can be chosen by convention, for the adjacent wedges:

$$t_F = t_R - i\frac{\pi}{2}\eta, \qquad z_F = e^{i(\pi/2)\eta} z_R, \tag{2.12}$$

$$t_L = t_F - i\frac{\pi}{2}\eta, \qquad z_L = e^{-i(\pi/2)\eta} z_F = z_R, \tag{2.13}$$

$$t_P = t_L + i\frac{\pi}{2}\eta = t_F, \qquad z_P = e^{-i(\pi/2)\eta} z_L, \tag{2.14}$$

$$t_R = t_P + i\frac{\pi}{2}\eta, \qquad z_R = e^{i(\pi/2)\eta} z_P, \tag{2.15}$$

$$z_R, z_F, z_L, z_P \geq 0. \tag{2.16}$$

One can easily check that these relations are compatible with the relations between the Minkowski variables and the Rindler wedge variables given above.

2.2. *Quantization in the Minkowski spacetime*

We now discuss the quantization of a massless scalar field ϕ in various coordinates.

In this subsection, just for setting the notation, we summarize the simple case for the Minkowski coordinate. The action, the canonical momentum, and the equation of motion are given by

$$S = -\frac{1}{2}\int dt_M dx^1 d^2x \left(-(\partial_{t_M}\phi^M)^2 + (\partial_{x^1}\phi^M)^2 + \sum_{i=2}^{3}(\partial_{x^i}\phi^M)^2\right), \tag{2.17}$$

$$\pi^M \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{t_M}\phi^M)} = \partial_{t_M}\phi^M, \tag{2.18}$$

$$\left(-\partial_{t_M}^2 + \partial_{x^1}^2 + \sum_{i=2}^{3}\partial_{x^i}^2\right)\phi^M = 0, \tag{2.19}$$

where we denote the fields and the time in the Minkowski frame with the super- and subscripts M. Now, ϕ^M can be expanded into Fourier modes as

$$\phi^M(t_M, x_M^1, x) = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}}\int \frac{d^2k}{2\pi} e^{ikx + ip^1x^1 - iE_{kp^1}t_M} a_{kp^1}^M + \text{h.c.}, \tag{2.20}$$

$$E_{kp^1} \equiv \sqrt{(k)^2 + (p^1)^2}. \tag{2.21}$$

Here and throughout, we often denote (x^2, x^3) simply by x, and similarly for the momenta for the corresponding dimensions by k, and write the inner product $\sum_{i=2}^{3} k_i x_i$ as kx. Canonical quantization is performed by demanding that[9]

$$[\pi^{\mathrm{M}}(t_{\mathrm{M}}, x^1, x), \phi^{\mathrm{M}}(t_{\mathrm{M}}, y^1, y)] = -i\delta(x^1 - y^1)\delta(x - y). \tag{2.22}$$

Using the orthogonality of the exponential function, we can easily extract the mode operators and check that they satisfy the usual commutation relations:

$$[a^{\mathrm{M}}_{kp^1}, a^{\mathrm{M}\dagger}_{k'p'^1}] = \delta(p^1 - p'^1)\delta(k - k'), \qquad \text{rest} = 0. \tag{2.23}$$

2.3. Quantization outside the Rindler horizon

Let us now begin the discussion of quantization in the Rindler coordinates in various wedges.

We first consider the Rindler wedges outside the horizon, namely $\mathrm{W_R}$ and $\mathrm{W_L}$. Since the metrics in these wedges take the same form in the respective variables, we will focus on $\mathrm{W_R}$. The action takes the form

$$\begin{aligned} S &= -\frac{1}{2}\int dt_{\mathrm{R}}dz_{\mathrm{R}}d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi^{\mathrm{R}}\partial_\nu\phi^{\mathrm{R}} \\ &= -\frac{1}{2}\int dt_{\mathrm{R}}dz_{\mathrm{R}}d^2x\left(-\frac{1}{z_{\mathrm{R}}}(\partial_{t_{\mathrm{R}}}\phi^{\mathrm{R}})^2 + z_R(\partial_{z_{\mathrm{R}}}\phi^{\mathrm{R}})^2 + z_{\mathrm{R}}\sum_{i=2}^{3}(\partial_{x^i}\phi^{\mathrm{R}})^2\right). \end{aligned} \tag{2.24}$$

The canonical momentum is given by

$$\pi^{\mathrm{R}} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_{t_M}\phi^{\mathrm{R}})} = \frac{1}{z_{\mathrm{R}}}\partial_{t_{\mathrm{R}}}\phi^{\mathrm{R}}, \tag{2.25}$$

which has an extra factor of $1/z_{\mathrm{R}}$ compared with the Minkowski case. Variation of the action yields the equation of motion

$$\left(\partial^2_{z_{\mathrm{R}}} + \frac{1}{z_{\mathrm{R}}}\partial_{z_{\mathrm{R}}} + \sum_{i=2}^{3}\partial^2_{x^i} - \frac{1}{z^2_{\mathrm{R}}}\partial^2_{t_{\mathrm{R}}}\right)\phi^{\mathrm{R}} = 0. \tag{2.26}$$

As it is a second-order differential equation, there are two independent solutions, which can be taken to be the exponential function times the modified Bessel functions, namely $e^{i(kx-\omega t)}I_{i\omega}(|k|z)$ and $e^{i(kx-\omega t)}K_{i\omega}(|k|z)$. The appropriate solution is the one which damps at $z \to \infty$, and we write it as[10]

$$f^{\mathrm{R}}_{k\omega}(t_{\mathrm{R}}, z_{\mathrm{R}}, x) = N^R_\omega K_{i\omega}(|k|z_{\mathrm{R}})e^{i(kx-\omega t_{\mathrm{R}})}, \tag{2.27}$$

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[9] $\delta(x - y)$ of course means the two-dimensional delta function $\delta^2(x - y)$. This abbreviation will be used throughout.

[10] Let us make a remark on the boundary condition at $z = 0$ for $\mathrm{W_R}$ (and similarly for $\mathrm{W_L}$). For the region $\mathrm{W_R}$, the point $z_{\mathrm{R}} = 0$ corresponds to the perpendicularly bent line consisting of the lightlike segments $t = -x$ and $t = x$ (in the Cartesian coordinate) for $x \geq 0$. This "point" should be defined as the limit $z_{\mathrm{R}} \to 0$. From the completeness relation for the solutions $K_{i\omega}(y)$ (where we have set $y = |k|z_{\mathrm{R}}$) of the equation of motion described in Appendix A.2, one easily sees that a function $f(y)$ can be expanded in terms of $K_{i\omega}(y)$ in the form $f(y) = \int_0^\infty d\omega\mu(\omega)C(\omega)K_{i\omega}(y)$, where the coefficient is given by $C(\omega) = \int_0^\infty \frac{du}{u}K_{i\omega}(u)f(u)$. This integral is convergent near $u \simeq 0$ if and only if $f(u) \to 0$ as $u \to 0$. Therefore, in order to be expandable into creation and annihilation parts in terms of $K_{i\omega}(|k|z_{\mathrm{R}})$ functions, the scalar field $\phi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x)$ should vanish as one approaches the $z_{\mathrm{R}} = 0$ boundary. This is, however, automatically built in due to the main dependence of $K_{i\omega}(|k|z_{\mathrm{R}})$ on z_{R} near $z_{\mathrm{R}} = 0$, which is the divergent phase $e^{\pm i\omega \ln_{\mathrm{R}}}$. Then, using the Riemann–Lebesgue lemma, the integral defining $\phi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x)$ vanishes as $z_{\mathrm{R}} \to 0$. For more discussions see Refs. [35] and [38].

where N_ω is a normalization constant given below. Thus, the scalar field in the right Rindler wedge can be expanded as

$$\phi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x) = \int_0^\infty d\omega \int d^2k N^{\mathrm{R}}_\omega \left[K_{i\omega}(|k|z_{\mathrm{R}}) e^{i(kx-\omega t_{\mathrm{R}})} a^{\mathrm{R}}_{k\omega} + \text{h.c.}\right],$$
$$N^{\mathrm{R}}_\omega = \frac{\sqrt{\sinh \pi\omega}}{2\pi^2}. \tag{2.28}$$

Let us make some remarks on this formula:

(1) For the Hermitian conjugate part, only the conjugation for the exponential part is needed since $K_{i\omega}(|k|z)$ is real.
(2) The normalization constant chosen here will lead to the canonical form of the commutation relations, as explained in Appendix B.1.1.
(3) The variable ω here is the *energy conjugate to the time-like variable* t_{R}, and hence *its range is* $\omega \geq 0$.

Canonical quantization is performed by imposing the following equal-time commutation relation:

$$[\pi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x), \phi^{\mathrm{R}}(t_{\mathrm{R}}, z'_{\mathrm{R}}, x')] = -i\delta(z_{\mathrm{R}} - z'_{\mathrm{R}})\delta(x - x'). \tag{2.29}$$

Using the orthogonality relation for the modified Bessel functions explained in Appendix A.1, it is straightforward to obtain the commutation relations for the mode operators:

$$[a^{\mathrm{R}}_{\omega k}, a^{\mathrm{R}\dagger}_{\omega' k'}] = \delta(\omega - \omega')\delta(k - k'), \qquad \text{rest} = 0. \tag{2.30}$$

For some details of the calculations, see Appendix B.1.1.

The quantization in $\mathrm{W_L}$ is essentially similar to that in $\mathrm{W_R}$ as shown above, except for one point that one must be careful about. Recall that as the Minkowski time t_{M} (and also t_{R}) goes from $-\infty$ to ∞, the time t_{L} in $\mathrm{W_L}$ runs oppositely, from ∞ to $-\infty$. This is due to the definition of t_{L} by a smooth analytic continuation and does not, of course, mean that a physical particle moves from the future to the past. After all, $\mathrm{W_L}$ is a part of the Minkowski space and all the particles and waves must evolve along the positive direction in Minkowski time. This applies to the $\mathrm{W_L}$ observer as well, who is under constant acceleration in the negative x^1 direction. The time that increases along the trajectory of the $\mathrm{W_L}$ observer is *not* t_{L} but $\tilde{t}_{\mathrm{L}} \equiv -t_{\mathrm{L}}$. Therefore, the quantization in this frame should be done with $\tilde{t}_{\mathrm{L}}$ regarded as time. Then, all the formulas for the quantization in the $\mathrm{W_R}$ frame hold for the $\mathrm{W_L}$ frame, with t_{R} replaced by $\tilde{t}_{\mathrm{L}}$. This means that *if one wishes to use the "time"* t_{L} to write the mode expansion of the field ϕ^{L} and define its conjugate momentum π^{L}, we have

$$\phi^{\mathrm{L}}(t_{\mathrm{L}}, z_{\mathrm{L}}, x) = \int_0^\infty d\omega \int d^2k N^{\mathrm{L}}_\omega \left[K_{i\omega}(|k|z_{\mathrm{L}}) e^{ikx+i\omega t_{\mathrm{L}}} a^{\mathrm{L}}_{k\omega} + \text{h.c.}\right],$$
$$N^{\mathrm{L}}_\omega = \frac{\sqrt{\sinh \pi\omega}}{2\pi^2}, \tag{2.31}$$

and

$$\pi^{\mathrm{L}}(t_{\mathrm{L}}, z_{\mathrm{L}}, x) \equiv -\frac{1}{z_{\mathrm{L}}}\partial_{t_{\mathrm{L}}}\phi^{\mathrm{L}}. \tag{2.32}$$

One can then check that the equal-time commutation relation $\left[\pi^{\mathrm{L}}(t_{\mathrm{L}}, z_{\mathrm{L}}, x), \phi^{\mathrm{L}}(t_{\mathrm{L}}, z'_{\mathrm{L}}, x')\right] = -i\delta(z_{\mathrm{L}} - z'_{\mathrm{L}})\delta(x - x')$ holds correctly.

2.4. Quantization inside the Rindler horizon

Next, consider the quantization in the Rindler wedges inside the horizon, i.e. in W_F and W_P. They can again be treated in parallel, and we focus on W_F. Compared to the previous analysis for the outside region, an important difference arises due to the interchange of the timelike and the spacelike coordinates.

The action in the W_F region is given by

$$S = -\frac{1}{2}\int dt_F dz_F d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi^F\partial_\nu\phi^F$$
$$= -\frac{1}{2}\int dt_F dz_F d^2x\left(-z_F(\partial_{z_F}\phi^F)^2 + \frac{1}{z_F}(\partial_{t_F}\phi^F)^2 + z_F\sum_{i=2}^{3}(\partial_{x^i}\phi^F)^2\right). \tag{2.33}$$

From the signs of various terms, it is clear that t_F *is the space coordinate and* z_F *is the time coordinate*. Therefore, the canonical momentum must be defined by

$$\pi^F \equiv \frac{\partial\mathcal{L}}{\partial(\partial_{z_F}\phi^F)} = z_F\partial_{z_F}\phi^F. \tag{2.34}$$

The equation of motion takes the form

$$\left(\partial_{z_F}^2 + \frac{1}{z_F}\partial_{z_F} - \sum_i \partial_i^2 - \frac{1}{z_F^2}\partial_{t_F}^2\right)\phi^F = 0. \tag{2.35}$$

There are again two independent solutions, which can be taken as

$$f_{k\omega}(t_F, z_F, x) = N_\omega^F H_{i\omega}^{(2)}(|k|z_F)e^{i(kx-\omega t_F)},$$
$$f^*_{k\omega}(t_F, z_F, x) = N_\omega^F e^{-\pi\omega}H_{i\omega}^{(1)}(|k|z_F)e^{-i(kx-\omega t_F)}, \tag{2.36}$$

where $H_{i\omega}^{(1)}$ and $H_{i\omega}^{(2)}$ are the Hankel functions of imaginary order and N_ω^F is the normalization constant, to be specified shortly.

To expand the scalar field in terms of these functions, care should be taken as to which function should be associated with the annihilation (resp. creation) modes. This is because, in contrast to the previous case, ω is conjugate to the spacelike variable t_F and hence it is not the energy but the usual momentum. Therefore, the range of ω is $-\infty \le \omega \le \infty$ and we cannot determine the positive (resp. negative) frequency mode from the exponential part of the functions above.

To guess which Hankel function should be taken as describing the positive frequency part, it is physically natural to first look at the asymptotic behavior of $H_\omega^{(1,2)}(|k|z_F)$ at late time, i.e. at very large positive z_F. Such behaviors are given by

$$H_{i\omega}^{(1)}(|k|z_F) \sim e^{i|k|z_F - i\pi/4}e^{\pi\omega/2}\sqrt{\frac{2}{\pi|k|z_F}}, \tag{2.37}$$

$$H_{i\omega}^{(2)}(|k|z_F) \sim e^{-i|k|z_F + i\pi/4}e^{-\pi\omega/2}\sqrt{\frac{2}{\pi|k|z_F}}. \tag{2.38}$$

We see that $H_{i\omega}^{(2)}(|k|z_F)$ behaves like $e^{-i|k|z_F}$, which corresponds to the positive frequency with respect to the "energy" $|k|$ (with an overall inessential damping behavior). This tells us that the the correct expansion is

$$\phi^{\mathrm{F}}(t_{\mathrm{F}}, z_{\mathrm{F}}, x) = \int_{-\infty}^{\infty} d\omega \int d^2k N_\omega^{\mathrm{F}} \left[e^{i(kx-\omega t_{\mathrm{F}})} H_{i\omega}^{(2)}(|k| z_{\mathrm{F}}) a_{k\omega}^{\mathrm{F}} + \text{h.c.} \right],$$

$$(N_\omega^{\mathrm{F}})^2 = \frac{e^{\pi\omega}}{8(2\pi)^2}, \tag{2.39}$$

where the factor N_ω^{F} is determined such that the commutator of the modes takes the canonical form as in Eq. (2.41) below. In the literature, the modes $a_{k\omega}^{\mathrm{F}}$ are often called the Unruh modes, whereas the modes $a_{k\omega}^{\mathrm{R}}$ are referred to as the Rindler modes.

To check that such an association is actually the correct one, one must compute the "equal-time" (i.e. equal-z_{F}) commutation relation. This indeed gives the right relation:

$$[\pi^{\mathrm{F}}(z_{\mathrm{F}}, t_{\mathrm{F}}, x), \phi^{\mathrm{F}}(z_{\mathrm{F}}, t'_{\mathrm{F}}, x')] = -i\delta(t_{\mathrm{F}} - t'_{\mathrm{F}})\delta(x - x'). \tag{2.40}$$

Using the orthogonality of the Hankel functions, we get the canonical form of the commutation relations for creation/annihilation operators:

$$[a_{k\omega}^{\mathrm{F}}, a_{k'\omega'}^{\mathrm{F}\dagger}] = \delta(\omega - \omega')\delta(k - k'), \qquad \text{rest} = 0. \tag{2.41}$$

See Appendix B.1.2 for some details of this computation.

2.5. *Hamiltonian in the future wedge*

We have seen that in $\mathrm{W_F}$ and $\mathrm{W_P}$ the timelike and the spacelike variables are swapped compared to the usual situations in $\mathrm{W_R}$ and $\mathrm{W_L}$, and this has made the identification of the positive and negative frequency modes somewhat non-trivial. In fact, this swapping makes the Hamiltonian in $\mathrm{W_F}$ and $\mathrm{W_P}$ *time dependent*. In this subsection, we briefly discuss the form of the Hamiltonian and its action as the proper time development operator.

From the action in Eq. (2.33) for the $\mathrm{W_F}$ region, the Hamiltonian is readily obtained as

$$H_{\mathrm{F}} = \frac{1}{2}\int dt_{\mathrm{F}} \left(\frac{1}{z_{\mathrm{F}}}(\pi^{\mathrm{F}})^2 + \frac{1}{z_{\mathrm{F}}}(\partial_{t_{\mathrm{F}}}\phi^{\mathrm{F}})^2 + z_{\mathrm{F}}(\sum_{i=2}^{3} \partial_{x^i}\phi^{\mathrm{F}})^2 \right), \qquad \pi^{\mathrm{F}} = z_{\mathrm{F}}\partial_{z_{\mathrm{F}}}\phi^{\mathrm{F}}. \tag{2.42}$$

Since z_{F} is the time variable, the Hamiltonian H_{F} is clearly time dependent. Therefore, the time development of a state $|\psi(z_{\mathrm{F}})\rangle$ is accomplished by the unitary operator $U(z_{\mathrm{F}})$ in the manner

$$|\psi(z_{\mathrm{F}})\rangle = U(z_{\mathrm{F}})|\psi(0)\rangle, \tag{2.43}$$

$$U(z_{\mathrm{F}}) = T\exp\left(-i\int_0^{z_{\mathrm{F}}} H_{\mathrm{F}}(z')dz'\right), \tag{2.44}$$

where $T\exp(\cdots)$ denotes the time-ordered exponential. Thus, for general z_{F} the time development is quite non-trivial.

We now wish to express H_{F} in terms of modes given in Eq. (2.39) and see how it simplifies for large z_{F}. The necessary computation is straightforward: Substitute the expansion in Eq. (2.39) and perform the space integral over t_{F}. Since the intermediate expressions are lengthy, we omit them and display the final form. It is given by

$$
\begin{aligned}
H_{\rm F} = \frac{\pi}{8}\int_{-\infty}^{\infty} d\omega d^2k \bigg[\left(\frac{\omega^2}{z_{\rm F}} + z_{\rm F}k^2\right) H^{(2)}_{i\omega}(|k|z_{\rm F})H^{(2)}_{-i\omega}(|k|z_{\rm F}) \\
+ z_{\rm F}\partial_{z_{\rm F}}H^{(2)}_{i\omega}(|k|z_{\rm F})\partial_{z_{\rm F}}H^{(2)}_{-i\omega}(|k|z_{\rm F})\bigg] a^{\rm F}_{k\omega}a^{\rm F}_{-k,-\omega} + \text{h.c.} \\
+ \frac{\pi}{4}\int_{-\infty}^{\infty} d\omega d^2k \bigg[\left(\frac{\omega^2}{z_{\rm F}} + z_{\rm F}k^2\right) H^{(2)}_{i\omega}(|k|z_{\rm F})H^{(1)}_{i\omega}(|k|z_{\rm F}) \\
+ z_{\rm F}\partial_{z_{\rm F}}H^{(2)}_{i\omega}(|k|z_{\rm F})\partial_{z_{\rm F}}H^{(1)}_{i\omega}(|k|z_{\rm F})\bigg] a^{\rm F\dagger}_{k\omega}a^{\rm F}_{k\omega},
\end{aligned}
\tag{2.45}
$$

where we have, as usual, discarded an infinite constant coming from the normal ordering of the last term.

Now let us consider the limit of large time, $z_{\rm F} \to \infty$. In this limit, since $t_{\rm M} = \sqrt{z_{\rm F}^2 + (x^1)^2}$, the line of equal time will approach that of equal Minkowski time $t_{\rm M}$ and hence we expect that $H_{\rm F}$ will take the form for the free scalar field. Using the formulas in Eqs. (2.37) and (2.38) for large z, we can drastically simplify the expressions for $H_{\rm F}$. The leading term that does not vanish as $z_{\rm F} \to \infty$ takes the form

$$
H_{\rm F}\big|_{z\to\infty} = \int_{-\infty}^{\infty} d\omega\, d^2k\, |k| a^{\rm F\dagger}_{k\omega}a^{\rm F}_{k\omega}. \tag{2.46}
$$

This is independent of $z_{\rm F}$, and indeed coincides with the form for the free scalar field in Minkowski space.

2.6. Relation between the quantizations in $W_{\rm R}$, $W_{\rm L}$, $W_{\rm F}$, and the Minkowski frames

We are ready to discuss the relation between the quantizations in $\text{W}_{\rm R}$, $\text{W}_{\rm F}$, and the Minkowski frames.

2.6.1. Minkowski and $W_{\rm R}$ frames

First, since $\text{W}_{\rm R}$ is contained in the Minkowski space, it should be possible to express the modes in the $\text{W}_{\rm R}$ frame in terms of the modes in the Minkowski frame. Using the Klein–Gordon inner product for $\text{W}_{\rm R}$ defined in Appendix B.1, we obtain the expression for the annihilation operator $a^{\rm R}_{k\omega}$ in the $\text{W}_{\rm R}$ frame as

$$
\begin{aligned}
a^{\rm R}_{k\omega} &= (f^{\rm R}_{k\omega}, \phi^{\rm M})^{\rm R}_{\rm KG} \\
&= i\int_0^{\infty} \frac{dz_{\rm R}}{z_{\rm R}} \int dx^2 (f^{\rm R*}_{k\omega} \overleftrightarrow{\partial_{t_{\rm R}}} \phi^{\rm M}) \\
&= \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{4\pi E_{kp^1}}} \frac{1}{\sqrt{\sinh \pi\omega}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1}\right)^{\frac{-i\omega}{2}} \left[e^{\pi\omega/2} a^{\rm M}_{kp^1} + e^{-\pi\omega/2} a^{\rm M\dagger}_{-kp^1}\right],
\end{aligned}
\tag{2.47}
$$

where $\omega \geq 0$. Some details of the calculations are given in Appendix B.2.

Actually, this expression for $a^{\rm R}_{k\omega}$ can be simplified rather drastically by introducing the rapidity variable u defined by

$$
u \equiv \frac{1}{2}\ln\left(\frac{E_{kp^1} + p^1}{E_{kp^1} - p^1}\right). \tag{2.48}
$$

Then we can immediately solve this relation for E_{kp^1} and p^1 in terms of u, and obtain

$$
E_{kp^1} = |k|\cosh u, \qquad p^1 = |k|\sinh u. \tag{2.49}
$$

Furthermore, the integration measures are related as

$$dp^1 = |k| \cosh u du = E_{kp^1} du, \tag{2.50}$$

with the identical range of integration $[-\infty, \infty]$ for both p^1 and u. Further, if we define the annihilation operator in the rapidity variable as

$$a^{\rm M}_{ku} \equiv \sqrt{|k| \cosh u}\, a^{\rm M}_{kp^1} = \sqrt{E_{kp^1}}\, a^{\rm M}_{kp^1}, \tag{2.51}$$

the commutation relation with its conjugate is

$$[a^{\rm M}_{ku}, a^{\rm M\dagger}_{k'u'}] = |k|\sqrt{\cosh u \cosh u'}\delta(p^1 - p'^1)\delta(k - k') = \delta(u - u')\delta(k - k'), \tag{2.52}$$

where we used $\delta(p^1 - p'^1) = \delta(|k| \sinh u - |k| \sinh u') = \delta(u - u')/(|k| \cosh u)$.

Using these definitions, the relation in Eq. (2.47) can be written as

$$a^{\rm R}_{k\omega} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{4\pi \sinh \pi\omega}} e^{i\omega u} \left[e^{\pi\omega/2} a^{\rm M}_{ku} + e^{-\pi\omega/2} a^{\rm M\dagger}_{-ku}\right]. \tag{2.53}$$

Note that, as is well known, the annihilation operator $a^{\rm R}_{k\omega}$ is composed of both the annihilation and the creation operators of the Minkowski frame. Another important fact is that there are no negative frequency modes $a^{\rm R}_{k\omega}$ (for $\omega < 0$) in the $\rm W_R$ frame since ω is the energy conjugate to $t_{\rm R}$. Consequently, it is *not possible to invert the relation above* to express the Minkowski annihilation/creation operators in terms of the ones in the $\rm W_R$ frame only. This means that *the number of degrees of freedom that the* $\rm W_R$ *observer sees is half as many as seen by the Minkowski observer.* Therefore, even when the $\rm W_R$ and the Minkowski observers[11] are within the same $\rm W_R$ region, the $\rm W_R$ observer cannot recognize half of the excitation modes that the Minkowski observer sees.

2.6.2. Minkowski and $\rm W_F$ *frames related by a Fourier transform*

The situation is different for the quantization in the $\rm W_F$ frame. By using the Klein–Gordon inner product for $\rm W_F$, we can obtain the relation between the annihilation operator $a^{\rm F}_{\omega k}$ in the $\rm W_F$ frame and the mode operators in the Minkowski frame. This time, what we obtain is the relation

$$\begin{aligned} a^{\rm F}_{k\omega} &= (f^{\rm F}_{k\omega}, \phi^{\rm M})^{\rm F}_{\rm KG} = i \int_{-\infty}^{\infty} z dt_{\rm F} dx^2 (f^{\rm F*}_{k\omega} \overleftrightarrow{\partial_{z_{\rm F}}} \phi^{\rm M}) \\ &= i \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi E_{kp^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1}\right)^{-\frac{i\omega}{2}} a^{\rm M}_{kp^1}, \end{aligned} \tag{2.54}$$

which requires only the annihilation operator in the Minkowski frame. Furthermore, since ω is conjugate to the *spacelike* coodinate $t_{\rm R}$ in this case, we *do* have negative frequency modes for $a^{\rm F}_{k\omega}$, $\omega < 0$, and hence *the number of degrees of freedom of the modes that the* $\rm W_F$ *observer sees is the same as for the Minkowski observer*.

As in the case of $a^{\rm R}_{k\omega}$, the relation in Eq. (2.54) above can be simplified using the rapidity variable u. It can be written as

$$a^{\rm F}_{k\omega} = i \int \frac{du}{\sqrt{2\pi}} e^{i\omega u} a^{\rm M}_{ku}. \tag{2.55}$$

[11] To avoid any confusion, let us stress that what we mean by a "Minkowski observer in $\rm W_R$" is an observer who is traveling along a constant $x_{\rm M}$ line (= along the flow of the Minkowski time $t_{\rm M}$) and happens to be in the $\rm W_R$ region at some time $t_{\rm M}$.

Apart from a factor of i, this is nothing but the Fourier transformation. Therefore, the inverse relation is trivial to obtain and we get

$$a^{\rm M}_{ku} = -i\int \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega u} a^{\rm F}_{k\omega}, \tag{2.56}$$

$$\Leftrightarrow \qquad a^{\rm M}_{kp^1} = -i\int \frac{d\omega}{\sqrt{2\pi E_{k'p^1}}} \left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{\frac{i\omega}{2}} a^{\rm F}_{k\omega}. \tag{2.57}$$

The fact that $a^{\rm M}_{kp^1}$ and $a^{\rm F}_{k\omega}$ are in one-to-one correspodence with no mixing of the creation and the annihilation operators tells us that the vacuum states of the two observers are the same, namely[12]

$$|0\rangle_{\rm M} = |0\rangle_{\rm F}. \tag{2.58}$$

The important difference, however, is that the entities recognized as "particles" by the two observers are quite distinct and their wave functions have "dual" profiles.

2.6.3. Fourier transform as a unitary transformation

We now make a useful observation that the Fourier transform exhibited above can be realized by a unitary transformation, in the sense to be described below.[13]

Define the Fourier transform $\tilde{g}(p)$ of a function $g(x)$ as

$$\int \frac{dx}{\sqrt{2\pi}} e^{ipx} g(x) = \tilde{g}(p). \tag{2.59}$$

The *functional forms* of $g(x)$ and $\tilde{g}(p)$ are, in general, different.

Let us look for a special class of functions for which the functional forms of their Fourier transform are the same up to a proportionality constant. The simplest such function is obviously the following Gaussian, for which the proportionality constant is unity:

$$f_0(x) \equiv \pi^{-1/4} e^{-x^2/2}, \qquad \tilde{f}_0(p) = \pi^{-1/4} e^{-p^2/2} = f_0(p^2). \tag{2.60}$$

We know that such a function is the coordinate representation of the ground state of the one-dimensional harmonic oscillators $\{a, a^\dagger\}$,

$$f_0(x) = \langle x|0\rangle, \tag{2.61}$$

where $|0\rangle$ denotes the oscillator ground state defined by

$$a|0\rangle = 0, \qquad \left[a, a^\dagger\right] = 1, \qquad \langle 0|0\rangle = 1, \tag{2.62}$$

and $|x\rangle$ is, as usual, the eigenstate of the operator $\hat{x}$ with the eigenvalue x, i.e. $\hat{x}|x\rangle = x|x\rangle$.

In what follows, we take the coordinate representations of a and $a^\dagger$ as

$$a = \frac{1}{\sqrt{2}}(x+ip) = \frac{1}{\sqrt{2}}\left(\frac{d}{dx}+x\right) = \frac{i}{\sqrt{2}}\left(\frac{d}{dp}+p\right), \tag{2.63}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x-ip) = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right) = \frac{(-i)}{\sqrt{2}}\left(-\frac{d}{dp}+p\right). \tag{2.64}$$

[12] The vacuum $|0\rangle_{\rm F}$ is called the "Unruh vacuum."
[13] For related references, see Refs. [35,39].

Now, as is well known, the x-representation of the excited states of the oscillator system

$$|n\rangle \equiv \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad \langle m|n\rangle = \delta_{m,n} \tag{2.65}$$

is given by

$$f_n(x) \equiv \langle x|n\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)\right]^n \langle x|0\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)\right]^n f_0(x). \tag{2.66}$$

Inserting the unity $\int (dp/\sqrt{2\pi})|p\rangle\langle p|$ and using $\langle x|p\rangle = e^{ipx}$, this can be written as the Fourier transform

$$\begin{aligned} f_n(x) = \langle x|n\rangle &= \int \frac{dp}{\sqrt{2\pi}}\langle x|p\rangle\langle p|n\rangle = \int \frac{dp}{\sqrt{2\pi}}\langle x|p\rangle(-i)^n\frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dp}+p\right)\right]^n\langle p|0\rangle \\ &= \int \frac{dp}{\sqrt{2\pi}}e^{ixp}(-i)^n f_n(p). \end{aligned} \tag{2.67}$$

Thus, the functional form of the Fourier transform $\tilde{f}_n(p)$ is the same as the original up to a constant, namely $\tilde{f}_n(p) = (-i)^n f_n(p)$.

Let us consider the number operator $\mathcal{N} = a^\dagger a$, for which $\mathcal{N}|n\rangle = n|n\rangle$. By using the p-representation of a and $a^\dagger$, as exhibited in Eqs. (2.63) and (2.64), this is written as

$$\mathcal{N}_p f_n(p) = \frac{1}{2}\left(-\frac{d^2}{dp^2}+p^2-1\right)f_n(p) = nf_n(p). \tag{2.68}$$

Therefore, we can express the Fourier transform $(-i)^n f_n(p)$ as

$$e^{-i\frac{\pi}{2}\mathcal{N}_p}f_n(p) = (-i)^n f_n(p). \tag{2.69}$$

Note that here the terminology "Fourier transform" refers to the transform of the *form of the function*, with the argument taken to be the same.

Exactly the same formulas hold for p replaced by x. Thus, as far as the set of functions $\{f_n(p)\}$ is concerned, the Fourier transform is realized by the operation on the left-hand side of Eq. (2.69).

Up to a constant, $f_n(x)$ is nothing but the Hermite polynomial $H_n(x)$ times the Gaussian $e^{-x^2/2}$. More precisely,

$$f_n(x) = \frac{1}{(2^n n!\sqrt{\pi})^{1/2}}H_n(x)e^{-x^2/2}, \tag{2.70}$$

where $H_n(x)$ is defined by[14]

$$H_n(x) \equiv e^{x^2/2}\left(-\frac{d}{dx}+x\right)^n e^{-x^2/2}. \tag{2.71}$$

Now, in order to apply this formalism to the oscillators such as a^{M}_{ku} and $a^{\mathrm{F}}_{k\omega}$, we consider a set of oscillators depending on a continuous variable and satisfying the following commutation relations:

$$\left[a(x), a^\dagger(y)\right] = \delta(x-y). \tag{2.72}$$

[14] There are different conventions for the normalization of the Hermite polynomials. Our definition is the most standard one.

Since so far we have realized the Fourier transform as a differential operation on the set of functions $f_n(x)$ only, in order to define the Fourier transform of such an oscillator function, we should first express $a(x)$ and $a^\dagger(x)$ in terms of $f_n(x)$. This can be done thanks to the following completeness relation:

$$\delta(x-y) = \sum_{n=0}^{\infty} f_n(x) f_n(y). \tag{2.73}$$

Thus, expanding

$$a(x) = \sum_{m=0}^{\infty} b_m f_m(x), \qquad a^\dagger(y) = \sum_{n=0}^{\infty} b_n^\dagger f_n(y), \tag{2.74}$$

the commutation relation can be reproduced as

$$\left[a(x), a^\dagger(y)\right] = \sum_{m,n} [b_m, b_n^\dagger] f_m(x) f_n(y) = \delta(x-y), \tag{2.75}$$

provided we take $[b_m, b_n^\dagger] \equiv \delta_{m,n}$. Therefore, since the Fourier transform is a linear operation, we can apply the formula in Eq. (2.69) to the operators $a(x)$ and $a^\dagger(x)$ as well. This can be implemented formally by a unitary transformation of the form

$$\tilde{a}(x) = U^\dagger a(x) U, \tag{2.76}$$

$$U = \exp\left(-\frac{i\pi}{2}\int dy a^\dagger(y)\mathcal{N}_y a(y)\right). \tag{2.77}$$

In fact, one can easily verify that

$$U^\dagger a(x) U = a(x) + \left[\frac{i\pi}{2}\int dy a^\dagger(y)\mathcal{N}_y a(y), a(x)\right] + \cdots = e^{-\frac{i\pi}{2}\mathcal{N}_x} a(x). \tag{2.78}$$

So, the Fourier transform for the form of the operator is indeed reproduced.

Applied to the oscillators a^{M}_{ku} and $a^{\mathrm{F}}_{k\omega}$, we have the relations

$$\begin{aligned} i a^{\mathrm{M}}_{ku} &= U_{\mathrm{F}} a^{\mathrm{F}}_{k\omega} U^\dagger_{\mathrm{F}}\big|_{\omega=u}, \\ a^{\mathrm{F}}_{k\omega} &= U^\dagger_{\mathrm{M}} i a^{\mathrm{M}}_{ku} U_{\mathrm{M}}\big|_{u=\omega}, \end{aligned} \tag{2.79}$$

where we defined

$$U_{\mathcal{I}} = \exp\left(\frac{i\pi}{2}\int d\omega' a^{\mathcal{I}\dagger}_{k\omega'}\mathcal{N}_{\omega'} a^{\mathcal{I}}_{k\omega'}\right), \quad \mathcal{I} = \mathrm{F}, \mathrm{M}. \tag{2.80}$$

In using the operators $U_{\mathcal{I}}$, one must make sure to use the differential operator $\mathcal{N}_\omega$ on any ω-dependent quantity, be it a function or an operator, to the right of it. Transformations using $U_{\mathcal{I}}$ are useful in converting various quantities in the Minkowski and the $\mathrm{W_F}$ frames, as will be demonstrated for the Poincaré generators in Appendix C.

2.6.4. Relations between W_R*,* W_L*,* W_F*, and Minkowski frames*

Finally, let us relate the modes in the W_R and W_L frames with those in the W_F frame. Combining Eqs. (2.47) and (2.54), and their Hermitian conjugates, and eliminating the Minkowski modes, we can obtain a simple *algebraic* relationship between $a^R_{k\omega}$ and $a^F_{k\omega}$:

$$a^R_{k\omega} = -\frac{i}{\sqrt{2\sinh \pi\omega}}\left[e^{\pi\omega/2}a^F_{k\omega} - e^{-\pi\omega/2}a^{F\dagger}_{-k,-\omega}\right], \qquad \omega \geq 0. \tag{2.81}$$

Again, since $a^R_{k\omega}$ exists only for $\omega \geq 0$, *this relation cannot be inverted*.

However, recall that the "full" Rindler spacetime has the left wedge W_L in addition to the right wedge W_R. By similar arguments we can obtain the relationship between $a^L_{k\omega}$ and $a^F_{k\omega}$ as

$$a^L_{k\omega} = -\frac{i}{\sqrt{2\sinh \pi\omega}}\left[e^{\pi\omega/2}a^F_{-k,-\omega} - e^{-\pi\omega/2}a^{F\dagger}_{k\omega}\right], \qquad \omega \geq 0. \tag{2.82}$$

Note that the right-hand side contains $a^F_{k,-\omega}$ instead of $a^F_{k\omega}$, in contrast to the expression of $a^R_{k\omega}$ given in Eq. (2.81). Therefore, combining Eqs. (2.81) and (2.82), one can express the modes in the future wedge W_F in terms of the modes in W_R and W_L in the following combinations:

$$a^F_{k\omega} = -\frac{i}{\sqrt{2\sinh \pi\omega}}\left[e^{\pi\omega/2}a^R_{k\omega} - e^{-\pi\omega/2}a^{L\dagger}_{k\omega}\right], \tag{2.83}$$

$$a^F_{-k,-\omega} = \frac{i}{\sqrt{2\sinh \pi\omega}}\left[e^{-\pi\omega/2}a^{R\dagger}_{k\omega} - e^{\pi\omega/2}a^L_{k\omega}\right]. \tag{2.84}$$

Intuitively, this is a reflection of the fact that the region W_F can be reached both from W_R by left-moving waves and from W_L by right-moving waves. Note that the right-hand sides contain both the creation and the annihilation operators, and hence these relations constitute the Bogoliubov transformations between the Rindler modes and the Unruh modes.

As an application of the formulas in Eqs. (2.83) and (2.84), let us express the W_F vaccum $|0\rangle_F$, which is the same as the Minkowski vacuum $|0\rangle_M$, in terms of the states in the W_R and W_L frames. The condition that $|0\rangle_F$ must be annihilated by $a^F_{k\omega}$ and $a^F_{k,-\omega}$ can be expressed in the form

$$a^L_{k\omega}|0\rangle_F = e^{-\pi\omega}a^{R\dagger}_{k\omega}|0\rangle_F, \qquad a^R_{k\omega}|0\rangle_F = e^{-\pi\omega}a^{L\dagger}_{k\omega}|0\rangle_F. \tag{2.85}$$

The solution is

$$|0\rangle_F = |0\rangle_M = \mathcal{N}\exp\left(\int d^2k \int_0^\infty d\omega e^{-\pi\omega}a^{L\dagger}_{k\omega}a^{R\dagger}_{k\omega}\right)|0\rangle_L \otimes |0\rangle_R, \tag{2.86}$$

where $\mathcal{N}$ is a normalization factor[15] and $|0\rangle_{L,R}$ are the vacua for the W_L and W_R frames defined by $a^L_{k\omega}|0\rangle_L = 0$, $a^R_{k\omega}|0\rangle_R = 0$, for all k and positive ω. They are known as the Rindler vacua.

Let us make a few remarks on the relation between the field expressed in the Minkowski frame and in the combined W_R and W_L frame.

- In the context of the discussions of the entanglement and the entropy thereof, instead of the expression in Eq. (2.86) for the Minkowski vacuum, a simpler formula of the form

$$|0\rangle_M = \sum_{n=0}^{\infty} e^{-\omega\pi n}|n\rangle_L \otimes |n\rangle_R \tag{2.87}$$

[15] The normalization constant $\mathcal{N}$ is divergent as it stands. To make it finite, one must discretize k and ω and regularize the infinite sum.

is often quoted in the literature. This, of course, is an expression for the two-dimensional toy model with only one frequency kept. The full expression in Eq. (2.86) for four dimensions can be written in a form similar to the above after discretizing the energy ω and the momenta k and expanding the exponential.

- By using the relations in Eqs. (2.57), (2.83), and (2.84), one can express $\phi^{\rm M}(t_{\rm M}, x^1, x)$ in terms of the modes of $\rm W_L$ and $\rm W_R$. An important check is if $\phi^{\rm M}(t_{\rm M}, x^1, x)$ so constructed depends only on the modes of $\rm W_L$ ($\rm W_R$) when $x^1 < 0$ ($x^1 > 0$). In Appendix B.3, we shall sketch a proof,[16] which turned out to require a careful treatment of the proper analytic continuation.

2.7. *Representation of Poincaré algebra for various observers*

2.7.1. *Poincaré algebra for the* $(1+1)$*-dimensional subspace*

Evidently, the Poincaré symmetry of the flat Minkowski space is a fundamental symmetry governing, above all, the structure of correlation functions. Although the quantum generators of the Poincaré algebra are well known in the ordinary Minkowski frame, their forms are non-trivial in terms of the modes of observers in the $\rm W_F$, $\rm W_R$, and $\rm W_L$ wedges, and have not been discussed in the literature. In this subsection, we shall construct them by using the relations among the modes of the various observers established in the previous subsections.

As described in the next section, the vicinity of the horizon of the four-dimensional Schwarzschild black hole we are interested in has the structure of the $(1+1)$-dimensional flat space $\mathbb{R}^{1,1}$. For that reason, in what follows we shall focus exclusively on the generators and the algebra pertaining to such a subspace of the four-dimensional Minkowski space. In terms of the coordinates of the aforementioned observers, the metric of the subspace $\mathbb{R}^{1,1}$ is expressed as

$$ds^2 = -(dt_{\rm M})^2 + (dx^1)^2 = z_{\rm F}^2 dt_{\rm F}^2 - dz_{\rm F}^2 = -z_{\rm R}^2 dt_{\rm R}^2 + dz_{\rm R}^2 = -z_{\rm L}^2 dt_{\rm L}^2 + dz_{\rm L}^2. \tag{2.88}$$

As usual, the Poincaré generators can be constructed in terms of the energy–momentum tensor, which for a scalar field takes the form

$$T^\mu_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} = -\partial^\mu \phi \partial_\nu \phi + \frac{1}{2} \delta^\mu_\nu \partial^\rho \phi \partial_\rho \phi. \tag{2.89}$$

Here, μ, ν, and ρ refer to general coordinates. Then, the generators of the Poincaré algebra of $\mathbb{R}^{1,1}$ are the energy H and the momentum P_1 in the first direction,

$$H \equiv P_0 = \int d^2x \int dx^1 T^0_0, \qquad P_1 = \int d^2x \int dx^1 T^0_1, \tag{2.90}$$

and the boost generator along the first direction M_{01},

$$M_{01} = \int d^2x \int dx^1 (x^0 T^{01} - x^1 T^{00}). \tag{2.91}$$

The subscripts 0 and 1 here refer, of course, to the directions in the Minkowski frame and when quantized the normal-ordering prescription for the modes is taken for granted. Then, in terms of the Minkowski modes, these generators are given by

[16] In the basic literature such as Refs. [17] and [15], this property appears to be put in by hand rather than derived.

$$H = \int dk^2 \int dp^1 \; E_{kp^1} a^{\mathrm{M}\dagger}_{kp^1} a^{\mathrm{M}}_{kp^1}, \tag{2.92}$$

$$P_1 = \int dk^2 \int dp^1 p_1 a^{\mathrm{M}\dagger}_{kp^1} a^{\mathrm{M}}_{kp^1}, \tag{2.93}$$

$$M_{01} = i \int dk^2 \int dp^1 E_{kp^1} \; a^{\mathrm{M}\dagger}_{kp^1} \frac{\partial}{\partial p_1} a^{\mathrm{M}}_{kp^1}, \tag{2.94}$$

and can be checked to form the $(1+1)$-dimensional Poincaré algebra

$$[H, M_{01}] = iP_1, \quad [P_1, M_{01}] = iH, \quad [H, P_1] = 0. \tag{2.95}$$

2.7.2. Poincaré generators for a W_{F} *observer*

Recall that the relation between the Minkowski modes $a^{\mathrm{M}}_{kp^1}$ and the (Unruh) modes $a^{\mathrm{F}}_{k\omega}$ for the $\mathrm{W_F}$ observer have been worked out in Eq. (2.57), reproduced here for convenience:

$$a^{\mathrm{M}}_{kp^1} = -i \int \frac{d\omega}{\sqrt{2\pi E_{k',p^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{i\omega}{2}} a^{\mathrm{F}}_{k\omega} = \frac{-i}{\sqrt{2\pi E_{k',p^1}}} \int d\omega e^{-i\omega u} a^{\mathrm{F}}_{k\omega}, \tag{2.96}$$

where $u \equiv \frac{1}{2} \ln \frac{E_{kp^1}+p^1}{E_{kp^1}-p^1}$ is the "rapidity" variable. Substituting this into Eq. (2.94), M_{01} can be rewritten in terms of the $\mathrm{W_F}$ modes as

$$M^{\mathrm{F}}_{01} = \int d^2k \int d\omega \; \omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}. \tag{2.97}$$

Note that this is diagonal in ω and hence interpretable as the "momentum" operator. In Appendix C.2, we show explicitly that by the unitary transformation constructed in Eq. (2.79), M_{01} and M^{F}_{01} are transformed into each other.

Next, let us rewrite the Hamiltonian operator in terms of the Unruh modes. Using the rapidity representation, with $E_{kp^1} = |k| \cosh u$, we get

$$\begin{aligned} H &= \int dk^2 \int dp^1 \; E_{k'p^1} a^{\mathrm{M}\dagger}_{kp^1} a^{\mathrm{M}}_{kp^1} \\ &= \int d^2k |k| \int d\omega d\omega' \int \frac{du}{2\pi} \cosh u e^{i(\omega'-\omega)u} a^{\mathrm{F}\dagger}_{k\omega'} a^{\mathrm{F}}_{k\omega}. \end{aligned} \tag{2.98}$$

Since the integral over u is divergent and behaves as $\sim e^{|u|}$ at large $|u|$, we should define this integral with a suitable regularization. We adopt the definition

$$\int \frac{du}{2\pi} \cosh u e^{i(\omega-\omega')u} \equiv \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cosh u e^{i(\omega-\omega')u}. \tag{2.99}$$

Then, expanding $\cosh u$ in powers, we can rewrite this integral as

$$\begin{aligned} &\lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cosh u e^{i(\omega'-\omega)u} \\ &\quad = \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \sum_n \frac{u^{2n}}{(2n)!} e^{-\epsilon u^2} e^{i(\omega'-\omega)u} \end{aligned}$$

$$= \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \sum_n \frac{(-1)^n}{(2n)!} \left(\frac{\partial}{\partial \omega} \right)^{2n} e^{i(\omega' - \omega)u}$$

$$= \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cos \left(\frac{\partial}{\partial \omega} \right) e^{i(\omega' - \omega)u}. \tag{2.100}$$

The remaining Gaussian integral produces a δ function $\delta(\omega' - \omega)$, and hence the Hamiltonian can be written as

$$H^{\mathrm{F}} = \int d^2k|k| \int d\omega d\omega' \cos \left(\frac{\partial}{\partial \omega} \right) \delta(\omega' - \omega) a^{\mathrm{F}\dagger}_{k\omega'} a^{\mathrm{F}}_{k\omega}$$

$$= \int d^2k|k| \int d\omega \, a^{\mathrm{F}\dagger}_{k\omega} \cos \left(\frac{d}{d\omega} \right) a^{\mathrm{F}}_{k\omega}. \tag{2.101}$$

In an entirely similar manner, the P_1 operator is expressed as

$$P^{\mathrm{F}}_1 = -i \int d^2k|k| \int d\omega \, a^{\mathrm{F}\dagger}_{k\omega} \sin \left(\frac{d}{d\omega} \right) a^{\mathrm{F}}_{k\omega}. \tag{2.102}$$

These operators are understood to be used within a matrix element such that the object is infinitely differentiable with respect to ω.

In Appendix C.1, we demonstrate that these operators M^{F}_{01}, H^{F}, and P^{F}_1 do satisfy the $(1 + 1)$-dimensional Poincaré algebra of Eq. (2.95).

2.7.3. On the Poincaré generators for W_{R} and W_{L} observers

Having derived the expressions of the generators in terms of W_{F} oscillators, we can now write them in terms of the W_{R} and W_{L} mode operators using the relations in Eqs. (2.83) and (2.84), that is,

$$a^{\mathrm{F}}_{k\omega} = -\frac{i}{\sqrt{2 \sinh \pi \omega}} \left[e^{\pi \omega/2} a^{\mathrm{R}}_{k\omega} - e^{-\pi \omega/2} a^{\mathrm{L}\dagger}_{k\omega} \right], \tag{2.103}$$

$$a^{\mathrm{F}}_{-k,-\omega} = \frac{i}{\sqrt{2 \sinh \pi \omega}} \left[e^{-\pi \omega/2} a^{\mathrm{R}\dagger}_{k\omega} - e^{\pi \omega/2} a^{\mathrm{L}}_{k\omega} \right]. \tag{2.104}$$

As for the generator of the angular momentum, it cleanly separates into a W_{R} part and a W_{L} part:

$$M_{01} = M^{\mathrm{R}}_{01} + M^{\mathrm{L}}_{01}, \tag{2.105}$$

where

$$M^{\mathrm{R}}_{01} = \int d^2k \int_0^\infty d\omega \, \omega a^{\mathrm{R}\dagger}_{k\omega} a^{\mathrm{R}}_{k\omega}, \qquad M^{\mathrm{L}}_{01} = -M^{\mathrm{R}}_{01}\big|_{a^{\mathrm{R}}_{k\omega} \to a^{\mathrm{L}}_{k\omega}}. \tag{2.106}$$

Two remarks are in order:

- First, M^{R}_{01} is diagonal in ω, which in this case has the meaning of the energy conjugate to the Rindler time t_{R}. This clearly shows that the boost generator M^{R}_{01} is the Hamiltonian for the W_{R} observer.
- Second, the relative minus sign between M^{R}_{01} and M^{L}_{01} simply means that the "time" flows in opposite directions in W_{R} and W_{L}.

These remarks are expressed by the following simple relations:

$$e^{i\xi M_{01}^{\mathrm{R}}} a_{k\omega}^{\mathrm{R}} e^{-i\xi M_{01}^{\mathrm{R}}} = e^{-i\omega\xi} a_{k\omega}^{\mathrm{R}}, \tag{2.107}$$

$$e^{i\xi M_{01}^{\mathrm{L}}} a_{k\omega}^{\mathrm{L}} e^{-i\xi M_{01}^{\mathrm{L}}} = e^{i\omega\xi} a_{k\omega}^{\mathrm{L}}. \tag{2.108}$$

Thus, acting on the field, the boost generator indeed induces the Rindler time evolution in each wedge as shown below:

$$e^{i\xi M_{01}^{\mathrm{R}}} \phi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x) e^{-i\xi M_{01}^{\mathrm{R}}} = \int_0^\infty d\omega \int d^2k N_\omega^{\mathrm{R}} [K_{i\omega}(|k|z_{\mathrm{R}}) e^{ikx} e^{-i\omega(t_{\mathrm{R}}+\xi)} a_{k\omega}^{\mathrm{R}} + \text{h.c.}],$$

$$e^{i\xi M_{01}^{\mathrm{L}}} \phi^{\mathrm{L}}(t_{\mathrm{L}}, z_{\mathrm{L}}, x) e^{-i\xi M_{01}^{\mathrm{L}}} = \int_0^\infty d\omega \int d^2k N_\omega^{\mathrm{L}} [K_{i\omega}(|k|z_{\mathrm{R}}) e^{-ikx} e^{i\omega(t_{\mathrm{L}}+\xi)} a_{k\omega}^{\mathrm{L}} + \text{h.c.}].$$

In contrast, the generators H and P_1 turned out not to factorize into a $\mathrm{W_R}$ part and a $\mathrm{W_L}$ part. They can be written as

$$\begin{aligned} H &= \int_{\omega>0} d\omega \int_{\omega'>0} d\omega' \int \frac{du}{2\pi} \frac{\cosh u e^{i(\omega-\omega')u}}{\sqrt{\sinh \pi\omega \sinh \pi\omega'}} \left[e^{-\frac{\pi(\omega+\omega')}{2}} \delta(\omega-\omega') \right. \\ &\quad \left. + \cosh \frac{\pi(\omega+\omega')}{2} \left(a_{k\omega}^{\mathrm{R}\dagger} a_{k\omega'}^{\mathrm{R}} + a_{k\omega'}^{\mathrm{L}\dagger} a_{k\omega}^{\mathrm{L}} \right) - \cosh \frac{\pi(\omega-\omega')}{2} \left(a_{k\omega'}^{\mathrm{L}\dagger} a_{k\omega}^{\mathrm{R}\dagger} + a_{k\omega}^{\mathrm{L}} a_{k\omega'}^{\mathrm{R}} \right) \right], \\ P_1 &= \int_{\omega>0} d\omega \int_{\omega'>0} d\omega' \int \frac{du}{2\pi} \frac{\sinh u e^{i(\omega-\omega')u}}{\sqrt{\sinh \pi\omega \sinh \pi\omega'}} \left[e^{-\frac{\pi(\omega+\omega')}{2}} \delta(\omega-\omega') \right. \\ &\quad \left. + \cosh \frac{\pi(\omega+\omega')}{2} \left(a_{k\omega}^{\mathrm{R}\dagger} a_{k\omega'}^{\mathrm{R}} + a_{k\omega'}^{\mathrm{L}\dagger} a_{k\omega}^{\mathrm{L}} \right) - \cosh \frac{\pi(\omega-\omega')}{2} \left(a_{k\omega'}^{\mathrm{L}\dagger} a_{k\omega}^{\mathrm{R}\dagger} + a_{k\omega}^{\mathrm{L}} a_{k\omega'}^{\mathrm{R}} \right) \right]. \end{aligned} \tag{2.109}$$

Note that the last term in both H and P_1 contains a mixture of the oscillators of $\mathrm{W_L}$ and $\mathrm{W_R}$ of the form $a_{k\omega'}^{\mathrm{L}\dagger} a_{k\omega}^{\mathrm{R}\dagger} + a_{k\omega}^{\mathrm{L}} a_{k\omega'}^{\mathrm{R}}$, which prevents the factorization. The basic reason is simple. As we already emphasized, while the oscillators of the $\mathrm{W_F}$ observer (and the Minkowski observer) can be expressed in terms of those of $\mathrm{W_R}$ and $\mathrm{W_L}$ observers, these relations are not invertible, reflecting the fact the number of degrees of freedom in each of the $\mathrm{W_R}$ and $\mathrm{W_L}$ frames is half that of the $\mathrm{W_F}$ (or the Minkowski) frame. Thus, the full Poincaré algebra does not exist separately in the $\mathrm{W_R}$ and $\mathrm{W_L}$ frames.

3. Quantization in an eternal Schwarzschild black hole by various observers

As emphasized in the introduction, the main aim of this paper is to study the structure of the Hilbert spaces of the scalar field near the horizon of the Schwarzschild black hole quantized in the frames of different observers. This is made possible largely because such near-horizon geometry has the structure of the flat Minkowski space, to be recalled shortly. This allows us to make use of the knowledge of the quantization in various frames which has been reviewed, with some additional new information, in the previous section. As we shall discuss, however, we must take due care that our computations should be performed in such a way that the approximation used is legitimate.

Now, in studying the quantization around the horizon of a black hole, it will be important to distinguish two cases, namely the case of the eternal (i.e. two-sided) black hole and the more physical one where the (one-sided) black hole is produced by a collapse of matter (or radiation). There are essential differerences between the two.

In this section, we analyze the simpler case of the eternal Schwarzschild black hole.

3.1. Flat geometry around the event horizon of a Schwarzschild black hole

Let us first recall how the flat geometry emerges in the vicinity of the event horizon of a Schwarzschild black hole.

We denote the metric for the four-dimensional Schwarzschild black hole of mass M in the asymptotic coordinates in the usual way:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.1}$$

The notations are standard, except that we set the Newton constant G to one and use $\tilde{t}$ to denote the Schwarzschild time. This will later be rescaled to t to denote the Minkowski time.

First, we consider the region $\mathrm{W_R}$ outside the horizon. It is convenient to introduce a positive coordinate z which measures the proper radial distance from the horizon:

$$z \equiv \int_{2M}^{r} \sqrt{g_{rr}(r')} dr' = \int_{2M}^{r} \frac{1}{\sqrt{1 - \frac{2M}{r'}}} dr'. \tag{3.2}$$

Near the horizon at $r = 2M$, we write r as $r = 2M + y$, expand z in powers of y in the form $z = ay^{1/2}(1 + by + \cdots)$, and then solve for y in terms of z. After a simple calculation we obtain

$$r = 2M + \frac{M}{8}\left(\frac{z}{M}\right)^2 - \frac{M}{384}\left(\frac{z}{M}\right)^4 + \mathcal{O}((z/M)^6). \tag{3.3}$$

Now, if we keep up to the second term of this expansion, the Schwarzschild metric becomes

$$ds^2 \simeq -z^2(dt)^2 + dz^2 + r^2(z)d\Omega^2, \tag{3.4}$$

$$t \equiv \frac{\tilde{t}}{4M}, \qquad d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2. \tag{3.5}$$

Further, focusing on the small two-dimensonal region perpendicular to the radial direction around $\theta = 0$, we can parametrize it by the coordinates

$$x^2 = 2M\theta\cos\phi, \qquad x^3 = 2M\theta\sin\phi. \tag{3.6}$$

Then in this region the metric further simplifies and becomes identical to the Rindler metric for $\mathrm{W_R}$ given in Eq. (2.5):

$$ds^2 = -z^2dt^2 + dz^2 + (dx^2)^2 + (dx^3)^2, \tag{3.7}$$

which expresses (a portion of) the flat spacetime.[17]

To see the region of validity of this approximation, let us find out the condition under which we can neglect the third term of the expansion in Eq. (3.3) compared to the second. A simple calculation shows that the condition is

$$\frac{z}{M} \ll 4\sqrt{3} \sim 7, \tag{3.8}$$

[17] The approximation of taking r to be the fixed value $2M$ in Eq. (3.6) is admissible, since in the expression $(dx^2)^2 + (dx^3)^2$ the radial coordinate appears in the forms $r^2d\theta^2$, $r^2d\phi^2$, dr^2, and $rdrd\theta$. For these expressions, the order $\mathcal{O}(z^2/M^2)$ terms are safely neglected.

showing that the flat space approximation is good for z up to the order of the Schwarzschild radius $\mathcal{O}(M)$ out from the horizon.

For the other regions W_L, W_F, and W_P, by appropriate analytic continutations we obtain a similar flat space form of the metric of appropriate signature, as already displayed in Sect. 2. In particular, we should remember that as we go from W_R to W_F the roles of the time and space variables are interchanged.

3.2. *Exact treatment for the transverse spherical space*

The approximation of the vicinity of the horizon as a four-dimensional flat Minkowski space is certainly a great advantage, as long as we are interested only in the quantities determined by the local properties of the fields. However, as we have repeatedly emphasized, in a quantum treatment the concept of states created by the mode operators is a global one, and that is precisely what we are interested in. It turns out that the inadequacy of the flat approximation is particularly troublesome for the two-dimensional transverse space, since the orthogonality relation needed to extract the modes from the fields requires integration over the entire range of (x^2, x^3) expressing the flat 2-space, which is unjustified for large values of these coordinates.

The obvious cure for this part of the problem is to replace the expansion in terms of the plane waves by the spherical harmonics $Y_{lm}(\theta,\varphi)$. Thus, instead of $M^{1,3}$, we will be dealing with the spacetime $\mathbb{R}^{1,1}\times\mathbb{S}^2_{2M}$, where the subscript $2M$ denotes the radius of the sphere.

Explicitly, we can write the general expansions of a massless scalar and its conjugate in the vicinity of the horizon in the form

$$\phi(t,x^1,\Omega)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}\frac{dp^1}{\sqrt{4\pi E_{k_lp^1}}}e^{ip^1x^1-iE_{k_lp^1}t}Y_{lm}(\Omega)a_{lmp^1}+\text{h.c.}, \tag{3.9}$$

$$\pi(t,y^1,\Omega')=-i\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}\int_{-\infty}^{\infty}\frac{dq^1}{\sqrt{4\pi E_{k_{l'}q^1}}}E_{k_{l'}q^1}e^{iq^1y^1-iE_{k_{l'}q^1}t}Y_{l'm'}(\Omega')a_{l'm'q^1}+\text{h.c.}, \tag{3.10}$$

where $\Omega=(\theta,\varphi)$. The energy $E_{k_lp^1}$ is determined in terms of p^1 and l by the equation of motion as

$$E^2_{k_lp^1}=(p^1)^2+k_l^2,\qquad k_l\equiv\frac{\sqrt{l(l+1)}}{2M}. \tag{3.11}$$

The equal-time canonical commutation relation takes the form

$$\left[\pi(t,x^1,\Omega),\phi(t,y^1,\Omega')\right]=-i\delta(x^1-y^1)\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'). \tag{3.12}$$

The orthogonality for $Y_{lm}(\Omega)$ is

$$\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\, Y^*_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi)=\delta_{ll'}\delta_{mm'}, \tag{3.13}$$

while the completeness reads

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y^*_{lm}(\theta,\varphi)Y_{lm}(\theta',\varphi')=\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'). \tag{3.14}$$

Using the orthogonality relation, we can extract the modes as

$$a_{lmp^1} = \int \frac{dx^1}{\sqrt{4\pi E_{k_l p^1}}} \int d\Omega Y^*_{lm}(\Omega) e^{-ip^1x^1 + iE_{k_l p^1} t} i \overleftrightarrow{\partial_t} \phi(t, x^1, \Omega), \tag{3.15}$$

$$a^\dagger_{lmp^1} = \int \frac{dx^1}{\sqrt{4\pi E_{k_l p^1}}} \int d\Omega Y_{lm}(\Omega) e^{ip^1x^1 - iE_{k_l p^1} t} \frac{1}{i} \overleftrightarrow{\partial_t} \phi(t, x^1, \Omega). \tag{3.16}$$

From the canonical commutation relation in Eq. (3.12), the modes satisfy

$$\left[a_{lmp^1}, a^\dagger_{l'm'q^1}\right] = \delta_{ll'}\delta_{mm'}\delta(p^1 - q^1), \qquad \text{rest} = 0. \tag{3.17}$$

In summary, the expansion in flat space described in Sect. 2 can be converted to the present case by the simple replacements

$$\int \frac{d^2k}{(2\pi)^2} e^{ikx}(\cdots) a_{kp^1} \quad \longrightarrow \quad \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\Omega)(\cdots) a_{lmp^1}, \tag{3.18}$$

$$E_{kp^1} = \sqrt{k^2 + (p^1)^2} \quad \longrightarrow \quad E_{k_l p^1} = \sqrt{k_l^2 + (p^1)^2}, \quad k_l^2 \equiv \frac{l(l+1)}{(2M)^2}, \tag{3.19}$$

$$a_{kp^1} \quad \longrightarrow \quad a_{lmp^1}. \tag{3.20}$$

As the behavior of the scalar field on the transverse spherical surface near the horizon is treated exactly as above, we need only be concerned with the dependence on the remaining two dimensions, (t_{M}, x^1). Thus from now on, we will use expressions such as "flat approximation" or "flat space" to refer only to the two-dimensional part near the horizon within $\mathbb{R}^{1,1}$.

3.3. *Quantization in the frame of a freely falling observer near the horizon*

Among the many interesting questions that stem from the observer dependence of the quantization around a black hole, perhaps the most provocative one is whether the freely falling observer, hereafter abbreviated as FFO, sees a different Hilbert space structure for the quantized scalar field before and after he/she passes through the horizon. In other words, whether the equivalence principle for the field is affected by the quantum effects or not.

In this subsection we will perform some preparatory computations in the frame of an FFO who crosses the horizon along various directions in the Penrose diagram, i.e. with various velocities.

First, let us briefly describe how the geodesic of a massive classical particle (which represents an FFO) near the horizon maps to the motion in the flat coordinate system obtained by the non-linear transformation of the previous subsection. Although the final answer should be a straight line in the flat coordinate system, as the geodesic should map to a geodesic, it is instructive to see the physical meaning of this mapping.

Consider first the motion in $\mathrm{W_R}$. The geodesic equation in the radial direction of a massive particle (with mass set to unity) in the Schwarzschild spacetime in the region $\mathrm{W_R}$ takes the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right) = \frac{1}{2}E^2, \tag{3.21}$$

where E is a constant of motion given by

$$E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}. \tag{3.22}$$

Here, t is the asymptotic time and τ is the proper time. Restricting to the region near the horizon, we can approximate r by $r \simeq 2M + (z_{\rm R}^2/8M)$, as worked out in Eq. (3.3). Then, from Eq. (3.22), one can express $d\tau$ in terms of t and $z_{\rm R}$, and, rescaling t like $t = 4Mt_{\rm R}$ as in Eq. (3.5), we can easily rewrite the geodesic equation as

$$\left(\frac{1}{z_{\rm R}}\frac{dz_{\rm R}}{dt_{\rm R}}\right)^2 + b^2 z_{\rm R}^2 = 1, \qquad b \equiv \frac{1}{4ME}. \tag{3.23}$$

This differential equation for $z_{\rm R}$ as a function of $t_{\rm R}$ is easily solved to give[18]

$$z_{\rm R} = \frac{2c}{c^2 e^{t_R} + b^2 e^{-t_{\rm R}}}, \qquad c > 0, \tag{3.24}$$

where c is a positive integration constant. This shows that $z_{\rm R}$ vanishes as $t_{\rm R} \to \pm\infty$, meaning that the trajectory starts and ends at the horizon. The physical picture is that, due to the gravitational attraction of the black hole, a trajectory which starts out at the horizon at $t_{\rm R} = -\infty$ with some initial velocity goes out to a certain maximum distance (actually $z_{\rm R} = 4ME$) away from the horizon where it stops and then gets pulled back to the horizon at $t_{\rm R} = \infty$.

Now let us rewrite this motion of Eq. (3.24) in terms of the flat Minkowski coordinates $(t_{\rm M}, x^1)$ related to $(z_{\rm R}, t_{\rm R})$ by $t_{\rm M} = z_{\rm R} \sinh t_{\rm R}$, $x^1 = z_{\rm R} \cosh t_{\rm R}$ as in Eq. (2.3). Then, we get

$$t_{\rm M} = -\frac{1}{\beta}x^1 + X, \tag{3.25}$$

$$\beta \equiv \frac{c^2 - b^2}{c^2 + b^2}, \qquad X \equiv \frac{2c}{c^2 - b^2}. \tag{3.26}$$

As expected, this describes a family of timelike straight line trajectories with velocity β. To construct an orthogonal coordinate system with the trajectories above as specifying the time direction, we must supply spacelike lines perpendicular (in the Lorentz sense) to them. Clearly they are of the form

$$t_{\rm M} = -\beta x^1 + T, \tag{3.27}$$

where T is a parameter. Thus, by changing the values of X and T we span (a part of) the Minkowski space. In other words, (T, X) serve as new orthogonal coordinates. In fact, better coordinates are the rescaled ones (t_β, x_β^1) defined in the following way:

$$t_\beta \equiv \gamma T, \qquad x_\beta^1 \equiv \gamma\beta X, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \tag{3.28}$$

$$ds^2 = -dt_\beta^2 + (dx_\beta^1)^2. \tag{3.29}$$

Then the relation to the canonical Minkowski variables $(t_{\rm M}, x^1)$ are obtained from Eqs. (3.25) and (3.27), and can be written as

$$\begin{pmatrix} t_\beta \\ x_\beta^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} t_{\rm M} \\ x^1 \end{pmatrix}. \tag{3.30}$$

This is nothing but the Lorentz boost by the velocity β in the negative x^1 direction.

[18] Actually, there is another solution with the sign in front of the $t_{\rm R}$ flipped. But since they are related simply by changing the sign of $t_{\rm R}$, we deal with the one displayed here without loss of generality.

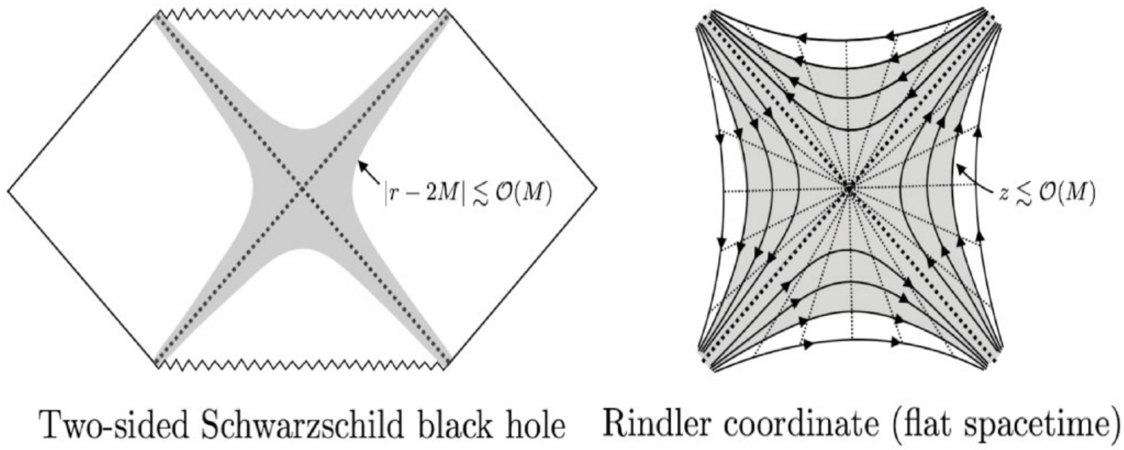

Fig. 2. Approximately flat regions, shown in gray, near the horizon of the two-sided Schwarzschild black hole and the corresponding region in the Rindler coordinates of the flat spacetime.

One can perform a similar analysis of the geodesic in the W_F region sharing the horizon as the boundary with W_R. The outcome of the study is that the geodesics which hit the same point on this common boundary from the inside and the outside with the same velocity β are actually one and the same straight line, which is obtained by the Lorentz boost of the trajectory along the time axis in the canonical Minkowski coordinate. Physically this must be the case since the FFO must be able to go through the horizon freely due to the *classical* equivalence principle.

With this preparation, let us now discuss the quantization and the mode expansion of the free scalar field by an FFO in the vicinity of the horizon where the flat space approximation for the dependence on (t_M, x^1) is valid. In the case of the two-sided eternal Schwarzschild black hole studied in this section we have both W_R and W_L regions, and the approximately flat region near the horizons can be depicted as the shaded region in Fig. 2.

In this region the general solution for a scalar field as seen by an FFO is

$$\phi^M(t_M, x^1, \Omega) = \sum_{l,m} \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2E_{k_l p^1}}} \left(e^{ip^1 x^1 - iE_{k_l p^1} t_M} Y_{lm}(\Omega) a^M_{lmp^1} + \text{h.c.} \right). \tag{3.31}$$

This expression is perfectly valid *locally*, but as we try to extract the mode operators $a^M_{lmp^1}$ and their conjugates and check that they obey the usual commutation relations, we encounter a problem of the same nature as occurred when using the flat coordinates (x^2, x^3). Namely, such an extraction requires orthogonality relations for the plane waves, which involves integration over an infinite range for the spatial variable x^1. As such a range is not within the flat space region, it appears to be quite difficult to solve this problem.

The observation that allows us to overcome this obstacle is that regions of infinite range do exist around the horizon along the lightcones in the (t_M, x^1) space. Technically, however, the quantization using the exact lightcone variable as the time is rather singular. Therefore, we shall make a very large (but not infinite) two-dimensional Lorentz boost so that the x^1-axis is rotated to the direction which is almost lightlike yet still slightly spacelike. Then, assuming the usual regularization that the scalar field vanishes at $t_M = \pm\infty$, we can integrate along this new x^1 axis, which is practically contained in the flat region, and extract the modes. Since what we used here is a Lorentz transformation, the exponent is invariant while the modes are transformed in a well-known simple way, namely

$$a'^{\mathrm{M}}_{lmp'^1}\sqrt{E'_{k_lp'^1}} = a^{\mathrm{M}}_{lmp^1}\sqrt{E_{k_lp^1}} \qquad \text{(same for the conjugates)}, \tag{3.32}$$

where the prime signifies the Lorentz-transformed quantities. The mode operators satisfy the usual commutation relations, i.e. $[a'_{lmp'^1}, a'^{\dagger}_{l'm'q'^1}] = \delta_{ll'}\delta_{mm'}\delta(p'^1 - q'^1)$, etc. This immediately tells us that the number of degrees of freedom observed by the FFO in the horizon region is exactly the same as that of the scalar field in the usual Minkowski space, and the structure of the Hilbert space is unchanged across the horizon. In this sense, *the equivalence principle is still valid quantum mechanically* around the eternal black hole.

3.4. Relation between the quantization by a freely falling observer and stationary observers in W_{F} and W_{P}

As the argument for $\mathrm{W_P}$ is the same as that for $\mathrm{W_F}$ we will concentrate on the case of a $\mathrm{W_F}$ observer.

In the approximately flat region near the horizon, the scalar field ϕ^{F} can be expanded in modes simply like

$$\phi^{\mathrm{F}}(t_{\mathrm{F}}, z_{\mathrm{F}}, \Omega) = \sum_{l,m}\int_{-\infty}^{\infty} d\omega N^{\mathrm{F}}_{\omega}\left(e^{-i\omega t_{\mathrm{F}}} H^{(2)}_{i\omega}(k_l z_{\mathrm{F}}) Y_{lm}(\Omega) a^{\mathrm{F}}_{lm\omega} + \text{h.c.}\right). \tag{3.33}$$

Contrary to the case of the Minkowski frame discussed above, the extraction of the modes in $\mathrm{W_F}$ is straightforward. This is because the equal-time spacelike lines near the horizon are entirely contained in the approximately flat region, as is clear from Fig. 2. Therefore, orthogonality relations for the Hankel functions can be used just as in the case of the entire Minkowski space described in Sect. 2.5.

This means that in the flat region around the horizon, the number of modes is the same between a $\mathrm{W_F}$ observer and the FFO. More explicitly, the relations between the mode operators are just as in the case of the Minkowski space (with k replaced by lm taken for granted). This is particularly clear in the rapidity representation given in Eqs. (2.55) and (2.56). Since $|k|\cosh u$ in the definition in Eq. (2.51) is the energy E, the operator a^{M}_{ku} is Lorentz invariant, as seen from Eq. (3.32), which means that $a'^{\mathrm{M}}_{ku'} = a^{\mathrm{M}}_{ku}$, where $u' = u + \xi$, where ξ is the rapidity for the boost. On the other hand, the invariance of ϕ^{F} and z_{F}, and $t_{\mathrm{F}} \to t_{\mathrm{F}} + \xi$, under the Lorentz transformation in Eq. (2.39) dictates that we should have $a'^{\mathrm{F}}_{k\omega} = e^{i\omega\xi} a^{\mathrm{F}}_{k\omega}$. With the angular-momentum indices explicitly implemented, we have, under the Lorentz transformation,

$$a'^{\mathrm{M}}_{lm,u+\xi} = a^{\mathrm{M}}_{lmu}, \qquad a'^{\mathrm{F}}_{lm\omega} = e^{i\omega\xi} a^{\mathrm{F}}_{lm\omega}. \tag{3.34}$$

It is easy to see that this is indeed compatible with the Fourier transform relation in Eq. (2.55), with k replaced by lm.

3.5. Relation between the quantization by a freely falling observer and stationary observers in W_{R} and W_{L}

We now come to the more difficult situation of the quantization from the viewpoints of a $\mathrm{W_R}$ (and $\mathrm{W_L}$) observer in the flat region. Expansion of the general solution into modes using the $K_{i\omega}$ functions is the same as in the Minkowski space, and the canonical quantization condition for the fields can be imposed. But the extraction of the mode operators $a^{\mathrm{M}}_{lm\omega}, a^{\mathrm{M}\dagger}_{lm\omega}$ and verifying that they satisfy the canonical commutation relations cannot be performed explicitly. In contrast to the case of $\mathrm{W_R}$ discussed in the previous subsection, there is no set of spacelike lines covering $\mathrm{W_F}$, such as described by t_R = constant, that are contained entirely within the flat region, and we cannot use the flat space orthogonality relation to express the mode operators in terms of the fields.

What we can check easily is that, if we assume the canonical form of the commutation relations for the modes as in the flat space, then by using the completeness relation, *which is a local relation*, the correct canonical commutation relations for the fields are reproduced. This shows the self-consistency of the assumption.

Actually, we can argue that the relation between $a^{\rm R}$ and $a^{\rm M}$ should be the same as in the full Minkowski space in the following two ways:

(1) In the flat region, using completeness, we can re-expand the field $\phi^{\rm R}$, which contains $a^{\rm R}$ and $a^{\rm R\dagger}$ in terms of the plane waves, i.e. in terms of the modes of $\phi^{\rm M}$. In this calculation we only need to use integration over the momenta. Now, as described in Sect. 3, we can use the orthogonality of the plane waves along the contour which, by a suitable Lorentz transformation, is brought within the flat region extending to infinity near the horizon, and extract the $a^{\rm M}$ modes. Along such a line, we can relate the $a^{\rm M}$ with the $a^{\rm R}$ as in the full Minkowski space. Then, Lorentz-transforming back this relation, we should be able to express the $a^{\rm R}$ in terms of the $a^{\rm M}$ in any flat region around the horizon.

(2) Another argument goes as follows: For simplicity, consider the case where we try to use the orthogonality integral along the spacelike straight line at $t_{\rm M} = 0$ extending from $x^1 = -\infty$ to $x^1 = \infty$. This passes both $W_{\rm L}$ and $W_{\rm R}$, and only a portion of the contour is within the flat region. Outside the flat region, the eigenfunctions $f_{k_l\omega}(z_{\rm R,L})$ satisfying the equation of motion start to differ *continuously* from the modified Bessel functions $K_{i\omega}(k_l z_{\rm R,L})$. But since the differential equation expressing the equation of motion does not acquire any new singularities, one expects that such deformed eigenfunctions continue to satisfy appropriate forms of orthogonality relations. Then, using them, one can extract the $a^{\rm R,L}$ from the fields and compute the commutation relations among them. These relations should reduce (continuously) to the usual commutation relations in the flat region, as they must lead, using the completeness relation, to the correct canonical commutation relations for the fields expandable in terms of the modified Bessel functions in such a region.

These arguments indicate that, as far as the flat region near the horizon is concerned, the relations between the modes for the FFO and the observers in various Rindler frames should be the same as those already exhibited in Sect. 2 for the fully flat case, with the replacement of the linear momentum label k by the angular momentum label lm.

4. Quantization in a Vaidya model of a physical black hole by various observers

Black holes of more physical interest are the ones formed by a collapse of matter, as actually occurs in nature. They are "one-sided" and have rather different spacetime structures compared with the two-sided eternal black holes discussed in the previous section.

In this section, we investigate how the observers in various frames quantize a massless scalar field in the simplest model of a Schwarzschild black hole of such a type, namely the so-called Vaidya spacetime [30–32],[19] created by the collapse of a thin spherical shell of matter at the speed of light, often referred to as a shock wave.

[19] For a review, see, for example, Ref. [40].

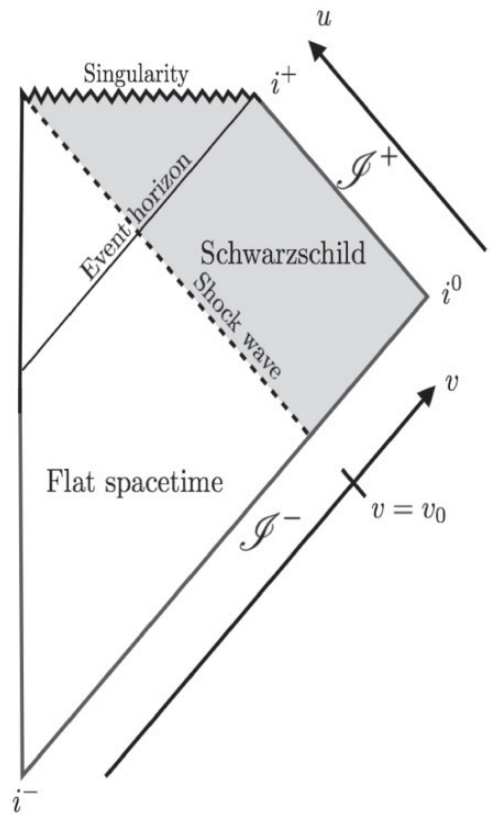

Fig. 3. Penrose diagram of the simplest Vaidya spacetime. It consists of two parts: one is the flat region inside the matter shell ($v < v_0$), shown in white; the other is the Schwarzschild spacetime outside the matter shell ($v > v_0$), shown in gray.

4.1. Vaidya model of a physical black hole and the effect of the shock wave on the field as a boundary condition

4.1.1. Vaidya model of a physical black hole

Let us begin by recalling the basics of such a Vaidya spacetime. After a black hole is formed by spherical collapse, by Birkohoff's theorem the metric outside the horizon is always that of the Schwarzschild black hole. On the other hand, for the simplest situation above, the metric inside is isomorphic to part of the flat Minkowski space. Thus the Penrose diagram of the entire spacetime is obtained by gluing these two types of geometries along the lightlike line representing the falling shell, as shown in Fig. 3.

The Vaidya metric is a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{4.1}$$

with the energy momentum tensor

$$T_{vv} = \frac{M}{4\pi r^2}\delta(v - v_0). \tag{4.2}$$

The delta function at $v = v_0$ represents a shock wave induced by the matter collapsing along the lightlike direction u. The metric of the Vaidya spacetime is described by

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2drdv + r^2 d\Omega^2, \tag{4.3}$$

where

$$\begin{cases} m(v) = 0 & \text{for } v < v_0, \\ m(v) = M & \text{for } v > v_0. \end{cases} \tag{4.4}$$

The metric above consists of two parts, one of which corresponds to the region inside the shock wave, $v < v_0$, and the other describes the outside, i.e. the region $v > v_0$. The solution inside the shock wave is actually a flat spacetime described by

$$ds^2 = -dv^2 + 2drdv + r^2 d\Omega^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{4.5}$$

where $v = t+r$ is a light-cone coordinate. This is expected from the spherical symmetry of the matter shell. The solution for the region $v > v_0$ is the Schwarzschild black hole in the Eddington–Finkelstein coordinate

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2, \tag{4.6}$$

as dictated by Birkhoff's theorem. This metric can be transformed into the Schwarzschild form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2 \tag{4.7}$$

by the coordinate transformation $v = \tilde{t} + r^*$. Here, $\tilde{t}$ is the Schwarzschild time and r^* is the tortoise coordinate defined by

$$r^* = \int \frac{dr}{1 - \frac{2M}{r}} = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{4.8}$$

Notice that the two time coordinates, t inside the shell and $\tilde{t}$ outside, are different. They are related as

$$\begin{aligned} t + r = \tilde{t} + r^* \qquad & \text{at } v = v_0, \\ \Leftrightarrow \quad \tilde{t} = t - 2M \ln & \left(\frac{v_0 - t}{2M} - 1\right). \end{aligned} \tag{4.9}$$

In the subsequent subsections we mainly focus on the special limit of the Vaidya spacetime, for which the collapse of the matter has taken place a very long time ago so that the flat space region in Fig. 3 is almost negligible. This is for technical simplicity, as will be explained more explicitly in Sect. 4.2.1, below Eq. (4.13).

4.1.2. Effect of the shock wave on the scalar field as a regularized boundary condition

In order to be able to study the quantization of a scalar field in an explicit manner, in what follows we shall (i) make a reasonable assumption about the effect of the matter shock wave on the field, and (ii) implement it in a well-defined way by making a regularization which replaces the lightlike trajectory by a slightly timelike one.

As for (i), since we focus on the Schwarzschild region outside of the locus of the shock wave, the effect of the shock wave on the scalar field $\phi(x)$ should be taken into account by an imposition of *an effective boundary condition* on $\phi(x)$ along the trajectory of the shock wave, which must be consistent with the bulk equation of motion. Such boundary conditions are either Dirichlet or

Neumann.[20] This depends on the nature of the interaction between the shock wave and the scalar field, and for definiteness in this work we adopt the Dirichlet condition and demand that $\phi(x)$ vanishes[21] along the boundary.[22]

Next, let us elaborate on point (ii). If we take the boundary to be strictly lightlike, i.e. along $t_{\mathrm{M}} = -x^1$, there is a complication for the spherical mode with zero angular momentum, for which $k_l = 0$. Thus this component of the scalar field becomes massless in two dimensions, and the future-directed massless field satisfying the Dirichlet condition along the lightlike line above can only be right-moving and hence chiral. As is well known, quantization of a chiral scalar in two dimensions is notoriously troublesome and we would like to avoid it. A physically natural regularization is to endow the falling matter with an infinitesimal mass so that the trajectory is slightly timelike. Then the boundary condition can be treated in a non-singular manner by the standard canonical quantization procedure.

Another advantage of such a regularization is the following. As will become evident, the effect of the boundary condition on the quantization can easily be taken into account in the frame of an FFO moving in the direction of the shock wave. When this direction is slightly timelike, we can change it by a Lorentz transformation into the case for a general FFO moving with any velocity. On the other hand, even if we could manage to treat the case of the strictly lightlike shock wave and an FFO moving along such a direction, we cannot relate such an observer by a Lorentz boost to a general FFO moving with a finite velocity.

4.2. Quantization of the scalar field with a boundary condition by a freely falling observer

In this section we explicitly perform the quantization of a scalar field with the boundary condition imposed along a slightly timelike line from the point of view of FFOs traversing the horizon with various velocities.

4.2.1. Three useful coordinate frames and the imposition of a boundary condition

In what follows we will concentrate on the flat two-dimensional portion in $\mathbb{R}^{1,1}$ and introduce three flat coordinates related by Lorentz transformations. One is the canonical coordinates (t, x^1) (where we use t for t_{M} for simplicity in this subsection), for which the t and x^1 axes respectively run vertically and horizontally. The second is the coordinates $(\hat{t}, \hat{x}^1)$, where the $\hat{t}$ axis runs almost lightlike but in a slightly timelike direction. To go from (t, x^1) to $(\hat{t}, \hat{x}^1)$ we make a large Lorentz transformation of

[20] Actually, there is another possibility: that the scalar field does not interact with the shock wave. In such a case, one needs to smoothly connect the solutions in the different spacetimes inside and outside the matter shell, and in $(1 + 3)$ dimensions this is a difficult task. For this reason, in this work we will not consider such a non-interacting case.

[21] For a massless scalar, by using the invariance of the action under a constant shift, we can do so without loss of generality.

[22] The boundary condition we introduce here should not be confused with the one considered in the so-called moving mirror model in the two-dimensional gravity theory discussed in the literature (see, for example, Sects. 4.3 and 4.4 of Ref. [41]). In the moving mirror model, the boundary condition is imposed on the field at the origin $r = 0$ of the Minkowski spacetime in Eq. (4.5) inside the matter shock wave of the Vaidya spacetime, and the field outside the shock wave $(v > v_0)$ is smoothly connected to the one inside $(v < v_0)$. On the other hand, in our treatment the interaction of the shock wave and the field is represented by an effective boundary condition imposed along the shock wave at $v = v_0$.

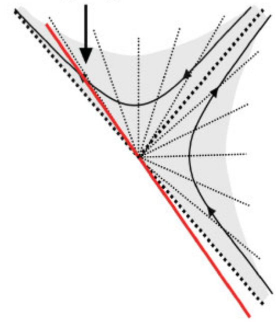

Fig. 4. Slightly timelike boundary line (shown in red).

the form

$$\begin{pmatrix} \hat{t} \\ \hat{x}^1 \end{pmatrix} = \Lambda_\epsilon \begin{pmatrix} t \\ x^1 \end{pmatrix}, \qquad \Lambda_\epsilon = \hat{\gamma} \begin{pmatrix} 1 & \hat{\beta} \\ \hat{\beta} & 1 \end{pmatrix}, \tag{4.10}$$

$$\hat{\beta} = 1 - \epsilon, \qquad \hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}} \simeq \frac{1}{\sqrt{2\epsilon}}, \tag{4.11}$$

where ϵ (> 0) is an infinitesimal parameter. Thus, the explicit transformations are

$$\hat{t} = \frac{1}{\sqrt{2\epsilon}}((1-\epsilon)x^1 + t), \qquad \hat{x}^1 = \frac{1}{\sqrt{2\epsilon}}((1-\epsilon)t + x^1). \tag{4.12}$$

We shall take the boundary line to be the one expressed by (see Fig. 4)

$$\hat{x}^1 = 0 \quad \Leftrightarrow \quad t = -\frac{1}{1-\epsilon}x^1, \tag{4.13}$$

and demand that $\phi(\hat{x}^1 = 0) = 0$.

We should remark that this corresponds to the case where the shell of matter collapses along the line for which the so-called tortoise light-cone coordinate $v^* = t + r^*$, where $r^* \equiv \int dr/(1-(2M/r))$, takes a very large negative constant value compared to the scale of the Schwarzschild radius $2M$. An example is the case where $t \to -\infty$. The reason for this rather special choice is strictly for technical convenience: such a trajectory is contained entirely within the region where the flat space approximation is valid and hence the computations can be done explicitly and reliably. We can deal with a general FFO later by making a Lorentz transformation, as explained below. As far as the qualitative conclusions are concerned, a constant shift in v^* should not affect the quantum property of the scalar field drastically, because the Dirichlet condition $\phi = 0$ along the matter trajectory, as we shall see, will act just like a reflecting wall for the scalar field.

The third set of coordinates to be introduced is $(\tilde{t}, \tilde{x}^1)$, where $\tilde{t}$ is the axis along which an FFO travels with a general velocity $\tilde{\beta}$, which can be positive or negative. He/she quantizes the scalar field with $\tilde{t}$ as the time. This frame is defined to be related to the canonical frame by a Lorentz transformation:

$$\begin{pmatrix} \tilde{t} \\ \tilde{x}^1 \end{pmatrix} = \tilde{\Lambda} \begin{pmatrix} t \\ x^1 \end{pmatrix} = \tilde{\gamma} \begin{pmatrix} 1 & \tilde{\beta} \\ \tilde{\beta} & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \end{pmatrix}, \tag{4.14}$$

$$\tilde{t} = \tilde{\gamma}(t + \tilde{\beta}x^1), \qquad \tilde{x}^1 = \tilde{\gamma}(x^1 + \tilde{\beta}t). \tag{4.15}$$

It will also be convenient to relate the frame $(\tilde{t}, \tilde{x}^1)$ with $(\hat{t}, \hat{x}^1)$ directly. We shall write this relation as

$$\begin{pmatrix} \hat{t} \\ \hat{x}^1 \end{pmatrix} = \Lambda \begin{pmatrix} \tilde{t} \\ \tilde{x}^1 \end{pmatrix}, \qquad \Lambda = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}, \tag{4.16}$$

$$\hat{t} = \gamma(\tilde{t} + \beta\tilde{x}^1), \qquad \hat{x}^1 = \gamma(\tilde{x}^1 + \beta\tilde{t}), \tag{4.17}$$

where Λ, in terms of the Lorentz transformations already introduced in Eqs. (4.10) and (4.14), is the combination $\Lambda = \Lambda_\epsilon \tilde{\Lambda}^{-1}$. For infinitesimal ϵ, the relations between (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$ can be approximated as

$$\beta \simeq 1 - \frac{1 + \tilde{\beta}}{1 - \tilde{\beta}}\epsilon, \qquad \gamma \simeq \frac{\tilde{\gamma}(1 - \tilde{\beta})}{\sqrt{2\epsilon}}. \tag{4.18}$$

4.2.2. Quantization of the scalar field satisfying the boundary condition by an FFO in the $(\hat{t}, \hat{x}^1)$ frame

We begin with the quantization in the $(\hat{t}, \hat{x}^1)$ frame. Since the boundary condition is imposed along the line $\hat{x}^1 = 0$, obviously the quantization is easiest in such a frame. More importantly, regularizing the scalar field to vanish at infinity, as usual, the trajectory of the FFO along the $\hat{t}$ axis is contained in the region where the flat space approximation is valid. Therefore, the following procedure is justified.

The quantized scalar field that vanishes for $\hat{x}^1 = 0$ is obtained by simply imposing such a condition on the one without the boundary condition, namely the expression given in Eq. (3.9) with unhatted variables replaced by hatted ones. Explicitly, setting the coefficient of $\cos\hat{p}^1\hat{x}^1$, which does not vanish for $\hat{x}^1 = 0$, in the expansion $\exp(i\hat{p}^1\hat{x}^1) = \cos\hat{p}^1\hat{x}^1 + i\sin\hat{p}^1\hat{x}^1$, we obtain the relation between the modes

$$\hat{a}_{lm,-\hat{p}^1} = -\hat{a}_{lm\hat{p}^1}. \tag{4.19}$$

This clearly shows that the mode with negative $\hat{p}^1$ is directly related to the mode with positive $\hat{p}^1$, and hence *the number of independent modes is halved* by the imposition of the boundary condition. Intuitively, the wave as seen in the hatted frame is reflected perpendicularly by the boundary line. Therefore, the scalar field as quantized by an FFO moving in the direction of the $\hat{t}$ axis and its conjugate momentum are of the form

$$\phi(\hat{t}, \hat{x}^1, \Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}\frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l\hat{p}^1}}}\left(e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)(\hat{a}_{lm\hat{p}^1} - \hat{a}_{lm-\hat{p}^1}) + \text{h.c.}\right)\sin\hat{p}^1\hat{x}^1, \tag{4.20}$$

$$\begin{aligned}\hat{\pi}(\hat{t}, \hat{x}^1, \Omega) &= \partial_{\hat{t}}\phi(\hat{t}, \hat{x}^1, \Omega) \\ &= -i\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}\frac{d\hat{p}^1\sqrt{E_{k_l\hat{p}^1}}}{\sqrt{4\pi}}\left(e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)(\hat{a}_{lm\hat{p}^1} - \hat{a}_{lm-\hat{p}^1}) - \text{h.c.}\right)\sin\hat{p}^1\hat{x}^1,\end{aligned} \tag{4.21}$$

where the notation $\hat{\pi}$ reminds us that the conjugate momentum in this frame is defined using the derivative with respect to $\hat{t}$. By using the commutation relation $\left[\hat{a}_{lm\hat{p}^1}, \hat{a}^\dagger_{l'm'\hat{q}^1}\right] = \delta_{ll'}\delta_{mm'}\delta(\hat{p}^1 - \hat{q}^1)$

and the formula $\int_0^\infty d\hat{p}^1 \sin\hat{p}^1\hat{x}^1 \sin\hat{p}^1\hat{y}^1 = (\pi/2)(\delta(\hat{x}^1-\hat{y}^1)-\delta(\hat{x}^1+\hat{y}^1))$, one can verify that the commutator of the conjugate fields takes the canonical form[23]

$$[\hat{\pi}(\hat{t},\hat{x}^1,\Omega),\phi(\hat{t},\hat{y}^1,\Omega')] = -i\delta(\hat{x}^1-\hat{y}^1)\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'). \tag{4.22}$$

4.2.3. *Comparison of the Hilbert space of an FFO and the genuine Minkowski Hilbert space*

Let us define for convenience the following combinations of the mode operators:

$$\hat{a}^{\pm}_{lm\hat{p}^1} \equiv \hat{a}_{lm\hat{p}^1} \pm \hat{a}_{lm,-\hat{p}^1}, \tag{4.23}$$

$$\hat{a}^{\pm\dagger}_{lm\hat{p}^1} \equiv \hat{a}^{\dagger}_{lm\hat{p}^1} \pm \hat{a}^{\dagger}_{lm,-\hat{p}^1}. \tag{4.24}$$

Clearly, the operators with plus and minus superscripts commute with each other. Then, from the discussion above, the Hilbert space $\mathcal{H}_{\rm FFO}$ of the FFO in the $(\hat{t},\hat{x}^1)$ frame is constructed upon the vacuum $|\hat{0}\rangle_-$, defined by

$$\hat{a}^{-}_{lm\hat{p}^1}|\hat{0}\rangle_- = 0, \tag{4.25}$$

by applying the operators $\hat{a}^{-\dagger}_{lm\hat{p}^1}$ repeatedly. In contrast, the genuine Minkowski Hilbert space $\mathcal{H}_{\rm M}$ is built upon the vacuum $|0\rangle_{\rm M}$, which is defined to be annihilated by $\hat{a}_{lm\hat{p}^1}$ for all values of l, m, and $\hat{p}^1$, by (repeated) applications of the $\hat{a}^{\dagger}_{lm\hat{p}^1}$. This means that $\mathcal{H}_{\rm M}$ can be written as the tensor product

$$\mathcal{H}_{\rm M} = \mathcal{H}^- \otimes \mathcal{H}^+, \tag{4.26}$$

where $\mathcal{H}^-$ stands for $\mathcal{H}_{\rm FFO}$, and the other half, $\mathcal{H}^+$, is constructed in an entirely similar manner to $\mathcal{H}^-$, using the a^+-type operators. From the point of view of the FFO $\mathcal{H}^+$ is unphysical, but it is needed for the construction of $\mathcal{H}_{\rm M}$. Note that this decomposition is completely *different* from the left–right decomposition $\mathcal{H}_{\rm M} = \mathcal{H}_{\rm W_L} \otimes \mathcal{H}_{\rm W_R}$.

This structure will be important in the discussion of the Unruh-like effect near the horizon of a physical Schwarzschild black hole in Sect. 5.

4.2.4. *Quantization by an FFO in a general frame $(\tilde{t},\tilde{x}^1)$ with the boundary condition*

We now consider the quantization by an FFO in a general frame $(\tilde{t},\tilde{x}^1)$ with the *same boundary condition* $\hat{x}^1 = 0$ along the shock wave. Since this boundary condition is simplest to describe in the $(\hat{t},\hat{x}^1)$ frame, the most efficient way to quantize in the $(\tilde{t},\tilde{x}^1)$ frame with such a boundary condition is to express the new conjugate momentum $\tilde{\pi} \equiv \partial_{\tilde{t}}\phi$ in terms of the quantities in the $(\hat{t},\hat{x}^1)$ frame by applying the relation

$$\frac{\partial}{\partial\tilde{t}} = \frac{\partial\hat{t}}{\partial\tilde{t}}\frac{\partial}{\partial\hat{t}} + \frac{\partial\hat{x}^1}{\partial\tilde{t}}\frac{\partial}{\partial\hat{x}^1} = \gamma\frac{\partial}{\partial\hat{t}} + \gamma\beta\frac{\partial}{\partial\hat{x}^1} \tag{4.27}$$

[23] Since $\hat{x}^1$ and $\hat{y}^1$ are both positive, we can discard $-\delta(\hat{x}^1+\hat{y}^1)$.

to $\phi(\hat{t},\hat{x}^1,\Omega)$, which we already have. Because $\partial_{\tilde{t}}$ contains the spatial derivative $\partial_{\hat{x}^1}$ as well, this leads to an important non-trivial change in the conjugate momentum, however. The result is[24]

$$\tilde{\pi}(\tilde{t},\tilde{x}^1,\Omega) = -i\widetilde{\mathcal{N}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty}\frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l\hat{p}^1}}}\Bigg[\Big((-i\gamma\hat{E}_{k_l\hat{p}^1})e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)a_{lm\hat{p}^1}+\text{h.c.}\Big)\sin\hat{p}^1\hat{x}^1 + \gamma\beta\hat{p}^1\Big(e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)a_{lm\hat{p}^1}+\text{h.c.}\Big)\cos\hat{p}^1\hat{x}^1\Bigg], \quad (4.28)$$

where we have denoted the normalization constant in this frame as $\widetilde{\mathcal{N}}$.

Now let us compute the *equal-$\tilde{t}$* canonical commutation relation between $\tilde{\pi}$ and $\tilde{\phi}$. In this process we need to take into account the following two points:

(1) Since $\tilde{t} = \gamma(\hat{t}-\beta\hat{x}^1)$, equal $\tilde{t}$ is equivalent to equal $\hat{t}-\beta\hat{x}^1$. In other words, if we denote the hatted time that appears in $\tilde{\pi}$ given by Eq. (4.28) by $\hat{t}$ and the one in ϕ by $\hat{t}'$, then the equal $\tilde{t}$ can be expressed as $\hat{t}-\beta\hat{x}^1 = \hat{t}'-\beta\hat{y}^1$, where $\hat{x}^1$ and $\hat{y}^1$ are the spatial coodinates that appear in $\tilde{\pi}$ and ϕ, respectively. Therefore we have the important relation

$$\hat{t}-\hat{t}' = -\beta(\hat{x}^1-\hat{y}^1) \quad \text{at equal } \tilde{t}. \quad (4.29)$$

(2) The second point to keep in mind is that from the Lorentz transformation we easily find that

$$\hat{x}^1-\hat{y}^1 = \gamma(\tilde{x}^1-\tilde{y}^1) \quad \text{at equal } \tilde{t}, \quad (4.30)$$

so that the difference in the spatial coodinates in the hatted frame can be rewritten as the rescaled difference in the tilded frame.

With these facts in mind, the equal-$\tilde{t}$ commutator $[\tilde{\pi},\phi]$ is given by

$$\begin{aligned}&\big[\tilde{\pi}(\tilde{t},\tilde{x}^1,\Omega),\phi(\tilde{t},\tilde{y}^1,\Omega')\big]\\ &= -i\widetilde{\mathcal{N}}^2\sum_{l,m}\sum_{l'm'}\int_0^{\infty}d\hat{p}^1\int_0^{\infty}d\hat{q}^1\frac{Y_{lm}(\Omega)Y^*_{l'm'}(\Omega')}{4\pi\sqrt{\hat{E}_{k_l\hat{p}^1}\hat{E}_{k_{l'}\hat{q}^1}}}\\ &\Bigg[(-i\hat{E}_{k_l\hat{p}^1}\gamma)\Big(e^{-i\hat{E}_{k_l\hat{p}^1}\hat{t}+i\hat{E}_{k_{l'}\hat{q}^1}\hat{t}'}+\text{h.c.}\Big)\Big[\hat{a}_{lm\hat{p}^1},\hat{a}^\dagger_{l'm'\hat{q}^1}\Big]\sin\hat{p}^1\hat{x}^1\ \sin\hat{q}^1\hat{y}^1\\ &+\gamma\beta\hat{p}^1\Big(e^{-i\hat{E}_{k_l\hat{p}^1}\hat{t}+i\hat{E}_{k_{l'}\hat{q}^1}\hat{t}'}-\text{h.c.}\Big)\Big[\hat{a}_{lm\hat{p}^1},\hat{a}^\dagger_{l'm'\hat{q}^1}\Big]\cos\hat{p}^1\hat{x}^1\ \sin\hat{q}^1\hat{y}^1\Bigg]\\ &= C_1 + C_2, \end{aligned} \quad (4.31)$$

where

$$C_1 = -\gamma\widetilde{\mathcal{N}}^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty}\frac{d\hat{p}^1}{4\pi}Y_{lm}(\Omega)Y^*_{lm}(\Omega')\Big(e^{i\beta\hat{E}_{k_l\hat{p}^1}(\hat{x}^1-\hat{y}^1)}+\text{h.c.}\Big)\sin\hat{p}^1\hat{x}^1\ \sin\hat{p}^1\hat{y}^1, \quad (4.32)$$

[24] To get the explicit form of $\tilde{\pi}(\tilde{t},\tilde{x}^1,\Omega)$, we must rewrite all the hatted quantities in terms of the tilded ones obtained by the Lorentz transformation of Eq. (4.16). This produces a rather involved expression. The procedure adopted here can avoid this complication.

$$C_2 = -i\gamma\beta\widetilde{\mathcal{N}}^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^\infty \frac{d\hat{p}^1}{4\pi\hat{E}_{k_l\hat{p}^1}} Y_{lm}(\Omega)Y^*_{lm}(\Omega')\left(e^{i\beta\hat{E}_{k_l\hat{p}^1}(\hat{x}^1-\hat{y}^1)} - \text{h.c.}\right)\cos\hat{p}^1\hat{x}^1\ \sin\hat{p}^1\hat{y}^1. \tag{4.33}$$

Note that for the the exponents involving $\hat{E}_{k_l\hat{p}^1}$ we used the relation in Eq. (4.29). The sum over m can be done by the well-known addition theorem for Y_{lm}, namely

$$\sum_{m=-l}^{l} Y_{lm}(\Omega)Y^*_{lm}(\Omega') = \frac{2l+1}{4\pi}P_l(\hat{n}\cdot\hat{n'}), \tag{4.34}$$

where $P_l(x)$ is the Legendre polynomial and $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ denotes the unit vector corresponding to the pair of angles $\Omega(\theta,\varphi)$. Furthermore, the integral over $\hat{p}^1$, after rewriting the product of trigonometric functions into a sum of them, can also be performed using the formulas in Eq. (3.961) of Ref. [47] (with the aid of the relation $\partial_z K_0(z) = -K_1(z)$):

$$\int_0^\infty e^{-b\sqrt{k^2+x^2}}\cos ax dx = \frac{bk}{\sqrt{a^2+b^2}}K_1(k\sqrt{a^2+b^2}), \tag{4.35}$$

$$\int_0^\infty \frac{x}{\sqrt{k^2+x^2}}e^{-b\sqrt{k^2+x^2}}\sin ax dx = \frac{ak}{\sqrt{a^2+b^2}}K_1(k\sqrt{a^2+b^2}), \tag{4.36}$$

where $K_1(z)$ is the Macdonald function (i.e. one of the modified Bessel functions) of order 1, and both formulas are valid for $\mathrm{Re}\, b > 0$, $\mathrm{Re}\, k > 0$. In particular, the convergence condition $b > 0$ is important since, in our case, $b = \pm i\beta(\hat{x}^1 - \hat{y}^1)$ and are pure imaginary. Thus, we must regularize them by introducing an infinitesimal positive parameter $\eta > 0$ and replace b by

$$b_- \equiv -i\beta(\hat{x}^1 - \hat{y}^1 + i\eta), \tag{4.37}$$

$$b_+ \equiv +i\beta(\hat{x}^1 - \hat{y}^1 - i\eta). \tag{4.38}$$

Since the rest of the calculations are somewhat tedious but more or less straightforward, we shall describe some intermediate steps in Appendix D and only list here the important structures that one will encounter as one proceeds:

- The terms which contain $\cos(\hat{x}^1 + \hat{y}^1)$ and $\sin(\hat{x}^1 + \hat{y}^1)$ produced from the product of sines and cosines turn out to cancel completely, because $\hat{x}^1 + \hat{y}^1$ is positive and generically finite, and the regulator η after performing the integrals can be ignored compared to them.
- On the other hand, for the terms containing the difference $\hat{x}^1 - \hat{y}^1$ there are two cases. When the difference is finite, and hence η in $b_\pm$ can be ignored, all the terms cancel just as in the case above and hence the commutator vanishes.
  In contrast, when the difference is of order η or smaller, then the contribution remains and becomes proportional to the structure $K_1(\alpha k_l\eta)$, with a finite constant α. Now, if we first make a cut-off on the angular momentum l so that $k_l = \sqrt{l(l+1)}/2M$ can be large but finite, then $\alpha k_l\eta \to 0$ as we send $\eta \to 0$. Then, from the behavior of $K_1(z)$ for small z, i.e. $K_1(z) \simeq 1/z$, we see that the contribution diverges like $1/\eta$. Thus, we see that as $\hat{x}^1 - \hat{y}^1 \to 0$ the commutator diverges as we remove the regulator.
  Together, this is nothing but the behavior of the δ-function $\delta(\hat{x}^1 - \hat{y}^1)$, which is proportional to $\delta(\tilde{x}^1 - \tilde{y}^1)$ due to the relation in Eq. (4.30).

- In the other limit, where l becomes so large that $k_l\eta$ is large, then $K_1(z)$ damps like $\sim e^{-z}/\sqrt{z}$ and such a region does not contribute. This indicates that we can effectively replace k_l by a large constant independent of l.
- Then, we are left with the sum over l, which produces the angular δ-functions in the manner

$$\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\vec{n}\cdot\vec{n}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\Omega)Y^*_{lm}(\Omega')$$
$$= \delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi') = \frac{1}{2\pi}\delta(\vec{n}\cdot\vec{n}'-1). \tag{4.39}$$

Combining, we find that the commutator is proportional to the desired product of δ-functions, and the quantization for an arbitrary FFO with the boundary condition along $\hat{x}^1 = 0$ in the vicinity of the horizon is achieved.

Two remarks are in order:

(1) Although the correct δ-function structure for the canonical commutation relation is confirmed, unfortunately we cannot compute the exact normalization constant because the relevant integrals and the infinite sum cannot be performed exactly. This is regrettable since such a constant must become singular as we let the almost lightlike trajectory approach exactly lightlike, and it would be interesting to see how this comes about.
(2) Nevertheless, the fact that the quantization for a general FFO with the boundary condition $\phi(\hat{x}^1 = 0) = 0$ can be carried out as we have shown shows that the number of modes that the FFO sees as he/she passes the horizon *does not change* and is naturally *half as many as for the case of the two-sided black hole.*

4.3. *Quantization of the scalar field by the observer in the* W_F *frame*

We now consider the quantization in the W_F frame with the same boundary condition along the slightly time-like line, namely $(1-\epsilon)t_M + x^1 = 0$. Expressed in terms of the W_F variables—see Eq. (2.8)—this becomes

$$z_F(e^{t_F} - \epsilon\cosh t_F) = 0. \tag{4.40}$$

Since z_F need not vanish, we should set $e^{t_F} = \epsilon\cosh t_F$. This can be easily solved for t_F as

$$t_F \simeq \frac{1}{2}\ln\frac{\epsilon}{2}, \tag{4.41}$$

which is very large and negative.

Now we impose the vanishing condition for ϕ^F along this line. An advantageous feature of the W_F region is that such a line is, practically, contained entirely in the flat region. Therefore we can make use of the expression in the flat spacetime, and the boundary condition on the field $\phi^F(t_F, z_F, \Omega)$ can be written as

$$\phi^F(z_F, t_F, \Omega) = \sum_{l,m}\int_{-\infty}^{\infty} d\omega N^F_\omega\left(e^{-i\omega t_F}H^{(2)}_{i\omega}(k_l z_F)Y_{lm}(\Omega)a^F_{lm\omega} + \text{h.c.}\right). \tag{4.42}$$

Using the relation

$$H^{(2)}_{-i\omega} = e^{\pi\omega}H^{(2)}_{i\omega}, \tag{4.43}$$

we can rewrite Eq. (4.42) as

$$\phi^{\rm F}(z_{\rm F}, t_{\rm F}, \Omega) = \sum_{l,m} \int_0^\infty d\omega \Big(N^{\rm F}_\omega e^{-i\omega t_{\rm F}} H^{(2)}_{i\omega}(k_l z_{\rm F}) Y_{lm}(\Omega) a^{\rm F}_{lm\omega}$$

$$+ N^{\rm F}_{-\omega} e^{i\omega t_{\rm F}} H^{(2)}_{-i\omega}(k_l z_{\rm F}) Y_{lm}(\Omega) a^{\rm F}_{lm-\omega} + \text{h.c.} \Big)$$

$$= \sum_{l,m} \int_0^\infty d\omega N^{\rm F}_\omega \left(H^{(2)}_{i\omega}(|k| z_{\rm F}) Y_{lm}(\Omega) (e^{-i\omega t_{\rm F}} a^{\rm F}_{lm\omega} + e^{i\omega t_{\rm F}} a^{\rm F}_{lm-\omega}) + \text{h.c.} \right). \quad (4.44)$$

We now impose the boundary condition

$$\phi^{\rm F}(z_{\rm F}, t_{\rm F} = t^{\rm B}_{\rm F}, \Omega) = 0, \quad (4.45)$$

where $t^{\rm B}_{\rm F} = \frac{1}{2} \ln \frac{\epsilon}{2}$; then the modes should satisfy the following relations:

$$a^{\rm F}_{lm\omega} = -e^{2i\omega t^{\rm B}_{\rm F}} a^{\rm F}_{lm-\omega}. \quad (4.46)$$

Notice that this argument is valid even when we take the value of ϵ very small.

Thus, the conclusion is that the boundary condition places the relations in Eq. (4.46) among the modes and hence the number of independent modes observed in the $W_{\rm F}$ frame is halved, just like the ones in the FFO frame $\hat{a}_{lm,\hat{p}^1}$.

4.4. Quantization of the scalar field by an observer in the $W_{\rm R}$ frame

Finally, let us consider the quantization by an observer in the $W_{\rm R}$ frame.

If we take the same special slightly timelike boundary line which goes through the origin of the coordinate frame $(t_{\rm M}, x^1)$, this line is outside of the region $W_{\rm R}$. Therefore, there is no boundary condition to impose and the modes which exist for the two-sided case are all present and independent.

Although this is a valid argument, it certainly depends crucially on the special choice of the boundary line. Therefore we should also consider the case where the boundary line is slightly shifted to the positive x^1 direction so that it passes inside $W_{\rm R}$, very close to its lightlike boundary (see Fig. 5). Explicitly, the boundary line is now taken to be along

$$t = -\frac{1}{1-\epsilon} x^1 + \delta, \quad (4.47)$$

where δ is a very small shift. In this case, the imposition of the boundary condition is meaningful and the argument to follow is of more general validity.

As discussed in Sect. 3.5, the expansion of $\phi^{\rm R}(z_{\rm R}, t_{\rm R}, \Omega)$ near the horizon is given by[25]

$$\phi^{\rm R}(z_{\rm R}, t_{\rm R}, \Omega) = \int_0^\omega d\omega N_\omega \sum_{l,m} Y_{lm}(\Omega) \left(K_{i\omega}(k_l z_{\rm R}) e^{-i\omega t_{\rm R}} a^{\rm R}_{lm\omega} + \text{h.c} \right), \quad (4.48)$$

$$N_\omega = \frac{\sqrt{\sinh \pi\omega}}{\pi}. \quad (4.49)$$

Since Eq. (4.47) can be rewritten as $x^+ = z_{\rm R} e^{t_{\rm R}} = \epsilon t + \delta$, $z_{\rm R}$ is small along the boundary line for finite $t_{\rm R}$. Now, to make use of the form of the $K_{i\omega}(z)$ for small z, we make a cut-off for the angular

[25] Due to the use of the spherical harmonics instead of the plane wave in the transverse directions, the normalization factor N_ω is slightly different from the one in Eq. (2.31).

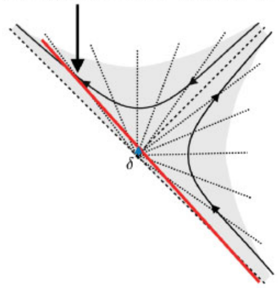

Fig. 5. A slightly timelike boundary line (shown in red), which is shifted infinitesimally in the positive x^1 direction compared to Fig. 4.

momentum l and consider the states for which k_l is bounded. Then, we can use the behavior of $N_\omega K_{i\omega}(y)$ for small y, which is given by

$$N_\omega K_{i\omega}(y) \sim \frac{\sqrt{\sinh \pi\omega}}{\pi} \times 2\sqrt{\frac{\pi}{\omega \sinh \pi\omega}} \cos\left(\omega \ln \frac{y}{2} - \arg \Gamma(i\omega)\right)$$
$$= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{\omega}} \cos\left(\omega \ln \frac{y}{2} - \arg \Gamma(i\omega)\right). \tag{4.50}$$

Note that this oscillates wildly as y becomes small. y here is $k_l z_{\mathrm{R}}$, and as we send the regulator ϵ to zero, it becomes of order δ, which is very small. Then, just as in the case of the quantization in W_{F}, by imposing a cut-off for the ω integral, we can apply the Riemann–Lebesgue lemma to conclude that the Fourier-type integral tends to vanish and the boundary condition $\phi^{\mathrm{R}} = 0$ is automatically satisfied along the matter trajectory.

Therefore, the conclusion should be the same as in the case of the W_{F} observer: If we exclude the highly excited states, the boundary condition does not impose any relations on the mode operators and the degrees of freedom remain the same as in the two-sided case. For the energetic states, more exact computation is needed to make definite statements.

5. Implications for the quantum equivalence principle, the firewall phenomenon, and the Unruh effect

Having analyzed and compared the quantizations of a scalar field by different natural observers in a concrete manner, we now consider the implications of our results.

5.1. Quantum equivalence principle and the firewall phenomenon

One of the clear results is that the degrees of freedom of the modes that the observer sees are in general different, both for the case of the two-sided eternal black hole and the more physical one-sided one. Explicitly, the FFO and the W_{F} observers see the same number of modes, while the observer in the W_{R} frame finds half as many in the two-sided case. On the other hand, in the one-sided case, the number of modes which the FFO and the W_{F} observers see are both halved by the imposition of the boundary condition, while the number seen by the W_{R} observers is the same as in the case without the boundary. Thus, in this case the number of modes that FFO, W_{F}, and W_{R} observers see are identical.

The fact that the sizes of the quantum Hilbert spaces are halved for FFO and W_F observers in the one-sided case and W_R observers in both cases is natural since such observers can only see a part of the spacetime due to the presence of the horizons for them.

Whether the equivalence principle holds quantum mechanically is quite a different question. It asks whether the FFO, upon crossing the “horizon,” which does not exist for him/her *classically*, sees extra or fewer degrees of freedom of the *quantum* excitation modes of a field. Our explicit computation shows that, for both the eternal and the physical black holes, *the quantum equivalence principle holds naturally*. This is essentially due to the fact that no new boundary conditions for the scalar field appear as seen by an FFO who goes through the “horizon.”

Needless to say, this conclusion is valid under the assumption that the metric of the interior of the Schwarzschild black hole is essentially given by the Vaidya-type metric. If the interior of the black hole is such that it cannot be specified just by the information of the metric, the conclusion may differ. However, as far as classical Schwarzschild black holes produced by the collapse of matter are concerned, our assumption is conservative and should be reasonable.

Thus, for a large enough black hole that itself can be treated classically, with a small value of curvature at the horizon, our explicit computations for the quantum effects of the massless scalar field as seen by the three types of observers should be reliable; in particular, the freely falling observer does not encounter the so-called firewall phenomenon.

5.2. Unruh-like effect near the horizon of a physical black hole

The Unruh effect [15] is the simplest example of non-trivial quantum phenomena due to the difference of the vacua for the relatively accelerated observers. In the original case treated by Unruh, a Rindler observer uniformly accelerated in the flat Minkowski space in the positive x^1 direction with accleration a (confined to the wedge W_R) sees in the Minkowski vacuum $|0\rangle_M$ a swarm of particles of energy ω with a number density distribution given by

$$\frac{\langle N^R_\omega \rangle}{\text{Vol.}} = \frac{{}_M\langle 0|a^{R\dagger}_\omega a^R_\omega|0\rangle_M}{{}_M\langle 0|0\rangle_M \text{Vol.}} \propto \frac{1}{e^{2\pi\omega/a}-1}. \tag{5.1}$$

Evidently, this coincides with the thermal distribution of bosons at temperature $a/2\pi$. In fact this computation is *truly thermal in nature* since $|0\rangle_M$ is an entangled state consisting of the states of W_L as well as of W_R, and one must take a trace over all the states of W_L to obtain the distribution above.

Although in this example the background is taken to be the flat spacetime to begin with, one might expect a similar phenomenon to be seen by a stationary observer just outside the horizon of a physical Schwarzschild black hole, since the spacetime there is well-approximated by the right Rindler wedge of a flat Minkowski space.

However, the analysis cannot be the same for the following reasons. First, there is no W_L region for the one-sided black hole and hence whatever distribution we obtain is *not* truly thermal in nature. It simply shows that the concept of “a particle” depends crucially on the vacuum state, even if it is a pure state. The second reason is the fact that, although the region of our interest is locally a flat Minkowski space, we must take into account the effect of the boundary condition for the scalar field and its vacuum seen by the FFO, who corresponds to the Minkowski observer in the Unruh setup. As discussed in Sect. 4.2.3, however, the vacuum $|\hat{0}\rangle_-$ for the FFO is *not* the genuine Minkowski vacuum. A related difference is that, as discussed in Sect. 4.4, in our setup the scalar field in the W_R frame is not affected by the boundary condition and hence the number of modes seen in that frame is the *same* as that of the FFO. This is in contrast to the case of the flat space, where the number of modes for the W_R observer is *half* that of the Minkowski observer.

Thus, the question of interest is the distribution of the W_R particles in the vacuum of the FFO. To answer this, we must express the mode operators $a^{R}_{lm\omega}$ and their conjugates of the W_R observer in terms of the field $\phi(\hat{t}, \hat{x}^1, \Omega)$ and its modes for an FFO.[26]

Unfortunately, in general this computation cannot be performed accurately due to our lack of knowledge of the fields outside the approximately flat region. The required calculation is of the form

$$a^{R}_{lm\omega} = i \int d\varphi \int \sin\theta d\theta Y^*_{lm}(\Omega) \int_0^\infty \frac{dz_R}{z_R} f^*_{\omega,l}(t_R, z_R) \overleftrightarrow{\partial_{t_R}} \phi^{\mathrm{FFO}}(\hat{t}, \hat{x}^1, \Omega), \tag{5.2}$$

where $f_{\omega,l}(t_R, z_R) Y_{lm}(\Omega)$ is the solution of the equation of motion , corresponding to the mode $a^{R}_{lm\omega}$ in the right Rindler wedge in the background of the Schwarzschild black hole. All we know is that this function takes the form $f_{\omega,l}(t_R, z_R) \simeq N_\omega K_{i\omega}(|k_l| z_R) e^{-i\omega t_R}$ in the approximately flat region where $z_R \lesssim M$. Therefore, integration over z_R, which extends outside such a region, cannot be performed explicitly.

There is, however, a class of modes for which the computation can be performed sufficiently accurately using the function for the flat space region. These are the ones with large angular momentum l such that $|k_l| M = \sqrt{l(l+1)}/2 \gg 1$. To see this, let us expand the scalar field into angular momentum eigenstates as $\phi(t, r, \Omega) = \sum_{l,m} \phi_{lm}(r) Y_{lm}(\Omega)$ and write down the equation of motion for $\phi_{lm}(t, r)$ in the Schwarzschild metric. It is given by

$$0 = -\frac{1}{1 - \frac{2M}{r}} \partial_t^2 \phi_{lm} + \frac{2(r - M)}{r^2} \partial_r \phi_{lm} + \left(1 - \frac{2M}{r}\right) \partial_r^2 \phi_{lm} - \frac{l(l+1)}{r^2} \phi_{lm}. \tag{5.3}$$

Now we look at the region $r \gg M$ where the flat space approximation is no longer valid. In such a region, writing $\phi_{lm}(t, r) = e^{\pm i\omega t} \tilde{\phi}_{lm}(r)$, the equation for $\tilde{\phi}_{lm}(r)$ simplifies to

$$0 = \left(\omega^2 + \frac{2}{r}\partial_r + \partial_r^2 - \frac{l(l+1)}{r^2}\right) \tilde{\phi}_{lm}(r). \tag{5.4}$$

The solution is well known and is given, with a certain normalization, by

$$\tilde{\phi}_{lm}(r) = \sqrt{\frac{\omega}{r}} J_{l+\frac{1}{2}}(\omega r), \tag{5.5}$$

where $J_{l+\frac{1}{2}}(\omega r)$ is the Bessel function. Its asymptotic form for large l can be obtained from the formula in Eq. (10.19.1) of Ref. [48] as

$$J_{l+\frac{1}{2}}(\omega r) \sim \frac{1}{\sqrt{(2l+1)\pi}} e^{-(l+\frac{1}{2}) \ln \frac{2l+1}{e\omega r}}. \tag{5.6}$$

This shows that for $l \gtrsim \omega r$, this expression is exponentially small in l and contributes negligibly to the integral over z_R. *Thus, for such modes with high angular momenta, we effectively need only the function in the flat region* and the computation is possible. Such a calculation is at the same time self-consistent because $K_{i\omega}(|k_l| z_R)$ damps exponentially for large $|k_l| z_R$, and for large enough l this

[26] The reason for focusing on the FFO in the $(\hat{t}, \hat{x}^1)$ frame is simply that the effect of the boundary condition is simplest in such a frame. For the other frames of an FFO, one can make a Lorentz transformation for the FFO *with the boundary condition kept intact.*

quantity is already large for $z_{\rm R} \simeq M$. Therefore, the contribution from $z_{\rm R} \gtrsim M$ region can safely be neglected.

To perform the computation of Eq. (5.2), first consider the projection of the angular part in Eq. (5.2) using the orthogonality of the spherical harmonics. Since $\phi^{\rm FFO}$ contains both Y_{lm} and Y^*_{lm}, the relevant formulas are $\int d\Omega Y^*_{lm}(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$ and $\int d\Omega Y^*_{lm}(\Omega) Y^*_{l'm'}(\Omega) = (-1)^m \delta_{ll'}\delta_{m,-m'}$. (The second formula follows from the first by using the relation $Y^*_{lm} = (-1)^m Y_{l,-m}$.)

Therefore, after the removal of the angular part, what we need to compute is

$$a^{\rm R}_{lm\omega} = i\int_0^\infty \frac{dz_{\rm R}}{z_{\rm R}} N_\omega K_{i\omega}(|k_l|z) e^{i\omega t_{\rm R}} \overleftrightarrow{\partial_{t_{\rm R}}} \int_{-\infty}^{\infty} \frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l\hat{p}^1}}}\Big(e^{-iE_{k_l\hat{p}^1}\hat{t}}\hat{a}^-_{lm\hat{p}^1} + (-1)^m e^{iE_{k_l\hat{p}^1}\hat{t}}\hat{a}^{-\dagger}_{l,-m\hat{p}^1}\Big)\sin\hat{p}^1\hat{x}^1. \tag{5.7}$$

To perform the differentiation with respect to $t_{\rm R}$, we must use the relation between $\hat{t}$ and $t_{\rm R}$ given by the Lorentz transformations

$$\hat{t} = \hat{\gamma}(t_{\rm M} + \hat{\beta}x^1) = \hat{\gamma} z_{\rm R}(\sinh t_{\rm R} + \hat{\beta}\cosh t_{\rm R}), \tag{5.8}$$

$$\hat{x}^1 = \hat{\gamma}(x^1 + \hat{\beta}t_{\rm M}) = \hat{\gamma} z_{\rm R}(\cosh t_{\rm R} + \hat{\beta}\sinh t_{\rm R}), \tag{5.9}$$

$$\hat{\beta} = 1 - \epsilon \equiv \tanh\xi, \qquad \hat{\gamma} = \frac{1}{\sqrt{1-\beta^2}} \equiv \cosh\xi, \tag{5.10}$$

where we have introduced the rapidity variable ξ. Then, the relevant part of Eq. (5.7) can be computed as

$$\begin{aligned} & e^{i\omega t_{\rm R}} \overleftrightarrow{\partial_{t_{\rm R}}} \left[e^{-iE_{k_l\hat{p}^1}\hat{t}} \sin(\hat{p}^1\hat{x}^1)\right] \\ &= e^{i\omega t_{\rm R}} e^{-iaz_{\rm R}} \left[-i(a'z_{\rm R} + \omega)\sin bz_{\rm R} + b' z_{\rm R}\cos bz_{\rm R}\right] \\ &= -\frac{1}{2}(a'z+\omega)e^{i\omega t}\left(e^{-i(a-b)z} - e^{-i(a+b)z}\right) + \frac{1}{2}b' z e^{i\omega t}\left(e^{-i(a-b)z} + e^{-i(a+b)z}\right), \end{aligned} \tag{5.11}$$

where

$$a \equiv E_{k_l\hat{p}^1}\hat{\gamma}(\sinh t_{\rm R} + \hat{\beta}\cosh t_{\rm R}), \qquad b \equiv \hat{p}^1\hat{\gamma}(\cosh t_{\rm R} + \hat{\beta}\sinh t_{\rm R}), \tag{5.12}$$

$$a' \equiv E_{k_l\hat{p}^1}\hat{\gamma}(\cosh t_{\rm R} + \hat{\beta}\sinh t_{\rm R}), \qquad b' \equiv \hat{p}^1\hat{\gamma}(\sinh t_{\rm R} + \hat{\beta}\cosh t_{\rm R}). \tag{5.13}$$

These expressions can be further simplified by introducing the parametrization

$$E_{k_l\hat{p}^1} = |k_l|\cosh\hat{u}, \qquad \hat{p}^1 = |k_l|\sinh\hat{u}. \tag{5.14}$$

Then, we can write

$$a \pm b = |k_l|\sinh\rho_\pm, \qquad a' \pm b' = |k_l|\cosh\rho_\pm, \tag{5.15}$$

$$\rho_\pm \equiv \xi + t_{\rm R} \pm \hat{u}. \tag{5.16}$$

Now we consider the integral over $z_{\rm R}$ in Eq. (5.7). The basic integrals we need are $A_1(c,k)$ and $A_2(c,k)$ given in Eqs. (B.24) and (B.25) in Appendix B.2.1. Specifically, the ones we need are with

$c = \pm(a \pm b)$ and $k = |k_l|$. When these parameters are substituted the integrals simplify drastically and we obtain

$$A_1(a \pm b, |k_l|) = C_\omega \left(e^{\pi\omega/2} e^{-i\omega\rho_\pm} + e^{-\pi\omega/2} e^{i\omega\rho_\pm}\right), \tag{5.17}$$

$$A_2(a \pm b, |k_l|) = \frac{\omega C_\omega}{|k_l| \cosh \rho_\pm} \left(e^{\pi\omega/2} e^{-i\omega\rho_\pm} - e^{-\pi\omega/2} e^{i\omega\rho_\pm}\right), \tag{5.18}$$

$$A_1(-(a \pm b), |k_l|) = C_\omega \left(e^{\pi\omega/2} e^{i\omega\rho_\pm} + e^{-\pi\omega/2} e^{-i\omega\rho_\pm}\right), \tag{5.19}$$

$$A_2(-(a \pm b), |k_l|) = \frac{\omega C_\omega}{|k_l| \cosh \rho_\pm} \left(e^{\pi\omega/2} e^{i\omega\rho_\pm} - e^{-\pi\omega/2} e^{-i\omega\rho_\pm}\right), \tag{5.20}$$

where

$$C_\omega \equiv \frac{\pi}{2\omega \sinh \pi\omega}. \tag{5.21}$$

Further, it is convenient to use the rapidity-based oscillators

$$\hat{a}^-_{lm\hat{u}} \equiv \sqrt{E_{k_l\hat{p}^1}}\, \hat{a}^-_{lm\hat{p}^1}, \tag{5.22}$$

similarly to Eq. (2.51).

Now, using these formulas, it is straightforward to compute the right-hand side of the formula in Eq. (5.7) and get the form of $a^{\mathrm{R}}_{lm\omega}$ (and its conjugate) in terms of the FFO mode operators $\hat{a}^-_{lm\rho}$ and $\hat{a}^{-\dagger}_{lm\rho}$. The answers take rather simple forms:

$$a^{\mathrm{R}}_{lm\omega} = \frac{1}{2i} \frac{e^{-i\omega\xi}}{\sqrt{\pi \sinh \pi\omega}} \int_{-\infty}^{\infty} d\rho \sin \omega\rho \left(e^{\pi\omega/2} \hat{a}^-_{lm\rho} - (-1)^m e^{-\pi\omega/2} \hat{a}^{-\dagger}_{l,-m\rho}\right), \tag{5.23}$$

$$a^{\mathrm{R}\dagger}_{lm\omega} = \frac{1}{2i} \frac{e^{i\omega\xi}}{\sqrt{\pi \sinh \pi\omega}} \int_{-\infty}^{\infty} d\rho \sin \omega\rho \left((-1)^m e^{-\pi\omega/2} \hat{a}^-_{l,-m\rho} - e^{\pi\omega/2} \hat{a}^{-\dagger}_{lm\rho}\right). \tag{5.24}$$

One can check that they satisfy the correct commutation relation, $[a^{\mathrm{R}}_{lm\omega}, a^{\mathrm{R}\dagger}_{l'm'\omega'}] = \delta_{ll'}\delta_{mm'}\delta(\omega - \omega')$.

Finally, with the expressions in Eq. (5.24), we can compute the expectation value of the number operator for the $\mathrm{W_R}$ "particles" in the FFO vacuum $|\hat{0}\rangle_-$. The result is

$$\frac{{}_-\langle\hat{0}|a^{\mathrm{R}\dagger}_{lm\omega} a^{\mathrm{R}}_{lm\omega}|\hat{0}\rangle_-}{{}_-\langle\hat{0}|\hat{0}\rangle_-} = \frac{1}{e^{2\pi\omega} - 1} \frac{2}{\pi} \int_{-\infty}^{\infty} d\rho \sin^2 \omega\rho. \tag{5.25}$$

Several remarks are in order:

(1) We recognize that the first factor is of the same form as the familiar "thermal" distribution. We emphasize, however, that in this case it is not genuinely thermal since $\mathrm{W_L}$ modes do not exist and hence no tracing over them is involved. The fact that the form looks thermal stems from the fact that the expression of $a^{\mathrm{R}}_{lm\omega}$ in terms of $\hat{a}^-_{lm\rho}$ and its conjugate in Eq. (5.23) is essentially the same as in Eq. (2.53), valid for the entire Minkowski space including the region $\mathrm{W_L}$.

(2) The last integral represents the coherent sum over an infinite number of rapidities which contribute to the $\mathrm{W_R}$ mode. Although it appears to depend on ω, this factor is divergent and, depending on how we cut it off, the ω dependence will be different. Moreover, as becomes clear from the comparison with the usual Unruh effect below, this factor comes from the nature of the boundary condition along the shock wave, i.e. it depends on the interaction between the falling matter and

the scalar field. Therefore, this integral is ambiguous and the form of its ω dependence should not be taken seriously. It indicates, however, that an extra ω dependence, other than the usual thermal factor, can be possible.

(3) As the last remark, note that the dependence on ξ, the Lorentz boost parameter, disappeared in the distribution. This is quite natural since the vacuum $|\hat{0}\rangle_-$ should be Lorentz invariant.

In any case, we have found that, even in the case of the one-sided black hole, the Unruh-like effect does exist.

It is instructive to compare this with the case of the usual Unruh effect. From Eq. (2.53), it is easy to find that

$$\frac{{}_{\mathrm{M}}\langle 0|a^{\mathrm{R}\dagger}_{k\omega}a^{\mathrm{R}}_{k\omega}|0\rangle_{\mathrm{M}}}{{}_{\mathrm{M}}\langle 0|0\rangle_{\mathrm{M}}} = \int_{-\infty}^{\infty}\frac{du}{4\pi\sinh\pi\omega}\int_{-\infty}^{\infty}du' e^{-i\omega(u-u')}e^{-\pi\omega}\frac{{}_{\mathrm{M}}\langle 0|[a^{\mathrm{M}}_{ku}, a^{\mathrm{M}\dagger}_{ku'}]|0\rangle_{\mathrm{M}}}{{}_{\mathrm{M}}\langle 0|0\rangle_{\mathrm{M}}}$$

$$= \frac{1}{2\pi\left(e^{2\pi\omega}-1\right)}\frac{V_{R^2}}{(2\pi)^2}\int_{-\infty}^{\infty}du, \tag{5.26}$$

where $\frac{V_{R^2}}{(2\pi)^2} = \delta^2(k-k)$ is the volume of the two-dimensional space and the divergent integral $\int_{-\infty}^{\infty} du$ counts all the modes with different rapidities making up a $\mathrm{W_R}$ particle wave. Note that in this flat space Unruh effect, the ω dependence $e^{-i\omega(u-u')}$ cancels out due to the appearance of $\delta(u-u')$ coming from the commutator $[a^{\mathrm{R}}_{ku}, a^{\mathrm{R}\dagger}_{ku'}]$ and we have exactly the thermal form, as is well known.

6. Summary and discussions

6.1. Brief summary

In this work we have made a detailed study of the issue of observer dependence for the quantization of fields in a curved spacetime, which is one of the crucial problems that one must deal with whenever one discusses quantum gravity. Understanding this issue is particularly important in cases where an event horizon exists for some of the observers. Explicitly, we have focused on the quantization of a scalar field in the most basic such configuration, namely the spacetime in the vicinity of the horizon of a four-dimensional Schwarzschild black hole, including the interior as well as the exterior. Detailed and comprehensive analyses were performed for the three typical observers, clarifying how the modes they observe are related. We studied both the two-sided eternal case and the more physical one-sided case produced by the falling shell, or a shock wave. For the latter, the effect of the collapsing matter upon the scalar field outside of the shell is represented by an effective boundary condition along the shock wave.

One important conclusion obtained from such explicit calculations is that as long as the interior of a large black hole can be described more or less by a metric like that of Vaidya, the free-falling ovserver sees no change in the Hilbert space structure of the quantized field as he/she crosses the horizon. In other words, the equivalence principle holds quantum mechanically as well, at least in the above sense.

Another result worth emphasizing is that in the one-sided case despite the fact that there are no counterparts of the left Rindler modes in the vacuum of the freely falling observer, and hence no tracing procedure over them is relevant, there still exists an Unruh-like effect. Namely, in such a vacuum the number density of the $\mathrm{W_R}$ modes contains the universal factor of “thermal” distribution in the frequency ω (apart from a divergent piece which depends on the interaction between the scalar field and the falling matter).

In addition to these results, comprehensive and explicit knowledge of the properties and the relations of the Hilbert spaces for the different observers have been obtained, and we believe this will be of use in better understanding the quantum properties of gravitational physics.

6.2. Discussion

Evidently, the problem of observer dependence that we studied in the semi-classical regime in this work is of universal importance in any attempt to understand quantum gravity. In particular, it would be extremely interesting to see how this problem appears and should be treated in the construction of the "bulk" from the "boundary" in the AdS/CFT correspondence, which is anticipated to give important hints for formulating quantum gravity and understanding quantum black holes. Although there have been some attempts to address this question, it is not well understood how the change of frame (i.e. the choice of "time") for the quantization, both in the bulk and the boundary, is expressed and controlled in the AdS/CFT context. The best place to look into would be the AdS_3/CFT_2 setting, where at least we have some knowledge of how the structure of CFT_2 changes under a redefinition of "time" by a conformal change of variable [49,50]. A further advantage to exploring the observer dependence in AdS_3/CFT_2 is that AdS_3 black holes (i.e. BTZ black holes) are locally equivalent to the pure AdS_3 spacetime and we can solve the equations of motion in the black hole spacetime in the same manner as for the pure AdS_3.

In this work we have concentrated on the relations between the modes seen by different observers and have not touched upon the correlation functions between the fields. Some two-point correlation functions in the Rindler wedges of the Minkowski space have been studied [51], but the most interesting question of whether one can extract physical information from behind the horizon or exchange information between different observers by quantum means is yet to be answered. We hope to study these and related questions and give a report in the near future.

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Appendix A. Orthogonality and completeness relations for the modified Bessel functions of imaginary order

For various basic computations performed in the main text using the expansions in terms of the eigenmodes, the orthogonality and the completeness of the modified Bessel functions of imaginary order are essential. In this appendix, we give some useful comments on such relations previously obtained in the literature and provide additional information.

Appendix A.1. Orthogonality

The orthogonality relations are needed in extracting each mode from the expansion of the scalar field appropriate for various coordinate frames. Such a relation for $K_{i\omega}(x)$ is proven in Refs. [42,43,52], and takes the form

$$\int_0^\infty \frac{dx}{x} K_{i\omega}(x) K_{i\omega'}(x) = \frac{1}{\mu(\omega)} \left(\delta(\omega - \omega') + \delta(\omega + \omega')\right), \quad \text{(A.1)}$$

where $\mu(\omega)$ here and below is given by

$$\mu(\omega) \equiv \frac{2\omega \sinh \pi\omega}{\pi^2}. \quad \text{(A.2)}$$

The corresponding relations for the Hankel functions $H^{(i)}_{i\omega}$ for $i = 1, 2$ have not been explicitly given in the literature but can be derived without difficulty, for example by the method described in Ref. [43]. The result is

$$\int_0^\infty \frac{dx}{x} H^{(i)}_{i\omega}(x) H^{(i)}_{i\omega'}(x) = \frac{4e^{\eta_i \pi\omega}}{\pi^2 \mu(\omega)} \left(\delta(\omega - \omega') + \delta(\omega + \omega')\right), \quad \eta_1 = +1, \quad \eta_2 = -1. \quad \text{(A.3)}$$

Appendix A.2. Completeness

The completeness relation for the function $K_{i\omega}(x)$ can be written as

$$\int_0^\infty d\omega \mu(\omega) K_{i\omega}(x) K_{i\omega}(y) = x\delta(x - y). \quad \text{(A.4)}$$

Since $K_{-i\omega} = K_{i\omega}$, we can, if we wish, extend the range of integration to $[-\infty \le \omega \le \infty]$ and multiply the right-hand side by a factor of 2.

This relation is equivalent to the inverse of the so-called Kontorovich–Lebedev (KL) transform [44] below. The KL transform $f(\omega, y)$ of a function $g(x, y)$ with respect to x (where y is a parameter) is defined by

$$f(\omega, y) = \mu(\omega) \int_0^\infty \frac{dx}{x} K_{i\omega}(x) g(x, y). \quad \text{(A.5)}$$

Then, $g(x, y)$ is obtained in terms of $f(\omega, y)$ by the formula

$$g(x, y) = \int_0^\infty d\omega K_{i\omega}(x) f(\omega, y). \quad \text{(A.6)}$$

If we take $g(x, y) = x\delta(x-y)$, then the formula in Eq. (A.5) gives $f(\omega, y) = \mu(\omega) K_{i\omega}(y)$. Substituting this into Eq. (A.6) then gives

$$x\delta(x - y) = \int_0^\infty d\omega \mu(\omega) K_{i\omega}(x) K_{i\omega}(y), \quad \text{(A.7)}$$

which is precisely the completeness relation of Eq. (A.4).

In fact, without resorting to the KL formula, there is a rather elementary derivation of Eq. (A.4), starting from the following integral formula [45]:

$$\int_0^\infty d\omega \cosh a\omega K_{i\omega}(x) K_{i\omega}(y) = \frac{\pi}{2} K_0(\sqrt{x^2 + y^2 + 2xy\cos a}), \quad \text{(A.8)}$$

valid for $x, y > 0$, $|\mathrm{Re}\, a| + |\arg x| < \pi$. (The second condition is stringent. We cannot set $a = \pi$ from the beginning.) First, by differentiating this with respect to a, we get

$$\int_0^\infty d\omega\omega \sinh a\omega K_{i\omega}(x) K_{i\omega}(y) = \frac{\pi xx' \sin a}{2\sqrt{x^2 + y^2 + 2xy\cos a}} K_1(\sqrt{x^2 + y^2 + 2xy\cos a}), \quad \text{(A.9)}$$

where we used the formula $\partial_z K_0(z) = -K_1(z)$. Now we set $a = \pi - \epsilon$, where ϵ is a positive infinitesimal quantity. Then the right-hand side becomes

$$\frac{\pi xy\epsilon}{2\sqrt{(x-y)^2 + xy\epsilon^2}} K_1(\sqrt{(x-y)^2 + xy\epsilon^2}). \tag{A.10}$$

For $x - y \neq 0$, this vanishes as $\epsilon \to 0$, i.e. as $a \to \pi$. On the other hand, for small $x - y$, using the small-argument expansion $K_1(z) \simeq \frac{1}{z}$, Eq. (A.10) becomes

$$\frac{\pi}{2} \frac{\epsilon xy}{(x-y)^2 + \epsilon^2 xy}. \tag{A.11}$$

By making a rescaling $x \to x/\sqrt{xy}$ and $y \to y/\sqrt{xy}$ in the well-known representation of the delta function, namely $\delta(x - y) = (\epsilon/\pi)/((x - y)^2 + \epsilon^2))$, we readily obtain

$$\delta((x-y)/\sqrt{xy}) = \sqrt{xy}\delta(x-y) = \frac{1}{\pi} \frac{\epsilon xy}{(x-y)^2 + \epsilon^2 xy}. \tag{A.12}$$

Comparing with Eq. (A.11), we obtain the completeness relation of Eq. (A.4).

The corresponding completeness relations for the Hankel functions are given by

$$\int_0^\infty d\omega \frac{\pi^2 \mu(\omega)}{4e^{\eta_i \pi\omega}} H^{(i)}_{i\omega}(x) H^{(i)}_{i\omega}(y) = x\delta(x-y), \tag{A.13}$$

where the sign η_i is as defined in Eq. (A.3).

Appendix B. Extraction of the modes in various wedges and their relations

In this appendix we provide some details of the computations concerning the extraction of the modes and their relations described in Sect. 2.

Appendix B.1. Klein–Gordon inner products and extraction of the modes

We are interested in a d-dimensional curved space with a metric of the form

$$ds^2 = -N(x)^2 dt^2 + g_{ab} dx^a dx^b, \tag{B.1}$$

where $N(x)$ is the lapse function. Let f_A, f_B be two independent solutions of the Klein–Gordon equation for this metric. Define the following current,

$$J^\mu_{f_A, f_B}(x) \equiv f_A^*(x) \overleftrightarrow{\nabla}^\mu f_B, \tag{B.2}$$

which is covariantly conserved, $\nabla_\mu J^\mu_{f_A, f_B}(x) = 0$. Let Σ be the constant-t surface. The conservation property above means that the Klein–Gordon inner product defined by

$$(f_A, f_B)_{\text{KG}} \equiv i \int_\Sigma d^{d-1}x \frac{\sqrt{g}}{N} n_\mu J^\mu_{f_A, f_B}, \tag{B.3}$$

where n^μ is the future-directed unit vector normal to Σ, is independent of t. This formula is useful in extracting the modes from the field expressed in various coordinates.

Appendix B.1.1. The right Rindler wedge

Hereafter, we will set $d = 4$. In the right Rindler wedge, the metric is given by

$$ds^2 = -z^2dt^2 + dz^2 + \sum_{i=2}^{3}(dx^i)^2. \tag{B.4}$$

In this case, we can identify $N = z$, $g_{zz} = 1$, $g_{ij} = \delta_{ij}$, and $\sqrt{g} = 1$, and hence the Klein–Gordon inner product in the right Rindler wedge is defined as

$$(f_A, f_B)^{\mathrm{R}}_{\mathrm{KG}} = i\int_0^\infty \frac{dz}{z}\int d^2x(f_A^* \overleftrightarrow{\partial_t} f_B). \tag{B.5}$$

The solutions of the Klein–Gordon equation in this coordinate frame are

$$f^{\mathrm{R}}_{k\omega}(t,z,x) = N^{\mathrm{R}}_{\omega}K_{i\omega}(|k|z)e^{i(kx-\omega t)},$$
$$(N^{\mathrm{R}}_{\omega})^2 = \frac{\sinh\pi\omega}{\pi^2(2\pi)^2}. \tag{B.6}$$

Let us compute the Klein–Gordon inner product of such functions explicitly. We get

$$\begin{aligned}(f^{\mathrm{R}}_{k\omega}, f^{\mathrm{R}}_{k'\omega'})^{\mathrm{R}}_{\mathrm{KG}} &= \int_0^\infty \frac{dz}{z}\int d^2x N^{\mathrm{R}}_{\omega}N^{\mathrm{R}}_{\omega'}K_{i\omega}(|k|z)K_{i\omega'}(|k'|z)(\omega+\omega')e^{i(k-k')x}e^{-i(\omega-\omega')t} \\ &= (2\pi)^2(\omega+\omega')\delta(k-k')N^{\mathrm{R}}_{\omega}N^{\mathrm{R}}_{\omega'}e^{-i(\omega-\omega')t}\int_0^\infty \frac{dz}{z}K_{i\omega}(|k|z)K_{i\omega'}(|k'|z) \\ &= \delta(k-k')\delta(\omega-\omega'),\end{aligned} \tag{B.7}$$

where, getting to the last line, we used the orthogonality of the modified Bessel function in Eq. (A.1) for $\omega, \omega' > 0$.

Recall that the scalar field in the right Rindler wedge can be expanded as

$$\phi^{\mathrm{R}}(t_{\mathrm{R}}, z_{\mathrm{R}}, x) = \int_0^\infty d\omega \int d^2k\left[f^{\mathrm{R}}_{k\omega}(t_{\mathrm{R}}, z_{\mathrm{R}}, x)a^{\mathrm{R}}_{k\omega} + \mathrm{h.c.}\right]. \tag{B.8}$$

The modes $a^{\mathrm{R}}_{k\omega}$ and $a^{\mathrm{R}\dagger}_{k\omega}$ are extracted using the Klein–Gordon inner product as

$$a^{\mathrm{R}}_{k\omega} = (f^{\mathrm{R}}_{k\omega}, \phi^{\mathrm{R}})^{\mathrm{R}}_{\mathrm{KG}}, \qquad a^{\mathrm{R}\dagger}_{k\omega} = -(f^{\mathrm{R}*}_{k\omega}, \phi^{\mathrm{R}})^{\mathrm{R}}_{\mathrm{KG}}. \tag{B.9}$$

Appendix B.1.2. The future Rindler wedge

In the future Rindler wedge, the metric is given by

$$ds^2 = -dz_{\mathrm{F}}^2 + z_{\mathrm{F}}^2dt_{\mathrm{F}}^2 + \sum_{i=2}^{3}(dx^i)^2. \tag{B.10}$$

In this case, z_{F} is the time variable, t_{F} is the space variable, and $N = 1$, $g_{t_{\mathrm{F}}t_{\mathrm{F}}} = z_{\mathrm{F}}^2$, $g_{ij} = \delta_{ij}$, $\sqrt{g} = z_{\mathrm{F}}$. The Klein–Gordon inner product in the future Rindler wedge is defined as

$$(f_A, f_B)^{\mathrm{F}}_{\mathrm{KG}} = i\int_{-\infty}^\infty dt_{\mathrm{F}}\int d^2x z_{\mathrm{F}}(f_A^* \overleftrightarrow{\partial_{z_{\mathrm{F}}}} f_B). \tag{B.11}$$

Then the solutions of the Klein–Gordon equation which damp at large $|k|z_{\mathrm{F}}$ are

$$f^{(2)}_{k\omega}(t_{\rm F},z_{\rm F},x)=N^{\rm F}_\omega H^{(2)}_{i\omega}(|k|z_{\rm F})e^{i(kx-\omega t_{\rm F})},$$
$$(N^{\rm F}_\omega)^2=\frac{e^{\pi\omega}}{8(2\pi)^2}. \tag{B.12}$$

The Klein–Gordon inner product of these functions is given by

$$\begin{aligned}
&(f^{(2)}_{k\omega},f^{(2)}_{k'\omega'})^{\rm F}_{\rm KG}\\
&=i\int_{-\infty}^{\infty}dt\int d^2xzN^{\rm F}_\omega N^{\rm F}_{\omega'}\left(H^{(1)}_{i\omega'}(|k'|z_{\rm F})\partial_{z_{\rm F}}H^{(2)}_{i\omega}(|k|z_{\rm F})-H^{(2)}_{i\omega}(|k|z_{\rm F})\partial_{z_{\rm F}}H^{(1)}_{i\omega'}(|k'|z_{\rm F})\right)\\
&\quad\cdot e^{-\pi\omega}e^{i(k-k')x}e^{-i(\omega-\omega')t}\\
&=i(2\pi)^3\delta(k-k')\delta(\omega-\omega')\\
&\quad\cdot z_{\rm F}N^{\rm F}_\omega N^{\rm F}_{\omega'}\left(H^{(1)}_{i\omega'}(|k'|z_{\rm F})\partial_{z_{\rm F}}H^{(2)}_{i\omega}(|k|z_{\rm F})-H^{(2)}_{i\omega}(|k|z_{\rm F})\partial_{z_{\rm F}}H^{(1)}_{i\omega'}(|k'|z)\right)e^{-\pi\omega}\\
&=\delta(k-k')\delta(\omega-\omega').
\end{aligned} \tag{B.13}$$

To get to the last line, we used the identity

$$H^{(1)}_{i\omega}(|k|z)\partial_z H^{(2)}_{i\omega}(|k|z)-H^{(2)}_{i\omega}(|k|z)\partial_z H^{(1)}_{i\omega}(|k|z)=-i\frac{4}{\pi z}. \tag{B.14}$$

By similar manipulations, it is easy to get the following inner products:

$$(f^{(2)*}_{k\omega},f^{(2)*}_{k'\omega'})^{\rm F}_{\rm KG}=-\delta(k-k')\delta(\omega-\omega'),\qquad (f^{(2)}_{k\omega},f^{(1)}_{k'\omega'})^{\rm F}_{\rm KG}=0. \tag{B.15}$$

Recall that the scalar field in the future Rindler wedge can be expanded as

$$\phi^{\rm F}(t_{\rm F},z_{\rm F},x)=\int_{-\infty}^{\infty}d\omega\int d^2k\left[f^{(2)}_{k\omega}(t_{\rm F},z_{\rm F},x)a^{\rm F}_{k\omega}+\text{h.c.}\right]. \tag{B.16}$$

Taking the inner product with $f^{(2)}_{k\omega}$, we obtain

$$a^{\rm F}_{k\omega}=(f^{(2)}_{k\omega},\phi^{\rm F})^{\rm F}_{\rm KG},\qquad a^{\rm F\dagger}_{k\omega}=-(f^{(2)*}_{k\omega},\phi^{\rm F})^{\rm F}_{\rm KG}. \tag{B.17}$$

Appendix B.2. Mode operators of $W_{\rm R}$ and $W_{\rm F}$ in terms of those of the Minkowski spacetime

Appendix B.2.1. Useful integrals involving $K_{i\omega}(z)$

We shall first derive several useful integrals involving $K_{i\omega}(z)$, which play important roles in Sects. B.2.2 and 5.2.

Formula I The first formula is

$$\int_0^\infty\frac{dz}{z}K_{i\omega}(z)e^{-iz\sinh t-\epsilon z}=\frac{\pi}{2\omega\sinh\pi\omega}\left(e^{i\omega t}e^{-\pi\omega/2}+e^{-i\omega t}e^{\pi\omega/2}\right), \tag{B.18}$$

where ϵ is an infinitesimal positive parameter, needed to make the integral convergent. To prove this formula, we start with the formula in Eq. (6.795-1) of Ref. [47], which can be expressed as

$$\frac{\pi}{2}e^{-z\cosh\tau} = \int_0^\infty d\omega \cos(\omega\tau)K_{i\omega}(z), \quad |\mathrm{Im}\,\tau| < \frac{\pi}{2}, \quad z > 0. \tag{B.19}$$

By extending the region of ω to $[-\infty, \infty]$ for convenience,[27] the integral on the right-hand side (RHS) can be rewritten as

$$\mathrm{RHS} = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega'\tau}K_{i\omega'}(z)d\omega'. \tag{B.20}$$

We now act $\int_0^\infty (dz/z)K_{i\omega}(z)$ on this expression, with ω non-negative. Then, using the orthogonality relation in Eq. (A.1) for $K_{i\omega}(z)$, we get

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} d\omega' e^{i\omega'\tau}\int_0^\infty \frac{dz}{z}K_{i\omega}(z)K_{i\omega'}(z) &= \frac{1}{2}\int_{-\infty}^{\infty} d\omega' e^{i\omega'\tau}\frac{1}{\mu(\omega)}\left(\delta(\omega'-\omega)+\delta(\omega'+\omega)\right)\\ &= \frac{1}{2\mu(\omega)}\left(e^{i\omega\tau}+e^{-i\omega\tau}\right) = \frac{1}{\mu(\omega)}\cos\omega\tau,\end{aligned} \tag{B.21}$$

where $\mu(\omega)$ is as given in Eq. (A.2). Performing the same integral for the left-hand side as well, Eq. (B.19) becomes

$$\int_0^\infty \frac{dz}{z}K_{i\omega}(z)e^{-z\cosh\tau} = \frac{2}{\pi\mu(\omega)}\cos\omega\tau. \tag{B.22}$$

We now make the substitution

$$\tau = t + ((\pi/2)-\epsilon)i, \tag{B.23}$$

where ϵ is an infinitesimal positive quantity. This is legitimate since $\mathrm{Im}\,\tau$ satisfies the condition for the formula in Eq. (B.19) to be valid. Then, by a simple calculation we get $\cosh\tau = \cosh\left(t + \frac{\pi}{2}i - i\epsilon\right) = i\sinh t + \epsilon$, where we re-expressed a positive infinitesimal quantity $\epsilon\cosh t$ as ϵ. Substituting Eq. (B.23) into the RHS of Eq. (B.22) and expanding $\cos\omega\tau$, we obtain Formula I.

Formula II The second formula is

$$\begin{aligned}A_1(c,k) &\equiv \int_0^\infty \frac{dz}{z}K_{i\omega}(kz)e^{-icz-\epsilon z}\\ &= \frac{\pi}{2\omega\sinh\pi\omega}\left\{e^{\pi\omega/2}\exp\left(-i\omega\ln\left(\frac{1}{k}\left(c+\sqrt{c^2+k^2}\right)\right)\right)\right.\\ &\quad \left.+\, e^{-\pi\omega/2}\exp\left(i\omega\ln\left(\frac{1}{k}\left(c+\sqrt{c^2+k^2}\right)\right)\right)\right\},\end{aligned} \tag{B.24}$$

where c is real and k is real positive. To prove this formula, we first rescale $z \to kz$ in Formula I and then set $c = k\sinh t$. Solving e^t in terms of c and substituting into the RHS of Formula I, we obtain the integral above.

[27] This is purely as a mathematical equality. The physical energy ω is, of course, non-negative.

Formula III Finally, a formula similar to II that we need is

$$\begin{aligned} A_2(c,k) &\equiv \int_0^\infty dz\, K_{i\omega}(kz) e^{-icz} \\ &= \frac{\pi}{2\sinh \pi\omega} \frac{1}{\sqrt{c^2+k^2}} \Big\{ e^{\pi\omega/2} \exp\left(-i\omega \ln\left(\frac{1}{k}\left(c+\sqrt{c^2+k^2}\right)\right)\right) \\ &\quad - e^{-\pi\omega/2} \exp\left(i\omega \ln\left(\frac{1}{k}\left(c+\sqrt{c^2+k^2}\right)\right)\right)\Big\}. \end{aligned} \tag{B.25}$$

This formula is obtained simply from $A_1(c,k)$ as $A_2(c,k) = i(\partial A_1(c,k)/\partial c)$.

Appendix B.2.2. $a^{\rm R}_{k\omega}$ *in terms of* $a^{\rm M}_{kp^1}$

The free scalar field in the right Rindler wedge can be expanded as in Eq. (B.8). On the other hand, in this region we should be able to express $\phi^{\rm R}$ in terms of $\phi^{\rm M}$ and hence $a^{\rm R}_{\omega k}$ in terms of the Minkowski modes $a^{\rm M}_{kp^1}$. In the Minkowski spacetime the scalar fields can be written in terms of the coordinates of $\mathrm{W_R}$ as

$$\begin{aligned} \phi^{\rm M}(z_{\rm R}, t_{\rm R}, x) &= \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k',p^1}}} \int \frac{d^2k'}{2\pi} e^{ik'x+ip^1x^1-iE_{k',p^1}t_{\rm M}} a^{\rm M}_{k'p^1} + \text{h.c.} \\ &= \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k',p^1}}} \int \frac{d^2k'}{2\pi}\, e^{ik'x+ip^1z_{\rm R}\cosh t_{\rm R}-iE_{k',p^1}z_{\rm R}\sinh t_{\rm R}} a^{\rm M}_{k'p^1} + \text{h.c.}, \end{aligned} \tag{B.26}$$

where in the second line we substituted $t_{\rm M} = z_{\rm R}\sinh t_{\rm R},\ x^1 = z_{\rm R}\cosh t_{\rm R}$. Thus, using the Klein–Gordon inner product we can extract the annihilation operators in the right Rindler coordinate from the expression of the scalar field in the Minkowski spacetime as

$$\begin{aligned} a^{\rm R}_{k\omega} &= (f^{\rm R}_{k\omega}, \phi^{\rm M})^{\rm R}_{\rm KG} \\ &= i\int_0^\infty \frac{dz}{z} \int d^2x (f^{{\rm R}*}_{k\omega} \overleftrightarrow{\partial_{t_{\rm R}}} \phi^{\rm M}) \\ &= i\int_0^\infty \frac{dz_{\rm R}}{z_{\rm R}} \int d^2x \int \frac{dp^1}{\sqrt{4\pi E_{k'p^1}}} \int \frac{d^2k'}{2\pi} N^{\rm R}_\omega K^*_{i\omega}(|k|z_{\rm R}) \\ &\times \left(e^{-i(kx-\omega t_{\rm R})} \overleftrightarrow{\partial_{t_{\rm R}}} [e^{ik'x+ip^1z_{\rm R}\cosh t_{\rm R}-iE_{k'p^1}z_{\rm R}\sinh t_{\rm R}} a^{\rm M}_{k'p^1} + \text{h.c.}]\right) \\ &= i\int \frac{2\pi\, dp^1 N^{\rm R}_\omega}{\sqrt{4\pi E_{k'p^1}}} e^{i\omega t_{\rm R}} \left(\overleftrightarrow{\partial_{t_{\rm R}}} \int_0^\infty \frac{dz_{\rm R}}{z_{\rm R}} K^*_{i\omega}(|k|z_{\rm R}) e^{-ip^1z_{\rm R}\cosh t_{\rm R}-iE_{kp^1}z_{\rm R}\sinh t_{\rm R}} a^{\rm M}_{kp^1} \right. \\ &\left. + \overleftrightarrow{\partial_{t_{\rm R}}} \int_0^\infty \frac{dz_{\rm R}}{z_{\rm R}} K_{i\omega}(|k|z_{\rm R}) e^{ip^1z_{\rm R}\cosh t_{\rm R}+iE_{-kp^1}z_{\rm R}\sinh t_{\rm R}} a^{\rm M\dagger}_{-kp^1}\right). \end{aligned}$$

Let us now use the convenient parametrization $E_{kp^1} = |k|\cosh\rho,\ p^1 = -|k|\sinh\rho$, such that $E^2_{kp^1} = (p^1)^2 + k^2$ is realized. Then the expression above can be written as

$$\begin{aligned} a^{\rm R}_{k\omega} = i\int \frac{2\pi\, dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} N^{\rm R}_\omega e^{i\omega t_{\rm R}} \overleftrightarrow{\partial_{t_{\rm R}}} \int_0^\infty \frac{dz_{\rm R}}{z_{\rm R}} K_{i\omega}(|k|z_{\rm R}) \\ \times \left[e^{-i|k|z_{\rm R}\sinh(t_{\rm R}+\rho)} a^{\rm M}_{kp^1} + e^{i|k|z_{\rm R}\sinh(t_{\rm R}+\rho)} a^{\rm M\dagger}_{-kp^1}\right], \end{aligned}$$

where we used the property $K_{i\omega}(z) = K_{-i\omega}(z)$.

Now, by using Formula I given in Eq. (B.18), we can perform the integral over $z_{\rm R}$ and get

$$a^{\rm R}_{k\omega} = i\int \frac{2\pi\, dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} N^{\rm R}_\omega \frac{\pi}{2\omega\sinh\pi\omega} e^{i\omega t_{\rm R}} \overleftrightarrow{\partial_{t_{\rm R}}} \Big[\Big(e^{-i\omega(t_{\rm R}+\rho)}e^{\pi\omega/2} + e^{i\omega(t_{\rm R}+\rho)}e^{-\pi\omega/2}\Big) a^{\rm M}_{kp^1}$$
$$+\Big(e^{-i\omega(t_{\rm R}+\rho)}e^{-\pi\omega/2} + e^{i\omega(t_{\rm R}+\rho)}e^{\pi\omega/2}\Big) a^{\rm M\dagger}_{-kp^1}\Big]$$
$$= \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{1}{\sqrt{\sinh\pi\omega}} \left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{\frac{-i\omega}{2}} \left[e^{\pi\omega/2}a^{\rm M}_{kp^1} + e^{-\pi\omega/2}a^{\rm M\dagger}_{-kp^1}\right]. \qquad \text{(B.27)}$$

This is the formula quoted in Eq. (2.47). Taking the Hermitian conjugate, we obtain the creation operator

$$a^{\rm R\dagger}_{k\omega} = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{1}{\sqrt{\sinh\pi\omega}} \left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{\frac{i\omega}{2}} \left[e^{\pi\omega/2}a^{\rm M\dagger}_{kp^1} + e^{-\pi\omega/2}a^{\rm M}_{-kp^1}\right]. \qquad \text{(B.28)}$$

Appendix B.2.3. $a^{\rm F}_{k\omega}$ *in terms of* $a^{\rm M}_{kp^1}$

As in $\rm W_R$, the free scalar field in the $\rm W_F$ frame should be describable in terms of the Minkowski modes. It is expanded as in Eq. (2.39) in terms of the Hankel functions $H^{(2)}_{i\omega}(|k|z_{\rm F})$, which is recalled in Eq. (B.16) for convenience. If we write such a field in the Minkowski spacetime in terms of the coordinates of $\rm W_F$, using the relation $t_{\rm M} = z_{\rm F}\cosh t_{\rm F}$, $x^1 = z_{\rm F}\sinh t_{\rm F}$, it reads

$$\phi^{\rm M}(z_{\rm F}, t_{\rm F}, x) = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} \int \frac{d^2k'}{2\pi} e^{ik'x_{\rm F}+ip^1 z\sinh t_{\rm R} - iE_{k'p^1}z\cosh t_{\rm F}} a^{\rm M}_{k'p^1} + \text{h.c.}. \qquad \text{(B.29)}$$

Using the Klein–Gordon inner product, we can extract $a^{\rm R}_{k\omega}$ from the Minkowski field $\phi^{\rm M}(z_{\rm F}, t_{\rm F}, x)$ as

$$a^{\rm F}_{k\omega} = (f^{\rm F}_{k\omega}, \phi^{\rm M})^{\rm F}_{\rm KG} = i\int_{-\infty}^{\infty} z dt_{\rm F} dx^2 (f^{\rm F*}_{k\omega} \overleftrightarrow{\partial_z} \phi^{\rm M})$$
$$= i\int_{-\infty}^{\infty} z_{\rm F} dt_{\rm F} dx^2 \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} \int \frac{d^2k'}{2\pi} N^{\rm F}_\omega e^{-i(kx_{\rm F}-\omega t_{\rm F})}$$
$$\times \Big(H^{(2)*}_{i\omega}(|k|z_{\rm F}) \overleftrightarrow{\partial_{z_{\rm F}}} [e^{ik'x_{\rm F}+ip^1 z_{\rm F}\sinh t_{\rm F} - iE_{k'p^1}z_{\rm F}\cosh t_{\rm F}} a^{\rm M}_{k'p^1} + \text{h.c.}]\Big)$$
$$= i\int_{-\infty}^{\infty} z_{\rm F} dt_{\rm F} \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{e^{\pi\omega/2}}{2\sqrt{2}} \Big(H^{(2)*}_{i\omega}(|k|z) \overleftrightarrow{\partial_{z_{\rm F}}} \big[e^{ip^1 z_{\rm F}\sinh t_{\rm F} - iE_{kp^1}z_{\rm F}\cosh t_{\rm F}} e^{i\omega t_{\rm F}} a^{\rm M}_{kp^1}$$
$$+ e^{-ip^1 z_{\rm F}\sinh t_{\rm F} + iE_{-kp^1}z_{\rm F}\cosh t_{\rm F}} e^{i\omega t_{\rm F}} a^{\rm M\dagger}_{-kp^1}\big]\Big). \qquad \text{(B.30)}$$

We now use the following integral representations [48]:

$$\text{(i)}\ \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^{\nu/2} H^{(1)}_\nu(\sqrt{\alpha^2-\beta^2}) = \frac{e^{-\nu\pi i/2}}{\pi i}\int_{-\infty}^{\infty} e^{i\alpha\cosh\tau + i\beta\sinh\tau - \nu\tau} d\tau, \quad \text{Im}(\alpha\pm\beta) > 0,$$
$$\text{(ii)}\ \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^{\nu/2} H^{(2)}_\nu(\sqrt{\alpha^2-\beta^2}) = -\frac{e^{\nu\pi i/2}}{\pi i}\int_{-\infty}^{\infty} e^{-i\alpha\cosh\tau - i\beta\sinh\tau - \nu\tau} d\tau, \quad \text{Im}(\alpha\pm\beta) < 0.$$
$$\text{(B.31)}$$

Note that formula (i) can be obtained by the analytic continuation $\alpha \to e^{i\pi}\alpha,\ \beta \to e^{i\pi}\beta$ from formula (ii).

For the part of Eq. (B.30) containing $a^{\rm M}_{kp^1}$, namely

$$H^{(2)*}_{i\omega}(|k|z)\overleftrightarrow{\partial_z}\int_{-\infty}^{\infty} dt_{\rm F} e^{ip^1 z_{\rm F}\sinh t_{\rm F} - iE_{k'p^1} z_{\rm F}\cosh t_{\rm F}} e^{i\omega t_{\rm F}} a^{\rm M}_{k'p^1}, \tag{B.32}$$

we can use formula (ii). On the other hand, for the part containing $a^{\rm M\dagger}_{kp^1}$, i.e.

$$H^{(2)*}_{i\omega}(|k|z)\overleftrightarrow{\partial_z}\int_{-\infty}^{\infty} dt_{\rm F} e^{-ip^1 z\sinh t_{\rm F} + iE_{k'p^1} z\cosh t_{\rm F}} e^{i\omega t_{\rm F}} a^{\rm M\dagger}_{k'p^1}, \tag{B.33}$$

it is convenient to use formula (i). In this way, we can compute Eq. (B.30) as

$$\begin{aligned}
a^{\rm F}_{\omega,k} &= -z_{\rm F}\int \frac{\pi\, dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}}\frac{e^{\pi\omega/2}}{2\sqrt{2}}\nu\Bigg(H^{(2)*}_{i\omega}(|k|z)\overleftrightarrow{\partial_z}\Bigg[H^{(2)}_{i\omega}(|k|z)e^{-\pi\omega/2}\left(\frac{E_{kp^1}-p}{E_{kp^1}+p}\right)^{-\frac{i\omega}{2}} a^{\rm M}_{kp^1} \\
&\quad - H^{(1)}_{i\omega}(|k|z)e^{\pi\omega/2}\left(\frac{E_{-kp^1}-p}{E_{-kp^1}+p}\right)^{\frac{i\omega}{2}} a^{\rm M\dagger}_{-kp^1}\Bigg]\Bigg) \\
&= i\int \frac{dp^1}{\sqrt{2\pi E_{-kp^1}}}\left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{-\frac{i\omega}{2}} a^{\rm M}_{kp^1}.
\end{aligned} \tag{B.34}$$

In the last step, we used the identity in Eq. (B.14).

Together with the similar result for $a^{\rm F\dagger}_{k\omega}$, we can summarize the results as

$$\begin{aligned}
a^{\rm F}_{\omega,k} &= i\int \frac{dp^1}{\sqrt{2\pi E_{kp^1}}}\left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{-\frac{i\omega}{2}} a^{\rm M}_{kp^1}, \\
a^{\rm F\dagger}_{\omega,k} &= -i\int \frac{dp^1}{\sqrt{2\pi E_{kp^1}}}\left(\frac{E_{kp^1}-p^1}{E_{kp^1}+p^1}\right)^{\frac{i\omega}{2}} a^{\rm M\dagger}_{kp^1}.
\end{aligned} \tag{B.35}$$

This is the relation quoted in Eq. (2.54), and its Hermitian conjugate. As shown in Eq. (2.55), in terms of the rapidity variable u defined in Eq. (2.48), these relations can be interpreted as Fourier transforms, and then it is practically trivial to check the desired commutation relations,

$$[a^{\rm F}_{k\omega}, a^{\rm F\dagger}_{k'\omega'}] = \delta(k-k')\delta(\omega-\omega'), \qquad \text{rest} = 0. \tag{B.36}$$

Appendix B.3. Sketch of the proof that $\phi^{\rm M}(t_{\rm M}, x^1, x)$ depends only on the modes of $W_{\rm L}$ ($W_{\rm R}$) for $x^1 < 0$ ($x^1 > 0$)

In this appendix we give a sketch of the proof that the scalar field in the Minkowski space $\phi^{\rm M}(t_{\rm M}, x^1, x)$, when expressed in terms of the oscillators of the Rindler wedge $\rm W_R$ and those of $\rm W_L$, receive only the contribution of the former (resp. the latter) in the region $\rm W_R$ (resp. $\rm W_L$).

As in Eq. (2.20), $\phi^{\rm M}(t_{\rm M}, x^1, x)$ is expanded in the plane wave basis as

$$\phi^{\rm M}(t_{\rm M}, x^1, x) = \int_{-\infty}^{\infty}\frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}}\int \frac{d^2k}{2\pi} e^{ikx + ip^1x^1 - iE_{kp^1}t_{\rm M}} a^{\rm M}_{kp^1} + \text{h.c.} \tag{B.37}$$

Now, substitute the expression of $a^{\rm M}_{kp^1} = a^{\rm M}_{ku}/\sqrt{E_{kp^1}}$ in terms of $a^{\rm F}_{k\omega}$ given in Eq. (2.56), and use the expressions of $a^{\rm F}_{k\omega}$ and $a^{\rm F}_{k,-\omega}$ in terms of $a^{\rm R}_{k\omega}$ and $a^{\rm L}_{k\omega}$ given in Eqs. (2.83) and (2.84). This gives $\phi^{\rm M}$ in terms of the modes of $\rm W_R$ and $\rm W_L$. After a simple rearrangement we obtain

$$\phi^{\rm M}(t_{\rm M}, x^1, x) = \int \frac{d^2k}{2\pi\sqrt{4\pi}} e^{ikx} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}\sqrt{2\sinh\pi\omega}} \cdot \left(-I(\omega)\left[e^{\pi\omega/2}a^{\rm R}_{k\omega} - e^{-\pi\omega/2}a^{\rm L\dagger}_{k\omega}\right] + I(-\omega)\left[e^{-\pi\omega/2}a^{\rm R\dagger}_{k\omega} - e^{\pi\omega/2}a^{\rm L}_{k\omega}\right]\right) + \text{h.c.}, \tag{B.38}$$

where

$$I(\omega) \equiv \int_{-\infty}^{\infty} du\, e^{i|k|x^1\sinh u - i|k|t_{\rm M}\cosh u - i\omega u}. \tag{B.39}$$

We must study the conditions under which this integral exists. First, for $u \to \infty$, the dominant part of the exponent is $i\frac{|k|}{2}e^u(x^1 - t_{\rm M})$. Thus, for the integral to converge in this region, we need the condition Im $(x^1 + t_{\rm M}) < 0$. On the other hand, for $u \to -\infty$ the dominant part of the exponent is $-i\frac{|k|}{2}(x^1 + t_{\rm M})$, and for convergence we need Im $(x^1 + t_{\rm M}) < 0$. These two conditions can be met simultaneously if we make the shift

$$t_{\rm M} \longrightarrow t_{\rm M} - i\epsilon, \qquad \epsilon > 0. \tag{B.40}$$

Then, we can make use of the formula in Eq. (10.9.16) of Ref. [48] and get

$$I(\omega) = -i\pi e^{\pi\omega/2}\left(\frac{t_{\rm M} - x^1 - i\epsilon}{t_{\rm M} + x^1 - i\epsilon}\right)^{i\omega/2} H^{(2)}_{i\omega}(((t_{\rm M} - i\epsilon)^2 - (x^1)^2)^{1/2}). \tag{B.41}$$

To express $\phi^{\rm M}(t_{\rm M}, x^1, x)$ in Eq. (B.38) it is clear that in addition to $I(\omega)$ we need the integrals $I(-\omega)$, $I(\omega)^*$, and $I(-\omega)^*$. To obtain them from $I(\omega)$, we need to make use of the well-known relations among the Hankel functions (see, for example, Eqs. (10.46) and (10.11.9) of Ref. [48]):

$$H^{(1)}_{-i\omega}(z) = e^{-\pi\omega}H^{(1)}_{i\omega}(z), \qquad H^{(2)}_{-i\omega}(z) = e^{\pi\omega}H^{(2)}_{i\omega}(z), \tag{B.42}$$

$$H^{(1)}_{i\omega}(z)^* = H^{(2)}_{-i\omega}(z^*) = e^{\pi\omega}H^{(2)}_{i\omega}(z^*), \tag{B.43}$$

$$H^{(2)}_{i\omega}(z)^* = H^{(1)}_{-i\omega}(z^*) = e^{-\pi\omega}H^{(1)}_{i\omega}(z^*). \tag{B.44}$$

We then get

$$I(\omega) = -i\pi e^{\pi\omega/2}\left(\frac{t_{\rm M} - x^1 - i\epsilon}{t_{\rm M} + x^1 - i\epsilon}\right)^{i\omega/2} H^{(2)}_{i\omega}(((t_{\rm M} - i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.45}$$

$$I(-\omega) = -i\pi e^{\pi\omega/2}\left(\frac{t_{\rm M} - x^1 - i\epsilon}{t_{\rm M} + x^1 - i\epsilon}\right)^{-i\omega/2} H^{(2)}_{i\omega}(((t_{\rm M} - i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.46}$$

$$I(\omega)^* = i\pi e^{-\pi\omega/2}\left(\frac{t_{\rm M} - x^1 + i\epsilon}{t_{\rm M} + x^1 + i\epsilon}\right)^{-i\omega/2} H^{(1)}_{i\omega}(((t_{\rm M} + i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.47}$$

$$I(-\omega)^* = i\pi e^{-\pi\omega/2}\left(\frac{t_{\rm M} - x^1 + i\epsilon}{t_{\rm M} + x^1 + i\epsilon}\right)^{i\omega/2} H^{(1)}_{i\omega}(((t_{\rm M} + i\epsilon)^2 - (x^1)^2)^{1/2}). \tag{B.48}$$

Now, rather than displaying the complete expression for $\phi^{\mathrm{M}}(t_{\mathrm{M}}, x^1, x)$, it should suffice to demonstrate that the coefficient of $a^{\mathrm{R}}_{k\omega}$ vanishes in the $\mathrm{W_L}$ region, as the rest of the calculations are entirely similar.

Let us note that $a^{\mathrm{R}}_{k\omega}$ appears in two places, namely in the $a^{\mathrm{F}}_{k\omega}$ part of $a^{\mathrm{M}}_{kp^1}$ and the $a^{\mathrm{F}\dagger}_{-\omega k}$ part of $a^{\mathrm{M}\dagger}_{kp^1}$. The total contribution for the coefficient of $a^{\mathrm{R}}_{k\omega}$ from these sources is proportional to $-I(\omega)e^{\pi\omega/2} + I(-\omega)^* e^{\pi\omega/2}$.

Consider now the region $\mathrm{W_L}$, where $t_{\mathrm{M}} + x^1 < 0$, $t_{\mathrm{M}} - x^1 > 0$, and, of course, $x_1 < 0$. Thus, apart from the $\pm i\epsilon$, we have $t_{\mathrm{M}}^2 - (x^1)^2 = -z_{\mathrm{L}}^2 < 0$ and we must choose the square root branch for the quantity $(-z^2)^{1/2}$. (Since $z_{\mathrm{L}} = z_{\mathrm{R}}$, we denote it by z for simplicity hereafter.) As a concrete choice, let us take the branch cut to be along $[-\infty, 0]$ in the z plane. This means that $(-z^2 \pm i\delta)^{1/2} = \pm iz$ for small positive δ.

First consider the region of $\mathrm{W_L}$ where $t_{\mathrm{M}} > 0$. Then we have $\delta = \epsilon t_{\mathrm{M}}$ and in the expressions of $-I(\omega)$ and $I(-\omega)^*$ we have, respectively, $H^{(2)}_{i\omega}(-iz)$ and $H^{(1)}_{i\omega}(iz)$. In this case, from the formula in Eq. (10.11.5) of Ref. [48], we have

$$H^{(2)}_{i\omega}(e^{-i\pi} iz) = -e^{-\pi\omega} H^{(1)}_{i\omega}(iz). \tag{B.49}$$

Using this relation, it is easy to see that $-I(\omega) + I(-\omega)^* = 0$, and the coefficient of $a^{\mathrm{R}}_{i\omega}$ vanishes in $\mathrm{W_L}$, as desired.

Next, consider the region in $\mathrm{W_L}$ where $t_{\mathrm{M}} < 0$. Then, we have instead $H^{(2)}_{i\omega}(iz)$ for $-I(\omega)$ and $H^{(1)}_{i\omega}(-iz)$ for $I(-\omega)^*$. Then, again from Eq. (10.11.5) of Ref. [48], we have

$$H^{(1)}_{i\omega}(e^{i\pi} iz) = -e^{\pi\omega} H^{(2)}_{i\omega}(iz), \tag{B.50}$$

and $-I(\omega)$ and $I(-\omega)^*$ cancel with each other in this case as well.

Combining, we have shown that $a^{\mathrm{R}}_{\omega k}$ does not contribute in the expansion in the region $\mathrm{W_L}$.

Appendix C. Poincaré algebra for the various observers

Appendix C.1. Proof of the Poincaré algebra in the W_{F} frame

In this appendix we shall demonstrate that the generators M^{F}_{01}, H^{F}, and P^{F}_1 constructed in Eqs. (2.97), (2.101), and (2.102) form the Poincaré algebra.

First, consider the commutator $[H^{\mathrm{F}}, M^{\mathrm{F}}_{01}]$. This can be computed as

$$\begin{aligned}
[H^{\mathrm{F}}, M^{\mathrm{F}}_{01}] &= -\int d^2k'\, |k'| \int d^{d-2}k \int d\omega' d\omega [a^{\mathrm{F}\dagger}_{k\omega'} \cos\left(\frac{d}{d\omega'}\right) a^{\mathrm{F}}_{k\omega'}, \omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}] \\
&= -\int d^2k\, |k| \int d\omega \omega a^{\mathrm{F}\dagger}_{k\omega} \left(\cos\left(\frac{d}{d\omega}\right)(\omega a^{\mathrm{F}}_{k\omega}) - \omega \cos\left(\frac{d}{d\omega}\right) a^{\mathrm{F}}_{k\omega}\right) \\
&= \int d^2k\, |k| \int d\omega a^{\mathrm{F}\dagger}_{k\omega} \sin\left(\frac{d}{d\omega}\right) a^{\mathrm{F}}_{k\omega} \\
&= iP^{\mathrm{F}}_1,
\end{aligned} \tag{C.1}$$

where in the second line we used the simple identity

$$\left(\frac{d}{d\omega}\right)^n (\omega a_\omega) = n \left(\frac{d}{d\omega}\right)^{n-1} a_\omega + \omega \left(\frac{d}{d\omega}\right)^n a_\omega. \tag{C.2}$$

In an entirely similar manner, with cos and sin interchanged, $[P^{\mathrm{F}}_1, M^{\mathrm{F}}_{01}] = iH^{\mathrm{F}}$ can be shown.

Finally, the fact that $[H^{\mathrm{F}}, P_1^{\mathrm{F}}]$ vanishes can be checked as

$$\begin{aligned}[H^{\mathrm{F}}, P_1^{\mathrm{F}}] &= -i\int d^{d-2}k'\ |k'| \int d^{d-2}k|k| \int d\omega' d\omega [a^{\mathrm{F}\dagger}_{k\omega'} \cos\left(\frac{d}{d\omega'}\right) a^{\mathrm{F}}_{k\omega'}, a^{\mathrm{F}\dagger}_{k\omega} \sin\left(\frac{d}{d\omega}\right) a^{\mathrm{F}}_{k\omega}] \\ &= -i\int d^{d-2}k\ |k|^2 \int d\omega\omega a^{\mathrm{F}\dagger}_{k\omega}\left(\sin\left(\frac{d}{d\omega}\right)\cos\left(\frac{d}{d\omega}\right) a^{\mathrm{F}}_{k\omega} - \cos\left(\frac{d}{d\omega}\right)\sin\left(\frac{d}{d\omega}\right) a^{\mathrm{F}}_{k\omega}\right) \\ &= 0. \end{aligned} \tag{C.3}$$

Appendix C.2. Poincaré generators for W_{F} by the unitary transformation

Recall that the unitary transformation U_{F} defined by

$$U_{\mathrm{F}} = e^{-\frac{i\pi}{2}A}, \qquad A = \frac{1}{2}\int_{-\infty}^{\infty} d\omega a^{\mathrm{F}\dagger}_{k\omega}(-\frac{d^2}{d\omega^2} + \omega^2 - 1) a^{\mathrm{F}}_{k\omega} \tag{C.4}$$

converts the mode operator $a^{\mathrm{F}}_{k\omega}$ into $a^{\mathrm{M}}_{kp^1}$ in the manner

$$U_{\mathrm{F}} a^{\mathrm{F}}_{k\omega} U_{\mathrm{F}}^\dagger = i\sqrt{E_{kp^1}}\, a^{\mathrm{M}}_{kp^1}. \tag{C.5}$$

As an application of this operation, let us show that it transforms M^{F}_{01} into M_{01}, namely

$$U_{\mathrm{F}}\left(\int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}\right) U_{\mathrm{F}}^\dagger = i\int d^2k \int_{-\infty}^{\infty} dp^1\, E_{kp^1}\, a^{\mathrm{M}\dagger}_{kp^1} \frac{\partial}{\partial p_1} a^{\mathrm{M}}_{kp^1}. \tag{C.6}$$

First, expand the unitary transformation as the sum of multiple commutators in the usual way:

$$U_{\mathrm{F}} M^{\mathrm{F}}_{01} U_{\mathrm{F}}^\dagger = M^{\mathrm{F}}_{01} - \frac{i\pi}{2}[A, M^{\mathrm{F}}_{01}] + \frac{1}{2!}\left(-\frac{i\pi}{2}\right)^2 [A, [A, M^{\mathrm{F}}_{01}]] + \cdots \tag{C.7}$$

The single commutator can be computed as

$$\begin{aligned}[A, M^{\mathrm{F}}_{01}] &= [A, \int d^2k \int_{-\infty}^{\infty} d\omega\omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}] \\ &= \frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' [a^{\mathrm{F}\dagger}_{k\omega'}\left(-\frac{d^2}{d\omega'^2}\right) a^{\mathrm{F}}_{k\omega'}, \omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}] \\ &= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \left[a^{\mathrm{F}\dagger}_{k\omega}\left(\frac{d^2}{d\omega^2}\right)\omega a^{\mathrm{F}}_{k\omega} - \omega a^{\mathrm{F}\dagger}_{k\omega}\frac{d^2}{d\omega^2} a^{\mathrm{F}}_{k\omega}\right] \\ &= -\int d^2k \int_{-\infty}^{\infty} d\omega a^{\mathrm{F}\dagger}_{k\omega}\frac{d}{d\omega} a^{\mathrm{F}}_{k\omega}. \end{aligned} \tag{C.8}$$

Based on this result, the double commutator is calculated as

$$\begin{aligned}[A, [A, M^{\mathrm{F}}_{01}]] &= [A, -\int d^2k \int_{-\infty}^{\infty} d\omega a^{\mathrm{F}\dagger}_{k\omega}\frac{d}{d\omega} a^{\mathrm{F}}_{k\omega}] \\ &= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' (\omega'^2 - 1)[a^{\mathrm{F}\dagger}_{k\omega'} a^{\mathrm{F}}_{k\omega'}, a^{\mathrm{F}\dagger}_{k\omega}\frac{d}{d\omega} a^{\mathrm{F}}_{k\omega}] \\ &= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega [(\omega^2 - 1) a^{\mathrm{F}\dagger}_{k\omega}\frac{d}{d\omega} a^{\mathrm{F}}_{k\omega} - a^{\mathrm{F}\dagger}_{k\omega}\frac{d}{d\omega}(\omega^2 - 1) a^{\mathrm{F}}_{k\omega}] \\ &= \int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a^{\mathrm{F}\dagger}_{k\omega} a^{\mathrm{F}}_{k\omega}. \end{aligned} \tag{C.9}$$

Since this is in the original form of $M^{\rm F}_{01}$, we see that the rest of the multiple commutators produce $\int d^2k \int_{-\infty}^{\infty} d\omega a^{\rm F\dagger}_{k\omega} \frac{d}{d\omega} a^{\rm F}_{k\omega}$ and $\int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a^{\rm F\dagger}_{k\omega} a^{\rm F}_{k\omega}$ alternately. The coefficients can be easily found in such a way that the series sum to

$$
\begin{aligned}
U_{\rm F} M^{\rm F}_{01} U^\dagger_{\rm F} &= \cos\frac{\pi}{2} \int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a^{\rm F\dagger}_{k\omega} a^{\rm F}_{k\omega} + i \sin\frac{\pi}{2} \int d^2k \int_{-\infty}^{\infty} d\omega\, a^{\rm F\dagger}_{k\omega} \frac{d}{d\omega} a^{\rm F}_{k\omega} \\
&= i \int d^2k \int_{-\infty}^{\infty} d\omega\, a^{\rm F\dagger}_{k\omega} \frac{d}{d\omega} a^{\rm F}_{k\omega}. \qquad \text{(C.10)}
\end{aligned}
$$

Making the replacements $\omega \to p^1$ and $a^{\rm F}_{k\omega} \to \sqrt{E_{kp^1}} a^{\rm M}_{kp^1}$, we obtain the desired result:

$$
U_{\rm F} M^{\rm F}_{01} U^\dagger_{\rm F} = i \int d^2k \int dp^1\, E_{kp^1}\, a^{\rm M\dagger}_{kp^1} \frac{\partial}{\partial p_1} a^{\rm M}_{kp_1} = M_{01}. \qquad \text{(C.11)}
$$

Appendix D. Quantization in different Lorentz frames with an almost lightlike boundary condition

Here we supply some details of the quantization in different Lorentz frames with a slightly timelike boundary condition as discussed in Sect. 4.2.4.

What we shall describe are the computations of the two terms in Eqs. (4.32) and (4.33) which constitute the commutator $\left[\tilde\pi(\tilde t, \tilde x^1, \Omega), \phi(\tilde t, \tilde y^1, \Omega')\right]$ given in Eq. (4.31). For the convenience of the reader let us display them again:

$$
C_1 = -\gamma \widetilde{\mathcal{N}}^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty \frac{d\hat p^1}{4\pi} Y_{lm}(\Omega) Y^*_{lm}(\Omega') \left(e^{i\beta \hat E_{k_l \hat p^1} (\hat x^1 - \hat y^1)} + \text{h.c.} \right) \sin \hat p^1 \hat x^1 \, \sin \hat p^1 \hat y^1, \quad \text{(D.1)}
$$

$$
C_2 = -i\gamma\beta \widetilde{\mathcal{N}}^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty \frac{d\hat p^1}{4\pi \hat E_{k_l \hat p^1}} Y_{lm}(\Omega) Y^*_{lm}(\Omega') \left(e^{i\beta \hat E_{k_l \hat p^1} (\hat x^1 - \hat y^1)} - \text{h.c.} \right) \cos \hat p^1 \hat x^1 \, \sin \hat p^1 \hat y^1. \qquad \text{(D.2)}
$$

First, the sum over m can be performed by the addition theorem for Y_{lm} as already described in Eq. (4.34). Next, we perform the integral over $\hat p^1$. Although the energy dependence in the exponent does not disappear at equal $\tilde t$, in contrast to the case for the frame $(\hat t, \hat x^1)$, such an integral can be performed, after expressing the product of trigonometric functions into a sum like $\sin \hat p^1 \hat x^1 \sin \hat p^1 \hat y^1 = \frac{1}{2}\left(\cos \hat p^1 (\hat x^1 - \hat y^1) - \cos \hat p^1 (\hat x^1 + \hat y^1)\right)$. The relevant formulas were given in Eqs. (4.35) and (4.36), with appropriate regularizations in Eqs. (4.37) and (4.38) for convergence.

Then, the result for $C_1 + C_2$ takes the form

$$
\begin{aligned}
C_1 + C_2 = &-i \frac{\beta\gamma \widetilde{\mathcal{N}}^2}{8\pi} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat n \cdot \hat{n'}) \\
&\times \Bigg\{ -\frac{a_- + i\eta}{\sqrt{a_-^2 + b_-^2}} K_1\left(k_l \sqrt{a_-^2 + b_-^2}\right) + \frac{a_- - i\eta}{\sqrt{a_-^2 + b_+^2}} K_1\left(k_l \sqrt{a_-^2 + b_+^2}\right) \\
&\quad + \frac{a_- + i\eta}{\sqrt{a_+^2 + b_-^2}} K_1\left(k_l \sqrt{a_+^2 + b_-^2}\right) - \frac{a_- - i\eta}{\sqrt{a_+^2 + b_+^2}} K_1\left(k_l \sqrt{a_+^2 + b_+^2}\right)
\end{aligned}
$$

$$+\frac{a_+}{\sqrt{a_+^2+b_-^2}}K_1(k_l\sqrt{a_+^2+b_-^2})-\frac{a_+}{\sqrt{a_+^2+b_+^2}}K_1(k_l\sqrt{a_+^2+b_+^2})$$
$$-\frac{a_-}{\sqrt{a_-^2+b_-^2}}K_1(k_l\sqrt{a_-^2+b_-^2})+\frac{a_-}{\sqrt{a_-^2+b_+^2}}K_1(k_l\sqrt{a_-^2+b_+^2})\Bigg\}, \tag{D.3}$$

where

$$a_\pm \equiv \hat{x}^1 \pm \hat{y}^1, \qquad b_\pm \equiv \pm i\beta(\hat{x}^1 - \hat{y}^1 \mp i\eta). \tag{D.4}$$

Consider first the four terms in the third and the fourth lines, which contain a_+^2 in the square roots of the denominator and in the argument of the K_1 functions. Since $\hat{x}^1$ and $\hat{y}^1$ are positive, a_+ is positive and generically finite. Therefore, we can ignore η for these terms. Then, $b_+^2 = b_-^2$ and hence the two terms in the third line cancel and similarly the two terms in the fourth line cancel. Therefore these four terms actually do not contribute and we can simplify $C_1 + C_2$ to

$$C_1 + C_2 = -i\frac{\beta\gamma\widetilde{\mathcal{N}}^2}{8\pi}\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\hat{n}\cdot\hat{n'})$$
$$\times\Bigg\{-\frac{a_-+i\eta}{\sqrt{a_-^2+b_-^2}}K_1(k_l\sqrt{a_-^2+b_-^2})+\frac{a_--i\eta}{\sqrt{a_-^2+b_+^2}}K_1(k_l\sqrt{a_-^2+b_+^2})$$
$$-\frac{a_-}{\sqrt{a_-^2+b_-^2}}K_1(k_l\sqrt{a_-^2+b_-^2})+\frac{a_-}{\sqrt{a_-^2+b_+^2}}K_1(k_l\sqrt{a_-^2+b_+^2})\Bigg\}. \tag{D.5}$$

To analyze this expression we must distinguish two regions:

(1) If a_-^2 is finite, then we can again ignore η and these four terms cancel in exactly the same fashion.
(2) Thus, a non-vanishing result can possibly be obtained if and only if $|a_-| \lesssim \eta$. In such a case, since a_- and $b_\pm$ are of the order of η, as long as k_l is not infinite, we can use the approximation $K_1(z) \simeq 1/z$ and hence each term diverges like $1/\eta$.

Combining, this shows that the sum of terms containing the K_1 function behaves precisely like $\sim\ \delta(\hat{x}^1 - \hat{y}^1)$. The rest of the argument is already given in the main text and the commutator $\left[\tilde{\pi}(\tilde{t},\tilde{x}^1,\Omega),\phi(\tilde{t},\tilde{y}^1,\Omega')\right]$ in the frame of an arbitrary FFO correctly behaves like the product of appropriate δ-functions.

Thus, we can write

$$\left[\tilde{\pi}(\tilde{t},\tilde{x}^1,\Omega),\phi(\tilde{t},\tilde{y}^1,\Omega')\right] = C_1 + C_2 = -iF\gamma\delta(\hat{x}^1-\hat{y}^1)\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'), \tag{D.6}$$

where we used the relation $\delta(\tilde{x}^1 - \tilde{y}^1) = \gamma\delta(\hat{x}^1 - \hat{y}^1)$ valid at equal $\tilde{t}$. F is a constant, which we want to set to unity by adjusting the normalization constant $\widetilde{\mathcal{N}}$. To find such an $\widetilde{\mathcal{N}}$, we need to carry out the integral $i\int d\hat{x}^1 d\cos\theta d\varphi(C_1 + C_2)$, perform the sum over l, and set the result to 1. This unfortunately is quite difficult and we have not been able to find the form of $\widetilde{\mathcal{N}}$ explicitly.

References

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200] [Search INSPIRE].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109] [Search INSPIRE].

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150] [Search INSPIRE].
[4] A. L. Fitzpatrick, J. Kaplan, and M. T. Walters, J. High Energy Phys. **1511**, 200 (2015) [arXiv:1501.05315 [hep-th]] [Search INSPIRE].
[5] A. L. Fitzpatrick, J. Kaplan, D. Li, and J. Wang, J. High Energy Phys. **1605**, 109 (2016) [arXiv:1603.08925 [hep-th]] [Search INSPIRE].
[6] C. T. Asplund, A. Bernamonti, F. Galli, and T. Hartman, J. High Energy Phys. **1502**, 171 (2015) [arXiv:1410.1392 [hep-th]] [Search INSPIRE].
[7] T. Anous, T. Hartman, A. Rovai, and J. Sonner, J. High Energy Phys. **1607**, 123 (2016) [arXiv:1603.04856 [hep-th]] [Search INSPIRE].
[8] S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[9] A. Kitaev, *Hidden correlations in the Hawking radiation and thermal noise* (University of California, Santa Barbara, 2015) (available at http://online.kitp.ucsb.edu/online/joint98/kitaev/, date last accessed December 31, 2018).
[10] A. Kitaev, *A simple model of quantum holography* (University of California, Santa Barbara, 2015) (available at: http://online.kitp.ucsb.edu/online/entangled15/kitaev/ and http://online.kitp.ucsb.edu/online/entangled15/kitaev2/, date last accessed December 31, 2018).
[11] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).
[12] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).
[13] W. Israel, Phys. Lett. A **57**, 107 (1976).
[14] J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976).
[15] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[16] P. C. W. Davies, J. Phys. A **8**, 609 (1975).
[17] S. A. Fulling, Phys. Rev. D **7**, 2850 (1973).
[18] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, J. High Energy Phys. **1302**, 062 (2013) [arXiv:1207.3123 [hep-th]] [Search INSPIRE].
[19] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford, and J. Sully, J. High Energy Phys. **1309**, 018 (2013) [arXiv:1304.6483 [hep-th]] [Search INSPIRE].
[20] D. Marolf and J. Polchinski, Phys. Rev. Lett. **111**, 171301 (2013) [arXiv:1307.4706 [hep-th]] [Search INSPIRE].
[21] L. Susskind, L. Thorlacius, and J. Uglum, Phys. Rev. D **48**, 3743 (1993) [arXiv:hep-th/9306069] [Search INSPIRE].
[22] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **74**, 066009 (2006) [arXiv:hep-th/0606141] [Search INSPIRE].
[23] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, arXiv:0710.4334 [hep-th] [Search INSPIRE].
[24] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **75**, 106001 (2007) [arXiv:hep-th/0612053] [Search INSPIRE].
[25] I. Heemskerk, D. Marolf, J. Polchinski, and J. Sully, J. High Energy Phys. **1210**, 165 (2012) [arXiv:1201.3664 [hep-th]] [Search INSPIRE].
[26] K. Papadodimas and S. Raju, J. High Energy Phys. **1310**, 212 (2013), [arXiv:1211.6767 [hep-th]] [Search INSPIRE].
[27] K. Papadodimas and S. Raju, Phys. Rev. D **89**, 086010 (2014) [arXiv:1310.6335 [hep-th]] [Search INSPIRE]. 60
[28] K. Papadodimas and S. Raju, Phys. Rev. D **93**, 084049 (2016) [arXiv:1503.08825 [hep-th]] [Search INSPIRE].
[29] D. L. Jafferis, arXiv:1703.01519 [hep-th] [Search INSPIRE].
[30] P. C. Vaidya, Curr. Sci. **12**, 183 (1943).
[31] P. C. Vaidya, Proc. IAS A **33**, 264 (1951).
[32] P. C. Vaidya, Nature **171**, 260 (1953).
[33] J. Louko and D. Marolf, Phys. Rev. D **58**, 024007 (1998) [arXiv:gr-qc/9802068] [Search INSPIRE].
[34] D. G. Boulware, Phys. Rev. D **11**, 1404 (1975).
[35] N. B. Narozhny, A. M. Fedotov, B. M. Karnakov, V. D. Mur, and V. A. Belinskii, Phys. Rev. D **65**, 025004 (2001) [arXiv:hep-th/9906181] [Search INSPIRE].
[36] M. Blasone, G. Lambiase, and G. G. Luciano, J. Phys.: Conf. Ser. **631**, 012053 (2015).
[37] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, Rev. Mod. Phys. **80**, 787 (2008) [arXiv:0710.5373 [gr-qc]] [Search INSPIRE].
[38] R. D. Richtmyer, *Principles of Advanced Mathematical Physics* (Academic Press, New York, 1975).

[39] V. Namias, J. Inst. Math. Appl. **25**, 241 (1980).
[40] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einsteins's General Relativity* (Cambridge University Press, Cambridge, 2009).
[41] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[42] A. Passian, H. Simpson, S. Kouchekian, and S. B. Yakubovich, J. Math. Anal. Appl. **360**, 380 (2009).
[43] R. Szmytkowski and S. Bielski, J. Math. Anal. Appl. **365**, 195 (2010).
[44] M. I. Kontorovich and N. N. Lebedev, Zh. Eksper. Teor. Fiz. **8**, 1192 (1938).
[45] A. Erdélyi (ed.), *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. 2, p. 175.
[46] T. M. Dunster, SIAM J. Math. Anal. **21**, 995 (1990).
[47] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, ed. A. Jeffery and D. Zwillinger (Academic Press, Orlando, FL, 2007).
[48] W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clar (eds.), *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, 2010).
[49] M. Natsuume and Y. Satoh, Int. J. Mod. Phys. A **13**, 1229 (1998).
[50] K. Goto and T.Takayanagi, J. High Energy Phys. **1710**, 153 (2017) [arXiv:1704.00053 [hep-th]] [Search INSPIRE].
[51] B. Michel, arXiv:1612.03158 [hep-th] [Search INSPIRE].
[52] S. B. Yakubovich, Opusc. Math. **26**, 161 (2006).